Hokkaido Mathematical Journal Vol. 37 (2008) p. 463–492

Remarks on the Levi conditions for differential systems

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(Received April 4, 2007; Revised October 24, 2007)

Abstract. In this paper we prove two results on the Levi conditions for weakly hyperbolic systems with characteristics of constant multiplicities.

A first result concerns scalar operators: we prove that Levi conditions defined by the second author in [29] are equivalent to the usual Levi conditions for scalar operator.

A second result concerns systems whose principal symbol has a Jordan form made of a large number of 2×2 blocks. For these systems we express the first Levi condition via an invariant constructed from the sub-characteristic matrix. Moreover we show that this condition is necessary for the $\mathcal{C}^\infty$ well-posedness.

Key words: Cauchy problem for systems with constant multiplicities, Levi conditions.

1. Notations and Hypothesis

Let Ω be an open neighborhood of 0 in $\mathbb{R}^{n+1}$, $x = (x_0, x') = (x_0, x_1, \ldots, x_n) \in \Omega$, we note $D = (D_0, D') = (D_0, D_1, \ldots, D_n)$ with $D_i = \partial/\partial x_i$, and $\xi = (\xi_0, \xi') = (\xi_0, \xi_1, \ldots, \xi_n)$.

Let $h(x, D) = a(x, D) + b(x)$ be an $N \times N$ system of order M, with analytic coefficients in $x \in \Omega$, $a(x, \xi)$ is its principal part; we consider the Cauchy problem for h:

$$\begin{cases} h(x; D)u = f(x), \\ D_0^j u|_{x_0=\underline{x_0}} = g_j(x'), \quad j = 0, 1, \ldots, M-1. \end{cases}$$

We assume that h is *hyperbolic with constant multiplicities*, with respect to $(1, 0, \ldots, 0)$, that is, we assume that there exist irreducible polynomials H_τ, $\tau = 1, \ldots, \tau_0$, homogeneous of degree s_τ in ξ, with analytic coefficients in x such that

$$\det a(x; \xi) = H_1^{m_1}(x; \xi) \cdots H_{\tau_0}^{m_{\tau_0}}(x; \xi), \tag{1}$$

where $m_1, \ldots, m_{\tau_0} \in \mathbb{N}$ do not depend on $(x, \xi) \in \Omega \times \mathbb{R}^{n+1}$, and the polynomial $H_1 \cdots H_{\tau_0}$ is strictly hyperbolic with respect to $(1, 0, \ldots, 0)$. We recall that, thanks to Matsuura [17], the decomposition in (1) is equivalent

2000 Mathematics Subject Classification : 35L45.

to say that the roots in ξ_0 of the equation $\det a(x;\xi) = 0$, are real and their multiplicity is constant, that is, we have:

$$\det a(x;\xi) = \prod_{j=1}^{r} \big(\xi_0 - \lambda_{(j)}(x;\xi')\big)^{m_{(j)}}, \tag{2}$$

where the $\lambda_{(j)}$ are real functions with $\lambda_{(j)}(x;\xi') \neq \lambda_{(k)}(x;\xi')$, for $j \neq k$, and the $m_{(j)}$ are constant on $\Omega \times \mathbb{R}^n$.

To simplify the presentation, in the following we assume that there is only one multiple factor H, of degree s and multiplicity m, and a simple factor K, of degree χ, but the general case can be treated in a similar way.

We recall the classification of systems introduced by Vaillant [24].

Let $\mathcal{P}^{(\nu)}$ be the set of homogeneous polynomials of degree ν in ξ, with analytic coefficients in x, and let $\mathcal{P} = \otimes_{\nu\in\mathbb{N}}\mathcal{P}^{(\nu)}$ be the ring of of the polynomials in ξ, with analytic coefficients in x, and let $\mathcal{P}_{(H)}$ be the localized ring of $\mathcal{P}$ with respect to (H), the prime ideal defined by H. $\mathcal{P}_{(H)}$ is a principal ring, and in $\mathcal{P}_{(H)}$ $a(x;\xi)$ is equivalent to a diagonal matrix [2, §4, no. 6, Prop. 5 and Cor. 1], [24]. More precisely there exist two matrices $P(x;\xi)$ and $Q(x;\xi)$ with entries in $\mathcal{P}_{(H)}$, with $\det P$ and $\det Q$ invertible in $\mathcal{P}_{(H)}$ and such that:

$$a(x;\xi) = P(x;\xi) \begin{pmatrix} H^p & 0 & \dots & & & & 0 \\ 0 & H^{q_1} & & & & & \\ \vdots & 0 & \ddots & & & & \\ & \vdots & & H^{q_\ell} & & & \vdots \\ & & & & 1 & & \\ & & & & & \ddots & 0 \\ 0 & & & & & 0 & 1 \end{pmatrix} Q(x;\xi), \tag{3}$$

where the integers $p = q_0, q_1, \dots, q_\ell$ are such that $p \geq q_1 \geq \dots \geq q_\ell > 0$, and $p + q_1 + \dots + q_\ell = m$.

We call the sequence $(H^p, H^{q_1}, \dots, H^{q_\ell}, 1, \dots, 1)$ the *type* of the system h with respect to the multiple factor H.

The sequence $p, q_1, \dots, q_\ell$ has the following property: $H^{q_1+\dots+q_\ell}$ is the biggest common divisor of the cofactors of a, $H^{q_2+\dots+q_\ell}$ is the biggest common divisor of the cofactors of order $M-2$ of $a, \dots$; there exists a cofactor of order $M-\ell-1$ not divisible by H [2, §4, no. 6, Prop. 6].

Let Γ be an open conic set of $\Omega \times \mathbb{R}^{n+1}$, we say that the *generalized* *rank* of h is constant on Γ, if $\det P$ and $\det Q$ in (3) never vanish in Γ.

If h is of type $(H^p, H^{q_1}, \ldots, H^{q_\ell}, 1, \ldots, 1)$, and the generalized rank is constant on Γ, we can reduce a, uniformly on the compacts of Γ, to its Jordan form, and the sequence $(p, q_1, \ldots, q_\ell)$ gives the dimensions of the Jordan blocks related to the characteristic roots of H. More precisely, if $\lambda_{(1)}, \ldots, \lambda_{(s)}$ are the zeroes in ξ_0 of $H(x; \xi_0, \xi') = 0$ and if $\lambda_{(s+1)}, \ldots, \lambda_{(s+\chi)}$ are the zeroes in ξ_0 of $K(x; \xi_0, \xi') = 0$, so that

$$\det a(x;\xi) = \prod_{j=1}^{s} \big(\xi_0 - \lambda_{(j)}(x;\xi')\big)^m \prod_{j=1}^{\chi} \big(\xi_0 - \lambda_{(s+j)}(x;\xi')\big),$$

then there exists a matrix $\Delta_0(x, \xi')$ with analytic coefficients, homogeneous of degree 0 in ξ such that

$$\Delta_0^{-1} a \Delta_0 = \bigotimes_{j=1}^{s+\chi} a_{(j)},$$

where $a_{(j)} = \big(\xi_0 - \lambda_{(j)}(x;\xi')\big) I_m + J$, if $j = 1, \ldots, s$, and $a_{(j)} = \xi_0 - \lambda_{(j)}(x;\xi')$, if $j = s+1, \ldots, s+\chi$.

Here I_m is the identity matrix of order m, $J = \bigotimes_{k=0}^{\ell} J_{q_k}$ and the J_{q_k} are the Jordan nilpotents blocks $q_k \times q_k$:

$$J_{q_k} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ & 0 & 1 & & 0 \\ & & 0 & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & & \ldots & 0 & 0 \end{pmatrix}.$$

On can define the type in a different way.

Let $\bar{x} \in \Omega$ be fixed, we consider the ring of polynomials in ξ, and we construct the type of Jordan as previously.

The type defined in this way depends on $\bar{x}$, and if we order the set of the m-ples $(p, q_1, \ldots, q_\ell, 0, \ldots, 0)$, by the lexicographic order, thanks to the analitycity of the coefficients, the maximal value is obtained on an open dense set, and this type coincide with the type defined previously.

This second method can be applied also to systems with non analytic coefficients. The systems with constant maximal rank correspond to the systems with maximal type and constant generalized rank. This allows us to define in the same way the conditions $\mathrm{L}(\mathfrak{q})$ for the systems with maximal type (analytic coefficients and variable generalized rank or $\mathcal{C}^\infty$ coefficients and constant generalized rank).

We remark that if $p = 1$, ($\ell = m - 1$, $q_1 = \cdots = q_\ell = 1$) the system is diagonalizable and strongly hyperbolic: the Cauchy Problem is well-posed in $\mathcal{C}^\infty$ [26] and in all Gevrey spaces [12].

2. Definition and properties of the conditions $\mathrm{L}(\mathfrak{q})$

Let A be the cofactor matrix of a, so that $aA = Aa = H^m K I_N$. Since A is divisible by H^{m-p} [2, §4, no. 6, Prop. 6], we set $\mathcal{A} = (1/H^{m-p})A$; $\mathcal{A}(x;\xi)$ is a matrix whose entries are polynomials in ξ with analytic coefficients in x, of degree $ps + \chi - M$.

Notations For a scalar or matrix-valued differential or classical pseudo-differential operator $\Lambda'(x;D)$, of order $\leq \nu$, we denote by $\Lambda = \sigma_\nu(\Lambda')$ the homogeneous symbol of order ν, which is equal to the principal part of Λ', if it is of order ν, and 0 if not. Note that σ_ν is an additive function.

Conversely, if $\Lambda(x;\xi) \in \mathcal{M}_N(\mathcal{P}^{(\nu)})$, where $\mathcal{M}_N(\mathcal{P}^{(\nu)})$ is the set of the $N \times N$ matrix, whose entries belong to $\mathcal{P}^{(\nu)}$, we denote $\Lambda'(x;D)$, any differential operator of order $\leq \nu$ such that $\sigma_\nu(\Lambda') = \Lambda$. If $\Lambda(x;\xi)$ is scalar, we associate to Λ a matrix operator $\Lambda'(x;D)$, such that $\sigma_\nu(\Lambda') = \Lambda I_N$.

For example, H' is an $N \times N$ operator of order s such that $\sigma_s(H') = HI_N$, K' is an $N \times N$ operator of order χ such that $\sigma_\chi(K') = KI_N$, $\mathcal{A}'$ is an $N \times N$ operator of order $ps + \chi - M$ such that $\sigma_{ps+\chi-M}(\mathcal{A}') = \mathcal{A}$.

