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Projectively flat connections and flat connections on homogeneous spaces

(Dedicated to the memory of Professor Masaru Takeuchi)

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Abstract. We show a correspondence between the set of all G-invariant projectively flat connections on a homogeneous space $M = G/K$, and the one of all $\widetilde{G}$-invariant flat connections on homogeneous spaces $\widetilde{M} = \widetilde{G}/K$, where $\widetilde{G}$ is a central extension of G (Theorem 3.3).

Key words: projectively flat connection, flat connection, reductive homogeneous space, symmetric space, simple Lie group.

1. Introduction and statement of results

Flat connections and projectively flat ones have been extensively studied by many authors (for examples, [1], [2], [3], [6], [8], [9], [11], [13]). Even though they are of course very different objects, but it seems that there would exist deep unknown relations between them each other. In this paper, we want to show some relation between the set of all G-invariant projectively flat connections on a homogeneous space $M = G/K$, and the one of all $\widetilde{G}$-invariant flat connections on homogeneous spaces $\widetilde{M} = \widetilde{G}/K$, where $\widetilde{G}$ is a central extension of G.

Indeed, let $\widetilde{G} \supset G \supset K$ be three Lie groups with Lie algebras $\widetilde{\mathfrak{g}} \supset \mathfrak{g} \supset \mathfrak{k}$, where $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}E$, and $[E, \widetilde{\mathfrak{g}}] = \{0\}$. Let us consider two homogeneous (not necessarily reductive) spaces $\widetilde{M} = \widetilde{G}/K$ and $M = G/K$, respectively. We consider the two sets of all $\widetilde{G}$-invariant *flat* affine connections on $\widetilde{M}$, and of all G-invariant *projectively flat* affine connections on M, which correspond to the sets $\mathcal{F}_0(\widetilde{\mathfrak{g}}, \mathfrak{k})$ and $\mathcal{PF}_0(\widetilde{\mathfrak{g}}, \mathfrak{k})$ of irreducible affine representations of $\widetilde{\mathfrak{g}}$, respectively (for more precise, see Section 3, Definitions 3.1, 3.2). Then,

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Theorem 1.1 (cf. Theorem 3.3) *It holds that*

$$\mathcal{F}_0(\widetilde{\mathfrak{g}}, \mathfrak{k}) = \mathcal{P}\mathcal{F}_0(\widetilde{\mathfrak{g}}, \mathfrak{k}) \cup \mathcal{F}_0^{II}(\widetilde{\mathfrak{g}}, \mathfrak{k}),$$

where $\mathcal{F}_0^{II}(\widetilde{\mathfrak{g}}, \mathfrak{k})$ *is the set of all real irreducible affine representations* $(\widetilde{f}, \widetilde{q}, \widetilde{V})$ *of* $\widetilde{\mathfrak{g}}$ *satisfying that*

(1) $\dim \widetilde{V} = \dim G/K + 1$,
(2) $\widetilde{V}$ *admits an* $\widetilde{f}(\widetilde{\mathfrak{g}})$*-invariant complex structure* J*, and*
(3) *there exists a non-zero element* $v_0 \in \widetilde{V}$ *satisfying that*

$$\widetilde{f}(\mathfrak{k})v_0 = \{0\}, \quad \textit{and} \quad \widetilde{V} = \widetilde{f}(\mathfrak{g})v_0 \oplus \mathbb{R}\widetilde{f}(E)v_0.$$

In particular, in the case that $\dim M = \dim G/K$ *is even, then,*

$$\mathcal{F}_0(\widetilde{\mathfrak{g}}, \mathfrak{k}) = \mathcal{P}\mathcal{F}_0(\widetilde{\mathfrak{g}}, \mathfrak{k}) \quad \textit{and} \quad \mathcal{F}_0^{II}(\widetilde{\mathfrak{g}}, \mathfrak{k}) = \emptyset.$$

Thus, G/K *admits a* G*-invariant projectively flat connection if and only if* $\widetilde{G}/K$ *admits a* $\widetilde{G}$*-invariant flat connection, in the case that* G/K *is of even dimension, and both the* G *and* $\widetilde{G}$ *are simply connected.*

Let us recall a classification of real simple Lie groups admitting a projectively flat connection ([1], [13], [3]).

Theorem 1.2 *Let* G *be a real semi-simple Lie group. Then,* G *admits a left invariant projectively flat connection if and only if the Lie algebra* $\mathfrak{g}$ *is one of the following*:

(1) $\mathfrak{sl}(n+1, \mathbb{R})$, $\quad n \geq 1$,
(2) $\mathfrak{su}^*(2n)$, $\quad n \geq 1$,

where $\mathfrak{su}^*(2n)$ *is the Lie algebra given by*

$$\mathfrak{su}^*(2n) = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix};\, Z_1,\ Z_2 \in M(n, \mathbb{C}),\ \mathrm{Tr} Z_1 + \mathrm{Tr}\overline{Z}_1 = 0 \right\}.$$

Since Theorems 1.1 and 1.2 (and also Remark 1.6) except the case of the real representation of $G = SL(n+1, \mathbb{R})$ on $\mathfrak{gl}(n+1, \mathbb{R})$ (cf. Theorem 4.3 in Chapter 3 in [9], see also [1]), we have

Corollary 1.3 *Let* G *be a real semi-simple Lie group with Lie algebra* $\mathfrak{g}$

of even dimension, *and let* $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}E$, *with* $[E, \widetilde{\mathfrak{g}}] = \{0\}$. *Let* $\widetilde{G}$ *be a simply connected Lie group with Lie algebra* $\widetilde{\mathfrak{g}}$. *Then,* $\widetilde{G}$ *admits a left invariant* flat *affine connection if and only if* G *admits a left invariant* projectively flat *affine connection. In this case,* $\mathfrak{g}$ *is* $\mathfrak{sl}(n+1, \mathbb{R})$, *where* $n \geq 1$ *is even.*

Let us recall a classification of irreducible Riemannian symmetric spaces admitting invariant projectively flat connections ([1], [13]).

