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Global existence and asymptotic behavior of solutions to systems of semilinear wave equations in two space dimensions

(Dedicated to Professor Rentaro Agemi on the occasion of his 70th birthday)

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Abstract. We consider the Cauchy problem for systems of semilinear wave equations in 2D with small initial data, and introduce a sufficient condition for global existence of small solutions. Our condition is weaker than the null condition for 2D wave equations, and it is motivated by Alinhac's condition for 3D. We also show that some global solutions under our condition are not asymptotically free.

Key words: system of nonlinear wave equations, null condition, weak null condition; grow-up of energy.

1. Introduction

Let $n = 2$ or 3. We consider the Cauchy problem for a system of semilinear wave equations of the following type:

$$\Box u_i = F_i(u, \partial u) \quad \text{for } (t, x) \in (0, \infty) \times \mathbb{R}^n \ (i = 1, 2, \ldots, N) \tag{1.1}$$

with initial data

$$u_i(0, x) = \varepsilon f_i(x),\ (\partial_t u_i)(0, x) = \varepsilon g_i(x) \quad \text{for } x \in \mathbb{R}^n \tag{1.2}$$

$(i = 1, \ldots, N)$, where $\Box = \partial_t^2 - \Delta_x$ is the d'Alembertian, $u = (u_j)_{1\le j\le N}$, and $\partial u = (\partial_a u_j)_{0\le a\le n, 1\le j\le N}$, while ε is a small positive parameter. Here we have used the notation $\partial_0 = \partial_t$ and $\partial_k = \partial_{x_k}$ for $1 \le k \le n$.

For simplicity, we suppose that each $F_i = F_i(u, \partial u)$ $(1 \le i \le N)$ is a homogeneous polynomial of degree p in its arguments.

We say that we have *small data global existence* (or we say that (SDGE) holds) if for any $f = (f_i)_{1\le i\le N}$ and $g = (g_i)_{1\le i\le N} \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^N)$, there exists a positive constant ε_0 such that the Cauchy problem (1.1)–(1.2) admits a unique global solution $u \in C^\infty([0, \infty) \times \mathbb{R}^n; \mathbb{R}^N)$ for any $\varepsilon \in (0, \varepsilon_0]$.

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We say that a nontrivial global solution u to (1.1)–(1.2) is *asymptotically free*, if there exists a function $\widetilde{u} = \widetilde{u}(t, x)$ ($\not\equiv 0$) solving $\Box\widetilde{u} = 0$ such that

$$\lim_{t\to\infty} \|u(t, \cdot) - \widetilde{u}(t, \cdot)\|_E = 0, \tag{1.3}$$

where $\|\cdot\|_E$ is the energy norm, that is

$$\|\varphi(t, \cdot)\|_E^2 = \int_{\mathbb{R}^n} (|\partial_t\varphi(t, x)|^2 + |\nabla_x\varphi(t, x)|^2)dx.$$

We say that (AF) holds when all the nontrivial global solutions to (1.1)–(1.2) with sufficiently small ε are asymptotically free.

Let us recall the known results for the three space dimensional case ($n = 3$) briefly. If the power of nonlinearity $p \geq 3$, then (SDGE) and (AF) hold. On the other hand, if $p = 2$, we do not have (SDGE) in general. Hence $p = 2$ is the critical power for $n = 3$. Klainerman [19] introduced a sufficient condition for (SDGE), which is called the null condition (see also Christodoulou [7]). If the null condition is satisfied, then each F_i can be written as a linear combination of $Q_0(u_j, u_k)$ and $Q_{ab}(u_j, u_k)$, where *the null forms* Q_0 and Q_{ab} are defined by

$$Q_0(\varphi, \psi) = (\partial_t\varphi)(\partial_t\psi) - (\nabla_x\varphi)\cdot(\nabla_x\psi), \tag{1.4}$$
$$Q_{ab}(\varphi, \psi) = (\partial_a\varphi)(\partial_b\psi) - (\partial_b\varphi)(\partial_a\psi) \quad \text{for } 0 \leq a, b \leq n, \tag{1.5}$$

respectively. It is easy to see that (AF) also holds under the null condition.

Alinhac [6] introduced a sufficient condition for (SDGE), which is weaker than the null condition. But Katayama–Kubo [18] showed that (AF) does not hold in general under the Alinhac condition. The simplest example satisfying the Alinhac condition is

$$\begin{cases} \Box u_1 = (\partial_1 u_1)(\partial_1 u_2 - \partial_2 u_1), \\ \Box u_2 = (\partial_2 u_1)(\partial_1 u_2 - \partial_2 u_1). \end{cases} \tag{1.6}$$

(AF) does not hold for (1.6), though (SDGE) holds.

Now we turn our attention to the two space dimensional case ($n = 2$). The critical power is $p = 3$ for $n = 2$. The null condition for $(n, p) = (2, 3)$ was also introduced, and (SDGE) under this null condition was obtained (see Godin [8], Hoshiga [11], and the author [16, 17] for the quasi-linear systems; see also Hoshiga–Kubo [14, 15] for the multiple propagation speeds case). More precisely, we say that the null condition for $(n, p) = (2, 3)$

holds, if each nonlinearity F_i can be written as a linear combination of $(\partial^\alpha u_j)Q_0(u_k, u_\ell)$ and $(\partial^\alpha u_j)Q_{ab}(u_k, u_\ell)$ with $|\alpha| \le 1$, $1 \le j, k, \ell \le N$, and $0 \le a < b \le 2$. Here and hereafter, for $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, ∂^α denotes $\partial_0^{\alpha_0}\partial_1^{\alpha_1}\partial_2^{\alpha_2}$. It is also easy to obtain (AF) under the null condition for $(n, p) = (2, 3)$.

This global existence result for $(n, p) = (2, 3)$ has been extended in various ways.

One way is to include nonlinear damping terms. Let us consider the equation $\Box u = -(\partial_t u)^3$ in $(0, \infty) \times \mathbb{R}^2$. It is well-known that the nonlinearity $-(\partial_t u)^3$ serves as a nonlinear damping term, and that there exists a global solution even for large data (see Lions–Strauss [26]). Since the nonlinear damping term makes the energy decrease, (AF) does not hold for this equation. In connection to this example, for single equations of the type $\Box u = F(\partial u)$ with $(n, p) = (2, 3)$, Agemi [1] introduced a condition which allows nonlinear damping terms as well as the terms satisfying the null condition (thus his condition is weaker than the null condition for $(n, p) = (2, 3)$ as far as we consider the single equation of the above type). He conjectured that (SDGE) holds under his condition. Recently, this conjecture turned out to be true (see Hoshiga [13] and Kubo [20]).

The other way is to include quadratic nonlinearities. Alinhac [2, 3] considered the (quasi-linear) systems for the case $(n, p) = (2, 2)$, and proved (SDGE) assuming that the quadratic nonlinearities (as well as the cubic ones if we consider higher perturbations) satisfy the null condition (see also Hoshiga [12] for the multiple speeds case). We can also show that (AF) holds under this assumption.

In this paper, we seek extension in another direction. Our aim here is to obtain the two space dimensional analogue to the three space dimensional results by Alinhac [6] and Katayama–Kubo [18], which we have mentioned above. In other words, we present a class of nonlinearity for which (SDGE) holds, but (AF) may fail to hold because the energy may increase as opposed to the nonlinear damping case.

In the following, for a family of functions $\{\varphi_\lambda\}_{\lambda\in\Lambda}$ and a function ψ, we write $\psi = {\sum}'_{\lambda\in\Lambda}\varphi_\lambda$ if there exist some constants c_λ ($\lambda \in \Lambda$) such that $\psi = \sum_{\lambda\in\Lambda} c_\lambda\varphi_\lambda$.