Let $\mathfrak{q} = (\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_\kappa)$, with $1 \leq \kappa \leq m - 1$ and $1 \leq \mathfrak{q}_j \leq p$. We set:

$$\mu_0 = ps + \chi - 1$$

$$\mu_k = \mu_0 + k(\chi - 1) + s \sum_{j=1}^{k} \mathfrak{q}_j, \quad \text{for } 1 \leq k \leq \kappa.$$

We construct the conditions $\mathrm{L}(\mathfrak{q})$ as the conditions L introduced in [29]. Anyway, here the sequence $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\ldots$, $\mathfrak{q}_\kappa$ is not necessarily the sequence of the invariant factors.

Let $\mathcal{A}'$, H', K' be matricial operators with principal symbols $\mathcal{A}$, HI and KI, the operator

$$S_1' := h\mathcal{A}' - H'^p K'$$

is of order μ_0, since the operator $h\mathcal{A}'$ and $H'^p K'$ have the same principal symbol.

We say that h verifies the condition $\mathrm{L}_1(\mathfrak{q})$ if there exists differential operators $\mathcal{A}'$, H', K', such that

$$\mathcal{A}\sigma_{\mu_0}(S'_1) \text{ is divisible by } H^{p-\mathfrak{q}_1}.$$

We remark that the condition $\mathrm{L}_1(\mathfrak{q})$ is equivalent to the following: there exists a symbol $\Lambda_1(x;\xi)$ homogeneous of order $\mu_1 - M + 1$ in ξ such that:

$$\mathcal{A}\sigma_{\mu_0}(S'_1) = H^{p-\mathfrak{q}_1}\Lambda_1.$$

If $\mathrm{L}_1(\mathfrak{q})$ is satisfied, we multiply on the left by a the previous identity, and we get:

$$\sigma_{\mu_0}(S'_1)H^{\mathfrak{q}_1}K = a\Lambda_1,$$

which implies that the operators $S'_1 H'^{\mathfrak{q}_1} K'$ and $h\Lambda'_1$ have the same principal symbol; hence the operator

$$S'_2 := h\Lambda'_1 - S'_1 H'^{\mathfrak{q}_1} K'$$

is of order μ_1.

We say that h verifies the condition $\mathrm{L}_2(\mathfrak{q})$ if there exists a differential operator Λ'_1, whose principal symbol is Λ_1, such that

$$\mathcal{A}\sigma_{\mu_1}(S'_2) \text{ is divisible by } H^{p-\mathfrak{q}_2}.$$

As before, $\mathrm{L}_2(\mathfrak{q})$ is verified if and only if there exists a symbol $\Lambda_2(x;\xi)$ homogeneous of order $\mu_2 - M + 1$ in ξ, such that

$$\mathcal{A}\sigma_{\mu_1}(S'_2) = H^{p-\mathfrak{q}_2}\Lambda_2,$$

and, if $\mathrm{L}_2(\mathfrak{q})$ is verified, we have:

$$\sigma_{\mu_1}(S'_2)H^{\mathfrak{q}_2}K = a\Lambda_2,$$

that is: the operator

$$S'_3 := h\Lambda'_2 - S'_1 H'^{\mathfrak{q}_2} K'$$

is of order μ_2.

We proceed by induction. Assume that h verifies the conditions $\mathrm{L}_1(\mathfrak{q})$,

$\ldots$, $\mathrm{L}_k(\mathfrak{q})$: there exists differential operators $\mathcal{A}'$, H', K', $\Lambda'_1, \ldots, \Lambda'_{k-1}$, whose principal symbols are $\mathcal{A}$, H, K, $\Lambda_1, \ldots, \Lambda_{k-1}$, ($\Lambda_j(x;\xi)$ is homogeneous of order $\mu_j - M + 1$ in ξ) and a symbol $\Lambda_k(x;\xi)$ homogeneous of order $\mu_k - M + 1$ in ξ, such that

$$\mathcal{A}\sigma_{\mu_{k-1}}(S'_k) = H^{p-\mathfrak{q}_k}\Lambda_k,$$

where the S'_j and the Λ_k are defined by:

$$\begin{aligned} S'_1 &:= h\mathcal{A}' - H'^p K' \\ S'_{k+1} &:= h\Lambda'_k - S'_k H'^{\mathfrak{q}_k} K', \quad k \geq 1, \end{aligned}$$

hence:

$$\begin{aligned} S'_{k+1} = &\sum_{s=0}^{k-1} (-1)^s h\Lambda'_{k-s} H'^{\mathfrak{q}_{k-s+1}} K' \cdots H'^{\mathfrak{q}_k} K' \\ &+(-1)^k h\mathcal{A}' H'^{\mathfrak{q}_1} K' \cdots H'^{\mathfrak{q}_k} K' + (-1)^{k+1} H'^p K' H'^{\mathfrak{q}_1} K' \cdots H'^{\mathfrak{q}_k} K'. \end{aligned}$$

The condition $\mathrm{L}_{k+1}(\mathfrak{q})$ is: there exists a matrix operator Λ'_k and a symbol $\Lambda_{k+1}(x;\xi)$ homogeneous of order $\mu_{k+1} - M + 1$ in ξ, such that

$$\mathcal{A}\sigma_{\mu_k}(S'_{k+1}) = H^{p-\mathfrak{q}_{k+1}}\Lambda_{k+1}.$$

The last condition is the condition $\mathrm{L}_\kappa(\mathfrak{q})$:

$\mathcal{A}\sigma_{\mu_{\kappa-1}}(S'_\kappa)$ is divisible by $H^{p-\mathfrak{q}_\kappa}$.

We resume the properties of the conditions $\mathrm{L}(\mathfrak{q})$ in the following Proposition:

Proposition 2.1

(i) *The conditions* $\mathrm{L}(\mathfrak{q})$ *are invariantly defined.*

(ii) *The conditions* $\mathrm{L}(\mathfrak{q})$ *do not depend on the choice of the operators* H', K', $\mathcal{A}'$, $\Lambda'_1, \ldots, \Lambda'_{\kappa-1}$.

(iii) *We can express the conditions* $\mathrm{L}(\mathfrak{q})$ *as differential relations between the coefficients of* h *(indeed thanks to the previous property, we can choose* $H' = I_N H(x;D)$, $K' = I_N K(x;D)$, $\mathcal{A}' = \mathcal{A}(x;D)$*).*

(iv) *Let* $\Delta(x;D')$ *be an elliptic classical pseudo-differential operator of order* 0, *if* h *verifies the conditions* $\mathrm{L}(\mathfrak{q})$, *the transformed operator* $\widetilde{h} := \Delta^{-1}h\Delta$ *verifies the same conditions* $\widetilde{\mathrm{L}}(\mathfrak{q})$.

(v) *If* h *verifies the conditions* $\mathrm{L}(\mathfrak{q})$, *its formal adjoint verifies* $\mathrm{L}(\mathfrak{q})$ *too.*

Proposition 2.1 has been proved in the case $p = m$ in [28, Chap. II]. The proof in the general case, which is essentially the same, will appear in a forthcoming paper.

We note that we can consider any sequence $\mathfrak{q}$ with $1 \le \mathfrak{q}_j \le p$. Anyway, only a finite number of conditions $\mathrm{L}(\mathfrak{q})$ are independent. For instance, for scalar operators the condition $\mathrm{L}(\mathfrak{q})$, with $\mathfrak{q} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_\kappa)$ and $\mathfrak{q}_1 + \cdots + \mathfrak{q}_\kappa \ge m$ is a consequence of the condition $\mathrm{L}(\mathfrak{q}')$ with $\mathfrak{q}' = (\mathfrak{q}_1, \ldots, \mathfrak{q}_{\kappa-1})$ (see below). The conditions $\mathrm{L}(\mathfrak{q})$ define a germ of Noetherian analytic set in x.

3. Results

The conditions $\mathrm{L}(\mathfrak{q})$ have been introduced to study the Cauchy Problem in $\mathcal{C}^\infty$ and in Gevrey spaces.

In general, if h is of type $(H^p, H^{q_1}, \ldots, H^{q_\ell})$, there exists a κ such that the conditions $\mathrm{L}(\mathfrak{q})$ with $\mathfrak{q} = (q_1, \ldots, q_\ell, 1, \ldots, 1)$ are necessary for the Cauchy Problem to be well-posed in $\mathcal{C}^\infty$ and moreover they are sufficient if the coefficients of h are analytic or the generalized rank of h is constant.

For example if h is of type (H^m) (maximal generalized rank: $p = m$ and $\ell = 0$) we get the above result with $\mathfrak{q} = (1, \ldots, 1)$ and $\kappa = m - 1$ [28]. Analogously, if h is of type $(H^p, H, \ldots, H)$ (that is $q_1 = \cdots = q_\ell = 1$ and $\ell = m - p$) we get the same result with the same $\mathfrak{q}$ [21]. In [23], it is considered the cases of multiplicity ≤ 5 and, in particular, it is proved the above result with $\mathfrak{q} = (2, 1, 1, 1, 1, 1)$ (that is $\kappa = 6$) if h is of type (H^3, H^2) and $\mathfrak{q} = (2, 1, 1, 1, 1, 1, 1, 1)$ (that is $\kappa = 8$) if h is of type (H^2, H^2, H).

Considering general $\mathfrak{q}$, we get the conditions for the well-posedness in Gevrey spaces. For the precise statements see [31] for the cases of multiplicity ≤ 5 and [22] if h is of type (H^m) or $(H^p, H, \ldots, H)$.

In this paper we prove two results regarding two different cases.

A first result concerns scalar operators:

Theorem 1 *If h is a scalar operator verifying* $\mathrm{L}(\mathfrak{q})$*, the Cauchy Problem for h is well-posed in all Gevrey spaces γ^d with $1 < d < d_0$, where*:

$$d_0 := \min\Big(\frac{\mathfrak{q}_1}{\mathfrak{q}_1 - 1}, \frac{\mathfrak{q}_1 + \mathfrak{q}_2}{\mathfrak{q}_1 + \mathfrak{q}_2 - 2}, \ldots, \frac{\mathfrak{q}_1 + \mathfrak{q}_2 + \cdots + \mathfrak{q}_\kappa}{\mathfrak{q}_1 + \mathfrak{q}_2 + \cdots + \mathfrak{q}_\kappa - \kappa}, \frac{m}{m - \kappa - 1}\Big). \tag{4}$$

Moreover, if $\mathfrak{q} = (1, \ldots, 1)$*, with* $\kappa = m - 1$*, the Cauchy Problem is well-posed in all the Gevrey spaces and in* $\mathcal{C}^\infty$*.*

To prove Theorem 1 we will show in Section §4 that the conditions $\mathrm{L}(\mathfrak{q})$ for h are equivalent to the fact that h have a decomposition with respect to H of type:

$$h = K'H'^m + l_1 H'^{\nu_1} + \cdots + l_r H'^{\nu_r} + \cdots + l_\kappa H'^{\nu_\kappa}, \tag{5}$$

with $\nu_r = m - \mathfrak{q}_1 - \mathfrak{q}_2 - \cdots - \mathfrak{q}_r$, for $r = 1, \ldots, \kappa$ and $l_r H'^{\nu_r}$ is of order $\leq M - r$ or zero.

