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Finite p-groups which determine p-nilpotency locally

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Abstract. Let G be a finite group, and let p be a prime number. It might happen that the p-Sylow normalizer $N_G(P)$, $P \in \mathrm{Syl}_p(G)$, of G is p-nilpotent, but G will not be p-nilpotent (see Example 1.1). However, under certain hypothesis on the structure of the Sylow p-subgroup P of G, this phenomenon cannot occur, e.g., by J. Tate's p-nilpotency criterion this is the case if P is a Swan group in the sense of H-W. Henn and S. Priddy. In this note we show that if P does not contain subgroups of a certain isomorphism type $Y_p(m)$ — in which case we call the p-group P *slim* — the previously mentioned phenomenon will not occur provided p is odd. For $p = 2$ the same remains true if P is D_8-free (see Main Theorem).

Key words: finite groups, p-nilpotency, slim p-groups, Sylow subgroups, p-nilpotent Sylow normalizer

1. Introduction

The three theorems of L. Sylow are certainly some of the central results in the theory of finite groups. From Sylow's first theorem grows the wish to detect properties of a finite group G analyzing a Sylow p-subgroup $P \in \mathrm{Syl}_p(G)$ or its normalizer $N_G(P)$. There are finite groups G which are not p-nilpotent, but $N_G(P)$ is p-nilpotent (see Example 1.1). Hence for the property p-nilpotency this wish cannot become a general theorem valid for all finite p-groups.

Example 1.1 (a) Let $p = 2$. Then for $n \geq 2$ a Sylow 2-subgroup of $G = \mathrm{SL}_n(2)$ is self-normalizing, but for $n \geq 3$ the group G is not 2-nilpotent. (b) Let T be a Coxeter torus of the Chevalley group $X = \mathrm{SL}_p(p)$. Then $T \cap \mathrm{Z}(X) = \{1\}$ and $N_X(T)/T \simeq \mathbf{C}_p$, where $\mathbf{C}_p$ denotes the cyclic group of order p. Let $\bar{P} \in \mathrm{Syl}_p(N_X(T))$. Then $N_{N_X(T)}(\bar{P}) = \bar{P}$. Let $V = \mathbb{F}_p^p$ denote the natural left $\mathbb{F}_p[\mathrm{SL}_p(p)]$-module. Then $G = V \rtimes N_X(T)$ is obviously not p-nilpotent, but $N_G(P) = P$ for $P \in \mathrm{Syl}_p(G)$ (see Fact 2.1(i)). Note that $P \simeq \mathbf{C}_p \wr \mathbf{C}_p$, and for $p = 2$ one has $G = S_4$.

Nevertheless, for some finite p-groups P this problem has an affirmative

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answer. We say that a finite p-group P *determines p-nilpotency locally*, if for every finite group G with $\bar{P} \in \mathrm{Syl}_p(G)$ isomorphic to P and $N_G(\bar{P})$ p-nilpotent follows that G is also p-nilpotent. The class of finite p-groups with this property will be denoted by $\mathfrak{DN}_p$. In particular, if $P \in \mathfrak{DN}_p$ and $\mathrm{Out}(P)$ is a p-group, then G is p-nilpotent whenever $\bar{P} \in \mathrm{Syl}_p(G)$ is isomorphic to P.

The first examples of finite p-groups with this property which attracted our attention are Swan groups. Following H.-W. Henn and S. Priddy (see [4]) one calls a finite p-group P a *Swan group*, if the restriction map in cohomology

$$\mathrm{res}^G_{N_G(\bar{P})} \colon H^\bullet(G, \mathbb{F}_p) \longrightarrow H^\bullet(N_G(\bar{P}), \mathbb{F}_p) \tag{1.1}$$

is an isomorphism whenever $\bar{P} \in \mathrm{Syl}_p(G)$ is isomorphic to P. From J. Tate's nilpotency criterion (see [7]) one concludes that every Swan group determines p-nilpotency locally.

In [8], J. Thévenaz pointed out — using a result of G. Mislin (see [6]) — that one can characterize Swan groups as those finite p-group P which have the property that for any finite group G with $\bar{P} \in \mathrm{Syl}_p(G)$ isomorphic to P follows that $N_G(\bar{P})$ controls p-fusion in G. Although all abelian p-groups and "most" finite p-groups are Swan groups, there are relatively small finite p-groups which are not Swan groups; e.g., for p odd, the extra special group $P = p^{1+2}$ of order p^3 and exponent p is isomorphic to the Sylow p-subgroup of $G = \mathrm{SL}_3(p)$, but $N_G(P)$ does not control p-fusion in G. Hence $P = p^{1+2}$ is not a Swan group. In [4], it was shown that for p odd every p-central p-group is indeed a Swan group.

The main purpose of this paper is to show that $\mathfrak{DN}_p$ contains a large subclass of the class of finite p-groups. For a prime number p and $m \geq 1$ let $\mathbf{C}_{p^m}$ denote the cyclic group of order p^m. Let $Y_p(1) = \mathbf{C}_p \wr \mathbf{C}_p$ denote the (regular) wreath product, and let $\beta \colon \mathbf{C}_p \wr \mathbf{C}_p \to \mathbf{C}_p$ denote the canonical homomorphism with elementary abelian kernel. We define the p-groups $Y_p(m)$, $m \geq 1$, as the pull-back of the diagram

$$\begin{tikzcd} \mathbf{C}_{p^m} \arrow[r] & \mathbf{C}_p \\ Y_p(m) \arrow[u, dashed] \arrow[r, dashed] & \mathbf{C}_p \wr \mathbf{C}_p \arrow[u, "\beta"'] \end{tikzcd} \tag{1.2}$$

A finite p-group will be called *slim*, if P contains no subgroup isomorphic to $Y_p(m)$ for all $m \geq 1$, and *extra slim*, if P contains no sub-quotient isomorphic to $\mathbf{C}_p \wr \mathbf{C}_p$. These groups are also called $\mathbf{C}_p \wr \mathbf{C}_p$*-free*. By $\mathfrak{slim}_p$ we denote the class of slim p-groups, and $\mathfrak{xslim}_p$ will denote the class of extra slim p-groups. Obviously, $\mathfrak{xslim}_p \subset \mathfrak{slim}_p$. In Section 4.2 we will proof the following.

Main Theorem *Every extra slim* 2*-group determines* 2*-nilpotency locally. For p odd every slim p-group determines p-nilpotency locally.*

Considering only certain subclasses of finite p-group, the Main Theorem can be considered to be best possible (see Proposition 4.3). The following examples of finite p-groups are extra slim:

(a) finite p-groups P of nilpotency class $\mathrm{cl}(P)$ less or equal to $p-1$;
(b) finite p-groups P of exponent p;
(c) regular finite p-groups in the sense of P. Hall (see [5, Kap. III, Section 10]);
(d) finite p-groups P with section rank $\mathrm{srk}(P) \leq p-1$,

where the *section rank* is defined by

$$\mathrm{srk}(P) = \max\big\{ \dim_{\mathbb{F}_p}(\mathrm{Hom}_{\mathrm{gr}}(U, \mathbf{C}_p)) \mid U \leq P \big\}. \tag{1.3}$$

For p odd, every p-central group of height $p-2$ is slim (see [3]).