We introduce the following assumption:

(H) By writing $u = (u_i)_{1\le i\le N} = \big((v_i)_{1\le i\le L}, (w_i)_{1\le i\le M}\big) = (v, w)$ with some $L \in \{1, \cdots, N-1\}$ and $M = N - L$ (to be specific, $v_i = u_i$ for $1 \le i \le L$, and $w_i = u_{i+L}$ for $1 \le i \le M$), each F_i $(1 \le i \le N)$ has the form

$$F_i(u, \partial u) = A_i(w, \partial v, \partial w) + N_i(u, \partial u) \quad \text{for } 1 \le i \le L, \tag{1.7}$$

$$F_i(u, \partial u) = N_i(u, \partial u) \quad \text{for } L+1 \le i \le N, \tag{1.8}$$

where

$$A_i(w, \partial v, \partial w) = \sideset{}{'}\sum_{\substack{1\le j\le L,\ 1\le k,\ell\le M\\ 0\le a\le 2,\ |\alpha|,|\beta|\le 1}} (\partial_a v_j)(\partial^\alpha w_k)(\partial^\beta w_\ell), \tag{1.9}$$

and

$$N_i(u, \partial u) = \sideset{}{'}\sum_{\substack{1\le j,k,\ell\le N\\ |\alpha|\le 1}} (\partial^\alpha u_j) Q_0(u_k, u_\ell) + \sideset{}{'}\sum_{\substack{1\le j,k,\ell\le N\\ |\alpha|\le 1,\, 0\le a<b\le 2}} (\partial^\alpha u_j) Q_{ab}(u_k, u_\ell). \tag{1.10}$$

In other words, (H) means that (1.1) can be written as

$$\begin{cases} \Box v_i = A_i(w, \partial v, \partial w) + N_i\big((v, w), (\partial v, \partial w)\big) & (1 \le i \le L),\\ \Box w_i = N_{i+L}\big((v, w), (\partial v, \partial w)\big) & (1 \le i \le M). \end{cases} \tag{1.11}$$

Remark The assumption (H) with $A_i \equiv 0$ for all $i \in \{1, \ldots, L\}$ coincides with the null condition for $(n, p) = (2, 3)$.

Theorem 1.1 *Let $n = 2$ and $p = 3$. Assume that* (H) *is fulfilled.*
Then (SDGE) *holds for the Cauchy problem* (1.1)–(1.2).
Moreover, there exists $(\widetilde{v}, \widetilde{w})$ solving

$$\Box \widetilde{v}_i = A_i(\widetilde{w}, \partial\widetilde{v}, \partial\widetilde{w}) \quad \textit{for } 1 \le i \le L, \tag{1.12}$$

$$\Box \widetilde{w}_i = 0 \quad \textit{for } 1 \le i \le M \tag{1.13}$$

such that

$$\lim_{t\to\infty} \big(\|v(t, \cdot) - \widetilde{v}(t, \cdot)\|_E + \|w(t, \cdot) - \widetilde{w}(t, \cdot)\|_E\big) = 0,$$

where $u = (v, w)$ is the global solution to (1.1)–(1.2).

There is a certain class of system which does not satisfy (H) explicitly, but can be reduced to other system satisfying (H). For example, consider

$$\begin{cases} \Box u_1 = (\partial_1 u_1)(\partial_1 u_2 - \partial_2 u_1)^2, \\ \Box u_2 = (\partial_2 u_1)(\partial_1 u_2 - \partial_2 u_1)^2, \end{cases} \tag{1.14}$$

which does not satisfy (H). Setting

$$v_1 = u_1,\ v_2 = u_2, \quad \text{and} \quad w = \partial_1 u_2 - \partial_2 u_1,$$

we find that solving (1.14) is equivalent to solving

$$\begin{cases} \Box v_1 = (\partial_1 v_1) w^2, \quad \Box v_2 = (\partial_2 v_1) w^2, \\ \Box w = 2w Q_{12}(w,\, v_1), \end{cases} \tag{1.15}$$

which satisfies the assumption (H). Observe that this example corresponds to (1.6) for $n = 3$.

More precisely, we can get a two dimensional analogue to the Alinhac condition in the following way: Suppose that each F_i $(1 \le i \le N)$ in (1.1) depends only on ∂u, i.e., $F_i = F_i(\partial u) = F_i\big((\partial_a u_j)_{0 \le a \le 2, 1 \le j \le N}\big)$. For $\omega = (\omega_1,\, \omega_2) \in S^1$ and $X = (X_j)_{1 \le j \le N}$, we define the *reduced* nonlinearity

$$F_i^{\mathrm{red}}(\omega,\, X) \equiv F_i\big((-\omega_a X_j)_{0 \le a \le 2,\, 1 \le j \le N}\big) \quad (1 \le i \le N),$$

whose right-hand side means that $-\omega_a X_j$ is substituted in place of $\partial_a u_j$ ("red" in F_i^{red} stands for "reduced"). Here and hereafter we put $\omega_0 = -1$. Now we introduce an alternative assumption as follows:

(H') There exist $\beta(\omega) = \big(\beta_i(\omega)\big)_{1 \le i \le N} \in \mathbb{R}^N$, a function $P(\omega,\, X)$, some number of bilinear forms

$$h_j = h_j(\omega,\, X) = \sum_{0 \le a \le 2,\, 1 \le k \le N} h_j^{ka} \omega_a X_k \quad (1 \le j \le M) \tag{1.16}$$

in $(\omega,\, X)$ (with real constants h_j^{ka}), and linear forms $g_i^{jk}(\omega,\, X)$ in X (with smooth coefficients in ω), satisfying

$$F_i^{\mathrm{red}}(\omega,\, X) = \beta_i(\omega) P(\omega,\, X) \ (1 \le i \le N,\ \omega \in S^1,\ X \in \mathbb{R}^N), \tag{1.17}$$

$$F_i^{\mathrm{red}}(\omega,\, X) = \sum_{1 \le j,k \le M} g_i^{jk}(\omega,\, X) h_j(\omega,\, X) h_k(\omega,\, X) \quad (1 \le i \le N, \omega \in S^1, X \in \mathbb{R}^N), \tag{1.18}$$

$$h_j\big(\omega,\,\beta(\omega)\big) \equiv 0 \quad (1 \le j \le M,\ \omega \in S^1). \tag{1.19}$$

We can easily check that the system (1.14) satisfies (H').

Remark (1.17), (1.18) and (1.19) yield

$$P\big(\omega,\,\beta(\omega)\big) = 0 \quad (\omega \in S^1) \tag{1.20}$$

if $\beta(\omega) \neq 0$, while (1.20) is triviality when $\beta(\omega) = 0$, because we can choose $P(\omega,\,X) = 0$ for such ω. The condition (AA) in [6] exactly coincides with (1.17) and (1.20), while the condition $(\overline{\text{AA}})$ in [6] corresponds to (1.18) and (1.19). In [6], as we have mentioned, it is proved that the Alinhac condition, which consists of (AA) and $(\overline{\text{AA}})$, implies (SDGE) for $(n,\,p) = (3,\,2)$, but Alinhac conjectures that (AA) would suffice for (SDGE) when $(n,\,p) = (3,\,2)$.

Theorem 1.2 *Let $n = 2$, $p = 3$ and $F_i = F_i(\partial u)$ for $1 \le i \le N$ in* (1.1). *Assume that* (H') *is fulfilled.*

Then (SDGE) *holds for the Cauchy problem* (1.1)–(1.2).

Concerning the asymptotic behavior of the solutions, we have the following:

Theorem 1.3 *Let $n = 2$, and consider* (1.14) *or* (1.15).

Then, there exist $f,\,g \in C_0^\infty(\mathbb{R}^2)$ and two positive constants C_0 and ε_1 such that we have

$$\|u(t,\,\cdot\,)\|_E \ge C_0\varepsilon(1+t)^{C_0\varepsilon^2}$$

for all $t \ge 0$ provided that $\varepsilon \in (0,\,\varepsilon_1]$, where $u = (u_1,\,u_2)$ (resp. $u = (v_1,\,v_2,\,w)$) is the global solution to (1.14) *(resp.* (1.15)*) with initial data $u = \varepsilon f$ and $\partial_t u = \varepsilon g$ at $t = 0$.*

If (AF) holds, then $\sup_{0\le t<\infty} \|u(t,\,\cdot\,)\|_E$ must be finite. Hence Theorem 1.3 shows that (AF) does not hold in general under the assumptions (H) or (H'), though Theorem 1.1 (resp. Theorem 1.2) ensures (SDGE) under (H) (resp. (H')).

Theorems 1.1, 1.2, and 1.3 will be proved in Sections 4, 5, and 6, respectively.

Throughout this paper, as usual, the letter C stands for a positive constant, which may change line by line.

2. Notation

We will use the notation given in this section throughout this paper.

Consider the Cauchy problem for the linear wave equation

$$\begin{cases} \Box\varphi(t, x) = \Phi(t, x) & \text{in } (0, \infty) \times \mathbb{R}^2, \\ \varphi(0, x) = \varphi_0(x),\ (\partial_t\varphi)(0, x) = \varphi_1(x) & \text{for } x \in \mathbb{R}^2. \end{cases} \tag{2.1}$$

We write $U_0[\varphi_0, \varphi_1]$ for the classical solution to (2.1) with $\Phi \equiv 0$, and $U[\Phi]$ for the classical solution to (2.1) with $\varphi_0 = \varphi_1 \equiv 0$, respectively.

For $\rho > 0$ and $y \in \mathbb{R}^2$, $B_\rho(y)$ denotes an open ball with radius ρ centered at y.