In particular, h has a *good decomposition* in the sense of De Paris [5], if and only if h verifies the conditions $\mathrm{L}(\mathfrak{q})$, with $\mathfrak{q} = (1, \ldots, 1)$, and $\kappa = m - 1$.

Theorem 1 follows from the results of [5, 3, 6].

A second result is about first order systems of the type

$$(\underbrace{H^2, H^2, \ldots, H^2}_{r \text{ times}}, H, \ldots, H). \tag{6}$$

Such operators are the hardest to deal with, due to the large number of Jordan nilpotent blocks appearing in the principal symbol.

For this kind of systems we study the first conditions $\mathrm{L}(\mathfrak{q})$ which is necessary for the well-posedness in $\mathcal{C}^\infty$. This condition is obtained with:

$$\mathfrak{q} = (\underbrace{2, 2, \ldots, 2}_{r-1 \text{ times}}, 1), \quad \kappa = r. \tag{7}$$

In particular the first order systems of type (H^2) (systems with double characteristic and maximal rank) has been extensively studied [4, 8, 14, 25, 32]. In this case the conditions $\mathrm{L}(\mathfrak{q})$ are reduced to the condition:

$$\mathrm{L}_1(\mathfrak{q})\colon \mathcal{A}\sigma_{\mu_0}(h\,\mathcal{A}' - H'^2 K') \quad \text{is divisible by } H.$$

For this kind of systems we prove that conditions $\mathrm{L}_1(\mathfrak{q})$ is equivalent to the usual condition on the sub-principal symbol. More precisely:

Proposition 3.1 *If* h *is of type* (H^2)*, then condition* $\mathrm{L}_1(\mathfrak{q})$ *is equivalent to each of the following conditions*:

$$A\Big[\mathcal{S}A + \frac{1}{2}\{a, A\}\Big] \text{ is divisible by } H, \tag{8}$$

$$\Big[A\mathcal{S} + \frac{1}{2}\{A,\, a\}\Big]A \text{ is divisible by } H, \tag{9}$$

where $\mathcal{S}$ *is the sub-characteristic matrix*:

$$\mathcal{S} := b - \frac{1}{2}\sum_{j=1}^{n} \partial_{\xi_j}\partial_{x_j} a, \tag{10}$$

and $\{\,\cdot\,,\,\cdot\,\}$ *is the Poisson bracket.*

More generally, we have:

Theorem 2 *If* h *is of type* (6) *and* $\mathfrak{q}$ *is as in* (7)*, then the conditions* $\mathrm{L}_1(\mathfrak{q}), \ldots, \mathrm{L}_{r-1}(\mathfrak{q})$ *are always satisfied and the condition* $\mathrm{L}_r(\mathfrak{q})$ *is equivalent to each of the following conditions*:

$$\mathcal{A}\Big[\mathcal{S}\mathcal{A} + \frac{1}{2}\{a,\, \mathcal{A}\}\Big]^r \quad \text{is divisible by } H, \tag{11}$$

$$\Big[\mathcal{A}\mathcal{S} + \frac{1}{2}\{\mathcal{A},\, a\}\Big]^r \mathcal{A} \quad \text{is divisible by } H, \tag{12}$$

where $\mathcal{S}$ *is the sub-characteristic matrix in* (10)*.*

Moreover, the condition $\mathrm{L}_r(\mathfrak{q})$ *is necessary for the Cauchy Problem to be well-posed in* $\mathcal{C}^\infty$*.*

The proof is standard: we reduce the operator to a simple microlocal form and then we construct an asymptotic solution which violates the a-priori inequality which is a consequence of the well-posedness. We give the essential part of the calculations in a simple case in paragraph §5, but the general case can be considered in a similar way.

Remark 3.2 By using the method of Ivrii-Komatsu one can prove that the condition $\mathrm{L}_r(\mathfrak{q})$ is necessary for the well-posedness in γ^s, with $s > 2$, but we will not develop the calculations here.

4. Scalar operators

We recall that if h is a differential operator of order M with analytic coefficients, H a irreducible factor of the principal symbol of h, with multiplicity m, De Paris showed that h admits a decomposition with respect to H [5, Prop. 1]:

$$h = \sum_{r=0}^{M} l_r {H'}^{\nu_r},$$

where $\nu_r \in \mathbb{N} \cup \{+\infty\}$ and the operators l_r, are with analytic coefficients, whose principal symbol is not divisible by H, and moreover $l_r H'^{\nu_r}$ is of order $M - r$ or zero (we will set $\nu_r = +\infty$ and $l_r H'^{\nu_r} \equiv 0$ if $l_r \equiv 0$).

There exists several way to decompose h with respect to H, and the ν_r depend on choice of the operator H' with principal symbol H; only $\nu_0 = m$ is an invariant.

We set:

$$\sigma(H) := \max_{1 \le r \le m} \Big(\frac{m - \nu_r}{r} \Big),$$

$$\alpha(H) := \begin{cases} \dfrac{\sigma(H)}{\sigma(H) - 1} & \text{if } \sigma(H) > 1 \\ +\infty & \text{if } \sigma(H) = 1. \end{cases}$$

Remark 4.1 Note that $\sigma(H) \le m$ and consequently $\alpha(H) \ge m/(m-1)$.

Remark 4.2 Let

$$\theta(r) := \min_{0 \le \rho \le r} (\nu_\rho + \rho),$$

θ is defined on $\{0, 1, \dots, m\}$, it is not increasing and it depends only on h and H [6, Prop. 1.1]; moreover:

$$\sigma(H) = 1 + \max_{1 \le r \le m} \Big(\frac{m - \theta(r)}{r} \Big).$$

Hence $\sigma(H)$ and $\alpha(H)$ depend only on h and H.

We recall that an operator h has a *good decomposition with respect to* H if $\sigma(H) = 1$, or, equivalently if $\alpha(H) = +\infty$ or $\theta(H) = m$. Moreover, h has a *good decomposition* if h has a good decomposition with respect to all its invariant factors.

The condition of good decomposition does not depend on the choice of H' with symbol H.

Let Ψ^ℓ be the set of the pseudo-differential operator of order ℓ and, for fixed H, let $\Psi^{\ell,m}$ be the set of the operators in Ψ^ℓ which have a good decomposition with respect to H with $\nu_0 \ge m$.

Lemma 4.3 *Let $A' \in \Psi^\alpha$ and $B' \in \Psi^\beta$, then $A'H'^p B' \in \Psi^{\alpha+\beta,p}$.*

Proof. It's enough to show that if $B' \in \Psi^{\beta}$, we have the following decomposition:

$$H'^{p}B' = B'H'^{p} + R'_{p,1}H'^{p-1} + \cdots + R'_{p,p-1}H + R'_{p,p}, \tag{13}$$

with $R'_{p,j} \in \Psi^{\beta+j(s-1)}$, for $j = 1, \ldots, p$.

If $p = 1$, we have:

$$H'B' = B'H' + [H', B'],$$

and $[H', B'] \in \Psi^{\beta+s-1}$.

We proceed by induction: we assume that (13) holds true for p and we prove it for $p + 1$. We have:

$$\begin{aligned} H'^{p+1}B' &= H'(H'^{p}B') \\ &= H'B'H'^{p} + H'R'_{p,1}H'^{p-1} + \cdots + H'R'_{p,p-1}H + H'R'_{p,p} \\ &= B'H'^{p+1} + [H', B']H'^{p} + R'_{p,1}H'^{p} + [H', R'_{p,1}]H'^{p-1} \\ &\quad + \cdots + R'_{p,p}H' + [H', R'_{p,p}], \end{aligned}$$

and choosing $R'_{p+1,j} = R'_{p,j} + [H', R'_{p,j-1}]$, we get the wished decomposition (we set $R'_{p,0} = B'$, $R'_{p,p+1} \equiv 0$, for any p et $R'_{p,j} \equiv 0$, for any p and $j < 0$). □

Lemma 4.4 *Let $A' \in \Psi^{\alpha}$ and $B' \in \Psi^{\beta}$, then $[A'H'^{p}, B']$, $[H'^{p}A', B'] \in \Psi^{\alpha+\beta-1,p-1}$.*

The proof follows from Lemma 4.3.

Proposition 4.5 *If h is a scalar operator satisfying* $\mathrm{L}(\mathfrak{q})$, *then $\alpha(H) = d_0$, where d_0 is defined in* (4).

Remark 4.6 If $N = 1$, then $p = m$ and we have $\mu_0 = M - 1$ and

$$\mu_k = M - 1 + k(\chi - 1) + s\sum_{j=1}^{k}\mathfrak{q}_j, \quad \text{for } 1 \le k \le \kappa.$$

Proof. Assume at first that the sequence $\mathfrak{q} = (\mathfrak{q}_1, \ldots, \mathfrak{q}_\kappa)$ verifies the condition

$$\mathfrak{q}_1 + \cdots + \mathfrak{q}_j + \mathfrak{q}_1 + \cdots + \mathfrak{q}_k \le \mathfrak{q}_1 + \cdots + \mathfrak{q}_{j+k}, \tag{14}$$

for any j, k such that $j + k \le \kappa$.

In particular, if (14) is verified, we have $\mathfrak{q}_1 \leq \mathfrak{q}_j$, for any $j = 2, \ldots, \kappa$.

We show that if h verifies $\mathrm{L}(\mathfrak{q})$, then h has a decomposition with respect to H with $\nu_r \geq M - \mathfrak{q}_1 - \mathfrak{q}_2 - \cdots - \mathfrak{q}_r$.

