Hokkaido Mathematical Journal Vol. 33 (2004) p. 369–381

The Lipschitz continuity of Neumann eigenvalues on convex domains

Marty ROSS

(Received July 17, 2002; Revised December 9, 2002)

Abstract. We consider the Neumann spectrum of the Laplacian on convex domains. Radially parametrizing these domains, we show that each Neumann eigenvalue is Lipschitz continuous with respect to the sup norm on the radial functions. We use this to prove that each Neumann eigenvalue is maximized on the class of convex domains with fixed volume.

Key words: Neumann spectrum, Laplace operator, eigenvalue.

1. Introduction

Suppose $\Omega \subset \mathbb{R}^n$ is a (sufficiently regular) bounded domain, and let $\{\mu_k\}_{k=1}^{\infty}$ be the Neumann spectrum for the Laplacian on Ω (see §2 for the functional analytic characterization of μ_k):

$$\begin{cases} \Delta v + \mu_k v = 0 & \text{on } \Omega, \\ \dfrac{\partial v}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here, $\mu_1(\Omega) = 0$, corresponding to $v = v_1 \equiv 1$. For $k \geq 2$, we are interested in maximizing $\mu_k \geq 0$ over classes of domains constrained to contain a specific volume V:

$$|\Omega| = V. \tag{V}$$

(The example of long, thin rectangles shows that no minimizer of μ_k exists).

Weinberger proved that amongst domains satisfying (V), μ_2 is maximized by the ball of the appropriate radius ([W]-see also [SY, pp. 140–142], [X, Th2]). For general k, one can easily prove that μ_k is bounded above by k and $|\Omega|$ (see §2). Therefore, it is reasonable to contemplate the existence of - much more ambitiously, the identity of - a maximizer of μ_k amongst

---

1991 Mathematics Subject Classification : Primary 35P15; Secondary 35J05, 58C40.

We thank the referee for bringing our attention to the article by P. Kröger, and for making a number of corrections.

domains satisfying (V).

We investigate this problem by the direct method (see, for example, [J]). So, we consider an extremizing sequence $\{\Omega_m\}$ of domains satisfying (V) and we hope to show, after passing to a subsequence, the existence of an extremal domain Ω:

$$\Omega_m \longrightarrow \Omega.$$

The issue is to determine a suitable notion of domain convergence. First of all, in order to obtain the existence of Ω, one needs the class of domains considered to be compact; secondly, for Ω to be the correct (extremal) limit, one needs Volume and μ_k to be continuous (or at least semicontinuous) with respect to the chosen convergence.

There are difficulties with implementing this program in general. Though there are many types of domain convergence which give compactness, it is often difficult or impossible to prove the corresponding continuity of geometric quantities. This is particularly true for μ_k, as even the discreteness of the Neumann spectrum is problematic for general, non-Lipschitz domains - see [F, §2.1] and [R]. And, unfortunately, such irregular domains can conceivably arise in the limit.

In this paper, we simplify the problem by restricting our attention to *convex domains* - this was the approach taken in [CR, §2] in the analysis of Dirichlet eigenvalues. Translating any such domain Ω to include the origin, we can introduce spherical coordinates (r, ω) and write Ω in terms of a *radial function* $f\colon \mathbb{S}^{n-1} \to \mathbb{R}^+$:

$$\Omega = \Omega_f \equiv \mathbb{R}^n \cap \{(r, \omega)\colon 0 \leq r < f(\omega),\ \omega \in \mathbb{S}^{n-1}\}.$$

f will be estimably Lipschitz: if

$$B_\rho(0) \subseteq \Omega \tag{ρ}$$

and

$$\Omega \subseteq B_R(0) \tag{R}$$

then, by elementary geometry,

$$\operatorname{Lip} f \leq \frac{R\sqrt{R^2 - \rho^2}}{\rho}. \tag{1.2}$$

($\partial\Omega$ will also be (strongly) Lipschitz in the sense of [EG, p. 127]; this can be shown, for instance, by introducing appropriately oriented spherical coordinates and applying (1.2)).

