Hokkaido Mathematical Journal Vol. 32 (2003) p. 117–126

Congruences for the Burnside module

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(Received August 13, 2001; Revised March 4, 2002)

Abstract. Let G be a finite group. Oliver-Petrie constructed a Π-complex for a finite G-CW-complex and defined a Burnside module $\Omega(G,\Pi)$ which consists of equivalent classes of all Π-complexes. It is well-known that a congruence holds for the Burnside ring. The purpose of this paper is to prove congruences for the Burnside module.

Key words: G-CW-complex, G-map, G-poset, Burnside module.

1. Introduction

Throughout this paper let G be a finite group. Let X be a finite G-CW-complex. T. tom Dieck [2, 3] proved that a congruence holds for the Burnside ring:

$$\sum_{(K)} n(H,K)\chi(X^K) \equiv 0 \mod |N_G(H)/H|, \tag{$*$}$$

where $N_G(H)$ is the normalizer of H in G, $|N_G(H)/H|$ is the order of $N_G(H)/H$, $\chi(X^K)$ is the Euler characteristic of X^K, the $n(H,K)$ are some integers, $n(H,H)=1$, and the sum is taken over all G-conjugacy classes (K) such that H is normal in K and K/H is cyclic. This congruence is called the *Burnside relation*.

Let $\Omega(G)$ be the Burnside ring, $\Phi(G)$ the conjugacy class set of G, and $C(G)$ the ring of functions from $\Phi(G)$ to $\mathbb{Z}$. Then we have

Theorem 1.1 [3, Chapter 4 (5.7)] *The congruences* $(*)$ *are a complete set of congruences for the image of* $\varphi : \Omega(G) \to C(G)$, *i.e. a function* $z \in C(G)$ *is contained in* $\mathrm{Im}(\varphi)$ *if and only if for all* $(H) \in \Phi(G)$ *the congruence* $(*)$

$$\sum_{(K)} n(H,K)z(K) \equiv 0 \mod |N_G(H)/H|$$

is satisfied.

2000 Mathematics Subject Classification : 57S17, 57S25.

On the other hand, E. Laitinen and W. Lück defined the Lefschetz ring [7]. Since it is well-known that the Burnside ring is isomorphic to the Lefschetz ring, the similar congruence holds for the Lefschetz ring [9].

Next we shall state fundmental definitions and properties on Π-complexes and the Burnside module. See our general reference R. Oliver-T. Petrie [10] for details. Suppose that Π is a partially ordered set and G acts on it preserving the partially order. Let $S(G)$ be the set of all subgroups of G. We regard $S(G)$ as a G-set via the action $(g, H) \mapsto gHg^{-1}$ ($g \in G$ and $H \in S(G)$) and as a partially ordered set via

$$H \leq K \text{ if and only if } H \supseteq K \quad (H, K \in S(G)).$$

For any $\alpha \in \Pi$, we set

$$\Pi_\alpha = \{\beta \in \Pi \mid \beta \geq \alpha\}, \quad \text{and}$$
$$G_\alpha = \{g \in G \mid g\alpha = \alpha\}.$$

In particular, G_α is called an *isotropy subgroup* of G at α. Let $\rho : \Pi \to S(G)$ be an order preserving G-map. A pair (Π, ρ) is called a *G-poset* if it is satisfying the following condition: for any $\alpha \in \Pi$,

$$\rho(\alpha) \lhd G_\alpha \quad \text{and} \quad \rho : \Pi_\alpha \to S(G)_{\rho(\alpha)} \text{ is injective.}$$

Note that $S(G)_{\rho(\alpha)} = S(\rho(\alpha)) \subset S(G_\alpha)$ and $G_\alpha \subset G_{\rho(\alpha)} = N_G(\rho(\alpha))$, the normalizer of $\rho(\alpha)$ in G. As example of a G-poset consider $(S(G), id)$. A G-poset (Π, ρ) is called *complete* if

$$\rho : \Pi_\alpha \to S(G)_{\rho(\alpha)} \text{ is bijective for all } \alpha \in \Pi.$$

There is a unique maximal element $\mathfrak{m} \in \Pi$ for a complete G-poset (Π, ρ).