Since $A = \mathcal{A} = 1$, the conditions $\mathrm{L}(\mathfrak{q})$ are

$$\begin{aligned}
&\mathrm{L}_1(\mathfrak{q})\colon S'_1 := h - H'^m K' && \sigma_{\mu_0}(S'_1) = H^{m-\mathfrak{q}_1}\Lambda_1,\\
&\mathrm{L}_2(\mathfrak{q})\colon S'_2 := h\Lambda'_1 - S'_1 H'^{\mathfrak{q}_1} K' && \sigma_{\mu_1}(S'_2) = H^{m-\mathfrak{q}_2}\Lambda_2,\\
&\quad\vdots\\
&\mathrm{L}_j(\mathfrak{q})\colon S'_j := h\Lambda'_{j-1} - S'_{j-1} H'^{\mathfrak{q}_{j-1}} K' && \sigma_{\mu_{j-1}}(S'_j) = H^{m-\mathfrak{q}_j}\Lambda_j,\\
&\quad\vdots\\
&\mathrm{L}_\kappa(\mathfrak{q})\colon S'_\kappa := h\Lambda'_{\kappa-1} - S'_{\kappa-1} H'^{\mathfrak{q}_{\kappa-1}} K' && \sigma_{\mu_{\kappa-1}}(S'_\kappa) = H^{m-\mathfrak{q}_\kappa}\Lambda_\kappa.
\end{aligned}$$

We show by induction that the conditions $\mathrm{L}_1(\mathfrak{q}), \ldots, \mathrm{L}_j(\mathfrak{q})$ are equivalent to the conditions

$$\begin{aligned}
\mathrm{L}^D_{1,j}(\mathfrak{q}):\ S'_1 &= \Lambda'_1 H'^{m-\mathfrak{q}_1} + R'_{1,1} H'^{m-\mathfrak{q}_1-\mathfrak{q}_2}\\
&\quad + \cdots + R'_{1,j-1} H'^{m-\mathfrak{q}_1-\cdots-\mathfrak{q}_j} + T'_{1,j},\\
&\vdots\\
\mathrm{L}^D_{k,j-k+1}(\mathfrak{q}):\ S'_k &= \Lambda'_k H'^{m-\mathfrak{q}_k} + R'_{k,1} H'^{m-\mathfrak{q}_k-\mathfrak{q}_{k+1}}\\
&\quad + \cdots + R'_{k,j-k} H'^{m-\mathfrak{q}_k-\cdots-\mathfrak{q}_j} + T'_{k,j-k+1},\\
&\vdots\\
\mathrm{L}^D_{j-1,2}(\mathfrak{q}):\ S'_{j-1} &= \Lambda'_{j-1} H'^{m-\mathfrak{q}_{j-1}} + R'_{j-1,1} H'^{m-\mathfrak{q}_{j-1}-\mathfrak{q}_j} + T'_{j-1,2},\\
\mathrm{L}^D_{j,1}(\mathfrak{q}):\ S'_j &= \Lambda'_j H'^{m-\mathfrak{q}_j} + T'_{j,1},
\end{aligned}$$

for some $R'_{\alpha,\beta}$ with $R'_{\alpha,\beta} H'^{m-\mathfrak{q}_\alpha-\cdots-\mathfrak{q}_{\alpha+\beta}} \in \Psi^{\mu_{\alpha-1}-\beta}$ and $T'_{\alpha,\beta} \in \Psi^{\mu_{\alpha-1}-\beta}$.

The conditions $\mathrm{L}_j(\mathfrak{q})$ and $\mathrm{L}^D_{j,1}(\mathfrak{q})$ are clearly equivalent, hence we assume the equivalence between $\mathrm{L}_1(\mathfrak{q}), \ldots, \mathrm{L}_j(\mathfrak{q})$ and $\mathrm{L}^D_{1,j}(\mathfrak{q}), \ldots, \mathrm{L}^D_{j,1}(\mathfrak{q})$ and we prove the equivalence of $\mathrm{L}_{j+1}(\mathfrak{q})$ and $\mathrm{L}^D_{1,j+1}(\mathfrak{q}), \ldots, \mathrm{L}^D_{j+1,1}(\mathfrak{q})$.

Let us show that the condition $\mathrm{L}^D_{j+1,1}(\mathfrak{q})$ is equivalent to the condition $\mathrm{L}^D_{j,2}(\mathfrak{q})$. We have:

$$\begin{aligned}
S'_{j+1} &= h\Lambda'_j - S'_j H'^{\mathfrak{q}_j} K'\\
&= (H'^m K' + S'_1)\Lambda'_j - (\Lambda'_j H'^{m-\mathfrak{q}_j} + T'_{j,1}) H'^{\mathfrak{q}_j} K'
\end{aligned}$$

$$
\begin{aligned}
&= (H'^m K' + \Lambda_1' H'^{m-\mathfrak{q}_1} + T_{1,1})\Lambda_j' - (\Lambda_j' H'^{m-\mathfrak{q}_j} + T_{j,1}')H'^{\mathfrak{q}_j} K' \\
&= [H'^m K', \Lambda_j'] + \Lambda_1' H'^{m-\mathfrak{q}_1}\Lambda_j' + T_{1,1}'\Lambda_j' - T_{j,1}' H'^{\mathfrak{q}_j} K'.
\end{aligned}
$$

Now, thanks to Lemmas 4.3 and 4.4:

$$
[H'^m K', \Lambda_j'] \in \Psi^{\mu_j, m-1}, \quad \Lambda_1' H'^{m-\mathfrak{q}_1}\Lambda_j' \in \Psi^{\mu_j, m-\mathfrak{q}_1},
$$

and moreover $T_{1,1}'\Lambda_j' \in \Psi^{\mu_j - 1}$, hence the condition $\mathrm{L}_{j+1}(\mathfrak{q})$ is equivalent to:

$$
\sigma_{\mu_j}(T_{j,1}' H'^{\mathfrak{q}_j} K') \text{ is divisible by } H^{m-\mathfrak{q}_{j+1}},
$$

that is:

$$
\sigma_{\mu_{j-1}-1}(T_{j,1}') \text{ is divisible by } H^{m-\mathfrak{q}_j-\mathfrak{q}_{j+1}},
$$

which is equivalent to:

$$
T_{j,1}' = R_{j,1}' H'^{m-\mathfrak{q}_j-\mathfrak{q}_{j+1}} + T_{j,2}',
$$

and hence:

$$
\mathrm{L}_{j,2}^D(\mathfrak{q})\colon S_j' = \Lambda_j' H'^{m-\mathfrak{q}_j} + R_{j,1}' H'^{m-\mathfrak{q}_j-\mathfrak{q}_{j+1}} + T_{j,2}'.
$$

Let us show that the condition $\mathrm{L}_{j,2}^D(\mathfrak{q})$ is equivalent to the condition $\mathrm{L}_{j-1,3}^D(\mathfrak{q})$. We have:

$$
\begin{aligned}
S_j' &= h\Lambda_{j-1}' - S_{j-1}' H'^{\mathfrak{q}_{j-1}} K' \\
&= (H'^m K' + \Lambda_1' H'^{m-\mathfrak{q}_1} + R_{1,1}' H'^{m-\mathfrak{q}_1-\mathfrak{q}_2} + T_{1,2}')\Lambda_{j-1}' \\
&\quad - (\Lambda_{j-1}' H'^{m-\mathfrak{q}_{j-1}} + R_{j-1,1}' H'^{m-\mathfrak{q}_{j-1}-\mathfrak{q}_j} + T_{j-1,2}')H'^{\mathfrak{q}_{j-1}} K' \\
&= [H'^m K', \Lambda_{j-1}'] + \big(\Lambda_1' H'^{m-\mathfrak{q}_1}\Lambda_{j-1}' - R_{j-1,1}' H'^{m-\mathfrak{q}_j} K'\big) \\
&\quad + \big(R_{1,1}' H'^{m-\mathfrak{q}_1-\mathfrak{q}_2}\Lambda_{j-1}' - T_{j-1,2}' H'^{\mathfrak{q}_{j-1}} K'\big) + T_{1,2}'\Lambda_{j-1}',
\end{aligned}
$$

and, using Lemmas 4.3 and 4.4:

$$
\begin{aligned}
[H'^m K', \Lambda_{j-1}'] &\in \Psi^{\mu_{j-1}, m-1}, \\
\Lambda_1' H'^{m-\mathfrak{q}_1}\Lambda_{j-1}' &\in \Psi^{\mu_{j-1}, m-\mathfrak{q}_1}, \\
R_{j-1,1}' H'^{m-\mathfrak{q}_j} K' &\in \Psi^{\mu_{j-1}, m-\mathfrak{q}_j}, \\
R_{1,1}' H'^{m-\mathfrak{q}_1-\mathfrak{q}_2}\Lambda_{j-1}' &\in \Psi^{\mu_{j-1}-1, m-\mathfrak{q}_1-\mathfrak{q}_2}, \\
T_{1,2}'\Lambda_{j-1}' &\in \Psi^{\mu_{j-1}-2}.
\end{aligned}
$$

If we compare with $\mathrm{L}^D_{j,2}(\mathfrak{q})$, we see:

$$\sigma_{\mu_{j-1}-1}(T'_{j-1,2}H'^{\mathfrak{q}_{j-1}}K') \text{ is divisible by } H^{m-\mathfrak{q}_j-\mathfrak{q}_{j+1}},$$

that is:

$$\sigma_{\mu_{j-2}-2}(T'_{j-1,2}) \text{ is divisible by } H^{m-\mathfrak{q}_{j-1}-\mathfrak{q}_j-\mathfrak{q}_{j+1}},$$

which is equivalent to:

$$T'_{j-1,2} = R'_{j-1,2}H'^{m-\mathfrak{q}_{j-1}-\mathfrak{q}_j-\mathfrak{q}_{j+1}} + T'_{j-1,3},$$

hence:

$$\begin{aligned}\mathrm{L}^D_{j-1,3}(\mathfrak{q})\colon S'_{j-1} = \Lambda'_{j-1}H'^{m-\mathfrak{q}_{j-1}} &+ R'_{j-1,1}H'^{m-\mathfrak{q}_{j-1}-\mathfrak{q}_j} \\ &+ R'_{j-1,2}H'^{m-\mathfrak{q}_{j-1}-\mathfrak{q}_j-\mathfrak{q}_{j+1}} + T'_{j-1,3}.\end{aligned}$$

We prove in a similar way that the condition $\mathrm{L}^D_{k+1,j-k+1}(\mathfrak{q})$ is equivalent to the condition $\mathrm{L}^D_{k,j-k+2}(\mathfrak{q})$. Indeed we have:

$$\begin{aligned}
&S'_{k+1} = h\Lambda'_k - S'_kH'^{\mathfrak{q}_k}K'\\
&= (H'^mK' + \Lambda'_1H'^{m-\mathfrak{q}_1} + \sum_{\ell=1}^{j-k} R'_{1,\ell}H'^{m-\mathfrak{q}_1-\cdots-\mathfrak{q}_{\ell+1}} + T_{1,j-k+1})\Lambda'_k\\
&-\Big(\Lambda'_kH'^{m-\mathfrak{q}_k} + \sum_{\ell=1}^{j-k} R'_{k,\ell}H'^{m-\mathfrak{q}_k-\mathfrak{q}_{k+1}-\cdots-\mathfrak{q}_{k+\ell+1}} + T_{k,j-k+1}\Big)H'^{\mathfrak{q}_k}K'\\
&= [H'^mK',\, \Lambda'_k] + \big(\Lambda'_1H'^{m-\mathfrak{q}_1}\Lambda'_k - R'_{k,1}H'^{m-\mathfrak{q}_{k+1}}K'\big)\\
&\quad + \sum_{\ell=1}^{j-k-1}\big(R'_{1,\ell}H'^{m-\mathfrak{q}_1-\cdots-\mathfrak{q}_{\ell+1}}\Lambda'_k - R'_{k,\ell}H'^{m-\mathfrak{q}_{k+1}-\cdots-\mathfrak{q}_{k+\ell+1}}K'\big)\\
&\quad + \big(R'_{1,j-k}H'^{m-\mathfrak{q}_1-\cdots-\mathfrak{q}_{j-k+1}}\Lambda'_k - T_{k,j-k+1}H'^{\mathfrak{q}_k}K'\big) + T_{1,j-k+1}\Lambda'_k.
\end{aligned}$$

