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A product formula for hypergeometric polynomials of type ${}_2F_0$

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Abstract. In this paper, we give a combinatorial proof to the following new product formula:

$$\prod_{i=1}^{m} {}_2F_0(-a_i,\, -b_i; z) = \prod_{r=0}^{n} p(r)\, {}_2F_0(-n,\, -r; z).$$

Key words: hypergeometric polynomial, product formula, hypergeometric distribution.

1. Main theorem

The generalized hypergeometric series

$${}_2F_0(\alpha,\, \beta; z) := \sum_{k=0}^{\infty} \frac{(\alpha)_k(\beta)_k}{k!} z^k$$

has the convergence radius 0 unless α, β are non-positive integers. The formal power series ${}_2F_0(\alpha,\, \beta; z)$ is a solution of the differential equation

$$z^2y'' + ((1+\alpha+\beta)z - 1)y' + \alpha\beta y = 0,$$

and satisfies the following recursion formula:

$$\frac{d}{dz} {}_2F_0(\alpha,\, \beta; z) = \alpha\beta {}_2F_0(\alpha+1,\, \beta+1; z).$$

T.W. Chaundy([3] (73)) showed the following product formula:

$$\begin{aligned} &{}_2F_0(\alpha,\, \beta; pz) {}_2F_0(\alpha',\, \beta'; qz) \\ &\qquad = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(pz)^n}{n!} {}_3F_2 \left[\begin{array}{c} \alpha',\, \beta',\, -n; -q/p \\ 1-\alpha-n,\, 1-\beta-n \end{array} \right]. \end{aligned}$$

When $-\alpha,\, -\beta$ are non-negative integers, ${}_2F_0(\alpha,\, \beta; z)$ is a polynomial of degree at most $\min(-\alpha,\, -\beta)$. In this paper, we study a new product

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formula for polynomial cases and give a combinatorial proof to it.

After this, we simply write

$$F_{a,b}(z) := {}_2F_0(-a,\, -b; z) = \sum_{k \geq 0} \binom{a}{k} \binom{b}{k} k! z^k$$

for nonnegative integers a, b. Furthermore, n denotes a non-negative integer; $\boldsymbol{a} = (a_1, \ldots, a_m)$ and $\boldsymbol{b} = (b_1, \ldots, b_m)$ vectors of non-negative integers; $\boldsymbol{x} = (x_{ij})_{i,j=1,\ldots,m}$ an $m \times m$-matrix whose entries are non-negative integers. Furthermore, we put

$$\boldsymbol{a}! := a_1!\, a_2! \cdots a_m!, \quad \boldsymbol{X}! := \prod_{i,j} x_{ij}!,$$
$$\overline{\boldsymbol{a}} := a_1 + a_2 + \cdots + a_m, \quad \overline{\boldsymbol{x}} := \sum_{i,j} x_{i,j}.$$

The multinomial coefficient used in this paper is defined as follows:

$$\binom{n}{\boldsymbol{a}} := \binom{n}{a_1,\, \ldots,\, a_m} := \frac{n!}{a_1! \cdots a_m! (n - \overline{\boldsymbol{a}})!}.$$

Only if $\overline{\boldsymbol{a}} = n$ holds, this notation is same as the usual one.

Now, let $\boldsymbol{\omega}$ be the set of non-negative integral solutions $(x_{ij})_{i,j=1,\ldots,m}$ of the inequalities

$$\sum_j x_{ij} \leq a_i, \ \sum_i x_{ij} \leq b_j, \ \sum_{ij} x_{ij} \geq \overline{\boldsymbol{a}} + \overline{\boldsymbol{b}} - n.$$

After this, we assume that n is greater than both of $\overline{\boldsymbol{a}}$ and $\overline{\boldsymbol{b}}$. We define an occurrence probability of an $\boldsymbol{x} = (x_{ij}) \in \boldsymbol{\omega}$ by

$$H(\boldsymbol{x}) := \frac{(n - \overline{\boldsymbol{a}})!\, (n - \overline{\boldsymbol{b}})!\, \boldsymbol{x}!}{n!\, (n - \overline{\boldsymbol{a}} - \overline{\boldsymbol{b}} + \overline{\boldsymbol{x}})!} \prod_i \binom{a_i}{x_{i1},\, \ldots,\, x_{im}} \times \prod_j \binom{b_j}{x_{1j},\, \ldots,\, x_{mj}},$$

Finallly let

$$p(r) := \sum_{\mathrm{Tr}(\boldsymbol{x}) = r} H(\boldsymbol{x}),$$

where the summation is taken over all $\boldsymbol{x} \in \boldsymbol{\omega}$ whose trace is equal to r.