The proof of the Main Theorem[1] uses G. Glauberman's version of J. G. Thompson's p-nilpotency criterion (see Theorem 4.1) and a detailed analysis of the possible counter examples, the pqp-sandwich groups (see Section 2), and their universal p-Schur-Frattini cover (see Section 3.4).

It should be remarked that the class of extra slim p-groups plays an important role in the analysis of the maximal p-abelian quotient of a finite group G. T. Yoshida showed in [11] that if $P \in \mathrm{Syl}_p(G)$ is extra slim one has an isomorphism

$$G/O^p(G).G' \simeq N_G(P)/O^p(N_G(P)).N_G(P)', \tag{1.4}$$

i.e., the maximal p-abelian quotients of G and $N_G(P)$ are isomorphic. In particular, if $N_G(P)$ is p-nilpotent one has an isomorphism $G/O^p(G)G' \simeq P/P'$. Indeed, the Main Theorem shows that in this case one has even an

[1]The classification of finite simple groups is not used in the proof of the Main Theorem.

isomorphism $G/O^p(G) \simeq P$. It would be very interesting to know whether some version of (1.4) also holds in case that P is slim provided p is odd.

2. *pqp*-sandwich groups

In this section we collect some basic facts about a particular class of p-soluble groups for a given prime number p. For a prime number q co-prime to p let Q be a non-trivial irreducible (left) $\mathbb{F}_q[\mathbf{C}_p]$-module, where $\mathbb{F}_q$ denotes the finite field with q elements. Then

$$G_0 = Q \rtimes \mathbf{C}_p \tag{2.1}$$

is a p-nilpotent group. For a non-trivial irreducible $\mathbb{F}_p[G_0]$-module P_0 we build the finite group

$$\mathcal{S}(P_0, Q) = P_0 \rtimes G_0 = P_0 \rtimes (Q \rtimes \mathbf{C}_p) \tag{2.2}$$

and call it a *pqp-sandwich group.* Let $\mathcal{S} = \mathcal{S}(P_0, Q)$ and $P \in \mathrm{Syl}_p(\mathcal{S})$. The following fact which easy proof we leave to the reader shows that $N_{\mathcal{S}}(P) = P$, but $\mathcal{S}$ is obviously not p-nilpotent.

Fact 2.1 *Let p be any prime number. Let G be a finite group, let $P \in \mathrm{Syl}_p(G)$, and let N be a normal subgroup of G.*

(i) *If N is of p-power order, one has $N_{G/N}(P/N) = N_G(P)/N$.*
(ii) *If N is of order co-prime to p one has $N_{G/N}(PN/N) = N_G(P)N/N$.*

2.1. Irreducible $\bar{\mathbb{F}}_p[G_0]$-modules

Let $\bar{\mathbb{F}}_p$ denote the algebraic closure of the finite field $\mathbb{F}_p$. Since G_0 is p-nilpotent, the only irreducible module in the principle $\bar{\mathbb{F}}_p[G_0]$-block $\bar{b}_0$ is the trivial module $\bar{\mathbb{F}}_p$ (see [1, Corollary 63.3]). Since $N_{G_0}(\mathbf{C}_p) = \mathbf{C}_p$, R. Brauer's first main theorem (see [1, Theorem 61.7]) implies that $\bar{b}_0$ is the only $\bar{\mathbb{F}}_p[G_0]$-block with defect group $\mathbf{C}_p$. Thus every non-principal block has trivial defect group. In particular, every non-trival irreducible $\bar{\mathbb{F}}_p[G_0]$-module is projective.

By hypothesis, $\mathbf{C}_p$ acts without non-trivial fixed points on Q, and thus also on the Pontrjagin dual $\mathrm{Hom}_{\mathrm{ab}}(Q, \bar{\mathbb{F}}_p^*)$. By Clifford theory, one has a natural one-to-one correspondence between the isomorphism types of non-trivial irreducible left $\bar{\mathbb{F}}_p[G_0]$-modules and $\mathbf{C}_p$-orbits on the set

$$\mathrm{Hom}_{\mathrm{ab}}\left(Q,\bar{\mathbb{F}}_p^*\right)^\sharp = \mathrm{Hom}_{\mathrm{ab}}\left(Q,\bar{\mathbb{F}}_p^*\right) \setminus \{0\}, \tag{2.3}$$

i.e., let $\bar{\mathbb{F}}_p(\bar{\theta})$ be the irreducible $\bar{\mathbb{F}}_p[Q]$-module corresponding to $\bar{\theta} \in \mathrm{Hom}_{\mathrm{ab}}(Q,\bar{\mathbb{F}}_p^*)^\sharp$, then

$$\bar{L}_{\mathbf{C}_p.\bar{\theta}} = \mathrm{ind}_Q^{G_0}\left(\bar{\mathbb{F}}_p(\bar{\theta})\right) \tag{2.4}$$

is irreducible and every non-trivial irreducibe $\bar{\mathbb{F}}_p[G_0]$-module can be obtained this way.

2.2. Irreducible $\mathbb{F}_p[G_0]$-modules

Let r denote the order of $q+\mathbb{Z}.p$ in $(\mathbb{Z}/\mathbb{Z}.p)^*$, where $\_^*$ denote the group of units in a ring. Then

$$p \mid q^r - 1 \quad \text{but} \quad p \nmid q^{r'} - 1 \quad \text{for } r' < r. \tag{2.5}$$

Note that $r \mid p-1$. In particular, every non-trivial $\mathbb{F}_q[\mathbf{C}_p]$-module has $\mathbb{F}_q$-dimension r. Hence $|Q| = q^r$.