Using Lemmas 4.3 and 4.4: we have

$$\begin{aligned}
&[H'^mK',\, \Lambda'_k] \in \Psi^{\mu_k,m-1}, \quad \Lambda'_1H'^{m-\mathfrak{q}_1}\Lambda'_k \in \Psi^{\mu_k,m-\mathfrak{q}_1},\\
&R'_{k,1}H'^{m-\mathfrak{q}_{k+1}}K' \in \Psi^{\mu_k,m-\mathfrak{q}_{k+1}}\\
&\vdots\\
&R'_{1,\ell}H'^{m-\mathfrak{q}_1-\cdots-\mathfrak{q}_{\ell+1}}\Lambda'_k \in \Psi^{\mu_k-\ell,m-\mathfrak{q}_1-\cdots-\mathfrak{q}_{\ell+1}},\\
&R'_{k,\ell}H'^{m-\mathfrak{q}_{k+1}-\cdots-\mathfrak{q}_{k+\ell+1}}K' \in \Psi^{\mu_k-\ell,m-\mathfrak{q}_{k+1}-\cdots-\mathfrak{q}_{k+\ell+1}}
\end{aligned}$$

$$R'_{1,j-k}H'^{m-\mathfrak{q}_1-\cdots-\mathfrak{q}_{j-k+1}}\Lambda'_k \in \Psi^{\mu_k-(j-k),m-\mathfrak{q}_1-\cdots-\mathfrak{q}_{j-k+1}},$$
$$T_{1,j-k+1}\Lambda'_k \in \Psi^{\mu_k-(j-k+1)}$$

hence, if we compare the last identity with $\mathrm{L}^D_{k+1,j-k+1}(\mathfrak{q})$, we see that:

$$\sigma_{\mu_k-(j-k)}(T'_{k,j-k+1}H'^{\mathfrak{q}_k}K') \text{ is divisible by } H^{m-\mathfrak{q}_{k+1}-\cdots-\mathfrak{q}_j-\mathfrak{q}_{j+1}},$$

that is:

$$\sigma_{\mu_{k-1}-(j-k+1)}(T'_{k,j-k+1}) \text{ is divisible by } H^{m-\mathfrak{q}_k-\cdots-\mathfrak{q}_j-\mathfrak{q}_{j+1}},$$

which is equivalent to:

$$T'_{k,j-k+1} = R'_{k,j-k}H'^{m-\mathfrak{q}_k-\cdots-\mathfrak{q}_j-\mathfrak{q}_{j+1}} + T'_{k,j-k+2},$$

hence:

$$\begin{aligned}\mathrm{L}^D_{k,j-k+2}(\mathfrak{q})\colon S'_k = {}&\Lambda'_k H'^{m-\mathfrak{q}_k} + R'_{k,1}H'^{m-\mathfrak{q}_k-\mathfrak{q}_{k+1}}\\ &+\cdots+R'_{k,j-k}H'^{m-\mathfrak{q}_k-\cdots-\mathfrak{q}_j-\mathfrak{q}_{j+1}}+T'_{k,j-k+2}.\end{aligned}$$

Condition $\mathrm{L}^D_{1,\kappa}(\mathfrak{q})$ is

$$\begin{aligned}h = {}&H'^m K' + \Lambda'_1 H'^{m-\mathfrak{q}_1} + R'_{1,1}H'^{m-\mathfrak{q}_1-\mathfrak{q}_2}\\ &+\cdots+R'_{1,\kappa-1}H'^{m-\mathfrak{q}_1-\cdots-\mathfrak{q}_\kappa}+T'_{1,\kappa},\end{aligned}$$

which gives:

$$\sigma(H) = \max\Big(\mathfrak{q}_1,\ \frac{\mathfrak{q}_1+\mathfrak{q}_2}{2},\ \ldots,\ \frac{\mathfrak{q}_1+\mathfrak{q}_2+\cdots+\mathfrak{q}_\kappa}{\kappa},\ \frac{m}{\kappa+1}\Big)$$

and since $\sigma\mapsto\sigma/(\sigma-1)$ is not increasing

$$\begin{aligned}\alpha(H) = \min\Big(&\frac{\mathfrak{q}_1}{\mathfrak{q}_1-1},\ \frac{\mathfrak{q}_1+\mathfrak{q}_2}{\mathfrak{q}_1+\mathfrak{q}_2-2},\\ &\ldots,\ \frac{\mathfrak{q}_1+\mathfrak{q}_2+\cdots+\mathfrak{q}_\kappa}{\mathfrak{q}_1+\mathfrak{q}_2+\cdots+\mathfrak{q}_\kappa-\kappa},\ \frac{m}{m-\kappa-1}\Big).\end{aligned}$$

If $\mathfrak{q}$ does not verify the condition (14), we prove as in [22] that we can find $\mathfrak{q}^\sharp$, with $\mathfrak{q}^\sharp_j \geq \mathfrak{q}_j$, for any $j = 1, \ldots, \kappa$, which verifies (14) and $\alpha_{\mathfrak{q}^\sharp}(H) = \alpha_{\mathfrak{q}}(H)$; moreover, if h verifies $\mathrm{L}(\mathfrak{q})$, then h verifies $\mathrm{L}(\mathfrak{q}^\sharp)$ too.

In fact, if $\mathfrak{q}$ does not verify the condition (14), let $\overline{\kappa}$ be the smallest integer such that there exists j and k for which $j+k=\overline{\kappa}$ and

$$\mathfrak{q}_1+\cdots+\mathfrak{q}_j+\mathfrak{q}_1+\cdots+\mathfrak{q}_k>\mathfrak{q}_1+\cdots+\mathfrak{q}_{j+k}.$$

Let $\bar{\jmath}_1$ and $\bar{\jmath}_2$ be such that

$$\begin{aligned}(\mathfrak{q}_1&+\cdots+\mathfrak{q}_{\bar{\jmath}_1})+(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_2})\\ &=\max\bigl\{(\mathfrak{q}_1+\cdots+\mathfrak{q}_j)+(\mathfrak{q}_1+\cdots+\mathfrak{q}_k)\bigm| j+k=\overline{\kappa}\bigr\},\end{aligned}$$

we can assume that

$$\frac{1}{\bar{\jmath}_1}(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1})\geq\frac{1}{\bar{\jmath}_2}(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_2}). \tag{15}$$

We set $\mathfrak{q}'_1=\mathfrak{q}_1$, ..., $\mathfrak{q}'_{\overline{\kappa}-1}=\mathfrak{q}_{\overline{\kappa}-1}$ and $\mathfrak{q}'_{\overline{\kappa}}=\mathfrak{q}_{\overline{\kappa}}+c_{\overline{\kappa}}$ where

$$c_{\overline{\kappa}}:=(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1})+(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_2})-\mathfrak{q}_1-\cdots-\mathfrak{q}_{\overline{\kappa}}.$$

We have:

$$\begin{aligned}&\frac{\mathfrak{q}'_1+\cdots+\mathfrak{q}'_j}{\mathfrak{q}'_1+\cdots+\mathfrak{q}'_j-j}=\frac{\mathfrak{q}_1+\cdots+\mathfrak{q}_j}{\mathfrak{q}_1+\cdots+\mathfrak{q}_j-j},\quad 1\leq j<\overline{\kappa},\\ &\frac{\mathfrak{q}'_1+\cdots+\mathfrak{q}'_{\overline{\kappa}}}{\mathfrak{q}'_1+\cdots+\mathfrak{q}'_{\overline{\kappa}}-\overline{\kappa}}\leq\frac{\mathfrak{q}_1+\cdots+\mathfrak{q}_{\overline{\kappa}}}{\mathfrak{q}_1+\cdots+\mathfrak{q}_{\overline{\kappa}}-\overline{\kappa}},\end{aligned}$$

which gives $\alpha_{\mathfrak{q}'}(H)\leq\alpha_{\mathfrak{q}}(H)$.

On the other side, thanks to (15), we have:

$$\begin{aligned}&\frac{\mathfrak{q}'_1+\cdots+\mathfrak{q}'_{\overline{\kappa}}}{\mathfrak{q}'_1+\cdots+\mathfrak{q}'_{\overline{\kappa}}-\overline{\kappa}}\\ &=\frac{\mathfrak{q}_1+\cdots+\mathfrak{q}_{\overline{\kappa}}+c_{\overline{\kappa}}}{\mathfrak{q}_1+\cdots+\mathfrak{q}_{\overline{\kappa}}+c_{\overline{\kappa}}-\overline{\kappa}}=\frac{(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1})+(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_2})}{(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1})+(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_2})-\overline{\kappa}}\\ &=\frac{\bar{\jmath}_1(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1})/\bar{\jmath}_1+\bar{\jmath}_2(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_2})/\bar{\jmath}_2}{\bar{\jmath}_1(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1})/\bar{\jmath}_1+\bar{\jmath}_2(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_2})/\bar{\jmath}_2-\overline{\kappa}}\\ &\geq\frac{(\bar{\jmath}_1+\bar{\jmath}_2)(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1})/\bar{\jmath}_1}{(\bar{\jmath}_1+\bar{\jmath}_2)(\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1})/\bar{\jmath}_1-(\bar{\jmath}_1+\bar{\jmath}_2)}=\frac{\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1}}{\mathfrak{q}_1+\cdots+\mathfrak{q}_{\bar{\jmath}_1}-\bar{\jmath}_1},\end{aligned}$$

which gives $\alpha_{\mathfrak{q}'}(H)\geq\alpha_{\mathfrak{q}}(H)$, and hence $\alpha_{\mathfrak{q}'}(H)=\alpha_{\mathfrak{q}}(H)$.

We proceed in the same way if $\mathfrak{q}'$ does not verify (14), for j, k, with $j+k=\overline{\kappa}+1$, replacing $\mathfrak{q}'$ with $\mathfrak{q}''$. We construct by induction $\mathfrak{q}^\sharp$, with the wished properties. □

Remark 4.7 We can give a different proof: we reduce the scalar operator to a first order system $\widetilde{h}$ by the standard method. We prove that $\widetilde{h}$ is of type H^m, and the conditions $\mathrm{L}(\mathfrak{q})$ for h are equivalent to the conditions $\widetilde{\mathrm{L}}(\mathfrak{q})$ for $\widetilde{h}$. The result follows then from [28] and [22]. In particular, we remark that if h verifies the conditions $\mathrm{L}(\mathfrak{q})$, with $\mathfrak{q} = (1, \ldots, 1)$ and $\kappa = m$, then $\widetilde{h}\mathcal{A}'$ has a good decomposition (cf. [28, Prop. 2.3]), for a suitable choice of $\mathcal{A}'$.