Theorem 1.4 *Let* $M = G/K$ *be an irreducible simply connected Riemannian symmetric space. Then,* M *admits a* G*-invariant* projectively flat *affine connection if and only if* $M = G/K$ *is one of the following*:

(1) $S^n = SO(n+1)/SO(n) \quad n \geq 2$,
(2) $SL(n+1, \mathbb{R})/SO(n+1) \quad n \geq 2$,
(3) $SU^*(2n)/Sp(n) \quad n \geq 3$,
(4) $SO_0(n, 1)/SO(n) \quad n \geq 2$,
(5) $SL(n+1, \mathbb{C})/SU(n+1) \quad n \geq 1$
(6) E_6/F_4 (*of non-compact type EIV*).

Since Theorems 1.1 and 1.4 (and also Remark 1.6), we have

Corollary 1.5 *Let* $M = G/K$ *be an irreducible simply connected Riemannian symmetric space of* even dimension, $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}E$ *with* $[E, \widetilde{\mathfrak{g}}] = \{0\}$, *and* $\widetilde{G}$, *a Lie group with Lie algebra* $\widetilde{\mathfrak{g}}$ *and* $\widetilde{M}$ *is simply connected. Then,* $M = G/K$ *admits a* G*-invariant* projectively flat *affine connection if and only if* $\widetilde{M} = \widetilde{G}/K$ *admits a* $\widetilde{G}$*-invariant* flat *affine connection. In this case,* G/K *is one of* (1) $\sim$ (6) *in Theorem* 1.4 *of even dimension.*

Remark 1.6 In Corollaries 1.3, and 1.5, Agaoka ([1]) showed that, if G (resp. $M = G/K$) admits a left invariant (resp. invariant) flat connection, then $\widetilde{G}$ (resp. $\widetilde{M} = \widetilde{G}/K$) admits a left invariant (resp. $\widetilde{G}$-invariant) flat connection. In this, paper, we show the reverse of his results in Corollaries 1.3 and 1.5 in the case of even dimension.

2. Preliminaries

In this section, we prepare materials and several facts on invariant connection on homogeneous spaces (cf. [5]) and also invariant flat connections and projectively flat invariant connections on homogeneous spaces (cf. [10]).

Let us consider a C^∞ affine connection D on a C^∞ manifold M. In this paper, we assume that everything is C^∞.

Definition 2.1 *A connection D is to be* flat *if it has a vanishing curvature tensor R^D and a vanishing torsion tensor T^D, where*

$$R^D(X,Y) = D_X D_Y - D_Y D_X - D_{[X,Y]},$$

and

$$T^D(X,Y) = D_X Y - D_Y X - [X,Y]$$

for all vector fields X, Y on M.

Let us recall also that the theory of invariant connections on homogeneous spaces (cf. [5, Vol. II, p. 188]). In case of affine connections, we have

Theorem 2.2 *Let M be a homogeneous space $M = G/K$. Then, there exists a one-to-one correspondence between the set of G-invariant affine connections on $M = G/K$ and the set of linear mappings $\Lambda : \mathfrak{g} \to \mathfrak{gl}(T_o(G/K))$ ($o = \{K\}$, the origin of G/K) such that*

$$\Lambda(X) = \lambda(X) \quad (X \in \mathfrak{k}), \tag{2.1}$$

$$\Lambda(\mathrm{Ad}(k)(X)) = \mathrm{Ad}(\lambda(k))(\Lambda(X)) \quad (k \in K, X \in \mathfrak{g}), \tag{2.2}$$

where λ is the isotropy representation of K, i.e., for $k \in K$, $\lambda(k) = k_* : T_o(G/K) \to T_o(G/K)$ denotes the differential of k at $o = \{K\}$.

To each G-invariant connection D on $M = G/K$, there corresponds the linear mapping Λ defined by

$$\Lambda(X) = -(A_{\widetilde{X}})_o \quad (X \in \mathfrak{g}). \tag{2.3}$$

Here, each $X \in \mathfrak{g}$ induces a tangent vector $X_o \in T_oM$, and also a vector field $\widetilde{X}$ on $M = G/K$, naturally. For all vector field $\widetilde{X}$ on M, $A_{\widetilde{X}}$ is the tensor field of type (1,1) on M defined by

$$A_{\widetilde{X}} = L_{\widetilde{X}} - D_{\widetilde{X}}, \tag{2.4}$$

where $L_{\widetilde{X}}$ is Lie derivative by $\widetilde{X}$.

Assume that a homogeneous space $M = G/K$ is *reductive*, i.e., the Lie algebra $\mathfrak{g}$ is decomposed into $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant subspace of $\mathfrak{g}$. Then, we have ([5, Vol. II, p. 191])

Theorem 2.3 *Assume that $M = G/K$ is a reductive homogeneous space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then, there is a one-to-one correspondence between the set of G-invariant affine connections on M and the set of linear mappings $\Lambda_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{gl}(T_o(G/K))$ such that*

$$\Lambda_{\mathfrak{m}}(\mathrm{Ad}(k)(X)) = \mathrm{Ad}(\lambda(k))(\Lambda_{\mathfrak{m}}(X)) \quad (X \in \mathfrak{m},\, k \in K), \tag{2.5}$$

where $\lambda(k)$ denotes the isotropy representation of K on G/K. The correspondence is given by

$$\Lambda(X) = \begin{cases} \lambda(X) & (X \in \mathfrak{k}), \\ \Lambda_{\mathfrak{m}}(X) & (X \in \mathfrak{m}). \end{cases} \tag{2.6}$$

To each G-invariant connection D, the correspondence (2.6) *is given by*

$$\Lambda_{\mathfrak{m}}(X) = -(A_{\widetilde{X}})_o \quad (X \in \mathfrak{m}). \tag{2.7}$$

The torsion tensor T^D and the curvature tensor R^D of a G-invariant connection D can be expressed in terms of Λ as follows ([5, Vol. II, p. 189]).