Let d denote the order of $p+\mathbb{Z}.q$ in $(\mathbb{Z}/\mathbb{Z}.q)^*$. Then

$$q \mid p^d - 1 \quad \text{but} \quad q \nmid p^{d'} - 1 \quad \text{for } d' < d \tag{2.6}$$

and $d \mid q-1$. Every non-trivial irreducible $\mathbb{F}_p[Q]$-module has $\mathbb{F}_p$-dimension d. Let $\mathcal{F} = \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^d})$. Then one can parametrize the irreducible $\mathbb{F}_p[Q]$-modules by the $\mathcal{F}$-orbits on $\mathrm{Hom}_{\mathrm{ab}}(Q,\mathbb{F}_{p^d}^*)^\sharp$. Let $M_{\theta.\mathcal{F}}$ denote the irreducible $\mathbb{F}_p[Q]$-module corresponding to the $\mathcal{F}$-orbit $\theta.\mathcal{F} \subset \mathrm{Hom}_{\mathrm{ab}}(Q,\mathbb{F}_{p^d}^*)^\sharp$. Then $M_{\theta.\mathcal{F}}$ can be extended to an $\mathbb{F}_p[G_0]$-module if, and only if, $p \mid d$. Hence in this case $r=1$, and Q is cyclic of prime order. Moreover, G_0 can be identified with a subgroup of $\mathbb{F}_{p^d}^* \rtimes \mathcal{F}$. Let

$$\bar{\_}\colon \mathrm{Hom}_{\mathrm{ab}}\left(Q,\mathbb{F}_{p^d}^*\right) \longrightarrow \mathrm{Hom}_{\mathrm{ab}}\left(Q,\bar{\mathbb{F}}_p^*\right) \tag{2.7}$$

denote the canonical isomorphism. Clifford theory yields the following description of irreducible $\mathbb{F}_p[G_0]$-modules:

Case (A): If $p \nmid d$, then for every $\mathcal{F}$-orbit $\theta.\mathcal{F} \subset \mathrm{Hom}_{\mathrm{ab}}(Q,\mathbb{F}_{p^d}^*)^\sharp$

$$L_{\mathcal{O}(\theta)} = \mathrm{ind}_Q^{G_0}(M_{\theta.\mathcal{F}}) \tag{2.8}$$

is irreducible. Here $\mathcal{O}(\theta) = \mathbf{C}_p.\theta.\mathcal{F}$ denote the $\mathbf{C}_p \times \mathcal{F}$-orbit containing θ. In particular, the isomorphism classes of irreducible $\mathbb{F}_p[G_0]$-modules can be parametrized by the $\mathbf{C}_p \times \mathcal{F}$-orbits on $\mathrm{Hom}_{\mathrm{ab}}(Q, \mathbb{F}_{p^d}^*)^\sharp$. Moreover,

$$L_{\mathcal{O}(\theta)} \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p = \coprod_{\eta \in \mathcal{R}(\theta)} \bar{L}_{\mathbf{C}_p.\bar{\eta}}, \tag{2.9}$$

where $\mathcal{R}(\theta)$ is a system of representatives of the $\mathbf{C}_p$-set $\mathcal{O}(\theta)$,

$$\mathrm{res}_Q^{G_0}(L_{\mathcal{O}(\theta)}) = \coprod_{\nu.\mathcal{F} \subset \mathcal{O}(\theta)} M_{\nu.\mathcal{F}}, \tag{2.10}$$

$$\mathrm{res}_Q^{G_0}(L_{\mathcal{O}(\theta)}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p = \coprod_{\mu \in \mathcal{O}(\theta)} \bar{\mathbb{F}}_p(\bar{\mu}). \tag{2.11}$$

Case (B): If $p \mid d$, there exists an irreducible $\mathbb{F}_p[G_0]$-module $L_{\mathcal{O}(\theta)}$ for every $\mathcal{F}$-orbit $\mathcal{O}(\theta) = \theta.\mathcal{F}$. Moreover, assuming $\mathbf{C}_p \leq \mathcal{F}$ one has

$$L_{\mathcal{O}(\theta)} \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p = \coprod_{\eta \in \mathcal{R}(\theta)} \bar{L}_{\mathbf{C}_p.\bar{\eta}}, \tag{2.12}$$

where $\mathcal{R}(\theta)$ is system of representatives of the $\mathbf{C}_p$-set $\mathcal{O}(\theta)$,

$$\mathrm{res}_Q^{G_0}(L_{\mathcal{O}(\theta)}) = M_{\theta.\mathcal{F}} \tag{2.13}$$

$$\mathrm{res}_Q^{G_0}(L_{\mathcal{O}(\theta)}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p = \coprod_{\mu \in \mathcal{O}(\theta)} \bar{\mathbb{F}}_p(\bar{\mu}). \tag{2.14}$$

An $\mathbb{F}_p[G_0]$-module L is projective if, and only if, $L \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ is a projective $\bar{\mathbb{F}}_p[G_0]$-module. Hence one has:

Fact 2.2 *A non-trivial irreducible $\mathbb{F}_p[G_0]$-module is projective.*

2.3. The second exterior square

Let p be odd. Put $\_^\vee = \mathrm{Hom}_{\mathbb{F}_p}(\_, \mathbb{F}_p)$. If R is a finite group of p'-order, and $M = \coprod_{1 \leq j \leq n} M_j$ is an $\bar{\mathbb{F}}_p[R]$-module, then one has an isomorphism of $\bar{\mathbb{F}}_p[R]$-modules

$$\Lambda_2(M) \simeq \coprod_{1\le j\le n} \Lambda_2(M_j) \oplus \coprod_{1\le i<k\le n} M_i \otimes_{\bar{\mathbb{F}}_p} M_k. \tag{2.15}$$

For $\theta \in \mathrm{Hom}_{\mathrm{ab}}(Q, \mathbb{F}_{p^d}^*)^\sharp$ and $\mu, \nu \in \mathcal{O}(\theta)$ one has $\bar{\mathbb{F}}_p(\bar{\mu}) \simeq \bar{\mathbb{F}}_p(\bar{\nu})^\vee$, if and only if $\mu = -\nu$. This happens only if $2 \mid d$ or $q = 2$. Since $\Lambda_2(\bar{\mathbb{F}}_p(\bar{\mu})) = 0$, and

$$\Lambda_2\big(\mathrm{res}_Q^{G_0}(L_{\mathcal{O}(\theta)})\big) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p \simeq \Lambda_2\big(\mathrm{res}_Q^{G_0}(L_{\mathcal{O}(\theta)}) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p\big), \tag{2.16}$$

one obtains the following:

Fact 2.3 *Let p be odd. Then*

$$\mathrm{Hom}_Q\left(\mathbb{F}_p, \Lambda_2\big(\mathrm{res}_Q^{G_0}(L_{\mathcal{O}(\theta)})\big)\right) \simeq \begin{cases} 0 & \text{if } 2 \nmid d, \\ \big(\mathbb{F}_p^{d/2}\big)^p & \text{if } 2 \mid d \text{ and } p \nmid d, \\ \mathbb{F}_p^{d/2} & \text{if } 2 \mid d \text{ and } p \mid d. \end{cases} \tag{2.17}$$

3. The universal *p*-Schur-Frattini cover of a *pqp*-sandwich group

In this section we will study certain p-Frattini extension for a profinite group G. For a profinite group G the *Frattini subgroup* $\Phi(G)$ is defined as the intersection of all maximal closed subgroups of G. A surjective map $\tau\colon X \to G$ of profinite groups is called a *p-Frattini extension*, if $\ker(\tau)$ is a pro-p group and $\ker(\tau) \le \Phi(X)$.