Example There are two familiar special cases:

(1) The case where $m = 1$ amd $\boldsymbol{x} = x$ (a non-negative integer).

$$H(x) = p(x) = \frac{a!\,b!\,(n-a)!\,(n-b)!}{n!\,(n-a-b+x)!\,(a-x)!\,(b-x)!\,x!}.$$

This is the density function of the hypergeometric distribution $H(n, a, b)$.

(2) The case where $\overline{\boldsymbol{a}} = \overline{\boldsymbol{b}} = n$. In this case, $\overline{\boldsymbol{x}} = n$ and

$$H(\boldsymbol{x}) = \frac{\boldsymbol{a}!\,\boldsymbol{b}!}{n!\,\boldsymbol{x}!} = \prod_i a_i! \prod_j b_j! \Big/ n! \prod_{ij} x_{ij}!.$$

Thus $H(\boldsymbol{x})$ is the occurence probability of a contingency table $\boldsymbol{x} = (x_{ij})$ with given marginal frequencies $\boldsymbol{a}$, $\boldsymbol{b}$.

The purpose of this paper is to give a combinatorial proof to the following product formula:

Theorem 1 $\prod_{i=1}^{m} F_{a_i,b_i}(z) = \sum_{r\geq 0} p(r) F_{n,r}(z)$.

2. Proof of Theorem 1

It is suffice to prove the theorem in the case where z is a non-negative integer; so we take a set Z with $|Z| = z$. We denote by Z^K the set of maps from a finite set K to Z.

Since n is greater than or equal to both of $\sum_i a_i$, $\sum_j b_j$, there are subsets $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ of N such that

$$|A_i| = a_i,\ |B_i| = b_i\ (1 \leq i \leq m);\ A_i \cap A_j = B_i \cap B_j = \emptyset\ (i \neq j).$$

We put

$$\overline{A} := \coprod_i A_i,\ \overline{B} := \coprod_i B_i,$$

where $\coprod$ stands for a disjoint union. Clearly, $|\overline{A}| = \overline{\boldsymbol{a}}$, $|\overline{B}| = \overline{\boldsymbol{b}}$. Then by the definition of $F_{a,b}(z)$, we have

$$F_{a_i,b_i}(z) = \sharp\{(K, L, \pi, f) \mid K \subset A_i,\ L \subset B_i,\ \pi\colon K \xrightarrow{\sim} L,\ f \in Z^K\}. \quad (1)$$

Thus

$$\prod_i F_{a_i,b_i}(z) = \sharp\left\{ (K_i,\, L_i,\, \pi_i,\, f_i)_i \;\middle|\; \begin{array}{l} K_i \subset A_i,\, L_i \subset B_i, \\ \pi_i\colon K_i \xrightarrow{\sim} L_i,\, f_i \in Z^{K_i} \end{array} \right\}$$
$$= \sharp\left\{ (K,\, L,\, \pi,\, f) \;\middle|\; \begin{array}{l} K \subset \overline{A},\, L \subset \overline{B},\, \pi\colon K \xrightarrow{\sim} L, \\ \pi(K \cap A_i) \subset B_i,\, f \in Z^K \end{array} \right\}.$$

Here, we put $K := \coprod_i K_i$ (a disjoint union) and $L := \coprod_i L_i$; and furthermore, we uniquely extended $(\pi_i)_i$ and $(f_i)_i$ to a bijection $\pi\colon K \xrightarrow{\sim} L$ and a map $f\colon K \longrightarrow Z$, respectively.

Now, note that every bijection $\pi\colon K \xrightarrow{\sim} L$ for $|K| = k$ has $(n-k)!$ extensions to permutations on N. Thus