5. Proof of Theorem 2

We prove at first Proposition 3.1.

To simplify the notations, in the following we note $P \sim Q$, to say that $P - Q$ is divisible by H. The condition $\mathrm{L}_1(\mathfrak{q})$ is:

$$A\sigma_{\mu_0}(hA' - H'^2K') \sim 0.$$

Thanks to Proposition 2.1-(iii) we can choose $A' := A(x, D)$, $H' := H(x, D)I$, $K' := K(x, D)I$, and we have:

$$\sigma_{\mu_0}(hA' - H'^2K') \sim \sum_{j=1}^{n} \partial_{\xi_j} a \partial_{x_j} A + bA - \sum_{j=1}^{n} \partial_{\xi_j} H \partial_{x_j} HKI.$$

Hence, in order to prove the equivalence between the condition $\mathrm{L}_1(\mathfrak{q})$ and the condition (8) it's enough to prove:

$$A\Big[\sum_{j=1}^{n} \partial_{\xi_j} a \partial_{x_j} A - \sum_{j=1}^{n} \partial_{\xi_j} H \partial_{x_j} HKI + \frac{1}{2}\sum_{j=1}^{n} \partial_{\xi_j}\partial_{x_j} aA - \frac{1}{2}\{a,\, A\}\Big] \sim 0. \quad (16)$$

Since $\partial_{\xi_j}\partial_{x_j}(H^2K) \sim 2\partial_{\xi_j} H \partial_{x_j} HK$, we have:

$$\begin{aligned}\partial_{\xi_j} H \partial_{x_j} HKI &\sim \frac{1}{2}\partial_{\xi_j}\partial_{x_j}(H^2KI) = \frac{1}{2}\partial_{\xi_j}\partial_{x_j}(aA)\\ &= \frac{1}{2}\big[\partial_{\xi_j}\partial_{x_j} aA + \partial_{\xi_j} a \partial_{x_j} A + \partial_{x_j} a \partial_{\xi_j} A + a\partial_{\xi_j}\partial_{x_j} A\big],\end{aligned}$$

which gives (16).

In order to prove the equivalence between the condition (8) and the condition (9), it's enough to remark that:

$$A\{a,\, A\} \sim \{A,\, a\}A.$$

In fact, since $Aa = H^2KI$, we have: $\partial_{\xi_j} aA \sim -a\partial_{\xi_j} A$ and $\partial_{x_j} aA \sim$

$-a\partial_{x_j}A$, hence:

$$\begin{aligned}&A\{a,\, A\} - \{A,\, a\}A\\&= A\partial_{\xi_j}a\partial_{x_j}A - A\partial_{x_j}a\partial_{\xi_j}A - \partial_{\xi_j}A\partial_{x_j}aA + \partial_{x_j}A\partial_{\xi_j}aA\\&\sim -\partial_{\xi_j}Aa\partial_{x_j}A + \partial_{x_j}Aa\partial_{\xi_j}A + \partial_{\xi_j}Aa\partial_{x_j}A - \partial_{x_j}Aa\partial_{\xi_j}A = 0.\end{aligned}$$

This completes the proof of the Proposition 3.1, and shows (11) and (12) for $r = 1$.

Now we assume that h is of type (6) with $r \geq 2$, and we prove that the conditions $\mathrm{L}_1(\mathfrak{q}), \ldots, \mathrm{L}_{r-1}(\mathfrak{q})$ are satisfied, and the condition $\mathrm{L}_r(\mathfrak{q})$ is:

$$\mathrm{L}_r(\mathfrak{q})\colon\ \mathcal{A}\sigma_{\mu_{r-1}}\big[(h\,\mathcal{A}' - {H'}^2K')^r\big] \quad \text{is divisible by } H. \tag{17}$$

If $p = q_1 = 2$, the first condition is trivially satisfied and we have:

$$\Lambda_1 = \mathcal{A}\sigma_{\mu_0}(S_1').$$

We choose $\Lambda_1' := \mathcal{A}'S_1'$, so that:

$$\begin{aligned}S_2' &= h\mathcal{A}'S_1' - S_1'{H'}^2K' = (h\mathcal{A}' - {H'}^2K')S_1' + [{H'}^2K',\, S_1']\\&= (h\mathcal{A}' - {H'}^2K')^2 + [{H'}^2K',\, S_1'].\end{aligned}$$

Now, from Lemma 4.4 we see that $\sigma_{\mu_1}\big([{H'}^2K',\, S_1']\big)$ is divisible by H, hence, if $r = 2$ the second condition is:

$$\mathcal{A}\sigma_{\mu_1}\big((h\mathcal{A}' - {H'}^2K')^2\big) \sim 0.$$

If $r > 2$ the second condition is always satisfied and:

$$\Lambda_2 = \mathcal{A}\sigma_{\mu_1}(S_2').$$

Hence we choose $\Lambda_2' := \mathcal{A}'S_2'$ so that:

$$\begin{aligned}S_3' &= h\mathcal{A}'S_2' - S_2'{H'}^2K' = (h\mathcal{A}' - {H'}^2K')S_2' + [{H'}^2K',\, S_2']\\&= (h\mathcal{A}' - {H'}^2K')^3 + (h\mathcal{A}' - {H'}^2K')[{H'}^2K',\, S_1'] + [{H'}^2K',\, S_2'].\end{aligned}$$

If $r = 3$, since $\sigma_{\mu_2}\big([{H'}^2K',\, S_1']\big)$ and $\sigma_{\mu_2}\big([{H'}^2K',\, S_2']\big)$ are divisible by H, the third condition can be written as:

$$\mathrm{L}_3(\mathfrak{q})\colon\ \mathcal{A}\sigma_{\mu_2}\big((h\mathcal{A}' - {H'}^2K')^3\big) \sim 0.$$

We easily prove by induction that

$$S'_k = (h\mathcal{A}' - H'^2K')^k + \sum_{l=1}^{k-1}(h\mathcal{A}' - H'^2K')^{k-1-l}[H'^2K', S'_l],$$

for $k = 1, \ldots, r$, where:

$$\sigma_{\mu_{k-1}}(S'_k) \sim \sigma_{\mu_{k-1}}\big((h\mathcal{A}' - H'^2K')^k\big),$$

since $\sigma_{\mu_{k-1}}\big([H'^2K', S'_l]\big) \sim 0$, which shows the equivalence between $\mathrm{L}_r(\mathfrak{q})$ and (17) for general r.

Since $\mu_j = (j+1)\mu_0$, for $j = 1, \ldots, r-1$, if $p = q_1 = \cdots = q_{r-1} = 2$, we have:

$$\sigma_{\mu_{r-1}}\big[(h\mathcal{A}' - H'^2K')^r\big] = \big[\sigma_{\mu_0}(h\mathcal{A}' - H'^2K')\big]^r,$$

hence we can write the condition $\mathrm{L}_r(\mathfrak{q})$ as:

$$\mathrm{L}_r(\mathfrak{q})\colon \mathcal{A}\big[\sigma_{\mu_0}(h\mathcal{A}' - H'^2K')\big]^r \quad \text{is divisible by } H. \tag{18}$$

In order to prove the equivalence between (11) and (18), we set:

$$\mathcal{B} := \sigma_{\mu_0}(h\mathcal{A}' - H'^2K'), \quad \mathcal{C} := \mathcal{S}\mathcal{A} + \frac{1}{2}\{a, \mathcal{A}\},$$

and we show by induction

$$\mathcal{A}\mathcal{B}^r \sim \mathcal{A}\mathcal{C}^r \quad \text{for } r \geq 1 \tag{19}$$

Thanks to Proposition 3.1, we have:

$$\mathcal{A}\mathcal{B} \sim \mathcal{A}\mathcal{C} \sim \mathcal{C}\mathcal{A}. \tag{20}$$

Now, assuming (19) for $r-1$, we show it for r. Using (20) and induction hypothesis, we have:

$$\mathcal{A}\mathcal{B}^r = \mathcal{A}\mathcal{B}\mathcal{B}^{r-1} \sim \mathcal{A}\mathcal{C}\mathcal{B}^{r-1} \sim \mathcal{C}\mathcal{A}\mathcal{B}^{r-1} \sim \mathcal{C}\mathcal{C}^{r-1}\mathcal{A} = \mathcal{C}^r\mathcal{A},$$

which gives the equivalence between (11) and (18).

Now we construct the asymptotic solution of h.

To simplify the presentation, here we consider here only the special case of a single space variable, that is $x = (x_0, x_1)$, and we assume that the system has only one characteristic root, which vanishes identically; the general case can be proved in a similar way (see [15], [23] and the reference therein cited).

Assume at first that $m = 2r$. Let:

$$h(x, D) = ID_0 + JD_1 + b(x), \tag{21}$$

where $x = (x_0, x_1)$, and

$$J = \begin{pmatrix} 0 & 1 & & & & & & \\ 0 & 0 & & & 0 & & & \\ & & 0 & 1 & & & & \\ & & 0 & 0 & & & & \\ & & & & \ddots & & & \\ & 0 & & & & & 0 & 1 \\ & & & & & & 0 & 0 \end{pmatrix}. \tag{22}$$

Proposition 5.1 *If h is as in* (21)*, then conditions* $\mathrm{L}_1(\mathfrak{q}), \ldots, \mathrm{L}_{r-1}(\mathfrak{q})$ *are satisfied, and moreover h verifies the condition* $\mathrm{L}_r(\mathfrak{q})$ *if and only if any of the following equivalent conditions are satisfied*:

(i) $J(bJ)^r \equiv 0$;

(ii) $\mathcal{B}^r \equiv 0$; *where* $\mathcal{B}^{\mathrm{I}}_{\mathrm{J}} = b^{\mathrm{I}+1}_{\mathrm{J}}$ *if* I *and* J *are odd and* $\mathcal{B}^{\mathrm{I}}_{\mathrm{J}} = 0$ *otherwise*:

$$\mathcal{B} = \begin{pmatrix} b_{2,1} & 0 & b_{2,3} & 0 & \cdots & b_{2,(2r-1)} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_{4,1} & 0 & b_{4,3} & 0 & \cdots & b_{4,(2r-1)} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & & \vdots \\ b_{2r,1} & 0 & b_{2r,3} & 0 & \cdots & b_{2r,(2r-1)} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix};$$

(iii) $\widetilde{\mathcal{B}}^r \equiv 0$; *where*

$$\widetilde{\mathcal{B}} = \begin{pmatrix} b_{2,1} & b_{2,3} & \cdots & b_{2,(2r-1)} \\ b_{4,1} & b_{4,3} & \cdots & b_{4,(2r-1)} \\ \vdots & & & \vdots \\ b_{2r,1} & b_{2r,3} & \cdots & b_{2r,(2r-1)} \end{pmatrix}.$$