3.1. *p*-Perfect profinite groups

Let G be a profinite group. Then

$$O^p(G) = \mathrm{cl}(\langle A \le G \mid A \text{ of } p'\text{-order}\rangle) \tag{3.1}$$

is a closed normal subgroup of G such that $G/O^p(G)$ is a pro-p group, and $O^p(G)$ is the minimal closed normal subgroup of G with this property. If $\phi\colon G \to H$ is a surjective mapping of profinite groups, then

$$\phi(O^p(G)) = O^p(H). \tag{3.2}$$

A profinite group G will be called *p-perfect*, if $G = O^p(G)$. Note that if $\phi\colon X \to G$ is a p-Frattini extension of the p-perfect group G, then X is also p-perfect. For p-perfect profinite groups one has also the following.

Proposition 3.1 *Let G be a p-perfect profinite group. Then $O_p(\mathrm{Z}(G)) \leq \Phi(G)$.*

Proof. Suppose there exists a maximal closed subgroup M of G not containing $A = O_p(\mathrm{Z}(G))$. Then $G = A.M$. Let r be a prime number co-prime to p, and let $R \in \mathrm{Syl}_r(M)$ be a Sylow pro-r subgroup of M. Then R is also a Sylow pro-r subgroup of G. Let ${}^gR$ be any Sylow pro-r subgroup of G. Since $g = a.m$ with $a \in A$ and $m \in M$, ${}^gR \leq M$ showing that M contains all Sylow pro-r subgroups of G. Hence M contains $O^p(G)$, a contradiction. $\square$

3.2. The universal p-Frattini extension

Every profinite group G has a *universal p-Frattini extension* $\pi_G\colon \tilde{G}_p \to G$, i.e., if $\tau\colon X \to G$ is p-Frattini extension, then there exists a homomorphism of profinite groups $\tau_\circ\colon \tilde{G}_p \to X$ making the diagram

$$\begin{tikzcd} \tilde{G}_p \arrow[rr, dashed, "\tau_\circ"] \arrow[dr, "\pi_G"'] & & X \arrow[dl, "\tau"] \\ & G & \end{tikzcd} \tag{3.3}$$

commute. Moreover, $\tau_\circ$ is surjective. The universal p-Frattini extension is unique up to isomorphism. For further details see [10, Section 2.2].

Let $\tilde{G}_p^c = \tilde{G}_p/[\tilde{G}_p, \ker(\pi_G)]$, and let $\pi_G^c\colon \tilde{G}_p^c \to G$ denote the induced map. Then $\pi_G^c\colon \tilde{G}_p^c \to G$ is the *universal central p-Frattini extension*, i.e., it has the universal property of (3.3) for every central p-Frattini extension $\tau\colon H \to G$. In general, this extension is quite difficult to analyze. Moreover, even if G is a finite group, $\tilde{G}_p^c$ can be infinite, e.g., for $G = \mathbf{C}_p$ one has $\tilde{G}_p^c \simeq \mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the additive group of the p-adic integers. However, if G is a finite p-perfect group, then $\pi_G^c\colon \tilde{G}_p^c \to G$ coincides with the *p-Schur cover* of G. In particular, in this case

$$\ker(\pi_G^c) \simeq H_2(G, \mathbb{Z}_p) \tag{3.4}$$

equals the p-Schur multiplier of G, and thus $\tilde{G}_p^c$ is also finite.

3.3. p-Schur-Frattini extension

A p-Frattini extension $\tau\colon X \to G$ is called a *p-Schur-Frattini extension*, if

$$\ker(\tau) \cap O^p(X) \leq \mathrm{Z}(O^p(X)). \tag{3.5}$$

Hence for such an extension the induced map $\tau_\circ\colon O^p(X) \to O^p(G)$ is a central p-Frattini extension (see Proposition 3.1).

Let $\tilde{G}_p^s = \tilde{G}_p/[O^p(\tilde{G}_p), \ker(\pi_G) \cap O^p(\tilde{G}_p)]$, and let $\pi_G^s\colon \tilde{G}_p^s \to G$ denote the induced map. Then $\pi_G^s\colon \tilde{G}_p^s \to G$ is a p-Schur Frattini extension, and it is universal, i.e., for any p-Schur-Frattini extension $\tau\colon X \to G$ there exists a map $\tau_\circ\colon \tilde{G}_p^s \to X$ making the corresponding diagram (3.3) commute. The map $\tau_\circ$ must be surjective. The usual standard argument shows that the extension $\pi_G^s\colon \tilde{G}_p^s \to G$ is unique up to isomorphism. The following proposition shows that it can be analyzed easily.

Proposition 3.2 *Let G be a profinite group, let $\pi_G^s\colon \tilde{G}_p^s \to G$ be the universal p-Schur-Frattini extension of G, and consider the commutative diagram*

$$\begin{tikzcd}
\{1\} \arrow[r] & O^p(\tilde{G}_p^s) \arrow[r] \arrow[d, "\pi_\circ^s"] & \tilde{G}_p^s \arrow[r] \arrow[d, "\pi_G^s"] & \tilde{G}_p^s/O^p(\tilde{G}_p^s) \arrow[r] \arrow[d, "\pi_q^s"] & \{1\} \\
\{1\} \arrow[r] & O^p(G) \arrow[r] & G \arrow[r] & G/O^p(G) \arrow[r] & \{1\}
\end{tikzcd} \tag{3.6}$$

Then $\pi_\circ^s$ coincides with the p-Schur cover of $O^p(G)$, and π_q^s coincides with the universal p-Frattini extension of $G/O^p(G)$.

Proof. The induced map $\tilde{G}_p/O^p(\tilde{G}_p) \to \tilde{G}_p^s/O^p(\tilde{G}_p^s)$ is an isomorphism. Hence $\mathrm{cd}_p(\tilde{G}^s/O^p(\tilde{G}_p^s)) \leq 1$ (see [10, Proposition 2.2]). As

$$\tilde{G}_p^s/O^p\big(\tilde{G}_p^s\big)/\Phi\big(\tilde{G}_p^s/O^p(\tilde{G}_p^s)\big) \longrightarrow G/O^p(G)/\Phi(G/O^p(G)) \tag{3.7}$$

is an isomorphism, $\pi_q^s\colon \tilde{G}_p^s/O^p(\tilde{G}_p^s) \to G/O_p(G)$ coincides with the universal p-Frattini extension.

