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# Boundedness of rough singular integral operators on the homogeneous Morrey-Herz spaces

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**Abstract.** Some boundedness results are established for some rough singular integral operators and fractional singular integral operators on the Homogeneous Morrey-Herz Spaces. And also we get some boundedness results on the weak homogeneous Morrey-Herz Spaces.

*Key words*: Lebesgue spaces, singular integral operator, fractional singular integral operator, power weight, homogeneous Morrey-Herz spaces.

## 1. Introduction

Let $\Omega$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and $\Omega \in L^r(S^{n-1})$ for some $r \in [1, \infty]$. If $f \in L^q_\omega(\mathbb{R}^n)$, that is

$$\|f\|_{L^q_\omega(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^q \omega(x)dx\right)^{1/q} < \infty.$$

Let $T$ satisfy

$$|Tf(x)| \leq c\int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy. \tag{1}$$

for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$. The main purpose of this paper is to establish the boundedness of the sublinear operator $T$ in the setting of the Morrey-Herz Spaces.

Let $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n \colon |x| \leq 2^k\}$ and $A_k = B_k/B_{k-1}$ for $k \in \mathbb{Z}$. Let $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$ be the characteristic function of the set $A_k$.

**Definition 1.1** Let $\alpha \in \mathbb{R}$, $0 < p \leq \infty$ $0 < q < \infty$ and $\lambda \geq 0$. The homogeneous Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ are defined by

$$M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n) = \left\{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n/\{0\}),\ \|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} < \infty\right\},$$

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where

$$\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)}\right)^{1/p}$$

with the usual modifications made when $p=\infty$.

Compare the homogeneous Morrey-Herz spaces $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ with the homogeneous Herz spaces $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ (see [1]) and the Morrey spaces $M^\lambda_q(\mathbb{R}^n)$ (see [2]), where $\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha,p}_q(\mathbb{R}^n) = \left\{f\in L^q_{\mathrm{loc}}(\mathbb{R}^n/\{0\})\colon \sum_{k=-\infty}^{\infty} 2^{kp\alpha}\|f\chi_k\|^p_{L^q(\mathbb{R}^n)} < \infty\right\},$$

and $M^\lambda_q(\mathbb{R}^n)$ is defined by

$$M^\lambda_q(\mathbb{R}^n) = \left\{f\in L^q_{\mathrm{loc}}(\mathbb{R}^n)\colon \sup_{r>0,\, x\in\mathbb{R}^n}\frac{1}{r^\lambda}\int_{|x-y|<r}|f(y)|^q dy < \infty\right\}.$$

Obviously, $M\dot{K}^{\alpha,0}_{p,q}(\mathbb{R}^n) = \dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ and $M^\lambda_q(\mathbb{R}^n)\subset M\dot{K}^{0,\lambda}_{q,q}(\mathbb{R}^n)$.

We can see that when $\lambda=0$, our results below coordinate with those in the setting of the Herz spaces, which proved by Hu, Lu and Yang in [3]. So in this paper, we only give the results when $\lambda>0$.

In what follows, for any $k\in\mathbb{Z}$ and any $\sigma>0$, let $m_k(\sigma, f) = \big|\{x\in A_k\colon |f(x)|>\sigma\}\big|$.

**Definition 1.2** Let $\alpha\in\mathbb{R}$, $0<p\le\infty$, $\lambda\ge 0$ and $0<q<\infty$. A measurable function $f$ on $\mathbb{R}^n$ is said to belong to the homogeneous weak Morrey-Herz spaces $WM\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$, if

$$\|f\|_{WM\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = \sup_{\gamma>0}\gamma\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0} 2^{k\alpha p} m_k(\gamma, f)^{p/q}\right)^{1/p} < \infty,$$

where the usual modifications are made when $p=\infty$.

Compare the homogeneous weak Morrey-Herz spaces $WM\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ with the homogeneous weak Herz spaces $W\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$ (see [7]). Clearly, $WM\dot{K}^{\alpha,0}_{p,q}(\mathbb{R}^n) = W\dot{K}^{\alpha,p}_q(\mathbb{R}^n)$. When $\lambda=0$, the boundedness is also proved by Hu, Lu and Yang in [3].

## 2. Main theorems and their proofs

In order to prove our main theorem, we begin with the following theorem which will be needed for proving our main theorem. And the following theorem indicates that the boundedness of operator $T$ on Lebesgue spaces with power weights implies its boundedness on homogeneous Morrey-Herz spaces.