Proof. We have $\mathcal{A} = ID_0 - JD_1$, hence: $S'_1 = bD_0 - bJD_1$; the condition L_r is then:

$$J(bJ)^r \equiv 0.$$

In order to show the equivalence between (i) and (ii), we remark that, since $J^{\mathrm{I}}_{\mathrm{J}} = 1$, if and only if I is odd and $\mathrm{J} = \mathrm{I} + 1$ and $J^{\mathrm{I}}_{\mathrm{J}} = 0$ otherwise, the element (I, J) of $J(bJ)^r$ is non zero only if I is odd and J is even, and in this case we have:

$$\begin{aligned}\big(J(bJ)^r\big)^{\mathrm{I}}_{\mathrm{J}} &= \sum J^{\mathrm{I}}_{\mathrm{K}_1} b^{\mathrm{K}_1}_{\mathrm{K}_2} J^{\mathrm{K}_2}_{\mathrm{K}_3} \cdots J^{\mathrm{K}_{2r-2}}_{\mathrm{K}_{2r-1}} b^{\mathrm{K}_{2r-1}}_{\mathrm{K}_{2r}} J^{\mathrm{K}_{2r}}_{\mathrm{J}} \\ &= \sum_{\mathrm{K}_2,\,\dots,\,\mathrm{K}_{2r-2}\ \text{odd}} b^{i+1}_{\mathrm{K}_2} b^{\mathrm{K}_2+1}_{\mathrm{K}_4} \cdots b^{\mathrm{K}_{2r-2}}_{j-1},\end{aligned}$$

where in the second sum all the K_2, K_4, $\dots$, K_{2r-2} are odd. This proves the equivalence between (i) and (ii).

The equivalence between (ii) and (iii) is straightforward. □

Example 1 ([27]) Let

$$h(x,\,D) = ID_0 + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} D_1 + \begin{pmatrix} b^1_1 & b^1_2 & b^1_3 & b^1_4 \\ b^2_1 & b^2_2 & b^2_3 & b^2_4 \\ b^3_1 & b^3_2 & b^3_3 & b^3_4 \\ b^4_1 & b^4_2 & b^4_3 & b^4_4 \end{pmatrix}.$$

We have:

$$\widetilde{\mathcal{B}} = \begin{pmatrix} b^2_1 & b^2_3 \\ b^4_1 & b^4_3 \end{pmatrix}$$

and h verifies the condition L_2 if and only if $\widetilde{\mathcal{B}}^2 = 0$, that is:

$$b^2_1 + b^4_3 \equiv 0 \quad \text{and} \quad b^2_1 b^4_3 - b^2_3 b^4_1 \equiv 0.$$

Thanks to Proposition 5.1, if the condition $\mathrm{L}_r(\mathfrak{q})$ is not verified, the matrix $\mathcal{B}$ has an eigenvalue λ which does not vanishes identically on Ω. We can then choose an open set $\omega \subseteq \Omega$ such that

$$\big|\lambda(x)\big| \geq \delta > 0, \quad \text{for any } x \in \omega.$$

We will construct a formal development of the type:

$$u(x) = \exp\big[x_1\eta + \psi(x)\eta^{1/2}\big] \sum_{j=0}^{\infty} Y_j(x)\eta^{-j/2},$$

which formally verifies $h(x,\,D)u(x) \equiv 0$. Here η is a complex number which will be suitably chosen in the following, and $\eta^{1/2}$ is one of the two square roots η (cf. (34)).

Notations For $V = (v_1, v_2, \ldots, v_{2r-1}, v_{2r})^t \in \mathbb{R}^{2r}$, we set:

$$V' = \begin{pmatrix} v_1 \\ 0 \\ v_3 \\ 0 \\ \vdots \\ v_{2r-1} \\ 0 \end{pmatrix} \qquad V'' = \begin{pmatrix} 0 \\ v_2 \\ 0 \\ v_4 \\ \vdots \\ 0 \\ v_{2r} \end{pmatrix}$$

so that $V = V' + V''$. Similarly $(hV)'$ and $(hV)''$ indicate respectively the vectors

$$(hV)' = \begin{pmatrix} \sum_{k=1}^{2r} h_k^1 V^k \\ 0 \\ \sum_{k=1}^{2r} h_k^3 V^k \\ 0 \\ \vdots \\ \sum_{k=1}^{2r} h_k^{2r-1} V^k \\ 0 \end{pmatrix} \qquad (hV)'' = \begin{pmatrix} 0 \\ \sum_{k=1}^{2r} h_k^2 V^k \\ 0 \\ \sum_{k=1}^{2r} h_k^4 V^k \\ \vdots \\ 0 \\ \sum_{k=1}^{2r} h_k^{2r} V^k \end{pmatrix}$$

Remark 5.2 Let J^* denote the transposed matrix of J, then we have:

$$V' = JJ^*V \text{ and } V'' = J^*JV, \quad \text{for every } V \in \mathbb{R}^{2r}.$$

0 Since $JV = JV''$, for any $V \in \mathbb{R}^{2r}$ the condition $JV = 0$ is equivalent to $V'' = 0$; moreover the condition $JV = U'$ is equivalent to $V'' = J^*U'$.

We have:

$$\begin{aligned} h(x, D)u(x) = {} & \exp\bigl[x_1\eta + \psi\eta^{1/2}\bigr] \\ & \times \Bigl[JY_0\eta + \bigl[JY_1 + D_0\psi Y_0 + D_1\psi JY_0\bigr]\eta^{1/2} \\ & + \sum_{j=0}^{\infty} \bigl[JY_{j+2} + D_0\psi Y_{j+1} + D_1\psi JY_{j+1} + hY_j\bigr]\eta^{-j/2}\Bigr]. \end{aligned}$$

Since all the terms in the development should be zero we get the following equations for Y_j:

$$JY_0 = 0 \tag{E_{-2}}$$

$$JY_1 + D_0\psi Y_0 + D_1\psi JY_0 = 0, \tag{E_{-1}}$$

$$JY_{j+2} + D_0\psi Y_{j+1} + D_1\psi JY_{j+1} + hY_j = 0, \qquad j = 0, 1, \ldots \tag{E_j}$$

Using the Remark 5.2, the equation (E_{-2}) gives

$$Y_0'' = 0 \tag{23}$$

(that is $Y_0^{2\ell} \equiv 0$, for $\ell = 1, \ldots, r$) and it gives no condition on Y_0'; the equation (E_{-1}) becomes:

$$JY_1'' + D_0\psi Y_0' = 0, \tag{$(E_{-1})'$}$$

(that is $Y_1^{2\ell} + D_0\psi Y_0^{2\ell-1} = 0$, for $\ell = 1, \ldots, r$) and it gives no condition on Y_1'.

(E_0) gives:

$$JY_2'' + D_0\psi Y_1' + D_1\psi JY_1'' + (hY_0)' = 0, \tag{$(E_0)'$}$$

$$D_0\psi Y_1'' + (hY_0)'' = 0. \tag{$(E_0)''$}$$

By multiplying $(E_0)''$ by J, and thanks to $(E_{-1})'$ we have:

$$(D_0\psi)^2 Y_0' - J(hY_0)'' = 0. \tag{24}$$

It's clear that:

$$J(hV')'' = JhV' = JbV' = \mathcal{B}V', \quad \text{for any } V \in \mathbb{R}^{2r}, \tag{25}$$

hence, from (24) we get:

$$\big[(D_0\psi)^2 I - \mathcal{B}\big] Y_0' = 0. \tag{26}$$

We choose ψ a solution of the problem

$$\begin{cases} (D_0\psi)^2 - \lambda = 0 \\ \psi|_{x_0 = \underline{x_0}} = 0 \end{cases}$$

where λ is a non zero eigenvalue of $\mathcal{B}$ and $\underline{x_0} \in \omega$. We choose Y_0' proportional to an eigenvector related to λ.

Now we consider (E_1):

$$JY_3'' + D_0\psi Y_2' + D_1\psi JY_2'' + (hY_1)' = 0, \tag{$(E_1)'$}$$

$$D_0\psi Y_2'' + (hY_1)'' = 0. \tag{$(E_1)''$}$$

By multiplying $(E_1)''$ by J and thanks to $(E_0)'$ and $(E_{-1})'$ we get:

$$(D_0\psi)^2 Y_1' - J(hY_1)'' + D_0\psi[(hY_0')' - D_0\psi D_1\psi Y_0'] = 0 \tag{27}$$

Using (25), we have:

$$\begin{aligned}J(hY_1)'' &= JhY_1 = JhY_1' + JhY_1'' = \mathcal{B}Y_1' + D_0JY_1'' + JbY_1'' \\ &= \mathcal{B}Y_1' - D_0(D_0\psi Y_0') - JbJ^*(D_0\psi Y_0') \\ &= \mathcal{B}Y_1' - D_0\psi D_0Y_0' - [D_0^2\psi I + D_0\psi JbJ^*]Y_0'.\end{aligned}$$

On the other side:

$$(hY_0')' = D_0Y_0' + (bY_0')' = D_0Y_0' + JJ^*bY_0',$$

hence (27) gives:

$$\big[(D_0\psi)^2 - \mathcal{B}\big]Y_1' + 2D_0\psi D_0Y_0' + Jc_0(x)Y_0' = 0, \tag{28}$$

where

$$c_0(x) := \big(D_0^2\psi - (D_0\psi)^2 D_1\psi\big)J^* + D_0\psi(bJ^* + J^*b).$$

Combining (26) and (28), we can see that Y_0' satisfies the conditions:

$$Y_0' \in \operatorname{Ker}\big[(D_0\psi)^2 I - \mathcal{B}\big], \tag{29}$$

$$2D_0\psi D_0Y_0' + Jc_0(x)Y_0' \in \operatorname{Im}\big[(D_0\psi)^2 I - \mathcal{B}\big]. \tag{30}$$

Reducing ω, we can assume that the dimension of $\operatorname{Ker}\big[(D_0\psi)^2 I - \mathcal{B}\big]$ is constant and equal to r_0, with $1 \le r_0 \le r$, and we choose a base $U_1(x), \ldots, U_{r_0}(x)$ of $\operatorname{Ker}\big[(D_0\psi)^2 I - \mathcal{B}\big]$ with $U_j \in \mathcal{C}^\infty(\omega)$. We choose

$$Y_0' := \sum_{\ell=1}^{r_0} y_{0,\ell}(x) U_\ell(x)$$

where the $y_{0,\ell} \in \mathcal{C}^\infty(\omega)$ are determined by integration of (30). Note that (30) is a system of r_0 ordinary differential equations, whose principal part is the matrix whose columns are the U_j, and hence its rank is r_0. We can determine completely Y_0'.