Let $\tau\colon B \to O^p(G)$ denote the universal p-Frattini extension of $O^p(G)$. By construction, $\pi_\circ^s\colon O^p(\tilde{G}_p^s) \to O^p(G)$ is a central extension, and thus a p-Frattini extension (see Proposition 3.1). Thus one has a surjective map

$$\tau_\circ\colon B/[B, \ker(\tau)] \longrightarrow O^p\big(\tilde{G}_p^s\big). \tag{3.8}$$

As $\mathrm{cd}_p(O^p(\tilde{G}_p)) \leq 1$, there exists a map $\gamma\colon O^p(\tilde{G}_p) \to B$ making the diagram

$$\begin{tikzcd} O^p(\tilde{G}) \arrow[r, "\gamma"] \arrow[d, "\pi^s_\circ"'] & B \arrow[d, "\tau"] \\ O^p(G) \arrow[r, equal] & O^p(G) \end{tikzcd} \tag{3.9}$$

commute. Since τ is a Frattini extension, γ is surjective. Hence the induced map

$$\gamma_\circ\colon O^p\big(\tilde{G}^s_p\big) \longrightarrow B/[B, \ker(\tau)] \tag{3.10}$$

is also surjective. The same argument as used in the proof of [10, Theorem 2.5(c)] then shows that $\gamma_\circ \circ \tau_\circ$ is an isomorphism. Hence $\gamma_\circ$ is an isomorphism. □

Remark 3.3 Note that $\tilde{G}^s_p/O^p(\tilde{G}^s_p)$ is a free pro-p group. Let $P \in \mathrm{Syl}_p(\tilde{G}^s_p)$. Then the induced map $P \to \tilde{G}^s_p/O^p(\tilde{G}^s_p)$ is a split surjection. Hence the upper row in the diagram (3.6) is a split short exact sequence of profinite groups.

3.4. The universal *p*-Schur-Frattini extension of a *pqp*-sandwich group

Let p be odd, and let $\mathcal{S} = \mathcal{S}(Q, P_0)$ be a pqp-sandwich group. Again we denote by d the order of $p + \mathbb{Z}.q$ in $(\mathbb{Z}/\mathbb{Z}.q)^*$. By construction, one has

$$O^p(\mathcal{S}) = P_0 \rtimes Q. \tag{3.11}$$

By $\mathbb{I}_p = \mathbb{Q}_p/\mathbb{Z}_p$ we denote the standard injective $\mathbb{Z}_p$-torsion module. For a finitely generated $\mathbb{Z}_p$-module W we denote by $W^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(W, \mathbb{I}_p)$ its Pontrjagin dual. We have to consider three cases separately.

Case 0: $2 \nmid d$. From the universal coefficient theorem, the Hochschild-Lyndon-Serre spectral sequence and Fact 2.3 one concludes that

$$\begin{aligned} H_2(O^p(\mathcal{S}), \mathbb{Z}_p)^\vee &= H^2(O^p(\mathcal{S}), \mathbb{I}_p) = H^2(P_0, \mathbb{I}_p)^Q \\ &= \mathrm{Hom}_Q(\mathbb{F}_p, \Lambda_2(P_0^\vee)) = 0. \end{aligned} \tag{3.12}$$

Hence $H_2(\mathcal{S}, \mathbb{Z}_p) = 0$. The universal p-Schur-Frattini cover of $\mathcal{S}$ coincides with the pull-back of the diagram

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r] & \mathbf{C}_p \\ \tilde{\mathcal{S}}_p^s \arrow[u, dashed] \arrow[r, dashed] & \mathcal{S} \arrow[u] \end{tikzcd} \tag{3.13}$$

Case 1.A: $p \nmid d$, $2 \mid d$. Put $e = d/2$. Then

$$\langle \_, \_ \rangle \colon \mathbb{F}_{p^d} \times \mathbb{F}_{p^d} \longrightarrow \mathbb{F}_{p^e}, \quad \langle x, y \rangle = x\, z^{p^e} - x^{p^e}\, z, \quad x, z \in \mathbb{F}_{p^d}, \tag{3.14}$$

is a skew-symmetric $\mathbb{F}_{p^e}$-linear form. Moreover, the set $\mathbb{F}_{p^d} \times \mathbb{F}_{p^e}$ with multiplication

$$(x, y) \cdot (x', y') = (x + x', y + y' + \langle x, x' \rangle), \quad x, x' \in \mathbb{F}_{p^d},\ y, y' \in \mathbb{F}_p^e, \tag{3.15}$$

is a group which we will denote by $\mathrm{UT}_3(\mathbb{F}_{p^e})$. It is isomorphic to the group of uni-triangular matrices of rank 3 over the field $\mathbb{F}_{p^e}$.

By definition, $q \mid p^e + 1$. Hence the cyclic group $\mathbf{C}_q$ has a natural action on $\mathrm{UT}_3(\mathbb{F}_{p^e})$. Moreover, every homomorphism $\theta \in \mathrm{Hom}_{\mathrm{ab}}(Q, \mathbb{F}_{p^d}^*)$ can be extended uniquely to a homomorphism $\tilde{\theta} \in \mathrm{Hom}_{\mathrm{gr}}(Q, \mathrm{Aut}(\mathrm{UT}_3(\mathbb{F}_{p^e})))$ given by

$$\tilde{\theta}(g).(x, y) = (\theta(g).x, y), \quad x \in \mathbb{F}_{p^d},\ y \in \mathbb{F}_{p^e}. \tag{3.16}$$

Let $H = (\mathrm{UT}_3(\mathbb{F}_{p^e}))^p \rtimes Q$ such that $P_0 \rtimes Q$ coincides with the canonical quotient $H/\,\mathrm{Z}(O_p(H))$, and put $\mathcal{S}_0 = H \rtimes \mathbf{C}_p$. The same calculation as in Case 0 shows that $\tilde{\mathcal{S}}_p^s$ coincides with the pull-back of the diagram

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r] & \mathbf{C}_p \\ \tilde{\mathcal{S}}_p^s \arrow[u, dashed] \arrow[r, dashed] & \mathcal{S}_0 \arrow[u] \end{tikzcd} \tag{3.17}$$

Case 1.B: $p \mid d$, $2 \mid d$. Put $f = d/p$ and $e = d/2$. As in Case B we may construct a group $H = \mathrm{UT}_3(\mathbb{F}_{p^e}) \rtimes Q$. Let g be a generator of $\mathbf{C}_p$. Then

$$g.(x,y) = (x^{p^f}, y^{p^f}), \quad g.a = a^{p^f}, \quad (x,y) \in \mathrm{UT}_3(\mathbb{F}_{p^e}),\ a \in Q, \tag{3.18}$$

defines a left action of $\mathbf{C}_p$ on H. Let $\mathcal{S}_0 = H \rtimes \mathbf{C}_p$. Then $\tilde{\mathcal{S}}_p^s$ coincides with the pull-back of the diagram (3.17).

3.5. The groups $Y_p(m)$ as subgroups

The following fact can be proved easily using the pull-back property.

Fact 3.4 *Let p be odd, and let P be a finite p-group containing a cyclic subgroup $C \simeq \mathbf{C}_{p^m}$ and an elementary abelian p-subgroup A with the following properties*:

(i) $\dim_{\mathbb{F}_p}(A) = p$;
(ii) $C \leq N_P(A)$;
(iii) $C^p \leq \mathrm{Cent}_P(A)$;
(iv) *as* $\mathbb{F}_p[C/C^p]$*-module A is projective*;
(v) $C \cap A = \{1\}$.