Theorem 2.4 *In the above theorems, the torsion tensor T^D and the curvature tensor R^D of a G-invariant connection D can be expressed as follows*:

$$T^D(X,Y)_o = \Lambda(X)(Y_o) - \Lambda(Y)(X_o) - [X,Y]_o \quad (X,\, Y \in \mathfrak{g}), \tag{2.8}$$

$$R^D(X,Y)_o = [\Lambda(X), \Lambda(Y)] - \Lambda([X,Y]) \quad (X,\, Y \in \mathfrak{g}). \tag{2.9}$$

For G-invariant flat connections on $M = G/K$, due to the above theorems, we have ([10])

Theorem 2.5 *Assume that a homogeneous space $M = G/K$ admits a G-invariant flat connection on $M = G/K$. Then, there exists an affine representation (f, q, V) of $\mathfrak{g}$ on V such that*

$$\begin{cases} \dim V = \dim M, \\ q : \mathfrak{g} \to V \text{ is surjective, and } \operatorname{Ker}(q) = \mathfrak{k}. \end{cases} \tag{2.10}$$

Conversely, if G is simply connected, and the Lie algebra $\mathfrak{g}$ admits an affine representation (f, q, V) satisfying (2.10)*, then $M = G/K$ admits a G-invariant flat affine connection. Here, two linear mappings $q : \mathfrak{g} \to V$, and $f : \mathfrak{g} \to \mathfrak{gl}(V)$ are given by $q(X) = X_o$ $(X \in \mathfrak{g})$, and $f(X) = \Lambda(X)$ $(X \in \mathfrak{g})$, where $V = T_oM$. Then, that (f, q, V) is an affine representation $\mathfrak{g}$ on V means that*

$$\begin{cases} [f(X), f(Y)] = f([X, Y]) \quad (X, Y \in \mathfrak{g}), \\ q([X, Y]) = f(X)q(Y) - f(Y)q(X) \quad (X, Y \in \mathfrak{g}). \end{cases} \tag{2.11}$$

Now, let us recall the notion of projectively flat connections.

Definition 2.6 *D is to be* projectively flat *if the Ricci tensor*

$$\operatorname{Ric}^D(Y, Z) := \operatorname{Tr}(\{T_pM \ni X \mapsto R^D(X, Y)Z \in T_pM\})$$

$(Y, Z \in T_pM\ p \in M)$, is symmetric*, i.e.,* $\operatorname{Ric}^D(Y, Z) = \operatorname{Ric}^D(Z, Y)$*, and for every $p \in M$, there exists a neighborhood U of p such that D is* equivalent *to a flat connection $\overline{D}$ on U, i.e., there exists a closed 1-form ρ on U such that*

$$D_XY = \overline{D}_XY + \rho(X)Y + \rho(Y)X, \quad \textit{for all } X, Y \in \mathfrak{X}(U).$$

A classical theorem says that

Theorem 2.7 *Assume that D is an affine connection of which* Ric^D *is symmetric and $T^D = 0$. Then, D is projectively flat if and only if the Weyl curvature tensor vanishes, i.e.,*

$$R^D(X, Y)Z = \frac{1}{n-1}\{\operatorname{Ric}^D(Y, Z)X - \operatorname{Ric}^D(X, Z)Y\}$$

and the Codazzi equation *holds, i.e.,*

$$(D_X\operatorname{Ric}^D)(Y, Z) = (D_Y\operatorname{Ric}^D)(X, Z)$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Let us consider a *centro-affine* immersion of an n-dimensional manifold M into $\mathbb{R}^{n+1}$, $\varphi: M \hookrightarrow \mathbb{R}^{n+1}$, that is,

$$T_{\varphi(p)}\mathbb{R}^{n+1} = \varphi_*(T_pM) \oplus \mathbb{R}\,\overrightarrow{o\,\varphi(p)}, \quad (p \in M).$$

Then, the induced connection D on M from the standard flat connection D^0 on $\mathbb{R}^{n+1}$ via φ, i.e.,

$$(D^0_X Y)_p = \varphi_*(D_X Y)_p + h(X,Y)(-\overrightarrow{o\,\varphi(p)}), \quad (X, Y \in \mathfrak{X}(M),\, p \in M),$$

is projectively flat. Furthermore, it holds ([11]) that

Theorem 2.8 *Let $M = G/K$ be a simply connected homogeneous space. Then, the following two conditions are equivalent*:

(1) *$M = G/K$ admits a G-invariant* projectively flat *connection.*
(2) *$M = G/K$ admits a G-equivariant centro-affine immersion.*

Furthermore, Shima showed ([11], see also [10, p. 228]) that

Theorem 2.9 *Let $M = G/K$ be an arbitrary homogeneous space.*

(1) *Assume that $M = G/K$ admits a G-invariant* projectively flat *connection. Let $\mathfrak{g}$ be the Lie algebra of G, and $\mathfrak{k}$ the Lie subalgebra corresponding to K, respectively. Let $\widetilde{\mathfrak{g}}$ be the central extension of $\mathfrak{g}$, i.e., $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}E$, where $[E, \widetilde{\mathfrak{g}}] = \{0\}$. Then, $\widetilde{\mathfrak{g}}$ admits an* affine representation $(\widetilde{f}, \widetilde{q}, \widetilde{V})$ *on a vector space $\widetilde{V}$ of dimension* $\dim M + 1$ *satisfying the following two conditions*:
   (i) *$\widetilde{q}: \widetilde{\mathfrak{g}} \to \widetilde{V}$ is surjective and* $\mathrm{Ker}(\widetilde{q})$ *is* $\mathfrak{k}$,
   (ii) *$\widetilde{f}(E)$ is the identity map of $\widetilde{V}$ and $\widetilde{q}(E) \neq 0$.*
(2) *Conversely, if $\widetilde{\mathfrak{g}}$ admits an affine representation $(\widetilde{f}, \widetilde{q}, \widetilde{V})$ on $\widetilde{V}$ of dimension* $\dim M + 1$ *satisfying* (i) *and* (ii), *then, $M = G/K$ admits a G-invariant* projectively flat *affine connection if G is simply connected.*

3. Invariant projectively flat connections and flat connections

Let us begin the following example due to Shima ([10, Ch. 9, Exercise 9.1.1]):

Example 3.1 Let $\widetilde{G} = GL(n, \mathbb{R})$,

$$K := \left\{ \begin{pmatrix} I_r & x \\ O & y \end{pmatrix} ; x \in M(r, n-r), y \in M(n-r, n-r) \right\}.$$