Of course the class of convex domains satisfying (V) does not uniformly satisfy such inradius and outradius constraints. However, from [Ch, §2] together with the techniques of [D], $\mu_k(\Omega)$ can be bounded in terms of k and $\operatorname{diam}\Omega$; below, we provide a weaker but more direct estimate, Proposition 2.1 (including a bound of $\mu_k(\Omega)$ for a not necessarily convex domain Ω in terms of k and $|\Omega|$). It follows that for the purposes of maximizing μ_k, we need not consider long, thin domains, and thus the uniform Lipschitz bound (1.2) holds for maximization candidates.

For any class of domains satisfying (ρ) and (R), the corresponding class of radial functions will be equicontinuous in the sup norm. Therefore, by the Arzela-Ascoli Theorem, we can obtain limiting radial functions, and the corresponding limit domains will be non-trivial and convex. Furthermore, since

$$|\Omega_f| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} f^n, \tag{1.3}$$

the constraint (V) will obviously hold in the limit.

The critical question is the behaviour of the eigenvalues with respect to this convergence. For Dirichlet eigenvalues, λ_k, the answer follows easily from *domain monotonicity*:

$$\Omega_1 \subseteq \Omega_2 \Longrightarrow \lambda_k(\Omega_1) \geq \lambda_k(\Omega_2). \tag{1.4}$$

Using (1.4) and a simple scaling argument, one can show ([CR, Lemma 2.1]):

$$|\lambda_k(\Omega_f) - \lambda_k(\Omega_g)| \leq \frac{3\lambda_k(B_\rho)}{\rho}\|f-g\|_\infty, \tag{1.5}$$

under the assumption that Ω_f and Ω_g are starlike (not necessarily convex) domains satisfying (ρ) with also $\|f-g\|_\infty \leq \rho$. Thus, for Dirichlet eigenvalues, one has the desired continuity, and the existence of extrema subject to appropriate geometric constraints follows readily ([CR, Th 2.2, 2.3]).

Such a simple argument cannot be applied here, however, as domain monotonicity is fundamentally false for Neumann eigenvalues; for example, if B is a ball and $R \subset B$ is a thin rectangle approximating a diameter of B then $\mu_2(B) > \mu_2(R)$. (Note, though, [X, Th3]).

As an alternative to the monotonicity argument, one can consider domains close in the C^1 sense, in which case [CH, ThI.VI.10] or [D, Th3] establishes the continuity of Neumann eigenvalues. However, though such arguments extend readily to the Lipschitz setting, *the class of convex domains is not compact in any C^1-like sense*, and thus the results and techniques of [D] are not of help here. This lack of compactness can be readily seen by considering domains with corners, for example a sequence of domains with two exterior angles θ converging in the C^0 sense to a domain with one exterior angle 2θ.

In §4, we prove the desired result on Neumann continuity, Theorem 4.2, and the consequent Maximization Theorem 4.3. The proof is similar in spirit to that for Dirichlet continuity: we are in effect establishing a form of local monotonicity, using the eigenfunctions of one domain as test functions for the other. The result obtained is similar to (1.5), but the constant is less explicit and more difficult to obtain.

The key estimate needed to establish this monotonicity result is a sup bound on the kth Neumann eigenfunction, Proposition 3.1 (a result which can be obtained in weaker form [D, Cor 5]). Such a bound holds for any Lipschitz domain, and in particular for starlike domains satisfying (ρ) and (1.2). Consequently, Neumann continuity (as well as the existence of extremizers over appropriate classes of domains) holds in this more general setting. (We believe that Neumann continuity should hold in general for starlike domains, without the Lipschitz assumption, but this appears to be considerably more difficult to prove).