**Theorem 2.1** *If a sublinear operator $T$ is bounded on $L^q_{|x|^\beta}(\mathbb{R}^n)$ for all $\beta \in (\beta_1, \beta_2)$, and some $q \in (1, \infty)$, where $\beta_1, \beta_2 \in \mathbb{R}$, then $T$ is also bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$ provided that $\alpha \in (\beta_1/q + \lambda, \beta_2/q + \lambda)$ and $0 < p \leq \infty$, $\lambda > 0$.*

*Proof.* We choose $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\beta_1/q < \alpha_1/q < \alpha - \lambda < \alpha_2/q < \beta_2/q$. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x).$$

Then,we have

$$\begin{aligned}
\|Tf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} &= \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|Tf(x)\chi_k\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p} \\
&\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{k+1} \|Tf_j(x)\chi_k\|_{L^q(\mathbb{R}^n)}\Big)^p\Big)^{1/p} \\
&\quad + C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty} \|Tf_j(x)\chi_k\|_{L^q(\mathbb{R}^n)}\Big)^p\Big)^{1/p} \\
&\equiv E_1 + E_2
\end{aligned}$$

For $E_1$, we have

$$\begin{aligned}
E_1 &\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k(\alpha-\alpha_2/q)p}\Big(\sum_{j=-\infty}^{k+1} \|Tf_j(x)\chi_k\|_{L^q_{|x|^{\alpha_2}}(\mathbb{R}^n)}\Big)^p\Big)^{1/p} \\
&\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k(\alpha-\alpha_2/q)p}\Big(\sum_{j=-\infty}^{k+1} \|f_j\|_{L^q_{|x|^{\alpha_2}}(\mathbb{R}^n)}\Big)^p\Big)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}\Big(\sum_{j=-\infty}^{k+1} 2^{j\alpha_2/q}2^{k(\alpha-\alpha_2/q)}\|f_j\|_{L^q(\mathbb{R}^n)}\Big)^p\bigg)^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}\Big(\sum_{j=-\infty}^{k+1} 2^{j\alpha_2/q}2^{k(\alpha-\alpha_2/q)}2^{-j\alpha}2^{j\lambda}2^{-j\lambda}\\
&\qquad\qquad\times\Big(\sum_{l=-\infty}^{j} 2^{l\alpha p}\|f_l\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}\Big)^p\bigg)^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}2^{kp\lambda}\Big(\sum_{j=-\infty}^{k+1}2^{(j-k)(-\alpha+\alpha_2/q+\lambda)}\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\Big)^p\bigg)^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}2^{kp\lambda}\bigg)^{1/p}\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}2^{k_0\lambda}\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}=C\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.
\end{aligned}
$$

For $E_2$, similar to $E_1$, we have

$$
\begin{aligned}
E_2&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}2^{k(\alpha-\alpha_1/q)p}\Big(\sum_{j=k+2}^{\infty}\|Tf_j(x)\chi_k\|_{L^q_{|x|^{\alpha_1}}(\mathbb{R}^n)}\Big)^p\bigg)^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}2^{k(\alpha-\alpha_1/q)p}\Big(\sum_{j=k+2}^{\infty}\|f_j\|_{L^q_{|x|^{\alpha_1}}(\mathbb{R}^n)}\Big)^p\bigg)^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}\Big(\sum_{j=k+2}^{\infty} 2^{j\alpha_1/q}2^{k(\alpha-\alpha_1/q)}\|f_j\|_{L^q(\mathbb{R}^n)}\Big)^p\bigg)^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}\Big(\sum_{j=k+2}^{\infty} 2^{j\alpha_1/q}2^{k(\alpha-\alpha_1/q)}2^{-j\alpha}2^{j\lambda}2^{-j\lambda}\\
&\qquad\qquad\times\Big(\sum_{l=-\infty}^{j} 2^{l\alpha p}\|f_l\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}\Big)^p\bigg)^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}2^{kp\lambda}\Big(\sum_{j=k+2}^{\infty}2^{(j-k)(-\alpha+\alpha_1/q+\lambda)}\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\Big)^p\bigg)^{1/p}\\
&\le C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}2^{kp\lambda}\bigg)^{1/p}\|f\|_{M\dot K^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}
\end{aligned}
$$

$$\leq C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}2^{k_0\lambda}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)} = C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 2.1. □

Let $\Omega$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and $\Omega \in L^r(S^{n-1})$ for some $r \in [1, \infty]$. The rough Hardy-Littlewood maximal function

$$M_\Omega f(x) = \sup_{t>0} \frac{1}{t^n} \int_{|y|<t} |\Omega(y')f(x-y)|dy,$$

where $y' = y|y|^{-1}$.