Then $A.C \leq P$ is a subgroup isomorphic to $Y_p(m)$.

From Fact 3.4 one concludes the following.

Proposition 3.5 *Let p be odd, and let $\tau\colon G \to \mathcal{S}$ be a finite p-Schur-Frattini extension of a pqp-sandwich group $\mathcal{S}$. Then G contains a subgroup isomorphic to $Y_p(m)$ for some $m \geq 1$.*

Proof. Let $\pi^s_{\mathcal{S}}\colon \tilde{\mathcal{S}}_p^s \to \mathcal{S}$ denote the universal p-Schur-Frattini extension. Then there exists a map $\tau_\circ\colon \tilde{\mathcal{S}}_p^s \to G$ making the diagram

$$\begin{tikzcd} \tilde{\mathcal{S}}_p^s \arrow[rr, "\tau_\circ"] \arrow[dr, "\pi^s_{\mathcal{S}}"'] & & G \arrow[dl, "\tau"] \\ & \mathcal{S} & \end{tikzcd} \tag{3.19}$$

commute. We have to deal with three cases separately.

Case 0: $2 \nmid d$. Let $C \leq \tilde{\mathcal{S}}_p^s$ be a complement to $O^p(\tilde{\mathcal{S}}_p^s)$ in $\tilde{\mathcal{S}}_p^s$, and let $B = O_p(O^p(\tilde{\mathcal{S}}_p^s))$. Put $C_\circ = \tau_\circ(C)$ and $B_\circ = \tau_\circ(B)$. Then $C_\circ$ is cyclic of order p^m, $C_\circ \leq N_G(B_\circ)$ and $C_\circ^p \leq \mathrm{Cent}_G(B_\circ))$. Then $B_\circ$ is a projective $\mathbb{F}_p[C_\circ/C_\circ^p]$-module. Let $A_\circ \leq B_\circ$ be a projective $\mathbb{F}_p[C_\circ/C_\circ^p]$-submodule of dimension p. If $m = 1$, $A_\circ.C_\circ \simeq Y_p(1)$. If $m > 1$, $B_\circ \cap \Omega_1(C_\circ^p)$ is a trivial $\mathbb{F}_p[O^p(G)]$-submodule of $B_\circ$, and thus $B_\circ \cap \Omega_1(C_\circ^p) = \{1\}$. Hence $C_\circ \cap A_\circ = \{1\}$, and, by Fact 3.4, $A_\circ.C_\circ \simeq Y_p(m)$.

Case 1.A: $2 \mid d$, $p \nmid d$. As in Case 0, let $C \leq \tilde{\mathcal{S}}_p^s$ be a complement to $O^p(\tilde{\mathcal{S}}_p^s)$ in $\tilde{\mathcal{S}}_p^s$. Let

$$X = \left\{(x,0) \in \mathrm{UT}_3\left(\mathbb{F}_p^{d/2}\right) \mid x \in \mathbb{F}_p^{d/2}\right\}, \tag{3.20}$$

and let $B = \langle X^c \mid c \in C \rangle \leq \tilde{\mathcal{S}}_p^s$. Put $B_\circ = \tau_\circ(B)$, $C_\circ = \tau_\circ(C)$. Then $C_\circ$ is cyclic of order p^m, $B_\circ$ is an elementary abelian p-subgroup of G, $C_\circ \leq N_G(B_\circ)$, $C_\circ^p \in \mathrm{Cent}_G(B_\circ)$. Moreover, as $\Omega_1(C_\circ^p) \leq \mathrm{Cent}_G(O^p(G))$ and $B_\circ \cap \mathrm{Z}(O^p(G)) = \{1\}$, one has $C_\circ \cap B_\circ = \{1\}$. By construction, one has an isomorphism

$$B_\circ \simeq \mathrm{ind}_{\{1\}}^{C_\circ/C_\circ^p}(X) \tag{3.21}$$

as $\mathbb{F}_p[C_\circ/C_\circ^p]$-modules. Hence $B_\circ$ is a projective $\mathbb{F}_p[C_\circ/C_\circ^p]$-module. By choosing an appropriate submodule $A_\circ$ as in Case 0, one shows that $B_\circ.C_\circ$ contains a subgroup isomorphic to $Y_p(m)$.

Case 1.B: $2 \mid d$, $p \mid d$. Put $f = d/p$. As in Case 0, let $C \leq \tilde{\mathcal{S}}_p^s$ be a complement to $O^p(\tilde{\mathcal{S}}_p^s)$ in $\tilde{\mathcal{S}}_p^s$. Let $F\colon \mathbb{F}_{p^d} \to \mathbb{F}_{p^d}$, $F(x) = x^p$, denote the standard Frobenius automorphism, and put $F_0 = F^f$. Then F_0 has order p and is acting non-trivially on $\mathbb{F}_{p^{d/2}}$. Let $x \in \mathbb{F}_{p^{d/2}}$, $F_0(x) \neq x$, and put $X = \mathrm{span}_{\mathbb{F}_p}(F_0^k(x) \mid k \geq 0)$. Then $A \leq \tilde{\mathcal{S}}_p^s$, where

$$A = \left\{(x,0) \in \mathrm{UT}_3\left(\mathbb{F}_{p^{d/2}}\right) \mid x \in X\right\}, \tag{3.22}$$

is an elementary abelian subgroup of order p. Moreover, $C \leq N_{\tilde{\mathcal{S}}_p^s}(A)$, $C^p \leq \mathrm{Cent}_{\tilde{\mathcal{S}}_p^s}(A)$, and A is a projective $\mathbb{F}_p[C/C^p]$-module. Using Fact 3.4 and the same argument used in Case 1.A one shows that $\tau_\circ(A.C)$ is isomorphic to $Y_p(m)$ for some $m \geq 1$. $\square$

4. The Main Theorem

4.1. G. Glauberman's version of J. G. Thompson's p-nilpotency criterion

Let P be a finite p-group. Then

$$J(P) = \langle A \leq P \text{ abelian} \bigm| |A| \text{ maximal} \rangle \tag{4.1}$$

is called the *Thompson subgroup*[2] of P. In [2, Theorem 14.11], G. Glauberman proved the following version of J. G. Thompson's p-nilpotency criterion (see [9]).

Theorem 4.1 (G. Glauberman, J. G. Thompson) *Let p be a prime number, let G be a finite group and let $P \in \mathrm{Syl}_p(G)$. If $p = 2$ assume also that G is S_4-free. Then G is p-nilpotent if, and only if, $\mathrm{Cent}_G(\mathrm{Z}(P))$ and $N_G(J(P))$ are p-nilpotent.*

From this nilpotency criterion one can deduce the following straight forward consequence.