We define

$$\mathcal{W}_\pm(t, x) = \langle t \pm |x|\rangle \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}^2, \tag{2.2}$$

where $\langle a\rangle = \sqrt{1 + |a|^2}$ for $a \in \mathbb{R}$.

We introduce vector fields

$$S = t\partial_t + x \cdot \nabla_x,\ L_j = x_j\partial_t + t\partial_j\ (j = 1, 2),\ \Omega_{12} = x_1\partial_2 - x_2\partial_1,$$

and we set

$$\Gamma_0 = S,\ \Gamma_j = L_j\ (j = 1, 2),\ \Gamma_3 = \Omega_{12},\ \Gamma_{a+4} = \partial_a\ (0 \le a \le 2).$$

It is well-known that we have $[S, \Box] = -2\Box$, $[\Gamma_i, \Box] = 0$ for $1 \le i \le 6$. We also have

$$[\Gamma_i, \Gamma_j] = \sum_{0\le k\le 6}{}' \Gamma_k, \quad [\partial_a, \Gamma_i] = \sum_{0\le b\le 2}{}' \partial_b$$

for $0 \le i, j \le 6$ and $0 \le a \le 2$. With a multi-index $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_6)$, we write $\Gamma^\alpha = \Gamma_0^{\alpha_0}\Gamma_1^{\alpha_1} \cdots \Gamma_6^{\alpha_6}$. For a nonnegative integer s, and a scalar or vector-valued smooth function $\varphi = \varphi(t, x)$, we define

$$|\varphi(t, x)|_s = \sum_{|\alpha|\le s} |\Gamma^\alpha\varphi(t, x)|,$$
$$\|\varphi(t, \cdot)\|_{s,q} = \big\||\varphi(t, \cdot)|_s\big\|_{L^q(\mathbb{R}^2)} \quad (1 \le q \le \infty).$$

We also introduce

$$Z_j = \frac{x_j}{|x|}\partial_t + \partial_j \quad (j = 1, 2). \tag{2.3}$$

Then we have

$$|\Gamma_a\varphi(t,\,x)| \le C\bigl(|x||Z\varphi(t,\,x)| + \langle t-|x|\rangle|\partial\varphi(t,\,x)|\bigr) \tag{2.4}$$

for $0 \le a \le 6$ and any smooth function φ, where $Z\varphi = (Z_1\varphi,\, Z_2\varphi)$. In fact, we have

$$S = \sum_{j=1}^{2} x_j Z_j + (t-|x|)\partial_t,\ L_j = |x|Z_j + (t-|x|)\partial_j \quad (j=1,\,2),$$
$$\Omega_{12} = x_1 Z_2 - x_2 Z_1,$$

while (2.4) is trivial for $4 \le a \le 6$.

On the other hand, we also have $|Z\varphi(t,\,x)| \le C|\partial\varphi(t,\,x)|$,

$$Z_1 = \frac{x_1}{|x|}(\partial_t+\partial_r) - \frac{x_2}{|x|^2}\Omega_{12},\ Z_2 = \frac{x_2}{|x|}(\partial_t+\partial_r) + \frac{x_1}{|x|^2}\Omega_{12},$$

and

$$(t+|x|)(\partial_t+\partial_r) = S + \sum_{j=1}^{2}\Bigl(\frac{x_j}{|x|}\Bigr)L_j,$$

where $\partial_r = \sum_{j=1}^{2}(x_j/|x|)\partial_j$ as usual. Hence we get

$$|Z\varphi(t,\,x)| \le C\langle |x|\rangle^{-1}\sum_{|\alpha|=1}|\Gamma^\alpha\varphi(t,\,x)|. \tag{2.5}$$

For a nonnegative integer s, and a scalar or vector-valued smooth function $\varphi = \varphi(t,\,x)$, we define

$$|\varphi(t,\,x)|_{Z,s} = \sum_{|\alpha|\le s}\sum_{j=1}^{2}|Z_j\Gamma^\alpha\varphi(t,\,x)|.$$

3. Preliminary Results

In this section, we state known estimates for linear wave equations, and we make some necessary estimates. In what follows, we always suppose that $\varphi_0,\,\varphi_1 \in C_0^\infty(\mathbb{R}^2)$, and that $\Phi = \Phi(t,\,x)$ is a smooth function decaying sufficiently fast at spatial infinity.

First of all, we introduce the improved energy estimate by Alinhac [5] (see also Alinhac [4, 6] and Lindblad–Rodnianski [25]).

Lemma 3.1 *Let* $\varphi = U_0[\varphi_0, \varphi_1] + U[\Phi]$.

Then, for $\lambda \geq 0$ *and* $\rho > 0$*, there exists a constant* C *depending only on* ρ *such that*

$$\begin{aligned}&\langle t\rangle^{-\lambda}\|\varphi(t, \cdot)\|_E + \Big(\sum_{j=1}^{2}\int_0^t\int_{\mathbb{R}^2}\frac{|Z_j\varphi(\tau, x)|^2}{\langle\tau\rangle^{2\lambda}\langle\tau-|x|\rangle^{1+\rho}}dxd\tau\Big)^{1/2}\\&\leq C\Big(\|\nabla_x\varphi_0\|_{L^2} + \|\varphi_1\|_{L^2} + \int_0^t\langle\tau\rangle^{-\lambda}\|\Phi(\tau, \cdot)\|_{L^2}d\tau\Big)\end{aligned} \tag{3.1}$$

for $t \geq 0$.

Outline of the proof. We set $\eta(s) = \int_{-\infty}^{s}\langle\tau\rangle^{-(\rho+1)}d\tau$ for $s \in \mathbb{R}$. Then following similar lines to the proof of the standard energy inequality, however multiplying $\square\varphi$ by $\langle t\rangle^{-2\lambda}e^{\eta(|x|-t)}(\partial_t\varphi)$ instead of $\partial_t\varphi$, we obtain

$$\begin{aligned}&2\int_{\mathbb{R}^2}\langle t\rangle^{-2\lambda}e^{\eta(|x|-t)}(\partial_t\varphi)\Phi dx\\&=\frac{d}{dt}\int_{\mathbb{R}^2}\langle t\rangle^{-2\lambda}e^{\eta(|x|-t)}\{(\partial_t\varphi)^2 + |\nabla_x\varphi|^2\}dx\\&\quad+\sum_{j=1}^{2}\int_{\mathbb{R}^2}\frac{e^{\eta(|x|-t)}|Z_j\varphi|^2}{\langle t\rangle^{2\lambda}\langle|x|-t\rangle^{1+\rho}}dx\\&\quad+2\lambda t\langle t\rangle^{-2\lambda-2}\int_{\mathbb{R}^2}e^{\eta(|x|-t)}\{(\partial_t\varphi)^2 + |\nabla_x\varphi|^2\}dx,\end{aligned} \tag{3.2}$$

which implies Lemma 3.1 (observe that we have $1 \leq e^{\eta(s)} \leq C_\rho$ for all $s \in \mathbb{R}$ with a constant C_ρ depending on ρ, and that the last term on the right-hand side of (3.2) is nonnegative). □

The following estimate is due to Hörmander [9] (see also the proof of Lemma 3.1 in the author [16]).

Lemma 3.2 *For* $\kappa \in [0, 1/2]$*, there exists a constant* C *depending only on* κ *such that we have*

$$\langle t+|x|\rangle^{1/2}\langle t-|x|\rangle^{\kappa}|U[\Phi](t, x)| \leq C\int_0^t\int_{\mathbb{R}^2}\frac{|\Phi(\tau, y)|_1}{\langle\tau+|y|\rangle^{(1/2)-\kappa}}dyd\tau$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^2$.

The following L^2-estimate will be used in the proof of Theorem 1.3.

Lemma 3.3 *For $0 < \rho \le 1$, there exists a constant C depending only on ρ such that we have*

$$\|U_0[\varphi_0, \varphi_1](t, \cdot)\|_{L^2(\mathbb{R}^2)} \le C(\|\varphi_0\|_{L^2(\mathbb{R}^2)} + t^{2\rho/(1+\rho)}\|\varphi_1\|_{L^{1+\rho}(\mathbb{R}^2)}), \quad (3.3)$$

$$\|U[\Phi](t, \cdot)\|_{L^2(\mathbb{R}^2)} \le Ct^{2\rho/(1+\rho)} \int_0^t \|\Phi(\tau, \cdot)\|_{L^{1+\rho}(\mathbb{R}^2)} d\tau \quad (3.4)$$

for $t \ge 0$.

For the proof, see Li–Zhou [23, Lemma 2.8], or the author [16, Proposition 3.2] for instance (see also Strichartz [29], Peral [28], Marshall–Strauss-Wainger [27], and Li–Yu–Zhou [22] for related results). Note that Lemma 3.3 fails to hold for $\rho = 0$ (see [16, Remark 3] for the counterexample).