For any G-space Y, we set

$$\Pi_G(Y) = \coprod_{H \in S(G)} \pi_0(Y^H) \quad \text{(the disjoint union of } \pi_0(Y^H)\text{'s).}$$

Here Y^H is the H-fixed point set of Y and $\pi_0(Y^H)$ is the set of all connected components of Y^H. For $\alpha \in \Pi_G(Y)$, there exists uniquely a subgroup $H \in S(G)$ such that $\alpha \in \pi_0(Y^H)$. Hence we can define a map $\rho_Y : \Pi_G(Y) \to S(G)$ by $\alpha \mapsto H$. In addition, $\Pi_G(Y)$ is equipped with a partial order $\leqq$ by

$$\alpha \leqq \beta \text{ if and only if } \rho(\alpha) \supseteqq \rho(\beta) \text{ and } |\alpha| \subseteqq |\beta| \quad (\alpha, \beta \in \Pi_G(Y))$$

where $|\alpha|$ is the underlying space for $\alpha \in \Pi_G(Y)$. Thus we get a G-poset $(\Pi_G(Y), \rho_Y)$, which is called a *G-poset associated to Y*. Note that $(\Pi_G(Y), \rho_Y)$ is complete.

Definition 1.2 Let a pair (Π, ρ) be a *G-poset*. A finite G-CW-complex X with a base point $*$ is called a *Π-complex* if it is equipped with a specified set $\{X_\alpha \mid \alpha \in \Pi\}$ of subcomplexes X_α of X, satisfying the following four conditions:

(i) $* \in X_\alpha$,

(ii) $gX_\alpha = X_{g\alpha}$ for $g \in G$, $\alpha \in \Pi$,

(iii) $X_\alpha \subseteqq X_\beta$ if $\alpha \leqq \beta$ in Π, and

(iv) for any $H \in S(G)$,

$$X^H = \bigvee_{\alpha \in \Pi \text{ with } \rho(\alpha) = H} X_\alpha \quad \text{(the wedge sum of } X_\alpha\text{'s).}$$

We shall give some examples of Π-complexes.

Example 1.3 Let (Π, ρ) be a G-poset. For each $\alpha \in \Pi$, we let the space $(G/\rho(\alpha))^+$ denote $G/\rho(\alpha) \coprod \{*\}$ (disjoint union) with

$$(G/\rho(\alpha))^+_\beta = \{g\rho(\alpha) \mid g\alpha \leq \beta,\, g \in G\} \coprod \{*\} \quad \text{for} \quad \beta \in \Pi.$$

Then $(G/\rho(\alpha))^+$ admits a Π-complex structure.

Example 1.4 Let $f : X \to Y$ be a G-map between finite G-CW-complexes. We define

$$\begin{aligned} Y_\alpha &= |\alpha|, \quad \text{and} \\ X_\alpha &= X^{\rho(\alpha)} \cap f^{-1}(Y_\alpha). \end{aligned}$$

Let X^+ denote the space X with the disjoint base point $*$. Similarly for Y^+. Then both X^+ and Y^+ are $\Pi_G(Y)$-complexes. Let C_f stand for the mapping cone of f and $f_\alpha : X_\alpha \to Y_\alpha$ be the restriction of f. By setting $(C_f)_\alpha = C_{f_\alpha}$, the space C_f can be also given the structure of $\Pi_G(Y)$-complex.

Let $\mathcal{F}$ denote the family of all Π-complexes and define the equivalence relation $\sim$ on $\mathcal{F}$ by

$$Z \sim W \text{ if and only if } \chi(Z_\alpha) = \chi(W_\alpha) \text{ for all } \alpha \in \Pi \quad (Z, W \in \mathcal{F})$$

where $\chi(Z_\alpha)$ is the Euler characteristic of Z_α.

The set $\Omega(G,\Pi) = \mathcal{F}/\sim$ is an abelian group via

$$[Z] + [W] = [Z \vee W] \quad (Z, W \in \mathcal{F}).$$

The zero element is the equivalence class of a point. We call $\Omega(G,\Pi)$ the *Burnside module associated with a G-poset* Π.

Let α be any element of Π and X a Π-complex. Construct a new space X' by attaching α-cells $G/\rho(\alpha) \times D^i$'s to X. Each attachment map

$$\varphi : G/\rho(\alpha) \times S^{i-1} \to X$$

is defined such that $\varphi(g\rho(\alpha) \times S^{i-1}) \subset X_{g\alpha}$. The space X' is equipped with a Π-complex structure:

$$(X')_\beta = X_\beta \cup \left(\bigcup \{g\rho(\alpha) \times D^i \mid g\alpha \leq \beta,\ g \in G\}\right) \quad \text{for} \quad \beta \in \Pi.$$

Any Π-complex is constructed from one point by attaching α-cells for $\alpha \in \Pi$.