Then, the quotient space $\widetilde{M} = \widetilde{G}/K$ admits a $\widetilde{G}$-invariant flat connection. Here, I_r is the identity matrix of degree r, and O is the $(n-r) \times r$ zero matrix where $r = 1, \ldots, n-1$. Then, $\widetilde{\mathfrak{g}} = \mathfrak{g}l(n, \mathbb{R})$,

$$\mathfrak{k} := \left\{ \begin{pmatrix} O & X \\ O & Y \end{pmatrix} ; X \in M(r, n-r), Y \in M(n-r, n-r) \right\},$$

and $T_0\widetilde{M}$ is isomorphic to V where

$$V = \left\{ \begin{pmatrix} X' & O \\ Y' & O \end{pmatrix} ; X' \in M(r, r), Y' \in M(n-r, r) \right\}.$$

Then, $\widetilde{\mathfrak{g}} = \mathfrak{k} \oplus V$, and for every $X \in \widetilde{\mathfrak{g}}$, $f(X) \in \mathfrak{g}l(V)$ and $q(X) \in V$ are defined by

$$f(X)v = Xv \in V, \quad q(X) = X_V \in V, \quad (X \in \widetilde{\mathfrak{g}},\, v \in V),$$

where Xv is the matrix multiplication of X and v, and $X_V \in V$ is the V-component of $X \in \widetilde{\mathfrak{g}}$ corresponding to the decomposition of $\widetilde{\mathfrak{g}} = \mathfrak{k} \oplus V$. Then, (f, q, V) is an affine representation of $\widetilde{\mathfrak{g}}$ on V which induces the $\widetilde{G}$-invariant flat connection on $\widetilde{M}$. But, $\widetilde{M} = \widetilde{G}/K$ is not a reductive homogeneous space since there is no $\mathrm{ad}(\mathfrak{k})$-invariant decomposition of $\widetilde{\mathfrak{g}}$ (see also [5, Vol. II, p. 199, Example 2.1]). The action of $\mathrm{ad}(\widetilde{\mathfrak{g}})$ on V is irreducible if and only if $r = 1$.

Example 3.2 $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, the upper triangular nilpotent Lie group N_1:

$$N_1 = \left\{ \begin{pmatrix} 1 & & & * \\ & 1 & & \\ & & \ddots & \\ O & & & 1 \end{pmatrix} ; * \text{ is arbitrary} \right\},$$

the $(2n+1)$-dimensional Heisenberg nilpotent Lie group N_2, and the solvable Lie group S:

$$S = \left\{ \begin{pmatrix} e^{\theta_1} & & & * \\ & e^{\theta_2} & & \\ & & \ddots & \\ O & & & e^{\theta_n} \end{pmatrix} ; \theta_1, \theta_2, \ldots, \theta_n \in \mathbb{R}, * \text{ is arbitrary} \right\}$$

admit left invariant flat connections.

But, $SU(2)$ admits no left invariant flat connection.

In this section, we do not assume that every homogeneous space is reductive. Our setting is as follows. Let $\widetilde{G} \supset G \supset K$ be three Lie groups with Lie algebras $\widetilde{\mathfrak{g}} \supset \mathfrak{g} \supset \mathfrak{k}$, respectively. Let us consider two quotient spaces $\widetilde{M} = \widetilde{G}/K$, and $M = G/K$, respectively. Assume that the Lie algebra $\widetilde{\mathfrak{g}}$ is a central extension of $\mathfrak{g}$ given by $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}E$, where $[E, \widetilde{\mathfrak{g}}] = \{0\}$. Then, $\dim \widetilde{M} = \dim M + 1$.

Definition 3.1 *Let us denote by $\mathcal{F}_{\widetilde{G}}(\widetilde{M})$ the set of all $\widetilde{G}$-invariant flat affine connections on $\widetilde{M}$, and $\mathcal{PF}_G(M)$, the set of all G-invariant projectively flat affine connections on M. Let us define $\mathcal{F}(\widetilde{\mathfrak{g}}, \mathfrak{k})$, the set of all affine representations $(\widetilde{f}, \widetilde{q}, \widetilde{V})$ of $\widetilde{\mathfrak{g}}$ on $\widetilde{V}$ satisfying that*

(1) $\dim \widetilde{V} = \dim \widetilde{M}$,
(2) $\widetilde{q} : \widetilde{\mathfrak{g}} \to \widetilde{V}$ *is a surjection, and* $\mathrm{Ker}(\widetilde{q}) = \mathfrak{k}$,

and also $\mathcal{PF}(\widetilde{\mathfrak{g}}, \mathfrak{k})$, the set of all affine representations $(\widetilde{f}, \widetilde{q}, \widetilde{V})$ of $\widetilde{\mathfrak{g}}$ on $\widetilde{V}$ satisfying that

(1) $\dim \widetilde{V} = \dim M + 1$,
(2) $\widetilde{q} : \widetilde{\mathfrak{g}} \to \widetilde{V}$ *is a surjection,* $\mathrm{Ker}(\widetilde{q}) = \mathfrak{k}$,
(3) $\widetilde{f}(E)$ *is a non-zero constant multiple of the identity transformation of* $\widetilde{V}$, *and* $\widetilde{q}(E) \neq 0$,

respectively.

Remark here that in the original definition of $\mathcal{PF}(\widetilde{\mathfrak{g}}, \mathfrak{k})$ in [10] corresponding to the above definition (3) was that:

(3′) $\widetilde{f}(E)$ *is the identity transformation of* $\widetilde{V}$, *and* $\widetilde{q}(E) \neq 0$.

But, if we replace E into a constant multiple of E, then we have (3′) from

(3), so that our $\mathcal{PF}(\widetilde{\mathfrak{g}},\mathfrak{k})$ corresponds bijectively to $\mathcal{PF}_G(M)$.

