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# Two variable subnormal completion problem

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**Abstract.** Given $m \geq 0$ and a finite collection of pairs of positive numbers $\mathcal{C} = \{(\alpha_1(k), \alpha_2(k))\}_{|k|\leq m}$ ($|k| := k_1 + k_2$). The two variable subnormal completion problem is to find necessary and sufficient conditions to guarantee the existence of a two variable subnormal weighted shift whose initial weighted are given by $\mathcal{C}$. Curto and Fialkow solved this problem in the case $m = 1$, using the solution of truncated complex moment problem. This paper obtained the representing measure in detail.

*Key words*: two variable weighted shift, subnormal completion problem, truncated complex moment problem.

## 1. Introduction and Preliminaries

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. For $S, T \in \mathcal{L}(\mathcal{H})$, we let $[S, T] := ST - TS$; $[S, T]$ is the *commutator* of $S$ and $T$. For $n \geq 1$ we let $\mathcal{H}^{(n)}$ denote the orthogonal direct sum of $\mathcal{H}$ with itself $n$ times. Given an $n$-tuple $\mathbb{T} = (T_1, \ldots, T_n)$ of operators on $\mathcal{H}$, we let $[\mathbb{T}^*, \mathbb{T}] \in \mathcal{L}(\mathcal{H}^{(n)})$ denote the *self-commutator* of $\mathbb{T}$, defined by $[\mathbb{T}^*, \mathbb{T}]_{ij} := [T_j^*, T_i]$ $(1 \leq i,\ j \leq n)$. For instance, if $n = 2$,

$$[\mathbb{T}^*, \mathbb{T}] = \begin{pmatrix} [T_1^*, T_1] & [T_2^*, T_1] \\ [T_1^*, T_2] & [T_2^*, T_2] \end{pmatrix}.$$

In analogy with the case $n = 1$, we shall say that $\mathbb{T}$ is *strongly hyponormal* (or simply *hyponormal*) if $([\mathbb{T}^*, \mathbb{T}]x, x) \geq 0$ for all $x \in \mathcal{H}^{(n)}$. Recall that $\mathbb{T}$ is said to be *normal* if $\mathbb{T}$ is commuting and each $T_i$ is normal operator. An $n$-tuple $\mathbb{S} = (S_1, \ldots, S_n)$ is *subnormal* if $\mathbb{S}$ is the restriction of a normal $n$-tuple to a common invariant subspace.

Let $\mathbb{Z}_+^2 := \mathbb{Z}_+ \times \mathbb{Z}_+$ and let $l^2(\mathbb{Z}_+^2)$ be the Hilbert space of square summable complex sequences indexed by $\mathbb{Z}_+^2$. For $\alpha = (\alpha_1, \alpha_2) \in l^2(\mathbb{Z}_+^2)$, let $W_\alpha \equiv (W_{\alpha_1}, W_{\alpha_2})$ be the associated 2-variable weighted shift, acting on $l^2(\mathbb{Z}_+^2)$ as follows:

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$$W_{\alpha_i} e_k := \alpha_i(k) e_{k+\epsilon_i} \quad (k \in \mathbb{Z}_+^2,\ i = 1, 2),$$

where $\alpha_i(k) > 0$ for all $k \in \mathbb{Z}_+^2$, $i = 1, 2$, and $\{e_k\}_{k \in \mathbb{Z}_+^2}$ is the canonical orthonormal basis for $l^2(\mathbb{Z}_+^2)$, $\epsilon_1 := (1, 0)$ and $\epsilon_2 := (0, 1)$. Assume that $W_\alpha$ is commuting, i.e., $\alpha_i(k + \epsilon_j)\alpha_j(k) = \alpha_j(k + \epsilon_i)\alpha_i(k)$ for all $k \in \mathbb{Z}_+^2$, $i, j = 1, 2$. By Theorem 6.1 of [Cu], we have the following