The following lemma for the boundedness of $M_\Omega$ on the Lebesgue spaces with power weights can be found in [4, P. 874].

**Lemma 2.1** [4] *If $\Omega \in L^r(S^{n-1})$, $1 \leq r \leq \infty$, $1 < q < \infty$ and*

$$\max\Big(-n,\, -1 - \frac{(n-1)q}{r'}\Big) < \beta < \min\Big(n(q-1),\, q-1+\frac{(n-1)q}{r'}\Big),$$

*then $M_\Omega$ is bounded on $L^q_{|x|^\beta}(\mathbb{R}^n)$.*

As a simple corollary of Theorem 2.1 and Lemma 2.1, we have the following corollary, which will be used in the proof of our main theorem.

**Corollary 2.1** *Let $\Omega \in L^r(S^{n-1})$, $1 \leq r \leq \infty$, $1 < q < \infty$, $0 < p \leq \infty$ and*

$$\begin{aligned}&\max\Big(-\frac{n}{q}+\lambda,\, -\frac{1}{q}-\frac{n-1}{r'}+\lambda\Big)\\ &<\alpha\\ &<\min\Big(n\Big(1-\frac{1}{q}\Big)+\lambda,\, 1-\frac{1}{q}+\frac{n-1}{r'}+\lambda\Big),\end{aligned}$$

*then $M_\Omega$ is bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$.*

To prove our main theorem, we also need the following lemma. It can be found in [5, P. 251].

**Lemma 2.2** [5] *Let $\Omega \in L^r(S^{n-1})$, $1 \leq r \leq \infty$. If $a > 0$, $0 < d \leq r$ and $-n + (n-1)d/r < \beta < \infty$, then*

$$\Big(\int_{|x|\leq a|y|} |\Omega(x-y)|^d |x|^\beta dx\Big)^{1/d} \leq C|y|^{(\beta+n)/d}\|\Omega\|_{L^r(\mathbb{R}^n)}.$$

In fact, when $1 < r \leq \infty$, this lemma is just a special case of [5, Lemma 1]. The case $r = 1$ is easily proved using polar coordinates. We omit the details.

**Theorem 2.2** *Let $\Omega \in L^r(S^{n-1})$ for $r > 1$ and let $0 < p < \infty$, $1 < q < \infty$, $\lambda > 0$ and*

$$\begin{aligned}&\max\Big(-\frac{n}{q}+\lambda,\ -\frac{1}{q}-\frac{n-1}{r'}+\lambda\Big)\\ <&\alpha\\ <&\min\Big(n\Big(1-\frac{1}{q}\Big)+\lambda,\ 1-\frac{1}{q}+\frac{n-1}{r'}+\lambda\Big).\end{aligned}$$

*If a sublinear operator $T$ is bounded on $L^q(\mathbb{R}^n)$ and there is a constant $C$ independent of $f$ such that*

$$|Tf(x)| \leq c\int_{\mathbb{R}^n}\frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy.$$

*for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$, then $T$ is also bounded on $M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)$.*

*Proof.* Write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x).$$

and

$$\begin{aligned}&\|Tf\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\\ &\quad=\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\|(Tf)\chi_k\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}\\ &\quad\leq C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big\|T\Big(\sum_{j=-\infty}^{k-2} f_j\Big)\chi_k\Big\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}\\ &\qquad+C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\sum_{j=k-1}^{k+1}\|(Tf_j)\chi_k\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}\\ &\qquad+C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big\|T\Big(\sum_{j=k+2}^{\infty} f_j\Big)\chi_k\Big\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}\end{aligned}$$

$$\equiv E_1 + E_2 + E_3.$$

For $E_2$, by the $L^q(\mathbb{R}^n)$-boundedness of $T$, we have

$$\begin{aligned}
E_2 \leq & C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \Big( \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \sum_{j=k-1}^{k+1} \|f_j\chi_k\|_{L^q(\mathbb{R}^n)}^p \Big)^{1/p} \\
\leq & C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \Big( \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\chi_k\|_{L^q(\mathbb{R}^n)}^p \Big)^{1/p} \\
= & C\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