Proposition 1.5 [10, Proposition 1.5] *One has*

$$\Omega(G,\Pi) \cong \bigoplus_{\alpha \in \mathcal{A}} \mathbb{Z}.$$

Any finite Π-complex X is equivalent in $\Omega(G,\Pi)$ to a sum of the form $\sum_{\alpha \in \mathcal{A}} a_\alpha[(G/\rho(\alpha))^+]$, and the map $[X] \to \{a_\alpha\}_{\alpha \in \mathcal{A}}$ defines the group isomorphism.

The purpose of this paper is to establish congruences for the Burnside module. The main two theorem in this paper are the following. Let $\Phi(G_\alpha/\rho(\alpha))$ be the conjugacy class set of $G_\alpha/\rho(\alpha)$. We define

$$\begin{aligned} &S((G),\alpha) \\ &\quad = \{K \in S(G) \mid (K/\rho(\alpha)) \in \Phi(G_\alpha/\rho(\alpha)) \text{ and } K/\rho(\alpha) \text{ is cyclic}\}. \end{aligned}$$

Theorem 1.6 *Let α be an element of Π. Then we have*

$$\begin{aligned} &\sum_{K \in S((G),\alpha)} \frac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \\ &\qquad \cdot \phi(|K/\rho(\alpha)|) \cdot \bar{\chi}(X_\alpha^K) \equiv 0 \mod |G_\alpha/\rho(\alpha)|, \end{aligned}$$

where $\phi(|K/\rho(\alpha)|)$ is the number of generators of the cyclic group $K/\rho(\alpha)$.

Moreover we define a group homomorphism

$$\bar{\chi}_\alpha : \Omega(G,\Pi) \to \mathbb{Z}$$

by $\bar{\chi}_\alpha([X]) = \bar{\chi}(X_\alpha)$ for $[X] \in \Omega(G,\Pi)$ and $\alpha \in \Pi$. Noting that $X_{g\alpha} = gX_\alpha$, a map $f : X_\alpha \to X_{g\alpha};\ x \mapsto gx$ is a homeomorphism. Now, a quotient set Π/G consists of all orbits of Π under G. Let $\mathcal{A} \subset \Pi$ be a complete set of representatives for Π/G. Then we introduce a new function defined by

$$\bar{\chi} = \bigoplus_{\alpha\in\mathcal{A}} \bar{\chi}_\alpha : \Omega(G,\Pi) \to \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z},$$

$$\bar{\chi}([X]) = \bigoplus_{\alpha\in\mathcal{A}} \bar{\chi}_\alpha([X]).$$

One verifies that the map $\bar{\chi}$ is a injective group homomorphism.

Theorem 1.7 *If a G-poset (Π,ρ) is complete, one has*

$$\begin{aligned}
&\mathrm{Im}\Big(\bar{\chi} : \Omega(G,\Pi) \to \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z}\Big) \\
&= \Big\{ (x_\alpha) \in \bigoplus_{\alpha\in\mathcal{A}} \mathbb{Z} \;\Big|\; \sum_{K\in S((G),\alpha)} \frac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \\
&\qquad \cdot \phi(|K/\rho(\alpha)|) \cdot x_{\alpha,(K)} \equiv 0 \mod |G_\alpha/\rho(\alpha)| \Big\},
\end{aligned}$$

where $x_{\alpha,(K)}$ is some integer such that

$$x_{\alpha,(K)} = \begin{cases} x_\alpha & (K = \rho(\alpha)) \\ \sum_\beta x_\beta & (K \neq \rho(\alpha),\ \beta \text{ is some element of } \Pi \text{ with} \\ & \qquad \rho(\beta) = K,\ \beta < \alpha). \end{cases}$$

This paper is organized as follows. In Section 2, we prove the main two theorem. Finally we give an example of Theorem 1.7.