**Lemma 1.1** *$W_\alpha$ is hyponormal if and only if*
(1) $\alpha_1(k + \epsilon_1) \geq \alpha_1(k)$,
(2) $\alpha_2(k + \epsilon_2) \geq \alpha_2(k)$,
(3) $\left(\alpha_1^2(k + \epsilon_1) - \alpha_1^2(k)\right)\left(\alpha_2^2(k + \epsilon_2) - \alpha_2^2(k)\right) \geq \left(\alpha_1(k + \epsilon_2)\alpha_2(k + \epsilon_1) - \alpha_1(k)\alpha_2(k)\right)^2$,
*for all $k \in \mathbb{Z}_+^2$.*

We now define

$$\tilde{\gamma}_k := \begin{cases} 1 & \text{if } \ k = (0, 0), \\ \alpha_1^2(0, 0) \cdot \ldots \cdot \alpha_1^2(k_1 - 1, 0) & \text{if } \ k_1 \geq 1,\ k_2 = 0, \\ \alpha_2^2(0, 0) \cdot \ldots \cdot \alpha_2^2(0, k_2 - 1) & \text{if } \ k_1 = 0,\ k_2 \geq 1, \\ \alpha_1^2(0, 0) \cdot \ldots \cdot \alpha_1^2(k_1 - 1, 0)\alpha_2^2(k_1, 0) \cdot \ldots \cdot \alpha_2^2(k_1, k_2 - 1) & \\ & \text{if } \ k_1, k_2 \geq 1. \end{cases}$$

For a positive finite weight sequence $\alpha$, if there exists a positive infinite sequence $\hat{\alpha}$ whose initial weights are $\alpha$, then we call $\hat{\alpha}$ is a *completion* of $\alpha$, or we say $\alpha$ has a completion. If $\hat{\alpha}$ is subnormal (or hyponormal), then we say $\alpha$ has a subnormal (or hyponormal) completion.

**Generalized Berger Theorem** ([CF4]) *$W_\alpha$ is subnormal if and only if there exists a compactly supported positive Borel measure $\mu$ on $\mathbb{R}_+^2$ such that*

$$\int t^k d\mu(t) := \int t_1^{k_1} t_2^{k_2} d\mu(t_1, t_2) = \tilde{\gamma}_k \quad (k \in \mathbb{Z}_+^2).$$

**Two Variable Subnormal Completion Problem** ([CF4]) *Given $m \geq 0$ and a finite collection of pairs of positive numbers $C = \{\alpha(k) \equiv (\alpha_1(k), \alpha_2(k))\}_{|k| \leq m}$ ($|k| := k_1 + k_2$), find necessary and sufficient conditions to guarantee the existence of a two variable subnormal weighted shift whose initial weights are given by $C$.*

Given a closed subset $K \subseteq \mathbb{C}$ and a doubly indexed finite sequence of complex numbers

$$\gamma : \gamma_{(0,0)}, \gamma_{(0,1)}, \gamma_{(1,0)}, \gamma_{(0,2)}, \gamma_{(1,1)}, \gamma_{(2,0)}, \cdots, \\ \gamma_{(0,2n)}, \gamma_{(1,2n-1)}, \cdots, \gamma_{(2n-1,1)}, \gamma_{(2n,0)}, \tag{1.1}$$
$$\text{where} \quad \gamma_{(0,0)} > 0 \quad \text{and} \quad \gamma_{(j,i)} = \overline{\gamma_{(i,j)}},$$

the *truncated K complex moment problem* entails finding a positive Borel measure $\mu$ such that

$$\gamma_{(i,j)} = \int \bar{z}^i z^j \, d\mu \quad (0 \le i+j \le 2n) \quad \text{and} \quad \operatorname{supp}\mu \subseteq K. \tag{1.2}$$

Any sequence $\gamma$ as in (1.1) is a *truncated moment sequence* and any measure $\mu$ as in (1.2) is a *representing measure* for $\gamma$.