For $E_1$, note that when $x \in A_k$, $j \leq k-2$, and $y \in A_j$, then $|x-y| \sim |x|$, $2|y| \leq |x|$. Therefore, for $x \in A_k$

$$\begin{aligned}
\Big|T\Big(\sum_{j=-\infty}^{k-2} f_j\Big)(x)\Big| \leq & c\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} \Big|\sum_{j=-\infty}^{k-2} f_j(y)\Big| dy \\
\leq & \frac{c}{|x|^n} \int_{|x-y|\leq 3|x|/2} |\Omega(x-y)| \Big|\sum_{j=-\infty}^{k-2} f_j(y)\Big| dy \\
\leq & CM_\Omega\Big(\sum_{j=-\infty}^{k-2} f_j\Big)(x).
\end{aligned}$$

Thus, from Corollary 2.1, it follows that

$$\begin{aligned}
E_1 \leq & C \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \Big( \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \Big\|\chi_k M_\Omega\Big(\sum_{j=-\infty}^{k-2} f_j\Big)\Big\|_{L^q(\mathbb{R}^n)}^p \Big)^{1/p} \\
= & C\Big\|M_\Omega\Big(\sum_{j=-\infty}^{k-2} f_j\Big)\Big\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} \\
\leq & C\Big\|\sum_{j=-\infty}^{k-2} f_j\Big\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} \\
\leq & C\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

Now, let us turn to estimate for $E_3$. Suppose that $r' \leq q$, $1 \leq r' \leq \infty$ and therefore, $\alpha > -n/q + \lambda$. Note that when $x \in A_k$, $j \geq k+2$, and $y \in A_j$, then $|x-y| \sim |y|$. Therefore, for $x \in A_k$, by Hölder's inequality, we have

$$
\begin{aligned}
\Big|T\Big(\sum_{j=k+2}^{\infty} f_j\Big)(x)\Big| &\le c\sum_{j=k+2}^{\infty}\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}|f_j(y)|dy\\
&\le C\sum_{j=k+2}^{\infty}2^{-jn/q}\|\Omega\|_{L^{q'}(s^{n-1})}\|f_j\|_{L^q(\mathbb{R}^n)}\\
&\le C\|\Omega\|_{L^r(s^{n-1})}\sum_{j=k+2}^{\infty}2^{-jn/q}\|f_j\|_{L^q(\mathbb{R}^n)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
E_3 &\le C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\|\Omega\|^p_{L^r(s^{n-1})}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)n/q}\|f_j\|_{L^q(\mathbb{R}^n)}\Big)^p\Big)^{1/p}\\
&\le C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\|\Omega\|^p_{L^r(s^{n-1})}\right.\\
&\quad\left.\times\left(\sum_{j=k+2}^{\infty}2^{(k-j)n/q}2^{-j\alpha}2^{j\lambda}2^{-j\lambda}\Big(\sum_{l=-\infty}^{j}2^{l\alpha p}\|f_l\|^p_{L^q(\mathbb{R}^n)}\Big)^{1/p}\right)^p\right)^{1/p}\\
&\le C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\|\Omega\|^p_{L^r(s^{n-1})}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)n/q}2^{-j\alpha}2^{j\lambda}\Big)^p\right)^{1/p}\\
&\qquad\times\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\\
&=C\|\Omega\|_{L^r(s^{n-1})}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\lambda p}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)(n/q+\alpha-\lambda)}\Big)^p\right)^{1/p}\\
&\qquad\times\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\\
&\le C\|\Omega\|_{L^r(s^{n-1})}\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\left(\sum_{k=-\infty}^{k_0}2^{k\lambda p}\right)^{1/p}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\\
&\le\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}2^{k_0\lambda}\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}\le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p,q}(\mathbb{R}^n)}.
\end{aligned}
$$