Corollary 4.2 *Let G be a finite group, and let $P \in \mathrm{Syl}_p(G)$, p odd. Assume that for all $R \lhd P$, $R \neq 1$, the subgroup $N_G(R)$ is p-nilpotent. Then G is p-nilpotent.*

4.2. The proof

Proof of the Main Theorem. For simplicity put $\mathfrak{X}_2 = \mathfrak{xslim}_2$ and $\mathfrak{X}_p = \mathfrak{slim}_p$ for p odd. Let $\mathcal{P} \in \mathfrak{X}_p$ be a p-group of minimal order with the property that $\mathcal{P} \notin \mathfrak{DN}_p$, and let G be a finite group of minimal order such that $P \in \mathrm{Syl}_p(G)$, $P \simeq \mathcal{P}$ and that $N_G(P)$ is p-nilpotent. From the minimality of G follows:

Conclusion 1 *Let $H \leq G$ be a proper subgroup of G containing P. Then H is p-nilpotent.*

Note that $\mathbf{C}_2 \wr \mathbf{C}_2$ is isomorphic to the Sylow 2-subgroup of S_4. Thus, by hypothesis, for $p = 2$ the group G is also S_4-free. The subgroups $\mathrm{Cent}_G(\mathrm{Z}(P))$ and $N_G(J)$ both contain the Sylow p-subgroup P. By J. G. Thompson's p-nilpotency criterion and the minimality of G not both of them can be proper (see Conclusion 1). Thus one has either

[2]This is the version of the Thompson subgroup used by G. Glauberman in [2].

$G = \mathrm{Cent}_G(\mathrm{Z}(P))$, or $G = N_G(J)$, where $J = J(P)$ and therefore:

Conclusion 2 $O_p(G) \neq \{1\}$.

From Fact 2.1(ii) one deduces:

Conclusion 3 $O_{p'}(G) = \{1\}$.

Since $\bar{G} = G/O_p(G)$ is non-trivial, there exists a normal subgroup M of G such that $M/O_p(G)$ is a minimal normal subgroup in $\bar{G}$. Thus, either $M/O_p(G)$ is of p'-order, or $M/O_p(G) \simeq S \times \cdots \times S$ for some non-abelian finite simple group S, which order is divisible by p. The second case cannot occur:

Conclusion 4 *$M/O_p(G)$ is of p'-order.*

Proof of Conclusion 4. Suppose $M/O_p(G) \simeq S \times \cdots \times S$ for some finite simple group S, which order is divisible by p. Note that for $p = 2$, $\mathfrak{X}_2$ is **q**-closed. Hence the minimality of P implies that $G/O_2(G)$ is 2-nilpotent. A contradiction, and hence we may assume that p is odd.

The subgroup $M.P$ is not p-nilpotent and contains P. Thus $G = M.P$ (see Conclusion 1). Let $\tilde{P} = P \cap M \in \mathrm{Syl}_p(M)$. Then $N_G(\tilde{P}) = P.N_M(\tilde{P})$, and by Conclusion 1, $N_M(\tilde{P})$ is p-nilpotent. Thus the minimality of P implies that $P = \tilde{P}$, and $G = M$. Thus by Fact 2.1(i) and the minimality of P, $\bar{G} = G/O_p(G)$ is simple and isomorphic to S. Let $\bar{P} = P/O_p(G)$. For $\bar{R} \lhd \bar{P}$, $\bar{R} \neq 1$, let $R \lhd P$ denote the normal subgroup of P containing $O_p(G)$ such that $\bar{R} = R/O_p(G)$. Then

$$N_G(R)/O_p(G) = N_{\bar{G}}(\bar{R}). \tag{4.2}$$

By construction, $N_G(R)$ is a proper subgroup of G containing P and thus p-nilpotent (see Conclusion 1). Hence $N_{\bar{G}}(\bar{R})$ is p-nilpotent. Thus by Corollary 4.2, $G/O_p(G)$ is p-nilpotent, a contradiction, and this yields the claim. $\square$

Note that $\mathrm{Cent}_M(O_p(G))$ is a normal subgroup, and

$$\mathrm{Cent}_M(O_p(G)) = \mathrm{Z}(O_p(G)) \times O_{p'}(\mathrm{Cent}_M(O_p(G))). \tag{4.3}$$

Thus Conclusion 3 implies that $\mathrm{Cent}_M(O_p(G)) \leq O_p(G)$. In particular, M is not p-nilpotent. Hence $G = M.P$ (see Conclusion 1).

Let $q \neq p$ be a prime number dividing the order of M, and let $Q \in \mathrm{Syl}_q(M)$. By the previously mentioned remark, $O_p(G).Q$ is not p-nilpotent. The Frattini argument implies that $G = M.N_G(Q)$, and hence $O_p(G).N_G(Q)$ contains a Sylow p-subgroup of G. By Conclusion 1, $G = O_p(G).N_G(Q)$. In particular, $M = O_p(G).Q$ and Q is an elementary abelian q-group.

Let $P_\circ \in \mathrm{Syl}_p(N_G(Q))$. Then $O_p(G).P_\circ$ is a Sylow p-subgroup of G, and we may assume that $P = O_p(G).P_\circ$. As M is not p-nilpotent, Conclusion 1 implies that $G = O_p(G).Q.P_\circ$.

Since $G \neq M$, there exists an element $g \in P_\circ \setminus M = P_\circ \setminus O_p(G)$ such that $g.M \in \mathrm{Z}(P_\circ.M/M)$ is order p. Put $C = \langle g \rangle \leq P_\circ$. By construction, $C^p \leq P_\circ \cap O_p(G)$ and $O_p(G).C$ is normal in P.

Suppose that Q is a trivial $\mathbb{F}_p[C]$-module. Then $Q \leq \mathrm{Cent}_G(C)$ and thus $P, Q \leq N_G(C.O_p(G))$. Hence $C.O_p(G)$ is normal in G, a contradiction. Thus there exists a non-trivial irreducible $\mathbb{F}_p[C]$-submodule $Q_0 \leq Q$. As $\mathrm{Cent}_G(O_p(G))$ is a p-group, Q_0 is acting non-trivially on $O_p(G)$. In particular, $O_p(G).Q_0$ is not p-nilpotent. Let $\tilde{G} = O_p(G).Q_0.C$. By construction, one has $N_{\tilde{G}}(O_p(G).C) = O_p(G).C$ (see Fact 2.1(i)). Thus the minimality of P and G imply that $G = \tilde{G}$ and $P = O_p(G).C$. Note that $N_G(P) = P$ and $G/M \simeq \mathbf{C}_p$.

Conclusion 5 *If $p = 2$, then G is a $2q2$-sandwich group. If p is odd, then $G/\Phi(G)$ is a pqp-sandwich group and*

$$\{1\} \longrightarrow \Phi(G) \longrightarrow G \longrightarrow G/\Phi(G) \longrightarrow \{1\} \tag{4.4}$$

is a p-Schur-Frattini extension.