If $\widetilde{G}$ is simply connected, there exists a one-to-one correspondence between $\mathcal{F}_{\widetilde{G}}(\widetilde{M})$ and $\mathcal{F}(\widetilde{\mathfrak{g}},\mathfrak{k})$, and also if G is simply connected, there exists a one-to-one correspondence between $\mathcal{PF}_G(M)$ and $\mathcal{PF}(\widetilde{\mathfrak{g}},\mathfrak{k})$ ([10]). By definition, $\mathcal{PF}(\widetilde{\mathfrak{g}},\mathfrak{k})\subset\mathcal{F}(\widetilde{\mathfrak{g}},\mathfrak{k})$, so that $\mathcal{PF}_G(M)\subset\mathcal{F}_{\widetilde{G}}(\widetilde{M})$.

To analyze them further, let us define

Definition 3.2

$$\mathcal{F}_0(\widetilde{g},\mathfrak{k})=\{(\widetilde{f},\widetilde{q},\widetilde{V})\in\mathcal{F}(\widetilde{\mathfrak{g}},\mathfrak{k});\ (\widetilde{f},\widetilde{q},\widetilde{V})\ \textit{is irreducible under}\ \widetilde{\mathfrak{g}}\},$$

and also

$$\mathcal{PF}_0(\widetilde{g},\mathfrak{k})=\{(\widetilde{f},\widetilde{q},\widetilde{V})\in\mathcal{PF}(\widetilde{\mathfrak{g}},\mathfrak{k});\ (\widetilde{f},\widetilde{q},\widetilde{V})\ \textit{is irreducible under}\ \widetilde{\mathfrak{g}}\},$$

respectively. Then, $\mathcal{PF}_0(\widetilde{g},\mathfrak{k})\subset\mathcal{F}_0(\widetilde{g},\mathfrak{k})$ *by definition.*

Then, we obtain

Theorem 3.3 *It holds that*

$$\mathcal{F}_0(\widetilde{\mathfrak{g}},\mathfrak{k})=\mathcal{PF}_0(\widetilde{\mathfrak{g}},\mathfrak{k})\cup\mathcal{F}_0^{II}(\widetilde{\mathfrak{g}},\mathfrak{k}),$$

where the set $\mathcal{F}_0^{II}(\widetilde{\mathfrak{g}},\mathfrak{k})$ *is a subset of* $\mathcal{F}_0(\widetilde{\mathfrak{g}},\mathfrak{k})$*, and coincides with the one of all real irreducible affine representations* $(\widetilde{f},\widetilde{q},\widetilde{V})$ *of* $\widetilde{\mathfrak{g}}$ *satisfying that*

(1) $\dim\widetilde{V}=\dim G/K+1$,
(2) $\widetilde{V}$ *admits an* $\widetilde{f}(\widetilde{\mathfrak{g}})$*-invariant complex structure* J*, and*
(3) *there exists a non-zero element* $v_0\in\widetilde{V}$ *satisfying that*

$$\widetilde{f}(\mathfrak{k})v_0=\{0\},\quad\textit{and}\quad\widetilde{V}=\widetilde{f}(\mathfrak{g})v_0\oplus\mathbb{R}\widetilde{f}(E)v_0.$$

In particular, in the case that $\dim M=\dim G/K$ *is even, then,*

$$\mathcal{F}_0(\widetilde{\mathfrak{g}},\mathfrak{k})=\mathcal{PF}_0(\widetilde{\mathfrak{g}},\mathfrak{k})\quad\textit{and}\quad\mathcal{F}_0^{II}(\widetilde{\mathfrak{g}},\mathfrak{k})=\emptyset.$$

Proof. Let $(\widetilde{f},\widetilde{q},\widetilde{V})\in\mathcal{F}_0(\widetilde{\mathfrak{g}},\mathfrak{k})$. Then, the affine representation $(\widetilde{f},\widetilde{q},\widetilde{V})$ satisfies that

(1) $(\widetilde{f},\widetilde{q},\widetilde{V})$ is an irreducible representation of $\widetilde{\mathfrak{g}}$,

(2) $\dim \widetilde{V} = \dim \widetilde{M} = \dim M + 1$,
(3) $\widetilde{q} : \widetilde{\mathfrak{g}} \to \widetilde{V}$ is a surjection, and $\mathrm{Ker}(\widetilde{q}) = \mathfrak{k}$.

By (1), $\widetilde{f}(E)$ is a semi-simple linear transformation of $\widetilde{V}$ because $(\widetilde{f}, \widetilde{q}, \widetilde{V})$ is a completely reducible representation of $\mathfrak{g}$ (see, for example, [12, p. 28, Theorem 2.9]). Then, $\widetilde{V}$ is decomposed into

$$\widetilde{V} = V_1 \oplus \cdots \oplus V_s \oplus V_{s+1} \oplus \cdots \oplus V_r,$$

where

(i) each V_i is $\widetilde{f}(E)$-invariant and the only $\widetilde{f}(E)$-invariant subspaces of V_i are V_i itself or $\{0\}$, and

(ii) for the complex extension $\widetilde{f}(E)^{\mathbb{C}}$ of $\widetilde{f}(E)$ to $V_i^{\mathbb{C}}$, there exist linearly independent vectors $\{v_j^i\}_{j=1}^{d_i}$ in the complexification $V_i^{\mathbb{C}}$ of V_i and some complex number λ_j^i such that

$$\widetilde{f}(E)^{\mathbb{C}} v_j^i = \lambda_j^i v_j^i \quad (j = 1, \ldots, d_i). \tag{3.1}$$

Notice here that $\dim V_i^{\mathbb{C}} = 1$, and $d_i = 1$ in this case due to (i).

(iii) For each $i = 1, \ldots, s$, $\lambda_j^i = a_i$ is a real number, and V_i itself is the eigenspace of $\widetilde{f}(E)$, and $\{v_j^i\}_{j=1}^{d_i}$ is a basis of $V_i^{\mathbb{C}}$.