When $r' > q$, and therefore, $\alpha > -1/q - (n-1)/r' + \lambda$. Let $s = 1 - r + r/q = 1 - r/q'$. Then $(1-s)q' = r$ and $sq = (1-r)q + r \le r$. When $x \in A_k$, $j \ge k+2$, and $y \in A_j$, by Hölder's inequality, we have

$$\Big|T\Big(\sum_{j=k+2}^{\infty} f_j\Big)(x)\Big| \le c \sum_{j=k+2}^{\infty} 2^{-jn/q}\|\Omega\|^{1-s}_{L^{(1-s)q'}(s^{n-1})} \times \Big(\int_{\mathbb{R}^n} |\Omega(x-y)|^{sq}|f_j(y)|^q dy\Big)^{1/q}$$
$$\le c\sum_{j=k+2}^{\infty} 2^{-jn/q}\Big(\int_{\mathbb{R}^n}|\Omega(x-y)|^{(1-r)q+r}|f_j(y)|^q dy\Big)^{1/q}.$$

We choose $\beta \in \mathbb{R}$ and $0 < \epsilon < 1$ such that $\alpha > \beta/q + \lambda > -1/q - (n-1)/r' + \lambda$ and $\alpha - \beta/q > (1-\epsilon)n/q + \lambda$. Then, by Hölder's inequality and Lemma 2.2, we obtain

$$\Big\|\chi_k T(\sum_{j=k+2}^{\infty} f_j)\Big\|_{L^q(\mathbb{R}^n)}$$
$$\le C\Big(\sum_{j=k+2}^{\infty} 2^{-jn(1-\epsilon)q'/q}\Big)^{1/q'} \times \Big\|\chi_k\Big[\sum_{j=k+2}^{\infty} 2^{-jn\epsilon}\Big(\int_{\mathbb{R}^n}|\Omega(x-y)|^{(1-r)q+r}|f_j(y)|^q dy\Big)\Big]^{1/q}\Big\|_{L^q(\mathbb{R}^n)}$$
$$\le C2^{-kn(1-\epsilon)/q}\Big\{\sum_{j=k+2}^{\infty} 2^{-jn\epsilon}2^{-k\beta}\Big(\int_{\mathbb{R}^n}|f_j(y)|^q \times \Big(\int_{|x|\le|y|/2}|\Omega(x-y)|^{(1-r)q+r}|x|^{\beta}dx\Big)dy\Big)^{1/q}\Big\}$$
$$\le C2^{-k(n(1-\epsilon)+\beta)/q}\Big\{\sum_{j=k+2}^{\infty} 2^{-jn\epsilon}\Big(\int_{\mathbb{R}^n}|f_j(y)|^q|y|^{\beta+n}dy\Big)^{1/q}\Big\}$$
$$\le C2^{-k(n(1-\epsilon)+\beta)/q}\sum_{j=k+2}^{\infty} 2^{-j(n\epsilon-\beta-n)/q}\|f_j\|_{L^q(\mathbb{R}^n)}.$$

Thus, we have

$$E_3 \le C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty} 2^{-k(n(1-\epsilon)+\beta)/q} \times 2^{-j(n\epsilon-\beta-n)/q}\|f_j\|_{L^q(\mathbb{R}^n)}\Big)^p\Big)^{1/p}$$

$$
\begin{aligned}
\leq & C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\lambda p}\Big(\sum_{j=k+2}^{\infty} 2^{(k-j)[(\alpha+n\epsilon-n)/q-\lambda]}\Big)^p\Big)^{1/p} \\
& \times\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} \\
\leq & \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}2^{k_0\lambda}\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\mathbb{R}^n)}.
\end{aligned}
$$

This finishes the proof of Theorem 2.2. □

Now, let us turn to consider the fractional rough singular integrals. We first have the following theorem similar to Theorem 2.1.

**Theorem 2.3** *Let $0 < l < n$ and $T_l$ be a sublinear operator. If $T_l$ is bounded on $L^{q_1}_{|x|^{\beta q_1}}(\mathbb{R}^n)$ into $L^{q_2}_{|x|^{\beta q_2}}(\mathbb{R}^n)$ for all $\beta \in (\beta_1, \beta_2)$, and some $q_1, q_2 \in (1, \infty)$, where $\beta_1, \beta_2 \in \mathbb{R}$ and $1/q_2 = 1/q_1 - l/n$, then $T_l$ is also bounded from $M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mathbb{R}^n)$ into $M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)$ provided that $\alpha \in (\beta_1+\lambda, \beta_2+\lambda)$ and $0 < p_1 \leq p_2 \leq \infty$, $\lambda > 0$.*