2. Upper bounds on Neumann eigenvalues

For $\Omega \subset \mathbb{R}^n$ a bounded domain, let $\mathcal{W} = W^{1,2}(\Omega)$ be the Sobolev space of L^2 functions on Ω with L^2 weak derivatives. We assume that $\partial\Omega$ is regular enough for the embedding $\mathcal{W} \hookrightarrow L^2(\Omega)$ to be compact, noting that $\partial\Omega$ being Lipschitz is sufficient for this ([EG, §4.6]). The Neumann spectrum on Ω is then discrete ([GT, §8.12], [CH, SSVI.2,3]), and we can characterize the kth Neumann eigenvalue by the *Poincaré Principle* ([P1, §1]):

$$\mu_k(\Omega) = \min_{V_k \subset \mathcal{W}} \max_{w \in V_k - \{0\}} \frac{\int_\Omega |Dw|^2}{\int_\Omega w^2}. \tag{2.1}$$

Here, V_k is an arbitrary k-dimensional subspace of $\mathcal{W}$; of course the minimum in (2.1) is attained by choosing $V_k = \langle v_1, \ldots, v_k \rangle$, the subspace spanned by the first k eigenfunctions. (As is standard, we make the convention that if $\mu_i = \mu_j$ for $i \neq j$ then v_i and v_j are chosen to be orthogonal in L^2).

As a simple consequence of (2.1), we note that if $\langle w_1, \ldots, w_k \rangle \subset \mathcal{W}$ is an orthonormal set then

$$\mu_k(\Omega) \leq \max_j \int_\Omega |Dw_j|^2 \quad w_1, \ldots, w_k \quad \text{orthonormal.} \tag{2.2}$$

Lemma 2.1 *Suppose that Ω is a bounded domain of $\mathbb{R}^n$, and write $|\Omega| = V$. Suppose there are constants $0 < \alpha < \beta < 1$, $\gamma > 0$ and suppose that Ω_1, $\ldots$, Ω_k are pairwise disjoint subdomains of Ω such that, for $j = 1, \ldots, k$,*

$$|\Omega_j| \geq \beta V$$

and

$$\left|\Omega_j \cap \left\{x \colon \operatorname{dist}(x, \partial\Omega_j \cap \Omega) \leq \gamma V^{1/n}\right\}\right| \leq \alpha V.$$

Then

$$\mu_k(\Omega) \leq \frac{\alpha}{\gamma^2(\beta - \alpha)V^{2/n}}.$$

Proof. For $j = 1, \ldots, k$, define

$$w_j(x) = \begin{cases} \min\left(\operatorname{dist}(x, \partial\Omega_j \cap \Omega), \gamma V^{1/n}\right) & x \in \Omega_j, \\ 0 & \text{otherwise.} \end{cases}$$

The hypotheses ensure

$$\int_\Omega |Dw_j|^2 \leq \alpha V$$

and

$$\int_\Omega w_j^2 \geq \gamma^2 V^{2/n}(\beta - \alpha)V.$$

The result is then immediate from (2.2). $\square$

Proposition 2.1 ([K], [Ch]) (a) *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and suppose $k \in \mathbb{Z}^+$. Then*

$$\mu_k(\Omega) \leq \frac{c_1}{|\Omega|^{2/n}} \quad c_1 = c_1(n, k). \tag{2.3}$$

(b) *If Ω is convex then*

$$\mu_k(\Omega) \leq \frac{c_2}{(\operatorname{diam}\Omega)^2} \quad c_2 = c_2(n, k). \tag{2.4}$$

Remark (a) The deeper argument in [K, Cor 2] proves (2.3) with $c_1 = ck^{2/n}$

(b) The proof below gives $c_2 = cn^2k^2$ in (2.4), as follows from [Ch, Th 2.1] and the techniques of [D], though the estimate on c there is sharper.

Proof. (a) We prove the result for $k = 2$: the same argument can be used inductively to prove the general case.