2. Proofs of the main results

Proof of Theorem 1.6. Let (Π,ρ) be a G-poset and G_α the isotropy subgroup at α. Given a Π-complex X, we see the $G_\alpha/\rho(\alpha)$-CW-complex $X^{\rho(\alpha)}$ is equipped with a Π-complex structure as following:

$$(X^{\rho(\alpha)})_\alpha = X_\alpha^{\rho(\alpha)} \quad \text{for all } \alpha \in \Pi.$$

By our definition of the Π-complex, it can be shown that $X_\alpha^{\rho(\alpha)} = X_\alpha$ for all $\alpha \in \Pi$. Let $\chi(X)$ be the Euler characteristic of X, and $\bar{\chi}(X) = \chi(X) - 1$. Note that a map $f : \mathcal{F}_c(G_\alpha/\rho(\alpha)) \to \mathbb{Z}$; $K/\rho(\alpha) \mapsto \bar{\chi}(X_\alpha^K)$ satisfies a Burnside relation. By Burnside's lemma [9, Lemma 4.1], we have the desired result. □

We need the following lemma to prove the Theorem 1.7.

Lemma 2.1 *Suppose that a G-poset* (Π, ρ) *is complete. Let* α *be an elemnet of* Π *and* K *a subgroup with* $K \supset \rho(\alpha)$. *For a* Π*-complex* X, *it holds that*

$$\bar{\chi}(X_\alpha^K) = \sum_{\beta \in \Pi \text{ with } \rho(\beta)=K, \beta<\alpha} \bar{\chi}(X_\beta).$$

Proof. Recall that

$$X^K = \bigvee_{\beta \in \Pi \text{ with } \rho(\beta)=K} X_\beta.$$

We set $H = \rho(\alpha)$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_m\} = \{\gamma \in \Pi \mid \rho(\gamma) = H\}$. After renumbering the α_i, we may assume that $\alpha_1 = \alpha$. Observe that

$$X^H = X_{\alpha_1} \vee X_{\alpha_2} \vee \cdots \vee X_{\alpha_m}.$$

Immediately,

$$X^K = X_{\alpha_1}^K \vee X_{\alpha_2}^K \vee \cdots \vee X_{\alpha_m}^K.$$

For an element β, since a G-poset (Π, ρ) is complete, there exists an element $\alpha \in \Pi$ such that $\rho(\alpha) = H$, $\beta \leq \alpha_1 = \alpha$. Hence we have

$$X_\alpha^K = X^K \cap X_\alpha = \bigvee_{\beta \in \Pi \text{ with } \rho(\beta)=K, \beta \leq \alpha} X_\beta,$$

and thereby prove our assertion. □

Recall that $\mathcal{A} \subset \Pi$ is a complete set of representatives for Π/G. Let α_i, α_j be elements of $\mathcal{A}$. Now, we give an order $\leq_*$ on $\mathcal{A}$:

$$\alpha_i \leq_* \alpha_j \text{ if and only if } g\alpha_i \leq \alpha_j \text{ for some } g \in G,$$

where $\leq$ is the order on Π. We write $\leq$ for $\leq_*$.

Proof of Theorem 1.7. First we use **S** for the right side, and **Im** for the left side in the equation of Theorem 1.7. Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_m\}$. By [5, Lemma 1.80], we can arrange elements of $\mathcal{A}$ such that

$$\alpha_i \le \alpha_j \Longrightarrow i \le j.$$

Define a map $P_{\le k} : \bigoplus_{i=1}^m \mathbb{Z}_{\alpha_i} \to \bigoplus_{i=1}^k \mathbb{Z}_{\alpha_i}$ by k coordinate maps $p_i : \bigoplus_{i=1}^m \mathbb{Z}_{\alpha_i} \to \mathbb{Z}_{\alpha_i}$ such that

$$P_{\le k}(x) = (p_1(x), \ldots, p_k(x)),$$

where each $\mathbb{Z}_{\alpha_i}$ is a copy of $\mathbb{Z}$. Note that $\mathbf{S} \subset \bigoplus_{i=1}^m \mathbb{Z}_{\alpha_i}$. It will now suffice to prove that

$$P_{\le m}(\mathbf{S}) = P_{\le m}(\mathbf{Im}).$$

We proceed by induction on k. In the case where $k = 1$, the map $P_{\le 1}$ means

$$\bigoplus_{i=1}^m \mathbb{Z}_{\alpha_i} \to \mathbb{Z}_{\alpha_1};\ (x_{\alpha_i}) \mapsto x_{\alpha_1}.$$

Take an element $[(G/G)^+] \in \Omega(G, \Pi)$. Then we have

$$\mathbf{Im} \ni \bar{\chi}([(G/G)^+]) = (\bar{\chi}_{\alpha_i}([(G/G)^+]))_{\alpha_i} = (1, 1, \ldots, 1).$$