To treat the null forms, we use the following:

Lemma 3.4 *Let s be a nonnegative integer, $u = (u_j)_{1\le j\le N}$ be a smooth function, and N_i be given by* (1.10)*. Then we have*

$$|N_i(u, \partial u)|_s \le C_s\langle t + |x|\rangle^{-1}|u|_{[s/2]+1} \times (|u|_{[s/2]+1}|\partial u|_s + |\partial u|_{[s/2]}|u|_{s+1}), \quad (3.5)$$

and

$$|N_i(u, \partial u)|_s \le C_s\langle t + |x|\rangle^{-1}|u|_{[s/2]+1} \times \big(|u|_{[s/2]+1} + \langle t - |x|\rangle|\partial u|_{[s/2]}\big)|\partial u|_s + C|u|_{[s/2]+1}|\partial u|_{[s/2]}|u|_{Z,s} \quad (3.6)$$

at $(t, x) \in [0, \infty) \times \mathbb{R}^2$, where C_s is a positive constant depending only on s.

Proof. For a null form Q, it is well known that we have

$$|Q(u_j, u_k)|_s \le C\langle t + |x|\rangle^{-1}(|u|_{[s/2]+1}|\partial u|_s + |\partial u|_{[s/2]}|u|_{s+1}) \quad (3.7)$$

(see Klainerman [19]), which immediately yields (3.5) (see also the author [16, 17]). Since we have $|u|_{s+1} \le |u| + \sum_{1\le|\alpha|\le s+1} |\Gamma^\alpha u|$, by using (2.4) to evaluate $|\Gamma^\alpha u|$ for $1 \le |\alpha| \le s + 1$, we obtain (3.6) from (3.5) (see also Alinhac [6]). □

For the proof of Theorem 1.3 we need the following:

Lemma 3.5 *There exists a positive constant C such that*

$$\int_{B_{4\delta}(0)\cap B_t(x)} \frac{dy}{\sqrt{t^2-|x-y|^2}} \geq C\frac{\delta^{3/2}}{(2\delta+t)^{1/2}} \tag{3.8}$$

for any $\delta > 0$ and any $(t,\, x) \in [0,\, \infty) \times \mathbb{R}^2$ satisfying

$$4\delta \leq t+\delta \leq |x| \leq t+2\delta. \tag{3.9}$$

Proof. By setting $a = |x| - t$, (3.9) implies

$$t \geq 3\delta \quad \text{and} \quad \delta \leq a \leq 2\delta. \tag{3.10}$$

Switching to the polar coordinates centered at x, we obtain

$$\begin{aligned}\int_{B_{4\delta}(0)\cap B_t(x)} \frac{dy}{\sqrt{t^2-|x-y|^2}} &\geq 2\theta_0 \int_{t-b}^{t} \frac{\lambda}{\sqrt{t^2-\lambda^2}} d\lambda \\ &= 2\theta_0 \sqrt{2bt-b^2},\end{aligned} \tag{3.11}$$

where $b = (4\delta - a)/2$, and $\theta_0 \in (0,\, \pi/2)$ is determined by

$$(t-b)^2 \sin^2\theta_0 + (t+a-(t-b)\cos\theta_0)^2 = (4\delta)^2. \tag{3.12}$$

From (3.12), we find

$$\theta_0^2 \geq \sin^2\theta_0 = \Big(2 - \frac{(4\delta-a)(12\delta+a)}{8(t+a)(t-b)}\Big)\frac{(4\delta-a)(12\delta+a)}{8(t+a)(t-b)}. \tag{3.13}$$

By (3.10), we obtain

$$\frac{13\delta^2}{4(2\delta+t)(t-b)} \leq \frac{(4\delta-a)(12\delta+a)}{8(t+a)(t-b)} \leq \frac{7}{8}. \tag{3.14}$$

On the other hand, (3.10) also leads to

$$\frac{2bt-b^2}{t-b} = 2b + \frac{b^2}{t-b} \geq 2b \geq 2\delta. \tag{3.15}$$

Now (3.11)–(3.15) imply

$$\int_{B_{4\delta}(0)\cap B_t(x)} \frac{dy}{\sqrt{t^2-|x-y|^2}} \geq \frac{3\sqrt{13}}{2}\frac{\delta^{3/2}}{(2\delta+t)^{1/2}}.$$

This completes the proof. □

Since it is well known that we have

$$U_0[0,\,\varphi_1](t,\,x)=\frac{1}{2\pi}\int_{B_t(x)}\frac{\varphi_1(y)}{\sqrt{t^2-|x-y|^2}}dy,$$

Lemma 3.5 immediately implies the following:

Corollary 3.6 *Fix $\omega^*\in S^1$ and a neighborhood Λ of ω^* on S^1. Set*

$$\Omega_\Lambda=\{y\in\mathbb{R}^2;\text{there exists }\eta\in\Lambda\text{ such that }y\cdot\eta\geq 0\}. \tag{3.16}$$

If $\varphi_1\in C_0^\infty(\mathbb{R}^2)$ satisfies

$$\varphi_1(y)\geq 0\text{ for }y\in\Omega_\Lambda,\ \text{ and }\varphi_1(y)\geq\zeta_0\text{ for }y\in\Omega_\Lambda\cap B_{4\delta}(0) \tag{3.17}$$

with some positive constants δ and ζ_0, then we have

$$U_0[0,\,\varphi_1](t,\,x)\geq\frac{C}{2\pi}\frac{\delta^{3/2}\zeta_0}{(2\delta+t)^{1/2}} \tag{3.18}$$

for any $(t,\,x)$ satisfying (3.9) *and $x/|x|\in\Lambda$, where C is the same constant as in* (3.8).

To prove Corollary 3.6, we only have to notice that (3.9) and $x/|x|\in\Lambda$ imply $B_t(x)\subset\Omega_\Lambda$.

Finally we recall the following Hardy type inequality.

Lemma 3.7 *Let $R>0$ be given. Then we have*

$$\left\|\frac{\varphi(t,\,\cdot\,)}{\mathcal{W}_-(t,\,\cdot\,)}\right\|_{L^2(\mathbb{R}^2)}\leq C_R\|\partial\varphi(t,\,\cdot\,)\|_{L^2(\mathbb{R}^2)} \tag{3.19}$$

for any smooth function φ satisfying $\operatorname{supp}\varphi(t,\,\cdot\,)\subset B_{t+R}(0)$*, where the constant C_R depends only on R.*

For the proof, see Lindblad [24] and the author [17].

4. Proof of Theorem 1.1

Suppose that all the assumptions in Theorem 1.1 are fulfilled.

Let $u=(v,\,w)\in C^\infty\big([0,\,T_0)\times\mathbb{R}^2;\mathbb{R}^N\big)$ be the local solution to (1.1)–(1.2) with some $T_0>0$. Assume that $\operatorname{supp} f\cup\operatorname{supp} g\subset B_R(0)$ with some $R>0$. Then it is well-known that we have $\operatorname{supp} u(t,\,\cdot\,)\subset B_{t+R}(0)$ for $t\in[0,\,T_0)$. Accordingly, we also find that $\Gamma^\alpha u$ is uniformly continuous on $[0,\,T]\times\mathbb{R}^2$ for any $T\in(0,\,T_0)$, and any multi-index α.

We define

$$d_k[u](t,\,x) = \langle t+r\rangle^{1/2} \times \big(\langle t+r\rangle^{-\gamma\varepsilon^2}|v(t,\,x)|_{k+1} + \langle t-r\rangle^{\nu}|w(t,\,x)|_{k+1}\big),$$

where $r=|x|$, k is a nonnegative integer, $1/4<\nu<1/2$ and $\gamma>0$. Since we have

$$\langle t-r\rangle|\partial\varphi(t,\,x)| \le C|\varphi(t,\,x)|_1 \tag{4.1}$$

for any smooth function φ (for the proof, see Lindblad [24] and the author [17]), we obtain

$$\langle t+r\rangle^{(1/2)-\gamma\varepsilon^2}\langle t-r\rangle|\partial v(t,\,x)|_k + \langle t+r\rangle^{1/2}\langle t-r\rangle^{1+\nu}|\partial w(t,\,x)|_k \le Cd_k[u](t,\,x) \quad \text{for any } (t,\,x)\in[0,\,T_0)\times\mathbb{R}^2,$$

where C is a positive constant independent of T_0. We set

$$E_{2k}[u](t) = \langle t\rangle^{-\gamma\varepsilon^2}\|\partial v(t,\,\cdot\,)\|_{2k,2} + \|\partial w(t,\,\cdot\,)\|_{2k,2} + \Big(\int_0^t\int_{\mathbb{R}^2}\langle\tau\rangle^{-4\gamma\varepsilon^2}\langle\tau-|x|\rangle^{-2}|u(\tau,\,x)|^2_{Z,2k}\,dx\,d\tau\Big)^{1/2}.$$