For $n \ge 1$, let $m \equiv m(n) = (n+1)(n+2)/2$. For $A \in M_m(\mathbb{C})$ (the $m \times m$ complex matrices), we denote the successive rows and columns according to the following lexicographic-functional ordering: $\mathit{1}, Z, \bar{Z}, Z^2, \bar{Z}Z, \bar{Z}^2, \ldots, Z^n, \ldots, \bar{Z}^n$; rows and columns indexed by $\mathit{1}, Z, Z^2, \ldots, Z^n$ are said to be *analytic*. For the truncated moment sequence (1.1), we define $M(n)(\gamma) \in M_m(\mathbb{C})$ as follows: for $0 \le i+j \le n$, $0 \le l+k \le n$, the entry in row $\bar{Z}^l Z^k$ and column $\bar{Z}^i Z^j$ is

$$M(n)_{(l,k)(i,j)} = \gamma_{(i+k,j+l)}. \tag{1.3}$$

For example, if $n = 1$, the *quadratic moment problem* for $\gamma : \gamma_{(0,0)}, \gamma_{(0,1)}, \gamma_{(1,0)}, \gamma_{(0,2)}, \gamma_{(1,1)}, \gamma_{(2,0)}$ corresponds to

$$M(1) = \begin{pmatrix} \gamma_{(0,0)} & \gamma_{(0,1)} & \gamma_{(1,0)} \\ \gamma_{(1,0)} & \gamma_{(1,1)} & \gamma_{(2,0)} \\ \gamma_{(0,1)} & \gamma_{(0,2)} & \gamma_{(1,1)} \end{pmatrix},$$

and if $n = 2$, the *quartic moment problem* for $\gamma : \gamma_{(0,0)}, \gamma_{(0,1)}, \gamma_{(1,0)}, \gamma_{(0,2)}, \gamma_{(1,1)}, \gamma_{(2,0)}, \gamma_{(0,3)}, \gamma_{(1,2)}, \gamma_{(2,1)}, \gamma_{(3,0)}, \gamma_{(0,4)}, \gamma_{(1,3)}, \gamma_{(2,2)}, \gamma_{(3,1)}, \gamma_{(4,0)}$ corresponds to

$$M(2) = \begin{pmatrix} \gamma_{(0,0)} & \gamma_{(0,1)} & \gamma_{(1,0)} & \gamma_{(0,2)} & \gamma_{(1,1)} & \gamma_{(2,0)} \\ \gamma_{(1,0)} & \gamma_{(1,1)} & \gamma_{(2,0)} & \gamma_{(1,2)} & \gamma_{(2,1)} & \gamma_{(3,0)} \\ \gamma_{(0,1)} & \gamma_{(0,2)} & \gamma_{(1,1)} & \gamma_{(0,3)} & \gamma_{(1,2)} & \gamma_{(2,1)} \\ \gamma_{(2,0)} & \gamma_{(2,1)} & \gamma_{(3,0)} & \gamma_{(2,2)} & \gamma_{(3,1)} & \gamma_{(4,0)} \\ \gamma_{(1,1)} & \gamma_{(1,2)} & \gamma_{(2,1)} & \gamma_{(1,3)} & \gamma_{(2,2)} & \gamma_{(3,1)} \\ \gamma_{(0,2)} & \gamma_{(0,3)} & \gamma_{(1,2)} & \gamma_{(0,4)} & \gamma_{(1,3)} & \gamma_{(2,2)} \end{pmatrix}. \tag{1.4}$$

The quadratic moment problem was solved completely. In fact, it was shown that $\gamma$ has a representing measure if and only if $M(1) \geq 0$ [CF4, Theorem 6.1]. But the quartic moment problem hasn't solved completely, so far (see [CF1-4], [JLLL], [Li] etc).