Thus we obtain

$$1 \in P_{\le 1}(\mathbf{Im}),$$

and so we get

$$P_{\le 1}(\mathbf{Im}) = \mathbb{Z}_{\alpha_1}.$$

Since $\mathbf{Im}$ is a subset of $\mathbf{S}$ by Theorem 1.6 and Lemma 2.1, it follows that $P_{\le 1}(\mathbf{Im}) \subset P_{\le 1}(\mathbf{S})$. Clearly $\mathbb{Z}_{\alpha_1} \supset P_{\le 1}(\mathbf{S})$. Therefore, $P_{\le 1}(\mathbf{Im}) = P_{\le 1}(\mathbf{S})$.

Suppose that $P_{\le k-1}(\mathbf{S}) = P_{\le k-1}(\mathbf{Im})$. Let $y = (y_{\alpha_1}, y_{\alpha_2}, \ldots, y_{\alpha_{k-1}}, y_{\alpha_k}, y_{\alpha_{k+1}}, \ldots, y_{\alpha_m})$ be an element of $\mathbf{S}$. By assumption, there exists an element

$$x = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_{k-1}}, x_{\alpha_k}, x_{\alpha_{k+1}}, \ldots, x_{\alpha_m}) \in \mathbf{Im}$$

such that $x_{\alpha_1} = y_{\alpha_1}, x_{\alpha_2} = y_{\alpha_2}, \ldots, x_{\alpha_{k-1}} = y_{\alpha_{k-1}}$. Then we have

$$\begin{aligned} z &= y - x \\ &= (0, 0, \ldots, 0, y_{\alpha_k} - x_{\alpha_k}, y_{\alpha_{k+1}} - x_{\alpha_{k+1}}, \ldots, y_{\alpha_m} - x_{\alpha_m}) \in \mathbf{S}. \end{aligned}$$

Here we let $z_{\alpha_i} = y_{\alpha_i} - x_{\alpha_i}$, and $n_{\alpha,K} = \frac{|G_\alpha/\rho(\alpha)|}{|N_{G_\alpha/\rho(\alpha)}(K/\rho(\alpha))|} \cdot \phi(|K/\rho(\alpha)|)$. Consider the case of $\alpha = \alpha_k$. Then we have

$$\sum_{K \in S((G),\alpha_k)} n_{\alpha_k,K} \cdot z_{\alpha_k,(K)} \equiv 0 \mod |G_{\alpha_k}/\rho(\alpha_k)|.$$

Observe that the coefficient $z_{\alpha_k,(K)}$ $(K \neq \rho(\alpha_k))$ is equal to $\sum_\beta z_\beta$, where β is some element of Π with $\rho(\beta) = K$, $\beta < \alpha_k$. Thus the above equation implies

$$n_{\alpha_k,\rho(\alpha_k)} \cdot z_{\alpha_k,(\rho(\alpha_k))} \equiv 0 \mod |G_{\alpha_k}/\rho(\alpha_k)|.$$

Note that

$$n_{\alpha_k,\rho(\alpha_k)} = \frac{|G_{\alpha_k}/\rho(\alpha_k)|}{|N_{G_{\alpha_k}/\rho(\alpha_k)}(\rho(\alpha_k)/\rho(\alpha_k))|} \cdot \phi(|\rho(\alpha_k)/\rho(\alpha_k)|) = 1.$$

That is,

$$z_{\alpha_k} \equiv 0 \mod |G_{\alpha_k}/\rho(\alpha_k)|.$$

On the other hand, we have

$$\begin{aligned} &\bigoplus_{\alpha \in \mathcal{A}} \bar{\chi}([(G/\rho(\alpha_k))^+]) \\ &= \left(\bar{\chi}_\alpha([(G/\rho(\alpha_k))^+])\right)_{\alpha \in \mathcal{A}} = (\overbrace{0, 0, \ldots, 0}^{k-1}, |G_{\alpha_k}/\rho(\alpha_k)|, \ldots). \end{aligned}$$

Hence there exists an integer $a \in \mathbb{Z}$ such that

$$y - x - a\left(\bar{\chi}_\alpha((G/\rho(\alpha_k))^+)\right) = (\overbrace{0, 0, \ldots, 0, 0}^{k}, \ldots).$$

That is,

$$y = x + a\left(\bar{\chi}_\alpha((G/\rho(\alpha_k))^+)\right) + (\overbrace{0, 0, \ldots, 0, 0}^{k}, \ldots).$$