We fix some $\nu\in(1/4,\,1/2)$ and $k\ge 5$. We assume that we have

$$\sup_{0\le t<T}\big\{\|d_k[u](t,\,\cdot\,)\|_{L^\infty(\mathbb{R}^2)} + E_{2k}[u](t)\big\} \le K\varepsilon \tag{4.2}$$

for some $K>0$ and some $T>0$ (note that we have $\|d_k[u](0,\,\cdot\,)\|_{L^\infty(\mathbb{R}^2)} + E_{2k}[u](0) \le K\varepsilon/2$ for sufficiently large K and consequently (4.2) is true for small T, because of the uniform continuity of $|u(t,\,x)|_{k+1}$ on $[0,\,T]\times\mathbb{R}^2$). We are going to prove that, if we choose sufficiently large K and γ, then (4.2) implies

$$\sup_{0\le t<T}\big\{\|d_k[u](t,\,\cdot\,)\|_{L^\infty(\mathbb{R}^2)} + E_{2k}[u](t)\big\} \le \frac{K}{2}\varepsilon \tag{4.3}$$

for sufficiently small ε. Once such an estimate is established, then by the well-known continuity argument (see the proof of Theorem 6.5.2 in Hörmander [10] for example), we obtain the global existence of the solution immediately.

Now we start the proof of (4.3). In the following, we always assume that K is large enough and ε is small enough.

By (3.6) in Lemma 3.4, we get

$$|N_i(u,\,\partial u)|_s \le C\mathcal{W}_+^{-1}|u|_{[s/2]+1}\big(|u|_{[s/2]+1}+\mathcal{W}_-|\partial u|_{[s/2]}\big)|\partial u|_s + C|u|_{[s/2]+1}(\mathcal{W}_-|\partial u|_{[s/2]})\frac{|u|_{Z,s}}{\mathcal{W}_-} \quad (4.4)$$

for $1 \le i \le N$ and a nonnegative integer s.

From (4.2) and (4.4), we get

$$\begin{aligned}&\int_0^t \|N_i(u,\,\partial u)(\tau,\,\cdot\,)\|_{2k,2}d\tau \\ &\le CK^3\varepsilon^3\int_0^t\langle\tau\rangle^{-2+3\gamma\varepsilon^2}d\tau + CK^3\varepsilon^3\Big(\int_0^t\langle\tau\rangle^{-2+8\gamma\varepsilon^2}d\tau\Big)^{1/2} \\ &\le CK^3\varepsilon^3 \end{aligned} \quad (4.5)$$

for $1 \le i \le N$, provided that $8\gamma\varepsilon^2 < 1/4$, say. Here we have evaluated the term coming from the last term on the right-hand side of (4.4) by

$$\begin{aligned}&\int_0^t \Big\| |u|_{k+1}(\mathcal{W}_-|\partial u|_k)\frac{|u|_{Z,2k}}{\mathcal{W}_-}\Big\|_{L^2}d\tau \\ &\le K^2\varepsilon^2\int_0^t\langle\tau\rangle^{-1+2\gamma\varepsilon^2}\Big\|\frac{|u|_{Z,2k}}{\mathcal{W}_-}\Big\|_{L^2}d\tau \\ &\le K^2\varepsilon^2\Big(\int_0^t\langle\tau\rangle^{-2+8\gamma\varepsilon^2}d\tau\Big)^{1/2}\Big(\int_0^t\langle\tau\rangle^{-4\gamma\varepsilon^2}\Big\|\frac{|u|_{Z,2k}}{\mathcal{W}_-}\Big\|_{L^2}^2d\tau\Big)^{1/2}.\end{aligned}$$

On the other hand, since we have

$$|A_i(w,\,\partial v,\,\partial w)|_s \le C|w|_{[s/2]+1}^2|\partial v|_s + C|w|_{[s/2]+1}|\partial v|_{[s/2]}|\partial w|_s + C|w|_{[s/2]+1}(\mathcal{W}_-|\partial v|_{[s/2]})\frac{|w|_s}{\mathcal{W}_-} \quad (4.6)$$

for $1 \le i \le L$ and a nonnegative integer s, we obtain

$$\int_0^t \|A_i(\tau,\,\cdot\,)\|_{2k,2}d\tau \le CK^3\varepsilon^3\int_0^t\langle\tau\rangle^{\gamma\varepsilon^2-1}d\tau \le C\frac{K^2}{\gamma}K\varepsilon\langle t\rangle^{\gamma\varepsilon^2} \quad (4.7)$$

with the help of (3.19). Therefore, (4.5) and (4.7) with the standard energy inequality lead to

$$\langle t\rangle^{-\gamma\varepsilon^2}\|\partial v(t,\,\cdot\,)\|_{2k,2} + \|\partial w(t,\,\cdot\,)\|_{2k,2} \le C\Big(\varepsilon + \frac{K^2}{\gamma}K\varepsilon + K^3\varepsilon^3\Big).$$

Similarly to (4.7), we get

$$\int_0^t \langle\tau\rangle^{-2\gamma\varepsilon^2}\|A_i(\tau,\,\cdot\,)\|_{2k,2}d\tau \le C\frac{K^2}{\gamma}K\varepsilon. \tag{4.8}$$

From Lemma 3.1, (4.5) and (4.8), we find

$$\Big(\int_0^t\int_{\mathbb{R}^2}\frac{|u(\tau,\,x)|^2_{Z,2k}}{\langle\tau\rangle^{4\gamma\varepsilon^2}\langle\tau-|x|\rangle^2}dxd\tau\Big)^{1/2} \le C\Big(\varepsilon+\frac{K^2}{\gamma}K\varepsilon+K^3\varepsilon^3\Big).$$

Summing up, we have shown

$$E_{2k}[u](t) \le C\Big(\varepsilon+\frac{K^2}{\gamma}K\varepsilon+K^3\varepsilon^3\Big) \tag{4.9}$$

for $0\le t<T$.

Now we turn our attention to $d_k[u]$. It is well-known that we have

$$\langle t+r\rangle^{1/2}\langle t-r\rangle^{1/2}|U_0[\varepsilon f_i,\,\varepsilon g_i](t,\,x)|_s \le C_s\varepsilon \tag{4.10}$$

for a nonnegative integer s (see Kubota [21] for instance). Since (3.5) of Lemma 3.4 implies

$$\begin{aligned}|N_i(u,\,\partial u)|_s \le{}& C\mathcal{W}_+^{-1}|u|^2_{[s/2]+1}|\partial u|_s\\ &+C\mathcal{W}_+^{-1}|u|_{[s/2]+1}(\mathcal{W}_-|\partial u|_{[s/2]})\Big(\frac{|u|_{s+1}}{\mathcal{W}_-}\Big)\end{aligned} \tag{4.11}$$

for a nonnegative integer s, we get

$$\begin{aligned}\Big\|\frac{|N_i(t)|_{2k-1}}{\mathcal{W}_+^{(1/2)-\nu}(t)}\Big\|_{L^1} &\le CK^2\varepsilon^2\Big\|\mathcal{W}_+^{-(5/2)+2\gamma\varepsilon^2+\nu}\Big\|_{L^2}\|\partial u\|_{2k,2}\\ &\le CK^3\varepsilon^3\langle t\rangle^{-(3/2)+3\gamma\varepsilon^2+\nu}\end{aligned} \tag{4.12}$$

for $1\le i\le N$. Hence by Lemma 3.2 we obtain

$$\langle t+r\rangle^{1/2}\langle t-r\rangle^{\nu}|w(t,\,x)|_{2k-2} \le C(\varepsilon+CK^3\varepsilon^3) \le CK\varepsilon, \tag{4.13}$$

provided that $3\gamma\varepsilon^2<(1/2)-\nu$.