A solution to the complex moment problem can provide a solution to the two variable subnormal completion problem ([CF4]). First, let $\mathbb{C}[t_1, t_2]_{m+1}$ be the set of complex polynomials in $t_1$ and $t_2$ of total degree at most $m+1$, and let $\tilde{\phi}$ be the complex linear functional on $\mathbb{C}[t_1, t_2]_{m+1}$ induced by $\tilde{\gamma} := \{\tilde{\gamma}_{|k| \leq m+1}\}$, i.e., $\tilde{\phi}(t_1^{k_1} t_2^{k_2}) := \tilde{\gamma}_{(k_1, k_2)}$, $0 \leq i+j \leq m+1$, define $\gamma_{(i,j)} := \tilde{\phi}((t_1 - it_2)^i (t_1 + it_2)^j)$. Then there exists a subnormal completion for $\mathcal{C}$ if and only if the associated truncated complex moment problem for $\{\gamma_k\}_{|k| \leq m+1}$ admits a solution. Curto and Fialkow solved this problem in the case $m = 1$ ([CF4]), using the solution of truncated complex moment problem. This paper obtained the representing measure in detail.

All of the calculations in this paper were obtained with the help of the software tool *Mathematica* [Wol].

## 2. Subnormality of $\mathcal{C}$ and the representing measure

First, if $m = 1$, for convenience, let $\mathcal{C} = \{(\sqrt{a}, \sqrt{b}), (\sqrt{c}, \sqrt{d}), (\sqrt{e}, \sqrt{f})\}$ where $a$, $b$, $c$, $d$, $e$, $f$ are positive numbers with $bc = af$. In this case we have

$$\begin{aligned} &\tilde{\gamma}_{(0,0)} = 1, \quad &&\tilde{\gamma}_{(1,0)} = a, \quad &&\tilde{\gamma}_{(0,1)} = b, \\ &\tilde{\gamma}_{(2,0)} = ae, \quad &&\tilde{\gamma}_{(1,1)} = af, \quad &&\tilde{\gamma}_{(0,2)} = bd. \end{aligned}$$

Now we can define

$$\begin{aligned} &\gamma_{(0,0)} = 1, \qquad &&\gamma_{(0,2)} = ae - bd + 2afi, \\ &\gamma_{(0,1)} = a + bi, \qquad &&\gamma_{(1,1)} = ae + bd, \\ &\gamma_{(1,0)} = a - bi, \qquad &&\gamma_{(2,0)} = ae - bd - 2afi. \end{aligned}$$

Hence,

$$M(1) = \begin{pmatrix} 1 & a+bi & a-bi \\ a-bi & ae+bd & ae-bd-2afi \\ a+bi & ae-bd+2afi & ae+bd \end{pmatrix}.$$

Using $\{\gamma_{(i,j)}\}_{0 \leq i+j \leq 2}$ as a data, assume that a compacted representing measure $\nu$ has been found, i.e.,

$$\int \bar{z}^i z^j d\nu(z,\bar{z}) = \gamma_{(i,j)} \quad (0 \le i+j \le 2).$$

Let $d\mu(t_1, t_2) := d\nu(t_1 + t_2 i, t_1 - t_2 i)$. It is easy to show that the following result.

**Proposition 2.1** *$\mu$ is a compactly supported positive Borel measure on $\mathbb{R}_+^2$ which interpolates $\tilde{\gamma}$.*

The solution of quadratic moment problem implies that $\mathcal{C}$ has a subnormal completion if and only if $M(1) \ge 0$. By Lemma 1.1, we first have the following

**Proposition 2.2** *Given $\mathcal{C} = \{(\sqrt{a}, \sqrt{b}), (\sqrt{c}, \sqrt{d}), (\sqrt{e}, \sqrt{f})\}$, where $a$, $b$, $c$, $d$, $e$, $f$ are positive numbers with $bc = af$. Then $\mathcal{C}$ has a hyponormal completion if and only if*

$$a \le e, \quad b \le d, \quad b(a-c)^2 \le a(d-b)(e-a). \tag{2.1}$$

In the sequel, we thus assume that the condition (2.1) is hold. For a positive $n \times n$ matrix $A$, we denote by $[A]_k$ $(k \le n)$ the compression of $A$ to the first $k$ rows and columns. We want to know the representing measure. Let $r$ be the rank of matrix $M(1)$.