*Proof.* Note that if $p_1 < p_2$, then $M\dot{K}^{\alpha,\lambda}_{p_1,q_2}(\mathbb{R}^n) \subset M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)$. We only need to prove the theorem in the case $p_1 = p_2$. We choose $\alpha_1, \alpha_2 \in \mathbb{R}^n$ such that $\beta_1 < \alpha_1 < \alpha - \lambda < \alpha_2 < \beta_2$. Write

$$
f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x),
$$

it follows that

$$
\begin{aligned}
\|T_l f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_2}(\mathbb{R}^n)} = & \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1}\|(T_l f)\chi_k\|^{p_1}_{L^{q_2}(\mathbb{R}^n)}\Big)^{1/p_1} \\
\leq & C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \\
& \times\Big(\sum_{j=-\infty}^{k+1} \|(T_l f_j)\chi_k\|_{L^{q_2}(\mathbb{R}^n)}\Big)^{p_1}\Big)^{1/p_1} \\
& + C \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1} \\
& \times\Big(\sum_{j=k+2}^{\infty} \|(T_l f_j)\chi_k\|_{L^{q_2}(\mathbb{R}^n)}\Big)^{p_1}\Big)^{1/p_1} \\
\equiv & E_1 + E_2
\end{aligned}
$$

Similar to the estimates for $E_1$, $E_2$ in Theorem 2.1, we can get

$$E_1 \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mathbb{R}^n)}; E_2 \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 2.3 . $\square$

The following lemma can be found in [5, P. 250].

**Lemma 2.3** *Let $0 < l < n$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $n/(n-1) \le r \le \infty$ and $\Omega \in L^r(S^{n-1})$. If there is a constant $C$ independent of $f$ such that the sublinear operator $T_l$ satisfies that*

$$|T_l f(x)| \le c\int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-l}}|f(y)|dy. \tag{2}$$

*for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x\bar{\in}\operatorname{supp} f$, then $T_l$ is also bounded from $L^{q_1}_{|x|^{\beta q_1}}(\mathbb{R}^n)$ into $L^{q_2}_{|x|^{\beta q_2}}(\mathbb{R}^n)$ provided that*

$$l + \max\Big(-\frac{n}{q_1}, -\frac{1}{q_1} - \frac{n-1}{r'}\Big) < \beta < -l + \min\Big(\frac{n}{q_2'}, \frac{q}{q_2'} + \frac{n-1}{r'}\Big).$$

As a simple corollary of Theorem 2.3 and Lemma 2.3, we have

**Corollary 2.2** *Assume that $0 < l < n$, $1 < q_1 < n/l$, $1/q_2 = 1/q_1 - l/n$, $n/(n-1) \le r \le \infty$, $0 < p_1 \le p_2 \le \infty$ and $\Omega \in L^r(S^{n-1})$. If a sublinear operator $T_l$ satisfies (2), then $T_l$ is also bounded from $M\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mathbb{R}^n)$ into $M\dot{K}^{\alpha,\lambda}_{p_2,q_2}(\mathbb{R}^n)$, provided that*

$$l + \max\Big(-\frac{n}{q_1}, -\frac{1}{q_1} - \frac{n-1}{r'}\Big) < \beta < -l + \min\Big(\frac{n}{q_2'}, \frac{q}{q_2'} + \frac{n-1}{r'}\Big).$$

Now we come to consider the end-point cases of Theorem 2.1, 2.2 and 2.3. First, we will establish the boundedness of $M_\Omega$ on weak Morrey-Herz spaces.

**Theorem 2.4** *Let $\Omega \in L^r(S^{n-1})$ and $1 \le r \le \infty$, $0 < p \le \infty$ and*

$$-n + \frac{n-1}{r} + \lambda < \alpha < 0.$$

*Then $M_\Omega$ is of type $(M\dot{K}^{\alpha,\lambda}_{p,1}(\mathbb{R}^n), WM\dot{K}^{\alpha,\lambda}_{p,1}(\mathbb{R}^n))$.*

*Proof.* Write

$$f(x)=\sum_{j=-\infty}^{\infty} f(x)\chi_j(x)\equiv \sum_{j=-\infty}^{\infty} f_j(x).$$

and

$$\begin{aligned}
&\gamma\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\big|\{x\in A_k\colon |M_\Omega f(x)|>\gamma\}\big|^p\Big)^{1/p}\\
\le{}& C\gamma\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big|\Big\{x\in A_k\colon \Big|M_\Omega\Big(\sum_{j=-\infty}^{k+1} f_j\Big)(x)\Big|>\gamma\Big\}\Big|^p\Big)^{1/p}\\
&+C\gamma\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big|\Big\{x\in A_k\colon \Big|M_\Omega\Big(\sum_{j=k+2}^{\infty} f_j\Big)(x)\Big|>\gamma\Big\}\Big|^p\Big)^{1/p}\\
\equiv{}& E_1+E_2.
\end{aligned}$$