By induction, we see immediately that

$$P_{\leq k}(y) = P_{\leq k}\left(x + a(\bar{\chi}_\alpha((G/\rho(\alpha_k))^+))\right) \in P_{\leq k}(\mathbf{Im}).$$

This completes the proof. $\square$

Finally we wish to give an example of Theorem 1.7. Let p be a prime number. We set $G = C_p$ (a cyclic group of order p). Since $S(G) = \{\{e\}, G\}$

(e is the unit element of G), and the G-action on $S(G)$ is trivial, a Burnside module $\Omega(G, S(G))$ is a free abelian group generated by $[(G/\{e\})^+]$, $[(G/G)^+]$. Clearly $\Phi(G) = \{\{e\}, G\}$.

First, consider the case of $\alpha = \{e\}$. Since $S((G), \alpha) = \{\{e\}, G\}$, we get

$$\frac{|G|}{|G|} \cdot 1 \cdot x_{\{e\},(\{e\})} + \frac{|G|}{|G|} \cdot (p-1) \cdot x_{\{e\},(G)} \equiv 0 \mod p.$$

That is,

$$x_{\{e\},(\{e\})} \equiv x_{\{e\},(G)} \mod p.$$

By Theorem 1.7, there exists a Π-complex X such that $\bar{\chi}(X_{\{e\}}) = x_{\{e\},(\{e\})}$ and $\bar{\chi}(X_G) = x_{\{e\},(G)}$. Thus we have

$$\bar{\chi}(X_{\{e\}}) \equiv \bar{\chi}(X_G) \mod p.$$

In particular, if X has a Π-complex structure as follows:

$$X_\alpha = \begin{cases} X & (\alpha = \{e\}) \\ X^G & (\alpha = G), \end{cases}$$

the previous expression implies

$$\chi(X) \equiv \chi(X^G) \mod p.$$

Next for $\alpha = G$, since $S((G), \alpha) = \{G\}$, we obtain

$$\frac{1}{1} \cdot 1 \cdot x_{G,(G)} \equiv 0 \mod 1.$$

Immediately,

$$x_{G,(G)} \equiv 0 \mod 1.$$

This equation is true for any integer, and so there is no relation for Π-complexes.

References

[1] Dovermann K.H. and Rothenberg M., *The generalized whitehead torsion of a G-fibre homotopy equivalence.* Transformation Groups, Kawakubo Katsuo 1987, Lecure Notes in Math. **1375**, Springer-Verlag, Berlin, 1989, pp. 60–88.

[2] tom Dieck T., *Transformation Groups and Representation Theory.* Lecture Notes in Math. **766**, Springer-Verlag, Berlin, 1978.

[3] tom Diec, T., *Transformation Groups*. de Gruyter Studies in Math **8**, Walter de Gruyter, Berlin, 1987.

[4] Fujita R., *The resolution module of a space and its universal covering space*. Journal of The Faculty of Environment Science and Technology, Okayama Univ. **5** (2000), 57–69.

[5] Kawakubo K., *The Theory of Transformation Groups*. Oxford University Press, London, 1991.

[6] Komiya K., *Congruences for the Burnside ring*. Transformation Groups, Kawakubo Katsuo 1987, Lecure Notes in Math. **1375**, Springer-Verlag, Berlin, 1989, pp. 191–197.

[7] Laitinen E. and Lück W., *Equivariant Lefschetz Classes*. Osaka J. Math. **26** (1989), 491–525.

[8] Morimoto M. and Iizuka K., *Extendibility of G-maps to pseudo-equivalences to finite G-CW-complexes whose fundamental groups are finite*. Osaka J. Math. **21** (1984), 59–69.

[9] Morimoto M., *The Burnside Ring Revisited*. Current Trends in Transformation Groups, Bak Anthony, Morimoto Masaharu and Ushitaki Fumihiro, K-Monographs in Mathematics **7**, Kluwer Academic Publishers, London, 2002, pp. 129–145.

[10] Oliver R. and Petrie T., *G-CW-surgery and* $K_0(\mathbb{Z}G)$. Math. Z. **179** (1982), 11–42.

[11] Rim D.S., *Modules over finite groups*. Ann. Math. **69** (1958), 700–712.

[12] Rotman J.J., *An introduction to Algebraic Topology*. Graduate Texts in Mathematics **119**, Springer-Verlag, New York, 1988.

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