**Theorem 2.3** *If $r = 1$, then the representing measure of $\mathcal{C}$ is $\mu = \delta_{(a,b)}$.*

*Proof.* If $r = 1$, then $\det M(1) = a(e-a)(d-b) - b(a-c) = 0$, and $\det[M(1)]_2 = a(e-a) + b(d-b) = 0$. Hence $a = c = e$, $b = d$. Since $bc = af$, we also have $b = d = f$. Hence $\mathcal{C} = \{(\sqrt{a}, \sqrt{b}), (\sqrt{a}, \sqrt{b}), (\sqrt{a}, \sqrt{b})\}$. So, $\nu = \gamma_{(0,0)} \delta_{\frac{\gamma_{(0,1)}}{\gamma_{(0,0)}}}$. Thus, $\mu = \delta_{(a,b)}$. □

If $r = 2$, then $\det M(1) = a(e-a)(d-b) - b(a-c) = 0$, and $\det[M(1)]_2 = a(e-a) + b(d-b) > 0$. Hence $e > a$ or $d > b$. Thus we have three cases.

(1) $e > a$ and $d = b$; It implies $a = c$ and $b = d = f$. Hence $\mathcal{C} = \{(\sqrt{a}, \sqrt{b}), (\sqrt{a}, \sqrt{b}), (\sqrt{e}, \sqrt{b})\}$.

(2) $e = a$ and $d > b$; It implies $a = c = e$ and $b = f$. Hence $\mathcal{C} = \{(\sqrt{a}, \sqrt{b}), (\sqrt{a}, \sqrt{d}), (\sqrt{a}, \sqrt{b})\}$.

(3) $e > a$ and $d > b$.

We first consider the cases (1) and (2).

**Theorem 2.4** *$\mathcal{C} = \{(\sqrt{a}, \sqrt{b}), (\sqrt{a}, \sqrt{b}), (\sqrt{e}, \sqrt{b})\}$ admits a subnormal completion if $e > a$, and the representing measure is*

$$\mu = \frac{a(e-a)}{(x-a)^2 + a(e-a)}\delta_{(x,b)} + \frac{(x-a)^2}{(x-a)^2 + a(e-a)}\delta_{\left(\frac{a(x-e)}{x-a},b\right)}$$

*(for any* $x \neq a$*).*

**Theorem 2.5** $\mathcal{C} = \{(\sqrt{a}, \sqrt{b}), (\sqrt{a}, \sqrt{d}), (\sqrt{a}, \sqrt{b})\}$ *admits a subnormal completion if* $d > b$*, and the representing measure is*

$$\mu = \frac{b(d-b)}{(y-b)^2 + b(d-b)}\delta_{(a,y)} + \frac{(y-b)^2}{(y-b)^2 + b(d-b)}\delta_{\left(a,\frac{b(y-d)}{y-b}\right)}$$

*(for any* $y \neq b$*).*

The proofs of Theorem 2.4 and Theorem 2.5 are similar to the proof of the following Theorem 2.6, which is the case (3).