On the other hand, since we have $k+3\le 2k-2$, (4.6) and (4.13) yield

$$\begin{aligned}\Big\|\frac{|A_i(t)|_{k+2}}{\mathcal{W}_+^{1/2}(t)}\Big\|_{L^1} \le{}& CK^2\varepsilon^2\|\mathcal{W}_+^{-3/2}\mathcal{W}_-^{-2\nu}\|_{L^2}\|\partial v\|_{2k,2}\\ &+CK^3\varepsilon^3\langle t\rangle^{\gamma\varepsilon^2}\|\mathcal{W}_+^{-2}\mathcal{W}_-^{-1-2\nu}\|_{L^1}\\ \le{}& CK^3\varepsilon^3\langle t\rangle^{\gamma\varepsilon^2-1},\end{aligned} \tag{4.14}$$

because we have

$$\|\mathcal{W}_+^{-3/2}(t)\mathcal{W}_-^{-2\nu}(t)\|_{L^2} + \|\mathcal{W}_+^{-2}(t)\mathcal{W}_-^{-1-2\nu}(t)\|_{L^1} \le C\langle t\rangle^{-1}$$

for $1/4 < \nu < 1/2$. By (4.12), (4.14), and Lemma 3.2 with $\kappa = 0$, we obtain

$$\langle t+r\rangle^{1/2}|v(t,\,x)|_{k+1} \le C\Big(\varepsilon + \frac{K^2}{\gamma}K\varepsilon\langle t\rangle^{\gamma\varepsilon^2} + K^3\varepsilon^3\Big). \tag{4.15}$$

Finally (4.9), (4.13) and (4.15) yield

$$\sup_{0\le t<T}\big\{\|d_k[u](t,\,\cdot\,)\|_{L^\infty(\mathbb{R}^2)} + E_{2k}[u](t)\big\} \le C_0\Big(\varepsilon + \frac{K^2}{\gamma}K\varepsilon + K^3\varepsilon^3\Big),$$

with some positive constant C_0. This inequality leads to (4.3), if we assume

$$K \ge 6C_0,\ \gamma \ge 6C_0K^2,\ C_0K^2\varepsilon^2 \le \frac{1}{6}.$$

This completes the proof for global existence of the solution.

Now we have

$$\|d_k[u](t,\,\cdot\,)\|_{L^\infty(\mathbb{R}^2)} + E_{2k}[u](t) \le C\varepsilon \quad \text{for all } t \in [0,\,\infty), \tag{4.16}$$

and a similar argument to the proof of Theorem 1.1 in [18] implies the existence of $\widetilde{v}$ and $\widetilde{w}$. We omit the details here.

5. Proof of Theorem 1.2

We are going to show that the proof of Theorem 1.2 can be essentially reduced to that of Theorem 1.1, by following the arguments in [6].

Assume that all the assumptions in Theorem 1.2 are fulfilled, and let u be the solution to (1.1)–(1.2). Since $F_i(\partial u)$ $(1 \le i \le N)$ are homogeneous polynomials of degree 3 with respect to ∂u, we can write them as

$$F_i(\partial u) = \sum_{\substack{1\le j\le k\le \ell\le N\\ 0\le a,b,c\le 2}} C_{abc}^{i,jk\ell}(\partial_a u_j)(\partial_b u_k)(\partial_c u_\ell) \tag{5.1}$$

with appropriate constants $C_{abc}^{i,jk\ell}$. We set

$$w_j = -\sum_{0\le a\le 2,\,1\le k\le N} h_j^{ka}\partial_a u_k \tag{5.2}$$

for $1 \le j \le M$, where the constants h_j^{ka} and M are from (1.16). We define $u^* = (v,\,w)$, where $v = (u,\,\partial u)$ and $w = (w_j)_{1\le j\le M}$. Then u^* satisfies the system

$$\begin{cases} \Box u_i = F_i(\partial u), \\ \Box(\partial_a u_i) = F_{i,a}(\partial v)\big(\equiv \partial_a\big(F_i(\partial u)\big)\big), \\ \Box w_j = G_j(\partial v)\Big(\equiv - \sum_{0\le a\le 2,\, 1\le k\le N} h_j^{ka}\partial_a\big(F_k(\partial u)\big)\Big). \end{cases} \tag{5.3}$$

In the following, we put $r = |x|$, and $\omega_j = x_j/|x|$ for $j = 1, 2$. We also set $\omega_0 = -1$, as before.

We assume that (4.2) with u replaced by $u^* = (v, w)$ holds. When $r < (1+t)/2$, since we have $\langle t+r\rangle \le C\langle t-r\rangle$, (4.1) and (4.2) yield

$$\begin{aligned} |F_i|_s + |F_{i,a}|_s + |G_j|_s &\le C|\partial v|^2_{[s/2]}|\partial v|_s \\ &\le CM^2\varepsilon^2\langle t+r\rangle^{-3+2\gamma\varepsilon^2}|\partial v|_s \end{aligned} \tag{5.4}$$

for $r < (1+t)/2$, if $s \le 2k$.

From now on, we suppose $r \ge (1+t)/2$. Note that we have $\langle t+r\rangle \le Cr$. Set $Z_0 = 0$. Then, using Z_j ($j = 1, 2$) defined in (2.3), we have

$$\partial_a = Z_a - \omega_a\partial_t \quad \text{for } 0 \le a \le 2. \tag{5.5}$$

We set

$$\begin{aligned} H_i(\omega, \partial u) &\equiv F_i(\partial u) - F_i^{\mathrm{red}}(\omega, \partial_t u) \\ &= \sum_{\substack{1\le j\le k\le \ell\le N \\ 0\le a,b,c\le 2}} C_{abc}^{i,jk\ell}\,\Xi_{abc}^{jk\ell}(\omega, \partial u), \end{aligned} \tag{5.6}$$

$$\Xi_{abc}^{jk\ell}(\omega, \partial u) = (\partial_a u_j)(\partial_b u_k)(\partial_c u_\ell) + \omega_a\omega_b\omega_c(\partial_t u_j)(\partial_t u_k)(\partial_t u_\ell). \tag{5.7}$$

By replacing ∂_a, ∂_b and ∂_c in (5.7) with (5.5), and remembering the definition of Z_a ($0 \le a \le 2$), we obtain

$$|\Xi_{abc}^{jkl}(\omega, \partial u)| = \sum_{\substack{1\le j',k',\ell'\le N \\ 0\le a',b',c'\le 2 \\ |\alpha|=|\beta|=1}}{}' \omega_{a'}\omega_{b'}(\partial^\alpha u_{j'})(\partial^\beta u_{k'})(Z_{c'}u_{\ell'}). \tag{5.8}$$

Observing that $[\Gamma_a, Z_j]$ ($0 \le a \le 6$, $j = 1, 2$) can be written as linear combinations of $\omega_b Z_k$, $(\omega_k\omega_\ell/r)\partial_t$ and $\big(\omega_k\omega_\ell(t-r)/r\big)\partial_t$ with $0 \le b \le 2$ and $1 \le k, \ell \le 2$, we obtain

$$|H_i|_s \le C\big(\langle t+r\rangle^{-1}\langle t-r\rangle|u|_{[s/2]+1}|\partial u|_{[s/2]}|\partial u|_s + |\partial u|^2_{[s/2]}|u|_{Z,s}\big) \quad (5.9)$$

in view of (2.5).

We define

$$A_i(\omega,\, w,\, \partial u) = \sum_{1\le j,k\le M} g_i^{jk}(\omega,\, \partial_t u) w_j w_k, \quad (5.10)$$

where g_i^{jk}'s are from (1.18). (5.5) leads to

$$h_j(\omega,\, \partial_t u) - w_j = \sum_{\substack{0\le a\le 2\\ 1\le k\le N}} h_j^{ka} Z_a u_k.$$

Hence, similarly to (5.9), by (1.18) we obtain

$$\begin{aligned}&|F_i^{\mathrm{red}}(\omega,\, \partial_t u) - A_i(\omega,\, w,\, \partial u)|_s \\ &\le C\big(\langle t+r\rangle^{-1}\langle t-r\rangle|u|_{[s/2]+1}|\partial u|_{[s/2]}|\partial u|_s + |\partial u|^2_{[s/2]}|u|_{Z,s}\big). \quad (5.11)\end{aligned}$$

From now on, for $\Phi = \Phi(\omega,\, \partial u^*)$ and $\Psi = \Psi(\omega,\, \partial u^*)$, we write $\Phi \approx \Psi$ if for any nonnegative integer s, there exists a positive constant C_s such that

$$|\Phi - \Psi|_s \le C_s |u^*|_{[s/2]+1}|\partial u^*|_{[s/2]}\Big(\frac{\langle t-r\rangle}{\langle t+r\rangle}|\partial u^*|_s + |u^*|_{Z,s}\Big). \quad (5.12)$$

Thanks to (2.5), if $\Phi \approx \Psi$, we get

$$\begin{aligned}|\Phi - \Psi|_s &\le C_s\langle t+r\rangle^{-1}|u^*|_{[s/2]+1}|\partial u^*|_{[s/2]} \\ &\qquad\times\big(\langle t-r\rangle|\partial u^*|_s + |u^*|_{s+1}\big) \\ &\le C_s\langle t+r\rangle^{-1}|u^*|_{[s/2]+1} \\ &\qquad\times\big(|u^*|_{[s/2]+1}|\partial u^*|_s + |\partial u^*|_{[s/2]}|u^*|_{s+1}\big), \quad (5.13)\end{aligned}$$

where we have used (4.1) to obtain the last inequality.