(iv) For each $i = s+1, \ldots, r$, $\lambda_j^i = a_j^i + \sqrt{-1} b_j^i$ where a_j^i and b_j^i are real numbers with $b_j^i \neq 0$, and $v_j^i = u_j^i + \sqrt{-1} w_j^i$ $(u_j^i, w_j^i \in V_i;\ j = 1, \ldots, d_i)$ such that

$$\begin{cases} \widetilde{f}(E) u_j^i &= a_j^i u_j^i - b_j^i w_j^i \\ \widetilde{f}(E) w_j^i &= b_j^i u_j^i + a_j^i w_j^i. \end{cases} \tag{3.2}$$

Then, by (i), it holds that $\dim V_i^{\mathbb{C}} = 2$, $d_i = 1$, and for $j = 1$,

$$\begin{cases} \widetilde{f}(E)^{\mathbb{C}} \overline{v_1^i} = \overline{\lambda_1^i}\, \overline{v_1^i}, \\ V_i^{\mathbb{C}} = \mathbb{C}\, v_1^i \oplus \mathbb{C} \overline{v_1^i}, \end{cases} \tag{3.3}$$

where $\{v_1^i, \overline{v_1^i}\}$ is a basis of $V_i^{\mathbb{C}}$.

Then we have two cases: $V_1 \oplus \cdots \oplus V_s$ is $\{0\}$ or not.

Case (a): $V_1 \oplus \cdots \oplus V_s \neq \{0\}$.

In this case, $V_1 \oplus \cdots \oplus V_s$ is a non-zero $\widetilde{f}(\widetilde{\mathfrak{g}})$-invariant subspace. Indeed, for each $v \in V_i$ and $\widetilde{X} \in \widetilde{\mathfrak{g}}$, since $[E, \widetilde{\mathfrak{g}}] = \{0\}$ and $\widetilde{f}$ is a Lie algebra homomorphism, we have

$$\widetilde{f}(E)(\widetilde{f}(\widetilde{X})v) = \widetilde{f}(\widetilde{X})(\widetilde{f}(E)v) = a_i \widetilde{f}(\widetilde{X})v,$$

which implies $\widetilde{f}(\widetilde{X})v$ belongs to $V_1 \oplus \cdots \oplus V_s$. Thus, $\widetilde{V} = V_1 \oplus \cdots \oplus V_s$ and $\widetilde{f}(E)$ has a unique real eigenvalue, say, $a \in \mathbb{R}$, since $(\widetilde{f}, \widetilde{V})$ is an irreducible representation of $\widetilde{\mathfrak{g}}$.

Furthermore, $a \neq 0$. Because if we assume that $a = 0$, then, $\widetilde{f}(E) = 0$. Then, for each $\widetilde{X} \in \widetilde{\mathfrak{g}}$, we have

$$\widetilde{f}(\widetilde{X})(\widetilde{q}(E)) = \widetilde{f}(E)(\widetilde{q}(\widetilde{X})) = 0, \tag{3.4}$$

which implies that $\widetilde{q}(E) = 0$. Because if we assume $\widetilde{q}(E) \neq 0$, then, $\{0\} \neq \mathbb{R}\,\widetilde{q}(E)$ $(\neq \widetilde{V})$ is $\widetilde{f}(\widetilde{\mathfrak{g}})$-invariant by means of (3.4). This contradicts the irreducibility of $(\widetilde{f}, \widetilde{q}, \widetilde{V})$. So we have $\widetilde{q}(E) = 0$. However, that $\widetilde{q}(E) = 0$ contradicts the assumption that $\mathrm{Ker}(\widetilde{q}) = \mathfrak{k}$. So, we have $a \neq 0$.

Notice here that, if we put $E' = \frac{1}{a}\,E$, then, we have also $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}E'$, and $[E', \widetilde{\mathfrak{g}}] = \{0\}$. Furthermore, the affine representation $(\widetilde{f}, \widetilde{q}, \widetilde{V})$ of $\widetilde{\mathfrak{g}}$ still satisfies all the conditions (1), (2) and (3) for the set $\mathcal{PF}(\widetilde{\mathfrak{g}}, \mathfrak{k})$ in Definition 3.1. Because, for (1), (2) they are the same, and for (3), we have that $\widetilde{f}(E') = \frac{1}{a}\,\widetilde{f}(E) = I$, and $\widetilde{q}(E') = \frac{1}{a}\,\widetilde{q}(E) \neq 0$. Thus, $(\widetilde{f}, \widetilde{q}, \widetilde{V}) \in \mathcal{PF}_0(\widetilde{\mathfrak{g}}, \mathfrak{k})$.

Case (b): $V_1 \oplus \cdots \oplus V_s = \{0\}$.

In this case, let λ be any non-zero λ^i_j in (ii), and consider a non-zero complex subspace

$$W := \sum_{\lambda^k_\ell = \lambda} \big(\mathbb{C}v^k_\ell \oplus \mathbb{C}\overline{v^k_\ell}\big)$$

of $\sum_{j=s+1}^{r} {V_j}^{\mathbb{C}}$, where λ^k_ℓ run over the set of all complex eigenvalues of $\widetilde{f}(E)^{\mathbb{C}}$ in (3.1) which are equal to λ. Then,

$$W \cap \widetilde{V} = \sum_{\lambda^k_\ell = \lambda} \big(\mathbb{R}u^k_\ell \oplus \mathbb{R}w^k_\ell\big), \tag{3.5}$$

where we denote $v^k_\ell = u^k_\ell + \sqrt{-1}w^k_\ell$, u^k_ℓ, $w^k_\ell \in \widetilde{V}$.

Then, for each $\widetilde{X} \in \widetilde{\mathfrak{g}}$,

$$\widetilde{f}(E)^{\mathbb{C}}\widetilde{f}(\widetilde{X})^{\mathbb{C}}v^k_\ell = \widetilde{f}(\widetilde{X})^{\mathbb{C}}\widetilde{f}(E)^{\mathbb{C}}v^k_\ell = \lambda\,\widetilde{f}(\widetilde{X})^{\mathbb{C}}v^k_\ell,$$

which implies that $\widetilde{f}(\widetilde{X})^{\mathbb{C}}v^k_\ell$ belongs to W. Furthermore, we have for each $\widetilde{X} \in \widetilde{\mathfrak{g}}$,

$$\widetilde{f}(\widetilde{X})(W \cap \widetilde{V}) \subset W \cap \widetilde{V}.$$