**Theorem 2.6** *If* $r = 2$ *and* $a < e$, $b < d$*, then the representing measure of* $\mathcal{C}$ *is*

$$\mu := \rho_0\delta_{(x,y)} + \rho_1\delta_{(s,t)},$$

*where*

$$b(d-b)x + a(b-f)y = ab(d-f), \tag{2.2}$$

$$s = \frac{(a-x)(ae-bd) + 2af(b-y) - x(a^2+b^2) + a(x^2+y^2)}{(a-x)^2 + (b-y)^2}, \tag{2.3}$$

$$t = \frac{(y-b)(ae-bd) + 2af(a-x) - y(a^2+b^2) + b(x^2+y^2)}{(a-x)^2 + (b-y)^2}, \tag{2.4}$$

$$\rho_0 = \frac{(s-a)(s-x) + (t-b)(t-y)}{(x-s)^2 + (y-t)^2}, \tag{2.5}$$

$$\rho_1 = \frac{(x-a)(x-s) + (y-b)(y-t)}{(x-s)^2 + (y-t)^2}. \tag{2.6}$$

*Proof.* If $r = 2$, then there exist $\alpha, \beta \in \mathbb{C}$ such that $\bar{Z} = \alpha \mathit{1} + \beta Z$. In fact,

$$\alpha = \frac{2ab((d-f) + (c-e)i)}{a(e-a) + b(d-b)} \quad \text{and}$$

$$\beta = \frac{(b^2 - a^2 + ae - bd) + 2a(b-f)i}{a(e-a) + b(d-b)}.$$

We must take an atom $z_0$ of the representing measure $\nu$ on the line $\overline{z} = \alpha + \beta z$ with $z_0 \neq a + bi$. If we let $z_0 = x + yi$, then we have (2.2). If we let the another atom of the representing measure $\nu$ is $z_1 = s + ti$, then we have (2.3), (2.4). Finally we can obtain the densities (2.5), (2.6). $\square$

**Example 2.7** Let $a = 1$, $b = 1$, $d = 2$, $e = 2$. If we choose $c = 2$, $f = 2$. Then all conditions of Proposition 2.2 are satisfied. Thus $\mathcal{C} = \{(1,1), (\sqrt{2}, \sqrt{2}), (\sqrt{2}, \sqrt{2})\}$ admits a subnormal completion. In this case, $\gamma_{(0,0)} = 1$, $\gamma_{(0,2)} = 4i$, $\gamma_{(0,1)} = 1+i$, $\gamma_{(1,1)} = 4$, $\gamma_{(1,0)} = 1-i$, $\gamma_{(2,0)} = -4i$. Thus

$$M(1) = \begin{pmatrix} 1 & 1+i & 1-i \\ 1-i & 4 & -4i \\ 1+i & 4i & 4 \end{pmatrix} \geq 0.$$

Since $\det[M(1)]_2 = 2$ and $\det M(1) = 0$, rank $M(1) = 2$, and $\alpha = 0$, $\beta = -i$. Take $z_0 \neq 1+i$, and satisfies $\overline{z} + iz = 0$. So if we choose $z_0 = \frac{1+i}{2}$. Then $z_1 = 3(1+i)$. $\rho_0 = \frac{4}{5}$, $\rho_1 = \frac{1}{5}$. Hence the associated measure is

$$\mu = \frac{4}{5}\delta_{(\frac{1}{2},\frac{1}{2})} + \frac{1}{5}\delta_{(3,3)}.$$

In order to find the representing measure in the case of rank $M(1) = 3$, i.e., $\det M(1) > 0$, first, we take $y$ that satisfies

$$\begin{aligned} 2\operatorname{Re}((\gamma_{(0,1)}\gamma_{(2,0)} - \gamma_{(1,1)}\gamma_{(1,0)})\gamma_{(2,0)}y) + (\gamma_{(1,1)} - |\gamma_{(0,1)}|^2)|y|^2 \\ = (\gamma_{(1,1)}^2 - |\gamma_{(0,2)}|^2)^2. \end{aligned} \tag{2.7}$$