For $r = 1, \ldots, n$, choose $\overline{x}_r$ such that

$$|\Omega \cap \{x \colon x_r < \overline{x}_r\}| = \frac{V}{2}.$$

For γ to be chosen, let

$$H_r(\gamma) = \left\{x \colon |x_r - \overline{x}_r| < 2\gamma V^{1/n}\right\}.$$

Then

$$\sum_{r=1}^{n} |\Omega \cap H_r(\gamma)| \leq ((n-1) + 4^n\gamma^n)\, V.$$

Setting $\gamma = 2^{-(2n+1)/n}$, there exists an r such that $|\Omega \cap H_r(\gamma)| \leq \big((2n-1)/(2n)\big)V$. and thus the volume of Ω in one or another half of $H_r(\gamma)$ is at most $\big((2n-1)/(4n)\big)V$. We can then apply Lemma 2.1 with $\alpha = (2n-1)/4n$ and $\beta - \alpha = 1/(4n)$.

(b) Let $d = \operatorname{diam}\Omega$. Slicing Ω by $k-1$ hyperplanes orthogonal to a diameter, we can split Ω into k slabs of equal volume. By the convexity of Ω, the $(n-1)$-volume of any slice orthogonal to the diameter is at most nV/d, and thus the width of each slab is at least $d/(kn)$. We can therefore apply Lemma 2.1 with $\beta = 1/k$, $\gamma V^{1/n} = d/(4kn)$ and $\alpha = 1/2k$, giving (2.4) with $c_2 = (4kn)^2$. $\square$

3. A sup bound for Neumann eigenfunctions.

In this section we assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and we take $v \in \mathcal{W}$ to be a Neumann eigenfunction on Ω with eigenvalue μ. The weak, variational characterization of v is:

$$\int_\Omega Dv \cdot Dw = \mu \int_\Omega vw \quad \text{for all} \quad w \in \mathcal{W}. \tag{3.1}$$

We emphasize that (3.1) is true by definition in the variational proof of the existence of Neumann eigenvalues; one then establishes the interior regularity of v, and one deduces that the (natural) boundary condition of (1.1) holds on suitably regular (e.g. $C^{(n-1)/2}$) pieces of $\partial\Omega$ ([GT, Corollary 8.11], [Ag, pp. 32, 129, 142]).

The estimate we want is a sup bound on v in terms of μ and $\|v\|_2$. We shall obtain this by the iteration technique of [Li1, Th 10] (see also [Ch, Th IV.8]), though the lack of regularity will necessitate a weak form of the argument, as displayed in [GT, Th 8.15].

The key ingredient in such an argument is a Sobolev inequality ([Ad, Lemma 5.14]):

$$\|w\|_p \le c_3\big(\|w\|_2 + \|Dw\|_2\big) \quad w \in \mathcal{W}(\Omega), \tag{3.2}$$

where $c_3 = c_3(\Omega)$ and

$$p = p(n) = \begin{cases} \dfrac{2n}{n-2} & n > 2, \\ 4 & n = 2. \end{cases} \tag{3.3}$$

(Note that $p(2)$ can be set as large as desired).

With this we can prove

Proposition 3.1 *Suppose Ω is a Lipschitz domain and suppose $v \in \mathcal{W}(\Omega)$ satisfies* (3.1). *Then*

$$\|v\|_\infty \le c_4\big((1+\sqrt{\mu})c_3\big)^r \|v\|_2\,, \tag{3.4}$$

where c_3 is from (3.2), *$c_4 = c_4(n)$, and*

$$r = r(n) = \begin{cases} \dfrac{n}{2} & n > 2, \\ 2 & n = 2. \end{cases} \tag{3.5}$$

Remarks (a) The existence of an inequality such as (3.2) depends upon Ω satisfying a uniform interior cone condition ([Ad, §4.3]), and then the constant c_3 depends only upon the defining cone. Any convex domain for which (ρ) and (R) hold satisfies this condition with cone of semi-vertex angle $\arcsin(\rho/R)$ and altitude $\rho/2$. Consequently, for our purposes, we can take

$$c_3 = c_3(\rho,\, R,\, n). \tag{3.6}$$

Similarly, (3.6) holds for starlike domains satisfying (ρ) and (1.2).

(b) Since $p(2)$ in (3.3) is arbitrary, $r(2) > 1$ is also arbitrary, where c_3 then also depends upon r.