Since we have

$$\partial_a h_j(\omega,\, \partial_t u) - \partial_a w_j = \sum_{\substack{0\le a\le 2\\ 1\le k\le N}} h_j^{kb}\big((\partial_a\omega_b)\partial_t u_k + Z_b(\partial_a u_k)\big)$$

and $\partial_a\omega_b = {\sum}'_{1\le j,k\le 2}\, \omega_j\omega_k/r$, following similar lines to (5.6)–(5.11), we can also obtain

$$\partial_a\big(F_i(\partial u)\big) \approx \partial_a\big(A_i(\omega,\, w,\, \partial u)\big). \tag{5.14}$$

Writing $P(\omega,\, X) = \sum_{1\le j\le k\le \ell\le N} P^{jk\ell}(\omega) X_j X_k X_\ell$, we define

$$\widetilde{F}_i^{\mathrm{red}}(\omega,\, X,\, Y) = -\sum_{\substack{1\le j\le k\le \ell\le N\\ 0\le a,b,c\le 2}} C^{i,jk\ell}_{abc}\omega_a\omega_b\omega_c[X,\, X,\, Y]_{j,k,\ell},$$

$$\widetilde{P}(\omega,\, X,\, Y) = \sum_{1\le j\le k\le \ell\le N} P^{jk\ell}(\omega)[X,\, X,\, Y]_{j,k,\ell}$$

for $X,\, Y \in \mathbb{R}^N$ and $\omega \in S^1$, where the constants $C^{i,jk\ell}_{abc}$ are from (5.1), and $[X,\, X,\, Y]_{j,k,\ell} = Y_j X_k X_\ell + X_j Y_k X_\ell + X_j X_k Y_\ell$. Since (1.17) implies

$$-\sum_{0\le a,b,c\le 2} C^{i,jk\ell}_{abc}\omega_a\omega_b\omega_c = \beta_i(\omega) P^{jk\ell}(\omega)$$

for any $1 \le i \le N$ and $1 \le j \le k \le \ell \le N$, we find

$$\widetilde{F}_i^{\mathrm{red}}(\omega,\, X,\, Y) = \beta_i(\omega)\widetilde{P}(\omega,\, X,\, Y) \tag{5.15}$$

for any $X,\, Y \in \mathbb{R}^N$ and $\omega \in S^1$.

By (5.5) we have

$$\partial_a\partial_b\varphi = (Z_a Z_b\varphi) - Z_a(\omega_b\partial_t\varphi) - \omega_a Z_b\partial_t\varphi + \omega_a\omega_b\partial_t^2\varphi,$$

which yields

$$\partial_a\big(F_i(\partial u)\big) \approx -\omega_a\widetilde{F}_i^{\mathrm{red}}(\omega,\, \partial_t u,\, \partial_t^2 u)$$

as before. Hence, by (5.15) and (1.19), we obtain

$$\begin{aligned} G_j &= -\sum_{k,a} h_j^{ka}\partial_a\big(F_k(\partial u)\big) \approx \sum_{k,a} h_j^{ka}\omega_a\widetilde{F}_k^{\mathrm{red}}(\omega,\, \partial_t u,\, \partial_t^2 u)\\ &= \sum_{k,a} h_j^{ka}\omega_a\beta_k(\omega)\widetilde{P}(\omega,\, \partial_t u,\, \partial_t^2 u) = h_j\big(\omega,\, \beta(\omega)\big)\widetilde{P}(\omega,\, \partial_t u,\, \partial_t^2 u)\\ &= 0. \end{aligned}$$

Summing up, we have proved

$$\begin{cases} \Box u_i = F_i(\partial u) \approx A_i(\omega,\, w,\, \partial u),\\ \Box(\partial_a u_i) = F_{i,a}(\partial v) \approx \partial_a\big(A_i(\omega,\, w,\, \partial u)\big),\\ \Box w_j = G_j(\partial v) \approx 0 \end{cases} \tag{5.16}$$

for $r \geq (1+t)/2$. Observe that (5.16) has a similar structure to that in (H). The only difference between these structures is dependence on ω, which causes no difficulty. Now, using (5.12) and (5.13) in place of Lemma 3.5, we can follow the proof of Theorem 1.1 to treat the nonlinearity in (5.16) for $r \geq (1+t)/2$, while (5.4) provides a far better estimate than we need for $r < (1+t)/2$. In this way, we obtain (4.3) with u replaced by u^*. This completes the proof.

6. Proof of Theorem 1.3

Suppose that all the assumptions in Theorem 1.3 are fulfilled. First we consider (1.15).

Let Λ be a small neighborhood of $(-1, 0)$ on S^1, and Ω_Λ be given by (3.16). Choosing some positive constants ζ, δ and δ_0 ($\leq \delta$), we give the following assumption on $f = (f_1, f_2, f_3)$ and $g = (g_1, g_2, g_3) \in C_0^\infty(\mathbb{R}^2; \mathbb{R}^3)$:

(i) $f_3 \equiv 0$. $g_3 \geq 0$ on Ω_Λ, and $g_3 \geq 2\zeta$ on $\Omega_\Lambda \cap B_{4\delta}(0)$.

(ii) $f_1 \equiv 0$. g_1 is radially symmetric,

$$\operatorname{supp} g_1 \subset X_{\delta_0} \equiv \{x \in \mathbb{R}^2; \delta \leq |x| \leq \delta + \delta_0 (\leq 2\delta)\},$$

and $\|g_1\|_{L^2(\Theta_0)} > 0$, where

$$\Theta_0 \equiv \left\{x \in \mathbb{R}^2; \delta \leq |x| \leq 2\delta, \frac{x}{|x|} \in \Lambda\right\}.$$

Let $u = (v_1, v_2, w)$ be the global solution to (1.15) with initial data $u = \varepsilon f$ and $\partial_t u = \varepsilon g$ at $t = 0$.

We fix some ζ and δ from now on, while δ_0 ($\leq \delta$) will be chosen later. In the following, C_* indicates a positive constant which may depend on some norms of g_1, while C is a constant independent of g_1 and δ_0.

By the assumption (i) and Corollary 3.6, we have

$$U_0[0, g_3](t, x) \geq 2C_1\zeta(1+t)^{-1/2} \tag{6.1}$$

for $t \geq 3\delta$ and $x \in \Theta_t$, where C_1 is a positive constant depending only on δ, and Θ_t is defined by

$$\Theta_t \equiv \left\{x \in \mathbb{R}^2; t+\delta \leq |x| \leq t+2\delta, \frac{x}{|x|} \in \Lambda\right\} \quad \text{for } t \geq 0. \tag{6.2}$$

Hence, by (4.12) and Lemma 3.2, we obtain

$$w(t, x) \geq 2C_1\zeta\varepsilon(1+t)^{-1/2} - C_*\varepsilon^3(1+t)^{-1/2}$$

$$\geq C_1\zeta\varepsilon(1+t)^{-1/2} \tag{6.3}$$

for $t \geq 3\delta$ and $x \in \Theta_t$, provided that ε is sufficiently small to satisfy $C_*\varepsilon^2 \leq C_1\zeta$.

We can decompose v_1 as

$$v_1(t, x) = U_0[0, \varepsilon g_1](t, x) + U[w^2\partial_1 v_1](t, x). \tag{6.4}$$

By (4.16), we have

$$\begin{aligned} \|(w^2\partial_1 v_1)(t)\|_{2,1+\rho} &\leq C_*\varepsilon^2\|\mathcal{W}_+^{-1}\mathcal{W}_-^{-2\nu}\|_{L^{2(1+\rho)/(1-\rho)}}\|\partial v_1(t)\|_{2,2} \\ &\leq C_*\varepsilon^3\langle t\rangle^{C_*\varepsilon^2-1+(1-\rho)(2+2\rho)^{-1}} \end{aligned} \tag{6.5}$$

for $\rho \in (0, 1)$. Therefore, Lemma 3.3 leads to

$$\begin{aligned} \|U[(w^2\partial_1 v_1)](t)\|_{2,2} &\leq C_*\varepsilon^3\langle t\rangle^{C_*\varepsilon^2+(1+3\rho)(2+2\rho)^{-1}} \\ &\leq C_*\varepsilon^3\langle t\rangle^{3/4}, \end{aligned} \tag{6.6}$$

if ε and ρ are sufficiently small.