In [6], Christ and Rubio de Francia have proved that when $r\ge 1$, $M_\Omega$ is of weak type $(1,1)$. By this and the hypothesis in the case $r=1$, for $E_1$, we have

$$\begin{aligned}
E_1\le{}& C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big\|\sum_{j=-\infty}^{k+1} f_j\Big\|_{L^1(\mathbb{R}^n)}^p\Big)^{1/p}\\
\le{}& C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p}\Big(\sum_{j=-\infty}^{k+1} \|f_j\|_{L^1(\mathbb{R}^n)}\Big)^p\Big)^{1/p}\\
\le{}& C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0}\Big[\sum_{j=-\infty}^{k+1} 2^{(k-j)\alpha}2^{j\lambda}2^{-j\lambda}\\
&\qquad\times\Big(\sum_{l=-\infty}^{j} 2^{l\alpha p}\|f_l\|_{L^1(\mathbb{R}^n)}^p\Big)^{1/p}\Big]^p\Big)^{1/p}\\
\le{}& C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{kp\lambda}\Big(\sum_{j=-\infty}^{k+1} 2^{(j-k)(-\alpha+\lambda)}\|f\|_{M\dot K_{p,1}^{\alpha,\lambda}(\mathbb{R}^n)}\Big)^p\Big)^{1/p}\\
\le{}& C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{kp\lambda}\Big)^{1/p}\|f\|_{M\dot K_{p,1}^{\alpha,\lambda}(\mathbb{R}^n)}\\
\le{}& C\sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}2^{k_0\lambda}\|f\|_{M\dot K_{p,1}^{\alpha,\lambda}(\mathbb{R}^n)}=C\|f\|_{M\dot K_{p,1}^{\alpha,\lambda}(\mathbb{R}^n)}.
\end{aligned}$$

To estimate for $E_2$, note that $-n+(n-1)/r+\lambda<\alpha<0$. We can choose $\alpha_1\in\mathbb{R}$ such that $0<\alpha_1<\alpha+n-(n-1)/r-\lambda$. When $j\geq k+2$, by Lemma 2.2, we have

$$
\begin{aligned}
&\Big\|\chi_k M_\Omega\Big(\sum_{j=k+2}^{\infty} f_j\Big)\Big\|_{L^1(\mathbb{R}^n)}\\
&\leq c\sum_{j=k+2}^{\infty}2^{-jn}\int_{A_k}\bigg(\int_{A_j}|\Omega(x-y)||f_j(y)|dy\bigg)dx\\
&\leq C\sum_{j=k+2}^{\infty}2^{-jn+k(\alpha_1-\alpha+\lambda)}\int_{A_j}|f_j(y)|\\
&\qquad\qquad\times\bigg(\int_{|x|\leq|y|/2}|\Omega(x-y)||x|^{\alpha-\alpha_1-\lambda}dx\bigg)dy\\
&\leq C\sum_{j=k+2}^{\infty}2^{-jn+k(\alpha_1-\alpha+\lambda)}\int_{A_j}|f_j(y)||y|^{\alpha-\alpha_1+n-\lambda}dy\\
&\leq C\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_1-\alpha+\lambda)}\|f_j\|_{L^1(\mathbb{R}^n)}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
E_2&\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\Big\|\chi_k M_\Omega\Big(\sum_{j=k+2}^{\infty}f_j\Big)\Big\|_{L^1(\mathbb{R}^n)}^p\bigg)^{1/p}\\
&\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_1-\alpha+\lambda)}\|f_j\|_{L^1(\mathbb{R}^n)}\Big)^p\bigg)^{1/p}\\
&\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\Bigg(\sum_{k=-\infty}^{k_0}2^{k\alpha p}\bigg(\sum_{j=k+2}^{\infty}2^{(k-j)(\alpha_1-\alpha+\lambda)}2^{-j\alpha}2^{j\lambda}2^{-j\lambda}\\
&\qquad\qquad\times\Big(\sum_{l=-\infty}^{j}2^{l\alpha p}\|f_l\|_{L^1(\mathbb{R}^n)}^p\Big)^{1/p}\bigg)^p\Bigg)^{1/p}\\
&\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}\bigg(\sum_{k=-\infty}^{k_0}2^{k\lambda p}\Big(\sum_{j=k+2}^{\infty}2^{(k-j)\alpha_1}\Big)^p\bigg)^{1/p}\|f\|_{M\dot{K}_{p,1}^{\alpha,\lambda}(\mathbb{R}^n)}\\
&\leq C\sup_{k_0\in\mathbb{Z}}2^{-k_0\lambda}2^{k_0\lambda}\|f\|_{M\dot{K}_{p,1}^{\alpha,\lambda}(\mathbb{R}^n)}=C\|f\|_{M\dot{K}_{p,1}^{\alpha,\lambda}(\mathbb{R}^n)}.
\end{aligned}
$$