Proof of Proposition 3.1. For $\beta \geq 1$ and $N > 0$, set

$$w = (\operatorname{sgn} v)\,(\min(|v|,\, N))^{2\beta-1}.$$

By the chain rule ([EG, §4.2.2]),

$$Dw = (2\beta - 1)\chi_{\Omega(N)}|v|^{2\beta-2}Dv,$$

where

$$\Omega(N) = \Omega \cap \{x \colon |v(x)| \leq N\}.$$

In particular, $w \in \mathcal{W}$, and thus w can be used as a test function in (3.1). After some simple manipulation, this gives

$$\int_{\Omega(N)} \left|D\left(|v|^\beta\right)\right|^2 = \frac{\mu\beta^2}{2\beta - 1}\int_\Omega \min\left(|v|^{2\beta},\, N^{2\beta-1}|v|\right).$$

Letting $N \to \infty$, we find

$$\int_\Omega \left|D\left(|v|^\beta\right)\right|^2 = \frac{\mu\beta^2}{2\beta - 1}\int_\Omega \left(|v|^\beta\right)^2 \quad \beta \geq 1.$$

As a consequence, and applying (3.2), we have the chain of implication

$$|v|^\beta \in L^2 \Longrightarrow |v|^\beta \in \mathcal{W} \Longrightarrow |v|^\beta \in L^p.$$

Specifically, we obtain the estimate

$$\begin{aligned} &\left\||v|^\beta\right\|_p \leq c_3\left(1 + \sqrt{\mu\beta}\right)\left\||v|^\beta\right\|_2 \\ \Longrightarrow &\|v\|_{p\beta} \leq \left(c_3(1 + \sqrt{\mu})\right)^{1/\beta}(\beta)^{1/2\beta}\,\|v\|_{2\beta}. \end{aligned}$$

We now iterate this estimate, choosing

$$\beta = \beta_k = \left(\frac{p}{2}\right)^k \quad k = 0,\, 1,\, 2,\, \ldots$$

Since $\|v\|_q \to \|v\|_\infty$ as $q \to \infty$, this gives

$$\|v\|_\infty \leq \left(c_3(1+\sqrt{\mu})\right)^{\sum 1/\beta_k} \left(\prod_{k=0}^{\infty} (\beta_k)^{1/2\beta_k}\right) \|v\|_2$$

In this expression, it is easy to show that the sum $\sum 1/\beta_k = r$, and that the product ($\equiv c_4$) is finite. □

Suppose now that $\Omega = \Omega_f$ is a convex domain satisfying (ρ) and (R). As a consequence of the above Proposition, an estimably small proportion of v lies in a thin boundary strip of Ω. For $0 \leq \alpha < \rho$, set

$$\Omega^\alpha = \Omega_{f-\alpha} = \mathbb{R}^n \cap \left\{(r,\, \omega) \colon 0 \leq r < f(\omega) - \alpha,\ \omega \in \mathbb{S}^{n-1}\right\}, \tag{3.7}$$

$$S^\alpha = \Omega - \Omega^\alpha. \tag{3.8}$$

Corollary 3.2 *Suppose $\Omega = \Omega_f \subset \mathbb{R}^n$ is a convex domain satisfying (ρ) and (R), and suppose $v \in \mathcal{W}$ satisfies (3.1). Then*

$$\int_{S^\alpha} |v|^2 \leq c_5 \alpha \|v\|_2^2, \tag{3.9}$$

where $c_5 = c_5(\rho,\, R,\, n,\, \mu)$.

Proof. This follows immediately from Proposition 3.1, together with (1.3) and (3.6). □

Remarks (a) This strip estimate will also hold for starlike domains as long as we hypothesize a Lipschitz estimate of the form (1.2).

(b) Without the hypothesis (3.1), one can obtain a bound

$$\int_{S^\alpha} |v|^2 \leq c\sqrt{\alpha}\, \|Dv\|_2^2$$

on Lipschitz domains for arbitrary $v \in \mathcal{W}$, where here S^α refers to the set of points in Ω within distance α of $\partial\Omega$ ([S]).