$$\prod_i F_{a_i,b_i}(z) = \sum_{k\geq 0} \sharp\left\{ (K,\, L,\, \pi) \;\middle|\; \begin{array}{l} K \subset \overline{A},\, L \subset \overline{B},\, |K| = k, \\ \pi\colon K \xrightarrow{\sim} L,\, \pi(A_i \cap K) \subset B_i \end{array} \right\} z^k$$
$$= \sum_{k\geq 0} \sharp\left\{ (K,\, L,\, \pi) \;\middle|\; \begin{array}{l} K \subset \overline{A},\, L \subset \overline{B},\, |K| = k, \\ \pi \in S_n,\, \pi(K) = L,\, A_i \cap K \subset \pi^{-1}B_i \end{array} \right\} \times \frac{z^k}{(n-k)!}$$
$$= \sum_{k\geq 0} \sharp\left\{ (K,\, \pi) \;\middle|\; \begin{array}{l} K \subset \overline{A},\, |K| = k, \\ \pi \in S_n,\, A_i \cap K \subset \pi^{-1}B_i \end{array} \right\} \frac{z^k}{(n-k)!}$$
$$= \sum_{k\geq 0} \sharp\left\{ (K,\, \pi) \;\middle|\; \begin{array}{l} \pi \in S_n,\, |K| = k \\ K \subset \coprod_i (A_i \cap \pi^{-1}B_i) \end{array} \right\} \frac{z^k}{(n-k)!}$$
$$= \sum_{\pi\in S_n} \sum_{k\geq 0} \binom{\sum_i |A_i \cap \pi^{-1}B_i|}{k} \frac{z^k}{(n-k)!}$$
$$= \frac{1}{n!} \sum_{\pi\in S_n} \sum_{k\geq 0} \binom{\sum_i |A_i \cap \pi^{-1}B_i|}{k} \binom{n}{k} k! z^k$$
$$= \frac{1}{n!} \sum_{\pi\in S_n} F_{n,\, \sum_i |A_i \cap \pi^{-1}B_i|}(z)$$
$$= \sum_{r\geq 0} \frac{\sharp\{\pi \in S_n \mid \sum_i |A_i \cap \pi^{-1}B_i| = r\}}{n!} F_{n,r}(z),$$

where S_n is the symmetric group.

For any $\pi \in S_n$, we define a $m \times m$-matrix $\boldsymbol{x}(\pi) := (x_{ij}(\pi))$ by

$$x_{ij}(\pi) := |A_i \cap \pi^{-1} B_j|.$$

Then we have

$$\prod_i F_{a_i,b_i}(z) = \sum_{r\geq 0} \frac{\sharp\{\pi \in S_n \mid \mathrm{Tr}(\boldsymbol{x}(\pi)) = r\}}{n!} F_{n,r}(z).$$

Note that the matrix $\boldsymbol{x}(\pi)$ is in $\boldsymbol{\omega}$. In fact,

$$\begin{aligned}
\sum_j x_{ij}(\pi) &= \sum_j |A_i \cap \pi^{-1} B_j| = |A_i \cap \pi^{-1}\overline{B}| \leq a_i,\\
\sum_i x_{ij}(\pi) &= \sum_i |A_i \cap \pi^{-1} B_j| = |\overline{A} \cap \pi^{-1} B_j| \leq |\pi^{-1} B_j| = b_j,\\
\sum_{i,j} x_{ij}(\pi) &= \sum_{i,j} |A_i \cap \pi^{-1} B_j| = |\overline{A} \cap \pi^{-1}(\overline{B})|\\
&= |\overline{A}| + |\pi^{-1}(\overline{B})| - |\overline{A} \cup \pi^{-1}(\overline{B})| \geq \overline{\boldsymbol{a}} + \overline{\boldsymbol{b}} - n.
\end{aligned}$$

We obtained the following equation:

$$\prod_i F_{a_i,b_i}(z) = \sum_{r\geq 0} \sum_{\mathrm{Tr}(X)=r} \frac{\sharp\{\pi \in S_n \mid \boldsymbol{x}(\pi) = \boldsymbol{x}\}}{n!} F_{n,r}(z)$$

Thus in order to finish the proof of the theorem, it will suffice to prove the following lemma:

Lemma $(1/n!)\sharp\{\pi \in S_n \mid \boldsymbol{x}(\pi) = \boldsymbol{x}\} = H(\boldsymbol{x})$ *for any* $\boldsymbol{x} \in \boldsymbol{\omega}$.

Proof of Lemma. Let $\boldsymbol{\Omega}$ be the set of families $(X_{ij})_{i,j=1,\ldots,m}$ of subsets of N satisfying the following condition:

$$X_{ij} \subset A_i,\ X_{ij} \cap X_{ij'} = \emptyset\ (j \neq j'), \quad (|X_{ij}|) \in \boldsymbol{\omega}.$$

For an $\boldsymbol{X} = (X_{ij}) \in \boldsymbol{\Omega}$, we put

$$\overline{\boldsymbol{X}} := \coprod_{i,j} X_{i,j} \subset N, \quad |\overline{\boldsymbol{X}}| := (|X_{ij}|) \in \boldsymbol{\omega}.$$

Let

$$X_{ij}(\pi) := A_i \cap \pi^{-1} B_j.$$

Then $\boldsymbol{X}(\pi) := (X_{ij}(\pi)) \in \boldsymbol{\Omega}$.