In the second-last inequality, we used $\alpha_1 > 0$.

This finishes the proof of Theorem 2.4. $\square$

By a proof similar to that of Theorem 2.4, we can prove the following theorem.

**Theorem 2.5** *Suppose that* $\Omega \in L^r(S^{n-1})$ *and* $1 \le r \le \infty$, $0 < p \le \infty$ *and*

$$-n + \frac{n-1}{r} + \lambda < \alpha < 0.$$

*If a sublinear operator* $T$ *is of weak type* $(1,1)$ *and satisfies* (1), *then* $T$ *is also bounded from* $M\dot{K}^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)$ *into* $WM\dot{K}^{\alpha,\lambda}_{p,1}(\mathbb{R}^n)$.

For fractional singular integrals, we have some theorems similar to Theorem 2.5.

**Theorem 2.6** *Let* $\Omega \in L^r(S^{n-1})$ *and* $0 < l < n$, $q_1 = n/(n-l) \le r \le \infty$, $0 < p \le \infty$, $0 < p_1 \le p_2 \le \infty$ *and*

$$l - n + \frac{n-1}{r} + \lambda < \alpha < 0.$$

*If a sublinear operator* $T_l$ *is of weak type* $(1,\, q_1)$ *and satisfies* (2), *then* $T_l$ *is also bounded from* $M\dot{K}^{\alpha,\lambda}_{p_1,1}(\mathbb{R}^n)$ *into* $WM\dot{K}^{\alpha,\lambda}_{p_2,q_1}(\mathbb{R}^n)$.

*Proof.* Note that if $p_1 < p_2$, then $WM\dot{K}^{\alpha,\lambda}_{p_1,q_1}(\mathbb{R}^n) \subset WM\dot{K}^{\alpha,\lambda}_{p_2,q_1}(\mathbb{R}^n)$. We only need to prove the theorem in the case $p_1 = p_2$. Write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j(x) \equiv \sum_{j=-\infty}^{\infty} f_j(x).$$

and

$$\begin{aligned}
&\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1}\big|\{x \in A_k\colon |T_l(f)(x)| > \gamma\}\big|^{p_1/q_1}\Big)^{1/p_1}\\
\le& C\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}\\
&\quad\times\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1}\Big|\Big\{x \in A_k\colon \Big|T_l\Big(\sum_{j=-\infty}^{k+1} f_j\Big)(x)\Big| > \gamma\Big\}\Big|^{p_1/q_1}\Big)^{1/p_1}\\
&+ C\gamma \sup_{k_0\in\mathbb{Z}} 2^{-k_0\lambda}
\end{aligned}$$

$$\times\Big(\sum_{k=-\infty}^{k_0} 2^{k\alpha p_1}\Big|\Big\{x\in A_k\colon \Big|T_l\Big(\sum_{j=k+2}^{\infty} f_j\Big)(x)\Big|>\gamma\Big\}\Big|^{p_1/q_1}\Big)^{1/p_1}$$
$$\equiv F_1+F_2.$$

Since $T_l$ is of weak type $(1,\, q_1)$, similar to the estimates for $E_1$, $E_2$ in Theorem 2.4, we get

$$F_1 \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,1}(\mathbb{R}^n)};\, F_2 \le C\|f\|_{M\dot{K}^{\alpha,\lambda}_{p_1,1}(\mathbb{R}^n)}.$$

This finishes the proof of Theorem 2.6. □

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