4. Continuity of Neumann Eigenvalues

The proof of Neumann continuity involves the uses of the eigenfunctions on one domain as test functions on the other. In order to do this, we need the following simple extension of (2.2):

Lemma 4.1 *Suppose $\Omega \subset \mathbb{R}^n$, and suppose that $\mu \geq 0$ and $w_1, \ldots, w_k \in \mathcal{W}$ satisfy*

$$\begin{cases} \|w_i\|_2 = 1 \\ \|Dw_i\|_2 \leq \sqrt{\mu} \end{cases} \qquad i = 1, \ldots, k.$$

Let

$$\delta = \max_{i \neq j} \left| \int_\Omega w_i w_j \right|.$$

Then there are constants $\delta_1 = \delta_1(k)$ and $c_6 = c_6(k)$ such that

$$\delta \leq \delta_1 \quad \Longrightarrow \quad \mu_k(\Omega) \leq (1 + c_6\delta)\mu. \tag{4.1}$$

Proof. We want to apply the Gram-Schmidt process to $\langle w_1, \ldots, w_k \rangle$, thereby obtaining an orthonormal sequence of functions $\widehat{w}_i$ to which (2.2) can be applied. By an obvious inductive argument, we find that if δ is small enough then this will be possible (i.e. the w_i will be linearly independent), and we obtain estimates of the form

$$\int_\Omega |D\widehat{w}_i|^2 \leq \big(1 + O(\delta)\big)\mu.$$

□

We can now prove:

Theorem 4.2 *Suppose that Ω_f and Ω_g are convex domains in $\mathbb{R}^n$ satisfying (ρ) and (R). Then*

$$|\mu_k(\Omega_f) - \mu_k(\Omega_g)| \leq c_7 \|f - g\|_\infty, \tag{4.2}$$

where $c_7 = c_7(\rho, R, n, k)$.

In combination with Proposition 2.1(b), this result immediately gives our maximization theorem:

Theorem 4.3 *For any $V > 0$ and $k \in \mathbb{Z}^+$ there is a domain maximizing μ_k amongst all convex domains in $\mathbb{R}^n$ satisfying (V).*

Proof of Theorem 4.2. Set

$$\alpha = \|f - g\|_\infty$$

and note that it is enough to prove (4.2) for small $\alpha = \alpha(\rho, R, n, k)$.
Special case.

First suppose that $f - \alpha \leq g \leq f$. Then $\mathcal{W}(\Omega_f) \subset \mathcal{W}(\Omega_g)$, and so we can use the first k eigenfunctions $v_1, \ldots, v_k$ of Ω_f to estimate $\mu_k(\Omega_g)$.

By Corollary 3.2,

$$\int_{\Omega_g} |v_i|^2 \geq \int_{\Omega_{f-\alpha}} |v_i|^2 \geq 1 - c_5\alpha \quad i = 1, \ldots, k$$

and

$$\left|\int_{\Omega_g} v_i v_j\right| = \left|\int_{\Omega_f - \Omega_g} v_i v_j\right| \leq c_5\alpha \quad i \neq j.$$

We also clearly have

$$\int_{\Omega_g} |Dv_i|^2 \leq \mu_k(\Omega_f) \quad i = 1, \ldots, k.$$

Thus, by Lemma 4.1, for small α we have

$$\mu_k(\Omega_g) \leq \left(1 + \frac{c_5 c_6 \alpha}{1 - c_5\alpha}\right) \frac{\mu_k(\Omega_f)}{1 - c_5\alpha},$$

which is a one-sided estimate of the form we want.
General case.

Given arbitrary f and g, set

$$\hat{g} = \frac{g}{1 + \alpha/\rho}.$$

Then, by (ρ),

$$f - \alpha\left(1 + \frac{1}{\rho}\right) \leq \hat{g} \leq f.$$

Thus we can apply the special case above to give

$$\mu_k(\Omega_g) = \left(1 + \frac{\alpha}{\rho}\right)^{-2} \mu_k(\Omega_{\hat{g}}) \leq (1 + c_7\alpha)\mu_k(\Omega_f) \tag{4.3}$$

for small α, and then c_7 can be adjusted to take care of large α.