Let

$$\begin{aligned} \beta_1 &:= (\gamma_{(1,1)}^2 - |\gamma_{(0,2)}|^2)\gamma_{(0,2)} + (\gamma_{(0,1)}\gamma_{(2,0)} - \gamma_{(1,1)}\gamma_{(1,0)})y \\ \beta_2 &:= (\gamma_{(2,0)}\gamma_{(0,1)} - \gamma_{(1,0)}\gamma_{(1,1)})\gamma_{(0,2)} + (\gamma_{(1,0)}^2 - \gamma_{(2,0)})y \\ \beta_3 &:= (\gamma_{(1,0)}\gamma_{(0,2)} - \gamma_{(0,1)}\gamma_{(1,1)})\gamma_{(0,2)} + (\gamma_{(1,1)} - |\gamma_{(0,1)}|^2)y \\ \beta_4 &:= (\gamma_{(1,1)}^2 - |\gamma_{(0,2)}|^2)\gamma_{(1,1)} \\ \beta_5 &:= (\gamma_{(2,0)}\gamma_{(0,1)} - \gamma_{(1,0)}\gamma_{(1,1)})\gamma_{(1,1)} \\ \beta_6 &:= (\gamma_{(1,0)}\gamma_{(0,2)} - \gamma_{(0,1)}\gamma_{(1,1)})\gamma_{(1,1)}. \end{aligned}$$

Then the atoms of $\mu(y)$ are the 3 distinct roots of

$$\begin{aligned} (\det M(1))^2 z^3 = \beta_3\beta_4 - \beta_1\beta_6 + ((\det M(1))\beta_1 + \beta_3\beta_5 - \beta_2\beta_6)z \\ + (\det M(1))(\beta_2 + \beta_6)z^2. \end{aligned} \tag{2.8}$$

**Example 2.8** If we choose $a = 1$, $b = 1$, $c = 1$, $d = 4$, $e = 4$, $f = 1$. Then all conditions of Proposition 2.2 are satisfied. Thus $\mathcal{C} = \{(1,1),(1,2),(2,1)\}$ admits a subnormal completion. In this case, $\gamma_{(0,0)} = 1$, $\gamma_{(0,1)} = 1+i$, $\gamma_{(1,0)} = 1-i$, $\gamma_{(0,2)} = 2i$, $\gamma_{(1,1)} = 8$, $\gamma_{(2,0)} = -2i$. Hence

$$M(1) = \begin{pmatrix} 1 & 1+i & 1-i \\ 1-i & 8 & -2i \\ 1+i & 2i & 8 \end{pmatrix}.$$

And $\det[M(1)]_2 = 6$ and $\det M(1) = 36$. So $r = 3$. To find the representing measure, first we take $y$ that satisfies (2.7), i.e.,

$$4\,\mathrm{Re}((1+i)y) + |y|^2 = 600.$$

So we let $y = 2(-1+i) + 4\sqrt{19}(1+i)$. Then

$$\begin{aligned} \beta_1 &= -48(\sqrt{19} - 2i), & \beta_4 &= 480, \\ \beta_2 &= -12(1+i), & \beta_5 &= -48(1-i), \\ \beta_3 &= 24\sqrt{19}(1+i), & \beta_6 &= -48(1+i). \end{aligned}$$

Then the atoms of $\mu(y)$ are the 3 distinct roots of

$$9z^3 + 15(1+i)z^2 + 4(7\sqrt{19} - 4i)z - 32(1+i)(2\sqrt{19} + i) = 0.$$

By using Mathematica, we can obtain the following atoms and densities

$$\begin{aligned} z_0 &\approx -1.50812 - 5.43922i, & \rho_0 &\approx 0.0591804, \\ z_1 &\approx -2.15249 + 2.69381i, & \rho_1 &\approx 0.189787, \\ z_2 &\approx 1.99394 + 1.07875i, & \rho_2 &\approx 0.751104. \end{aligned}$$

Thus the representing measure is

$$\mu := \rho_0 \delta_{(\mathrm{Re}\, z_0, \mathrm{Im}\, z_0)} + \rho_1 \delta_{(\mathrm{Re}\, z_1, \mathrm{Im}\, z_1)} + \rho_2 \delta_{(\mathrm{Re}\, z_2, \mathrm{Im}\, z_2)}.$$

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