Proof of Conclusion 5. Let $p = 2$. Since Q acts non-trivially on $O_2(G)$, Q acts non-trivially on $O_2(G)/\Phi(O_2(G))$. Thus $\bar{G} = G/\Phi(O_2(G))$ is not p-nilpotent, and for $\bar{P} = P/\Phi(O_2(G))$, one has $N_{\bar{G}}(\bar{P}) = \bar{P}$. Thus the minimality of P implies that $\Phi(O_2(G)) = \{1\}$, and $O_2(G)$ is elementary abelian. Let $G_0 = G/O_2(G)$. The $\mathbb{F}_2[Q]$-module $A = O_2(G)$ must have a non-trivial composition factor (see Conclusion 3). Hence the $\mathbb{F}_2[G_0]$-module $B = O_2(G)$ must have a non-trivial irreducible composition factor B_0 (see Section 2.2). Since B_0 is projective and injective, it is a direct summand of B. Thus the previously mentioned argument shows that $O_2(G) = B$. Since B is a projective $\mathbb{F}_2[G_0]$-module, the extension $\{1\} \to B \to G \to G_0 \to \{1\}$ must split. Hence G is a $2q2$-sandwich group.

Let p be odd. By Conclusion 3, $\Phi(G)$ is a p-group. As $G = O_p(G).N_G(Q)$, there exists a maximal subgroup M_1 containing $N_G(Q)$ and $D = M_1 \cap O_p(G) \neq O_p(G)$. Since $\Phi(O_p(G)) \leq \Phi(G)$ (see [5, Section 3, Hilfsatz 3.3]), M_1 contains $\Phi(O_p(G))$. In particular, D is normal in $O_p(G)$. As $G = O_p(G).M_1$, D is normal in G. Assume there exists a normal subgroup N of G satisfying $M_1 \cap O_p(G) \lneq N \lneq O_p(G)$. Then N is not contained in M_1 and thus $N.M_1 = G$. In particular, for all $g \in O_p(G)$ there exists $x \in N$, and $y \in M_1$ such that $g = xy$. But $y = x^{-1}g \in M_1 \cap O_p(G)$. Thus $g \in N$, a contradiction.

Hence D is a maximal normal subgroup of G properly contained in $O_p(G)$. In particular, $B = O_p(G)/D$ is an irreducible $\mathbb{F}_p[G_0]$-module, where $G_0 = G/O_p(G)$. Let $P_1 \in \mathrm{Syl}_p(M_1)$. Since $M_1/D \simeq G_0$, $N_{M_1}(P_1) = P_1$. Thus by the minimality of P, M_1 is p-nilpotent, and therefore $M_1 = N_G(Q)$. As D is normal in M_1, $Q \leq \mathrm{Cent}_G(D)$. Since D is normal in G, one has ${}^gQ \leq \mathrm{Cent}_G(D)$, or equivalently $D \leq \mathrm{Cent}_G({}^gQ)$ for all $g \in G$. In particular,

$$D \cap O^p(G) \leq \mathrm{Z}(O^p(G)). \tag{4.5}$$

Suppose that B is a trivial $\mathbb{F}_p[G_0]$-module. Then $Q \leq \mathrm{Cent}_G(O_p(G)/D)$, and therefore $Q \leq \mathrm{Cent}_M(O_p(G))$ and $O_{p'}(G) \neq \{1\}$, a contradiction (see Conclusion 3). This implies that the irreducible $\mathbb{F}_p[G_0]$-module B is non-trivial. In particular, it is projective (see Section 2.2) and the extension $\{1\} \to B \to G/D \to G_0 \to \{1\}$ splits. Thus G/D is a pqp-sandwich group. Hence $\Phi(G/D) = \{1\}$.

Suppose there exists a maximal subgroup M_2 of G not containing D. Then $G = D.M_2$, and $M_2/(M_2 \cap D) \simeq G/D$. In particular, for $P_2 \in \mathrm{Syl}_2(M_2)$ one has $N_{M_2}(P_2) = P_2$ (see Fact 2.1(i)), and M_2 is not p-nilpotent, a contradiction to the minimality of P. Hence D is contained in every maximal subgroup of G, i.e., $D \leq \Phi(G)$. By (4.5), the extension (4.4) is a p-Schur-Frattini extension. □

By Fact 2.2, every $2q2$-sandwich group contains a subgroup isomorphic to the dihedral group D_8, a contradiction.

For p odd, every finite p-Schur-Frattini extension of a pqp-sandwich group contains a subgroup isomorphic to $Y_p(m)$ for some $m \geq 1$ (see Proposition 3.5). Again a contradiction, and the theorem is proved. □

4.3. Subclasses of $\mathfrak{DN}_p$

Let $\mathfrak{X}$ be a class of finite groups. Then $\mathfrak{X}$ is called **s**-*closed*, if $G \in \mathfrak{X}$ and H a subgroup of G implies that $H \in \mathfrak{X}$. Simarly, $\mathfrak{X}$ is called *q-closed*, if for all $G \in \mathfrak{X}$ and normal subgroups N of G follows that $G/N \in \mathfrak{X}$. From the counterexamples mentioned in the introduction (see Example 1.1(b)) one concludes the following.

Proposition 4.3 *The class $\mathfrak{xslim}_2$ is a maximal* **s**- *and* **q**-*closed subclass of $\mathfrak{DN}_2$; for p odd the class $\mathfrak{slim}_p$ is a maximal* **s**-*closed subclass of $\mathfrak{DN}_p$.*

Proof. Let p be odd, and let Let $\mathfrak{X}_p$ be a **s**-closed class of finite p-groups containing $\mathfrak{slim}_p$ and being contained in $\mathfrak{DN}_p$. Suppose there exists $P \in \mathfrak{X}_p \setminus \mathfrak{slim}_p$. Then P contains a subgroup isomorphic to $Y_p(m)$, $m \geq 1$. Since $\mathfrak{X}_p$ is **s**-closed, $Y_p(m) \in \mathfrak{X}_p$ and thus $Y_p(m) \in \mathfrak{DN}_p$. Let G be the group described in Example 1.1(b) and let $\beta\colon G \to \mathbf{C}_p$ denote the canonical homomorphism. The pull-back H of the diagram

$$\begin{tikzcd} \mathbf{C}_{p^m} \arrow[r] & \mathbf{C}_p \\ H \arrow[u, dashed] \arrow[r, dashed] & G \arrow[u, "\beta"'] \end{tikzcd} \tag{4.6}$$

is not p-nilpotent, for $P \in \mathrm{Syl}_p(H)$ one has $N_G(P) = P$ and $P \simeq Y_p(m)$. Thus $Y_p(m) \notin \mathfrak{DN}_p$, a contradiction. The case $p = 2$ follows by a similar argument. $\square$

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