Because, since it holds that

$$\widetilde{f}(\widetilde{X})u^k_\ell + \sqrt{-1}\widetilde{f}(\widetilde{X})w^k_\ell = \widetilde{f}(\widetilde{X})^{\mathbb{C}}v^k_\ell = \sum_{\lambda^p_q=\lambda} \big(\alpha_{pq}v^p_q + \beta_{pq}\overline{v^p_q}\big),$$

for some complex numbers α_{pq}, and β_{pq}, we have

$$\widetilde{f}(\widetilde{X})u^k_\ell = \sum_{\lambda^p_q=\lambda} \big\{\Re e(\alpha_{pq} + \beta_{pq})\,u^p_q + \Im m(-\alpha_{pq} + \beta_{pq})\,w^p_q\big\} \in W \cap \widetilde{V},$$

$$\widetilde{f}(\widetilde{X})w^k_\ell = \sum_{\lambda^p_q=\lambda} \big\{\Im m(\alpha_{pq} + \beta_{pq})\,u^p_q + \Re e(\alpha_{pq} - \beta_{pq})\,w^p_q\big\} \in W \cap \widetilde{V}.$$

Thus, together with (3.5), we have $\widetilde{f}(\widetilde{X})(W \cap \widetilde{V}) \subset W \cap \widetilde{V}$.

Since $(\widetilde{f}, \widetilde{q}, \widetilde{V})$ is an irreducible representation of $\widetilde{\mathfrak{g}}$, we have

$$\widetilde{V} = W \cap \widetilde{V} = \sum_{\lambda^k_\ell=\lambda} \big(\mathbb{R}u^k_\ell + \mathbb{R}w^k_\ell\big),$$

which means that ${V_{s+1}}^{\mathbb{C}} \oplus \cdots \oplus {V_r}^{\mathbb{C}}$ is the sum of the two eigenspaces of $\widetilde{f}(E)^{\mathbb{C}}$ with the eigenvalues λ and $\overline{\lambda}$ for some complex number $\lambda = a + \sqrt{-1}b$ $(a, b \in \mathbb{R})$ with $b \neq 0$.

By means of (3.2), $\widetilde{f}(E)$ can be written as

$$\widetilde{f}(E) = a\,I + b\,J, \tag{3.6}$$

where I is the identity transformation of $\widetilde{V}$, and J is the transformation of $\widetilde{V}$ of the form:

$$J(u_j) = -w_j, \quad J(w_j) = u_j \quad (j = 1, \ldots, d), \tag{3.7}$$

where $\{u_1, \ldots, u_d, w_1, \ldots, w_d\}$ is a basis of $\widetilde{V}$ ($\dim \widetilde{V} = 2d$). Then, J is a complex structure of $\widetilde{V}$, i.e., $J^2 = -I$.

The complex structure J of $\widetilde{V}$ is $\widetilde{f}(\widetilde{\mathfrak{g}})$-invariant, i.e.,

$$J\,\widetilde{f}(\widetilde{X}) = \widetilde{f}(\widetilde{X})\,J \quad (\forall\ \widetilde{X} \in \widetilde{\mathfrak{g}}). \tag{3.8}$$

Because, since E is central in $\widetilde{\mathfrak{g}}$, we have, for each $\widetilde{X} \in \widetilde{\mathfrak{g}}$,

$$\widetilde{f}(E)\widetilde{f}(\widetilde{X}) = \widetilde{f}(\widetilde{X})\widetilde{f}(E). \tag{3.9}$$

The left hand side of (3.9) coincides with

$$(a\,I + b\,J)\,\widetilde{f}(\widetilde{X}) = a\,\widetilde{f}(\widetilde{X}) + b\,J\,\widetilde{f}(\widetilde{X}).$$

The right hand side of (3.9) is equal to

$$\widetilde{f}(\widetilde{X})(a\,I + b\,J) = a\,\widetilde{f}(\widetilde{X}) + b\,\widetilde{f}(\widetilde{X})\,J.$$

Since $b \neq 0$, we have (3.8).

Notice that $\widetilde{q}(E) \neq 0$ since $\mathrm{Ker}(\widetilde{q}) = \mathfrak{k}$. Since

$$\widetilde{q}\big([\widetilde{X}, \widetilde{Y}]\big) = \widetilde{f}(\widetilde{X})\big(\widetilde{q}(\widetilde{Y})\big) - \widetilde{f}(\widetilde{Y})\big(\widetilde{q}(\widetilde{X})\big) \quad (\widetilde{X}, \widetilde{Y} \in \widetilde{\mathfrak{g}}),$$

we have

$$\widetilde{f}(\widetilde{Y})(\widetilde{q}(E)) = \widetilde{f}(E)\big(\widetilde{q}(\widetilde{Y})\big) = (a\,I + b\,J)\big(\widetilde{q}(\widetilde{Y})\big). \tag{3.10}$$

Then, we have, for all $\widetilde{Y} \in \widetilde{\mathfrak{g}}$,

$$\begin{aligned}
\widetilde{q}(\widetilde{Y}) &= \frac{1}{a^2 + b^2}(a\,I - b\,J)\big(\widetilde{f}(\widetilde{Y})(\widetilde{q}(E))\big) \\
&= \widetilde{f}(\widetilde{Y})\left(\frac{1}{a^2 + b^2}(a\,I - b\,J)(\widetilde{q}(E))\right) \\
&= \widetilde{f}(\widetilde{Y})v_0 \qquad (3.11)
\end{aligned}$$

where

$$0 \neq v_0 := \frac{1}{a^2+b^2}(a\,I - b\,J)(\widetilde{q}(E)) \in \widetilde{V}. \tag{3.12}$$

By (3.11), $\widetilde{f}(\mathfrak{k})v_0 = \{0\}$ since $\mathrm{Ker}(\widetilde{q}) = \mathfrak{k}$. Since $\widetilde{q} : \widetilde{\mathfrak{g}} \to \widetilde{V}$ is a surjection and $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}E$, and (3.11), we also have

$$\widetilde{V} = \widetilde{f}(\mathfrak{g})v_0 \oplus \mathbb{R}\,\widetilde{f}(E)v_0. \tag{3.13}$$

Thus, we have $(\widetilde{f}, \widetilde{q}, \widetilde{V}) \in \mathcal{F}_0^{II}(\widetilde{\mathfrak{g}}, \mathfrak{k})$. □