Now, using these notations, the number $\sharp$ of permutations π such that $\boldsymbol{x}(\pi) = \boldsymbol{x}$ in the left hand side of the lemma is presented as follows:

$$\begin{aligned}\sharp &:= \sharp\{\pi \in S_n \mid \boldsymbol{x}(\pi) = \boldsymbol{x}\}\\ &= \sum_{|\boldsymbol{X}|=\boldsymbol{x}} \sharp\{\pi \in S_n \mid \boldsymbol{X}(\pi) = \boldsymbol{X}\}.\end{aligned}$$

where the summation is taken over $\boldsymbol{X} \in \boldsymbol{\Omega}$ such that $|\boldsymbol{X}| = \boldsymbol{x}$.

Let $\boldsymbol{X} = (X_{ij}) \in \Omega$ with $|\boldsymbol{X}| = \boldsymbol{x} = (x_{ij})$. We first note that the number of such $\boldsymbol{X}$'s is

$$\prod_i \binom{a_i}{x_{i1},\,\ldots,\,x_{im}}.$$

Now, a permutation $\pi \in S_n$ satisfies $\boldsymbol{X}(\pi) = \boldsymbol{X}$ if and only if

$$\begin{aligned}\pi\Big(\coprod_i X_{ij}\Big) &\subset B_j,\\ \pi(\overline{A} - \overline{X}) &\subset \overline{B}^c,\\ \pi(\overline{A}^c) &\subset N.\end{aligned}$$

Thus the number of such permutations π is given by

$$\prod_j \binom{b_j}{x_{1j},\,\ldots,\,x_{mj}} x_{1j}! \cdots x_{mj}! \times \binom{n - \overline{\boldsymbol{b}}}{\overline{\boldsymbol{a}} - \overline{\boldsymbol{x}}} (\overline{\boldsymbol{a}} - \overline{\boldsymbol{x}})! \times (n - \overline{\boldsymbol{a}})!.$$

Hence

$$\begin{aligned}\sharp = \prod_i \binom{a_i}{x_{i1},\,\ldots,\,x_{im}} \times \prod_j \binom{b_j}{x_{1j},\,\ldots,\,x_{mj}}\\ \times \binom{n - \overline{\boldsymbol{b}}}{\overline{\boldsymbol{a}} - \overline{\boldsymbol{x}}} (\overline{\boldsymbol{a}} - \overline{\boldsymbol{x}})!(n - \overline{\boldsymbol{a}})!\boldsymbol{x}!,\end{aligned}$$

is now equal to $n!H(\boldsymbol{X})$, which proves the lemma and then the theorem. □

Remark The lemma can be extended to those for non-squared matrices.

3. Inversion formula

The coefficient $p(r)$ in Theorem 1 can be calculated from the expansion of the left hand side by using the following theorem:

Theorem 2 *Let $G(z) = \sum_{k=0}^{n} q_k z^k$ be a polynomial of degree at most n. Then for a series $\{p_r\}_{r=0,1,\ldots,n}$, the following are equivalent*:
(a) $G(z) = \sum_{r=0}^{n} p_r F_{n,r}(z)$.
(b) $p_r = \sum_{k=0}^{n} (-1)^{k-r} \binom{k}{r} q_k \Big/ \binom{n}{k} k!$.

Proof. We write

$$q_k = \binom{n}{k} k! \widetilde{q}_k, \quad (k = 0, 1, \ldots, n).$$

Since

$$\begin{aligned} G(z) &= \sum_{k\geq 0} \widetilde{q}_k \binom{n}{k} k! z^k = \sum_{r=0}^{n} p_r F_{n,r}(z) \\ &= \sum_{r=0}^{n} \sum_{k=0}^{r} p_r \binom{n}{k} \binom{r}{k} k! z^k \\ &= \sum_{k=0}^{n} \left[\sum_{r=k}^{n} p_r \binom{r}{k} \right] \binom{n}{k} k! z^k, \end{aligned}$$

the condition (a) is written as

$$\widetilde{q}_k = \sum_{r=k}^{n} p_r \binom{r}{k} \quad (k = 0, 1, \ldots, n).$$

Clearly, this is equivalent to the condition (b)

$$p_r = \sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r} \widetilde{q}_k.$$

The theorem is proved. □

Corollary $z^n = (1/n!) \sum_{r=0}^{n} (-1)^{n-r} \binom{n}{r} F_{n,r}(z)$.

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References

[1] Aigner M., *Combinatorial Theory*. Springer, 1979.
[2] Bishop Yvonne M.M., *Discrete Multivarate Analysis*: *Theory and Practice*. MIT Press, 1975.

[3] Chaundy T.W., *Expansions of hypergeometric functions* (I). Quart. J. Math. **14** (1943), 55–78.

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