Interchanging the roles of f and g, we obtain the desired result. $\square$

Remarks (a) A similar estimate holds for more general Lipschitz domains, and in particular for Ω_f and Ω_g starlike domains satisfying (ρ) and (1.2).

(b) Notice that (4.3), giving an upper bound for $\mu_k(\Omega_g)$ in terms of $\mu_k(\Omega_f)$, only uses the regularity of Ω_f: the result holds for an (essentially) arbitrary domain Ω_g close to Ω_f in a reasonable sense.

References

[Ad] Adams R.A., *Sobolev Spaces.* Academic Press, Orlando, Florida, 1975.

[Ag] Agmon S., *Lectures on Elliptic Boundary Value Problems.* Van Nostrand, Princeton, 1965.

[Ch] Cheng S.-Y., *Eigenvalue comparison theorems and its geometric applications.* Math. Z. **143** (1975), 289–297.

[CH] Courant R. and Hilbert D., *Methods of Mathematical Physics, I.* Interscience, New York, 1953.

[CR] Cox S.J., and Ross M., *Extremal eigenvalue problems for starlike planar domains.* J. Diff. Eq. **120** (1995), 174–197.

[D] Davies E.B., *Spectral properties of compact manifolds and changes of metrics.* Amer. J. Math. **112** (1990), 15–39.

[EG] Evans L.C. and Gariepy R.F., *Measure Theory and Fine Properties of Functions.* CRC Press, Boca Raton, Florida, 1992.

[F] Fraenkel L.E., *On regularity of the boundary in the theory of Sobolev spaces.* Proc. London Math. Soc. **39** (1979), 385–427.

[GT] Gilbarg D. and Trudinger N.S., *Elliptic Partial Differential Equations of Second Order, 2nd ed.*. Springer-Verlag, New York, 1983.

[J] Jerison D., *The Direct Method in the Calculus of Variations for Convex Bodies.* Adv. Math. **122** (1996), 262–279.

[K] Kröger P., *Upper Bounds for the Neumann Eigenvalues on a bounded domain in Euclidean Space.* J. Funct. Anal. **106** (1992), 353–357.

[Li1] Li P., *On the Sobolev constant and the p-spectrum of a compact Riemannnian manifold.* Ann. Scient. Éc. Norm. Sup. 4^e série. **13** (1980), 451–469.

[Li2] Li P., *A lower bound for the first eigenvalue of the Laplacian on a compact manifold.* Indiana U. Math. J. **28** (1979), 1013–1019.

[LY] Li P. and Yau S.-T., *Estimates of eigenvalues of compact Riemannian manifold.* Proc. Sym. Pure Math. **36** (1980), 205–239.

[P1] Pólya G., *Estimates for eigenvalues in "Collected Works III".* MIT Press, 1984, pp. 394–401.

[P2] Pólya G., *Mathematics and Plausible Reasoning Vol.* 2. Princeton Univ. Press, Princeton, 1973.

[R] Ross M., *A Rellich-Kondrachov theorem for spiky domains.* Indiana Math. J. **47** (1998), 1497–1509.

[S] Simon L., *Lectures on partial differential equations.* (unpublished).

[SY] Schoen R. and Yau S.-T, *Lectures on Differential Geometry.* International Press, Cambridge, Mass, 1995.

[W] Weinberger H.F., *On the isoperimetric inequality for the n-dimensional free membrane problem.* J. Ration. Mech. Anal. **5** (1956), 633–636.

[X] Xu Y., *The first nonzero eigenvalue of Neumann problem on Riemannian manifolds.* J. Geom. Anal. **5** (1995), 151–165.

[Z] Ziemer W.P., *Weakly Differentiable Functions.* Springer-Verlag, New York, 1980.

University of Godknowswhere
Australia
E-mail: martinskirossinski@yahoo.com