Remark 3.4

(1) The set $\mathcal{PF}_0(\widetilde{\mathfrak{g}}, \mathfrak{k})$ corresponds ([13, Theorem 1.3]) to the set of all real irreducible representations $(f, \widetilde{V})$ of $\mathfrak{g}$ of dimension $\dim M + 1$ which admits a nonzero vector $v_0 \in \widetilde{V}$ such that $f(\mathfrak{k})v_0 = \{0\}$ and $\widetilde{V} = f(\mathfrak{g})v_0 \oplus \mathbb{R}v_0$. Each $(f, \widetilde{V})$ induces a G-equivariant centro-affine immersion $\varphi : M = G/K \to \widetilde{V}$ by $\varphi(xK) = f(xK)v_0$ $(xK \in G/K)$ with transversal vector field $\xi_{xK} = -\overrightarrow{o\,\varphi(xK)}$ $(xK \in G/K)$ ([9]). Our Theorem 3.3 suggests that the set $\mathcal{F}_0^{II}(\widetilde{\mathfrak{g}}, \mathfrak{k})$ would correspond to the set of all G-equivariant affine immersions given by $\varphi'(xK) = \widetilde{f}(xK)v_0$ $(xK \in G/K)$ with the transversal vector field $\xi'_{xK} = \overrightarrow{\varphi'(xK)\,\widetilde{f}(E)\varphi'(xK)}$ $(xK \in G/K)$. Then, $M = G/K$ would admit a G-invariant affine connection via the immersion φ' (cf. [9]).

(2) There is an example belonging to $\mathcal{F}_0^{II}(\widetilde{\mathfrak{g}}, \mathfrak{k})$. Indeed, let us recall Example 11.1 in the book of Takeuchi [12, p. 119], and let $\mathfrak{g} = \mathfrak{su}(2)$, $\mathfrak{k} = \{0\}$, $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}E$, with $E = I_2$. Let $V = \mathbb{C}^2 = \left\{\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right);\, z, w \in \mathbb{C}\right\}$, and $\widetilde{V}$, the real 4-dimensional space V restricted to the field $\mathbb{R}$, and $\mathfrak{g}$ acts on $\widetilde{V}$ by the matrix multiplications $\widetilde{f}(X)v = Xv \in \widetilde{V}$, $(X \in \mathfrak{g}, v \in \widetilde{V})$, and $\widetilde{f}(E)v = I_2 v = v$ $(v \in \widetilde{V})$. Then, $\widetilde{V}$ admits the complex structure J defined by

$$Jv = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \overline{v}, \quad v \in \widetilde{V},$$

which satisfies $J(Xv) = XJv$, $(X \in \mathfrak{g}, v \in \widetilde{V})$. The linear mapping $\widetilde{q} : \widetilde{\mathfrak{g}} \to \widetilde{V}$ is given by $\widetilde{q}\left(\left(\begin{smallmatrix} i\theta & \alpha \\ -\overline{\alpha} & -i\theta \end{smallmatrix}\right) + \xi E\right) = \left(\begin{smallmatrix} i\theta \\ -\overline{\alpha} \end{smallmatrix}\right) + \xi\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, and satisfies

$$\widetilde{q}\big([\widetilde{X},\widetilde{Y}]\big) = \widetilde{f}(\widetilde{X})\widetilde{q}(\widetilde{Y}) - \widetilde{f}(\widetilde{Y})\widetilde{q}(\widetilde{X}), \quad (\widetilde{X},\widetilde{Y} \in \widetilde{\mathfrak{g}}).$$

Then, it turns out that $(\widetilde{f},\widetilde{q},\widetilde{V}) \in \mathcal{F}_0^{II}(\widetilde{\mathfrak{g}},\mathfrak{k})$.

(3) The real representations in $\mathcal{F}_0^{II}(\widetilde{\mathfrak{g}},\mathfrak{k})$ in the case $\mathfrak{k} = \{0\}$ were treated, called the representations of $\widetilde{\mathfrak{g}}$ *of class II*, in the book of Takeuchi ([12, pp. 85–92]).

(4) The union in the right hand side for $\mathcal{F}_0(\widetilde{\mathfrak{g}},\mathfrak{k})$ in Theorem 3.3 seems to be a disjoint union.

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References

[1] Agaoka Y., *Invariant flat projective structures on homogeneous spaces.* Hokkaido Math. J. **11** (1982), 125–172.

[2] Doi H., *Non-existence of torsion free flat connections on reductive homogeneous spaces.* Hiroshima Math. J. **9** (1979), 321–322.

[3] Elduque A., *Invariant projectively flat affine connections on Lie groups.* Hokkaido Math. J. **30** (2001), 231–239.

[4] Helgason S., *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press, New York, 1978.

[5] Kobayashi S. and Nomizu K., *Foundation of Differential Geometry, Vol. I, II*, (1963), (1969), John Wiley and Sons, New York.

[6] Matsushima H. and Okamoto K., *Non-existence of torsion free flat connections on a real semisimple Lie group.* Hiroshima Math. J. **9** (1979), 59–60.

[7] Matsushima Y. and Murakami S., *On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds.* Ann. Math. **78** (1963), 365–416.

[8] Nomizu K., *Invariant affine connections on homogeneous spaces.* Amer. J. Math. **76** (1954), 33–65.

[9] Nomizu K. and Sasaki T., *Affine Differential Geometry, Geometry of Affine Immersions*, (1994), Shokabo, Tokyo; (1994), Cambridge Univ. Press, Cambridge.

[10] Shima H., *The Geometry of Hessian Structures*, (2001), Shokabo, Tokyo; (2007), World Sci. Publ., Hackensack.

[11] Shima H., *Homogeneous spaces with invariant projectively flat affine connections*. Trans. Amer. Math. Soc. **351** (1999), 4713–4726.

[12] Takeuchi M., *Real Irreducible Representations of Real Simple Lie Algebras* (*in Japanese*). Lecture Notes Series, Vol. 3, 1996, Osaka University.

[13] Urakawa H., *On invariant projectively flat affine connections.* Hokkaido Math. J. **28** (1999), 333–356.

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