We determine then Y_1'' thanks to $(E_{-1})'$.

Now we consider (E_2):

$$JY_4'' + D_0\psi Y_3' + D_1\psi JY_3'' + (hY_2)' = 0, \tag*{$(E_2)'$}$$

$$D_0\psi Y_3'' + (hY_2)'' = 0. \tag*{$(E_2)''$}$$

By multiplying $(E_2)''$ by J and thanks to $(E_1)'$ and $(E_0)'$ we get:

$$(D_0\psi)^2 Y_2' - J(hY_2)'' - (D_0\psi)^2 D_1\psi Y_1' \\ + D_0\psi(hY_1)' + JP_1(\psi,\, D_0)Y_0' = 0, \quad (31)$$

where, here and in the following, $P_1(\psi,\, D_0)$, $P_2(\psi,\, D_0)$, …, are differential operators in D_0, whose coefficients depend on ψ and its derivatives, which is useless to specify.

Now:

$$\begin{aligned} J(hY_2)'' &= JhY_2 = JhY_2' + JhY_2'' = \mathcal{B}Y_2' + JhJ^*JY_2'' \\ &= \mathcal{B}Y_2' - JhJ^*\big(D_0\psi Y_1' + JP_2(\psi,\, D_0)Y_0'\big) \\ &= \mathcal{B}Y_2' - D_0(D_0\psi Y_1') \\ &\qquad - D_0\psi JbJ^*Y_1' - JhJ^*JP_2(\psi,\, D_0)Y_0' \\ &= \mathcal{B}Y_2' - D_0\psi D_0 Y_1' - D_0^2\psi Y_1' \\ &\qquad - D_0\psi JbJ^*Y_1' - JhJ^*JP_2(\psi,\, D_0)Y_0' \\ D_0\psi(hY_1)' &= D_0\psi(D_0Y_1)' + D_0\psi(JD_1Y_1)' + D_0\psi(bY_1)' \\ &= D_0\psi D_0 Y_1' + D_0\psi JD_1 Y_1'' \\ &\qquad + D_0\psi JJ^*bY_1' + D_0\psi JJ^*bY_1'' \\ &= D_0\psi D_0 Y_1' + D_0\psi JJ^*bY_1' \\ &\qquad + D_0\psi D_0 D_1 Y_0' + D_0\psi JJ^*bJ^*D_0\psi Y_0'. \end{aligned}$$

Equation (31) is then:

$$\big[(D_0\psi)^2 I - \mathcal{B}\big]Y_2' + 2D_0\psi D_0 Y_1' + Jc_0(x)Y_1' + JP_3(\psi,\, D_0)Y_0' = 0,$$

Let $\mathring{Y}_1'$ be a solution of (28) (which exists thanks to (30)), we choose:

$$Y_1' := \mathring{Y}_1' + \sum_{\ell=1}^{r_0} y_{1,\ell}(x)U_\ell(x),$$

we determine the $y_{1,\ell}(x)$ so that:

$$2D_0\psi D_0 Y_1' + Jc_0(x)Y_1' + JP_3(\psi,\, D_0)Y_0' \in \operatorname{Im}\big[(D_0\psi)^2 I - \mathcal{B}\big],$$

and we show as before that we can determine Y_1', and hence Y_2'' thanks to $(E_0)'$.

The components Y_j'' can be determined by $Y_0, \ldots, Y_{j-1}$, using $(E_{j-2})''$, hence we need only to determine the components Y_j'.

We consider (E_j):

$$JY''_{j+2} + D_0\psi Y'_{j+1} + D_1\psi JY''_{j+1} + (hY_j)' = 0, \qquad (E_j)'$$

$$D_0\psi Y''_{j+1} + (hY_j)'' = 0. \qquad (E_j)''$$

By multiplying $(E_j)''$ by J and thanks to $(E_{j-1})''$ and $(E_{j-2})''$ we get:

$$\begin{aligned}(D_0\psi)^2 Y'_j - J(hY_j)'' - (D_0\psi)^2 D_1\psi Y'_{j-1}&\\ + D_0\psi(hY_{j-1})' + JP_{j,1}(\psi,\, D_0)Y'_{j-2} &= 0.\end{aligned}$$

Now:

$$\begin{aligned}
J(hY_j)'' &= JhY_j = JhY'_j + JhY''_j = \mathcal{B}Y'_j + JhJ^*JY''_j\\
&= \mathcal{B}Y'_j - JhJ^*\big(D_0\psi Y'_{j-1} + JP_{j,2}(\psi,\, D_0)Y'_{j-2}\big)\\
&= \mathcal{B}Y'_j - D_0(D_0\psi Y'_{j-1})\\
&\qquad - D_0\psi JbJ^*Y'_{j-1} - JhJ^*JP_{j,2}(\psi,\, D_0)Y'_{j-2}\\
&= \mathcal{B}Y'_j - D_0\psi D_0 Y'_{j-1} - D_0^2\psi Y'_{j-1}\\
&\qquad - D_0\psi JbJ^*Y'_{j-1} - JhJ^*JP_{j,2}(\psi,\, D_0)Y'_{j-2}\\
D_0\psi(hY_{j-1})' &= D_0\psi(D_0Y_{j-1})' + D_0\psi(JD_1Y_{j-1})' + D_0\psi(bY_{j-1})'\\
&= D_0\psi D_0Y'_{j-1} + D_0\psi JD_1Y''_{j-1}\\
&\qquad + D_0\psi JJ^*bY'_{j-1} + D_0\psi JJ^*bY''_{j-1}\\
&= D_0\psi D_0Y'_{j-1} + D_0\psi JJ^*bY'_{j-1} + P_{j,3}(\psi,\, D_0)Y'_{j-2}.
\end{aligned}$$

Hence equation $(E_j)''$ is equivalent to:

$$\begin{aligned}\big[(D_0\psi)^2 I - \mathcal{B}\big]Y'_j + 2D_0\psi D_0 Y'_{j-1}&\\ + Jc_0(x)Y'_{j-1} + JP_{j,4}(\psi,\, D_0)Y'_{j-2} &= 0. \qquad (32)\end{aligned}$$

Assume by induction that

$$\begin{aligned}2D_0\psi D_0 Y'_{j-1} + Jc_0(x)Y'_{j-1}&\\ + JP_{j,4}(\psi,\, D_0)Y'_{j-2} &\in \operatorname{Im}\big[(D_0\psi)^2 I - \mathcal{B}\big],\end{aligned}$$

and let $\mathring{Y}'_j$ be a solution of (32), we choose:

$$Y'_j := \mathring{Y}'_j + \sum_{\ell=1}^{r_0} y_{j,\ell}(x)U_\ell(x),$$

and we determine the $y_{j,\ell}(x)$ so that:

$$2D_0\psi D_0 Y'_j + Jc_0(x)Y'_j + JP_{j+1,4}(\psi,\, D_0)Y'_{j-1} \in \operatorname{Im}\bigl[(D_0\psi)^2 I - \mathcal{B}\bigr],$$

which is obtained by $(E_{j+1})''$.

Using the same argument we can determine all the Y_j.

If $r < m/2$, we have

$$J = \begin{pmatrix} 0 & 1 & & & & & & \\ 0 & 0 & & & 0 & & & \\ & & \ddots & & & & & \\ & & & 0 & 1 & & & \\ & & & 0 & 0 & & & \\ & 0 & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{pmatrix},$$

instead of (22).

In this case the asymptotic solution can be constructed in the same way, but we choose at each step the last $m-2r$ components of Y_j identically equal to zero.

Now we show, by standard argument, that the existence of the formal asymptotic solution contradicts the well-posedness in $\mathcal{C}^\infty$ of the Cauchy problem. Indeed, if the Cauchy Problem is well-posed in $\underline{x} = (\underline{x_0},\, \underline{x_1})$, a consequence of the Closed Graph Theorem is that for any neighborhood ω of $\underline{x}$, there exist ε, $\delta > 0$ and $k \in \mathbb{N}$ such that, if we set $K_1 = [\underline{x_1} - \varepsilon,\, \underline{x_1} + \varepsilon]$ and $K = [\underline{x_0},\, \underline{x_0} + \delta] \times K_1$, we have $K \subset \omega$ and the following estimate holds true:

$$\|u\|_{0,K} \le C\Bigl[\|hu\|_{k,K} + \bigl\|u(\underline{x_0},\, \cdot\,)\bigr\|_{k,K_1}\Bigr], \tag{33}$$

for any $u = (u_1,\, \ldots,\, u_N) \in [\mathcal{C}^\infty(U)]^N$, where

$$\|u\|_{k,K} = \max_{j=1,\,\ldots,\,N}\ \max_{|\alpha|\le k}\ \max_{x\in K}\bigl|D_x^\alpha u_j(x)\bigr|$$

is a semi-norm of $\mathcal{C}^\infty(\omega)$.

Let $\eta = i\widetilde{\eta}$, with $\widetilde{\eta} \in \mathbb{R}$, we choose the sign of ψ and $\eta^{1/2}$ so that

$$\operatorname{Re}\bigl(D_0\psi(x)\eta^{1/2}\bigr) > 0, \quad \text{for all } x \in K. \tag{34}$$

For $\kappa \in \mathbb{N}$, let u_κ be:

$$u_\kappa(x) = \exp\big[x_1\eta + \psi(x)\eta^{1/2}\big] \sum_{j=0}^{\kappa} Y_j(x)\eta^{-j/2};$$

we have:

$$\|u_\kappa\|_{0,K} = \big[U + o(\widetilde{\eta})\big] \cdot \exp\Big[\sup_{x_1 \in K_1} \big(\operatorname{Re}\psi(\underline{x_0} + \delta,\, x_1)\eta^{1/2}\big)\Big], \qquad \text{if } |\eta| \to +\infty,$$

for some $U \neq 0$, thanks to the choice of $Y_0 \not\equiv 0$, and

$$\big\|u_0(\underline{x_0},\,\cdot\,)\big\|_{k,K_1} = O(\widetilde{\eta}^k)\big[U + o(\widetilde{\eta})\big] \cdot \exp\Big[\sup_{x_1 \in K_1} \big(\operatorname{Re}\psi(\underline{x_0},\, x_1)\eta^{1/2}\big)\Big], \qquad \text{if } |\eta| \to +\infty,$$

$$\|hu_\kappa\|_{k,K} = O(\widetilde{\eta}^{k-N_\kappa}) \cdot \exp\Big[\sup_{x_1 \in K_1} \big(\operatorname{Re}\psi(\underline{x_0} + \delta,\, x_1)\eta^{1/2}\big)\Big], \qquad \text{if } |\eta| \to +\infty,$$

for some $N_\kappa > 0$.

It is clear that thanks to the choice of ψ and η, the sequence of the u_κ does not verify uniformly the inequality (33).

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