On the other hand, Lemma 3.3 also implies

$$\|U_0[0, \varepsilon g_1](t)\|_{L^2} \leq C\varepsilon\langle t\rangle^{2\rho/(1+\rho)}\|g_1\|_{L^{1+\rho}} \leq C\varepsilon\langle t\rangle^{1/4}\|g_1\|_{L^{1+\rho}}, \tag{6.7}$$

$$\|\Omega_{12}^2 U_0[0, \varepsilon g_1](t)\|_{L^2} = 0, \tag{6.8}$$

$$\|U_0[0, \varepsilon g_1](t)\|_{1,2} \leq C_*\varepsilon\langle t\rangle^{2\rho/(1+\rho)} \leq C_*\varepsilon\langle t\rangle^{1/4} \tag{6.9}$$

for small $\rho \in (0, 1/7)$, where we have used the assumption (ii).

We define $D_\pm = \partial_t \pm \partial_r$, and set $V(t, r, \omega) = r^{1/2}v_1(t, r\omega)$ for $(t, r) \in [0, \infty) \times [0, \infty)$ and $\omega = (\omega_1, \omega_2) \in S^1$. We also define

$$\widetilde{E}(t) = \Big(\int_{t+\delta}^{t+2\delta}\int_\Lambda |(D_-V)(t, r, \omega)|^2 dS_\omega dr\Big)^{1/2},$$

where dS_ω is the surface measure on S^1. We have

$$\Box = \partial_t^2 - \partial_r^2 - r^{-1}\partial_r - r^{-2}\Omega_{12}^2,$$

$$\partial_1 = \omega_1\partial_r - \frac{\omega_2}{r}\Omega_{12} = \frac{\omega_1}{2}(D_+ - D_-) - \frac{\omega_2}{r}\Omega_{12}.$$

Therefore we find

$$D_+D_-V = -\frac{\omega_1}{2}w^2D_-V + \frac{r^{1/2}}{2}(P_1 + P_2), \tag{6.10}$$

where

$$P_1 = \frac{v_1 + 4\Omega_{12}^2 v_1}{2r^2}, \ P_2 = w^2\Big(\omega_1 D_+ v_1 - \frac{2\omega_2\Omega_{12}v_1}{r} - \frac{\omega_1 v_1}{2r}\Big). \quad (6.11)$$

By integrating (6.10) multiplied by $D_- V$, we get

$$\frac{d}{dt}\widetilde{E}^2(t) = \int_{t+\delta}^{t+2\delta}\int_\Lambda \big(-\omega_1 w^2 |D_- V|^2 + r^{1/2}(P_1 + P_2)D_- V\big) dS_\omega dr.$$

Since we may assume $\omega_1 \le -1/2$ for $\omega \in \Lambda$, by (6.3) we obtain

$$2\frac{d}{dt}\widetilde{E}(t) \ge \frac{C_1^2\zeta^2\varepsilon^2}{2}(1+t)^{-1}\widetilde{E}(t) - \|P_1(t)\|_{L^2(\Theta_t)} - \|P_2(t)\|_{L^2(\Theta_t)} \quad (6.12)$$

for $t \ge 3\delta$. We also have

$$2\frac{d}{dt}\widetilde{E}(t) \ge -\|P_1(t)\|_{L^2(\Theta_t)} - \|P_2(t)\|_{L^2(\Theta_t)} \quad (6.13)$$

for $t \ge 0$.

Observing that $r \ge C\langle t\rangle$ in Θ_t, from (6.6), (6.7) and (6.8), we obtain

$$\|P_1(t)\|_{L^2(\Theta_t)} \le C\varepsilon\langle t\rangle^{-7/4}\|g_1\|_{L^{1+\rho}} + C_*\varepsilon^3\langle t\rangle^{-5/4}, \quad (6.14)$$

in view of (6.4). Since $D_+ = (t+r)^{-1}(S + \omega_1 L_1 + \omega_2 L_2)$, by (4.16), (6.6) and (6.9) we obtain

$$\|P_2(t)\|_{L^2(\Theta_t)} \le C\langle t\rangle^{-1}\|w(t)\|_{L^\infty}^2\|v_1(t)\|_{1,2} \le C_*\varepsilon^3\langle t\rangle^{-5/4}. \quad (6.15)$$

Now (6.12), (6.14) and (6.15) lead to

$$\frac{d}{dt}\widetilde{E}(t) \ge C_0\varepsilon^2(1+t)^{-1}\widetilde{E}(t) - C\varepsilon(1+t)^{-7/4}\|g_1\|_{L^{1+\rho}} - C_*\varepsilon^3(1+t)^{-5/4} \quad (6.16)$$

for $t \ge 3\delta$ with $C_0 = C_1^2\zeta^2/4$, which yields

$$\begin{aligned}(1+t)^{-C_0\varepsilon^2}\widetilde{E}(t) &\ge \widetilde{E}(3\delta)(1+3\delta)^{-C_0\varepsilon^2} - \frac{4C\varepsilon}{3}\|g_1\|_{L^{1+\rho}} - 4C_*\varepsilon^3 \\ &\ge \frac{\widetilde{E}(3\delta)}{4} - \frac{4C\varepsilon}{3}\|g_1\|_{L^{1+\rho}} - 4C_*\varepsilon^3\end{aligned}$$

for $t \ge 3\delta$, provided that $\delta \le 1$ and $C_0\varepsilon^2 \le 1$.

Similarly, using (6.13) instead of (6.12), we get

$$\widetilde{E}(3\delta) \geq \widetilde{E}(0) - \frac{4C\varepsilon}{3}\|g_1\|_{L^{1+\rho}} - 4C_*\varepsilon^3.$$

Hence we obtain

$$(1+t)^{-C_0\varepsilon^2}\widetilde{E}(t) \geq \frac{\widetilde{E}(0)}{4} - C\varepsilon\|g_1\|_{L^{1+\rho}(\mathbb{R}^2)} - C_*\varepsilon^3 \tag{6.17}$$

for $t \geq 3\delta$ with appropriate positive constants C and C_*.

Since g_1 is radially symmetric and supported on $X_{\delta_0}(\subset \Theta_0)$, we have

$$\|g_1\|_{L^2(\mathbb{R}^2)} = C\|g_1\|_{L^2(\Theta_0)}$$

with some constant C determined only by the size of Λ. Now it follows from the support condition on g_1 and Hölder's inequality that

$$\begin{aligned}\|g_1\|_{L^{1+\rho}(\mathbb{R}^2)} &\leq C\{(\delta+\delta_0)^2 - \delta^2\}^{(1-\rho)/(2+2\rho)}\|g_1\|_{L^2(\mathbb{R}^2)}\\ &\leq C\delta_0^{(1-\rho)/(2+2\rho)}\|g_1\|_{L^2(\Theta_0)}.\end{aligned} \tag{6.18}$$

Since we have $\widetilde{E}(0) = \varepsilon\|g_1\|_{L^2(\Theta_0)} > 0$, we obtain

$$\begin{aligned}\frac{\widetilde{E}(0)}{4} - C\varepsilon\|g_1\|_{L^{1+\rho}(\mathbb{R}^2)} &\geq \Big(\frac{1}{4} - C\delta_0^{(1-\rho)/(2+2\rho)}\Big)\varepsilon\|g_1\|_{L^2(\Theta_0)}\\ &\geq \frac{\varepsilon}{8}\|g_1\|_{L^2(\Theta_0)},\end{aligned} \tag{6.19}$$

provided that δ_0 was chosen to be sufficiently small.

Now, by (6.17) and (6.19), we get

$$\begin{aligned}\widetilde{E}(t) &\geq \Big(\frac{\varepsilon}{8}\|g_1\|_{L^2(\Theta_0)} - C_*\varepsilon^3\Big)(1+t)^{C_0\varepsilon^2}\\ &\geq \frac{\varepsilon}{16}\|g_1\|_{L^2(\Theta_0)}(1+t)^{C_0\varepsilon^2}\end{aligned} \tag{6.20}$$

for $t \geq 3\delta$, provided that ε satisfies $16C_*\varepsilon^2 \leq \|g_1\|_{L^2(\Theta_0)}$.

Switching to the polar coordinates, and then by direct calculations, we have

$$\begin{aligned}\|v_1(t)\|_E^2 &\geq \int_{t+\delta}^{t+2\delta}\int_\Lambda (|\partial_t v_1|^2 + |\nabla v_1|^2)(t, r\omega)rdS_\omega dr\\ &= \frac{1}{2}\widetilde{E}^2(t) + \int_{t+\delta}^{t+2\delta}\int_\Lambda P_3(t, r, \omega)rdS_\omega dr,\end{aligned} \tag{6.21}$$

where

$$P_3 = \frac{(D_+v_1)^2}{2} + \frac{v_1(D_-v_1)}{2r} - \frac{v_1^2}{8r^2} + \frac{(\Omega_{12}v_1)^2}{r^2}.$$

As before, from (4.16), (6.6) and (6.9), we get

$$\begin{aligned}\int_{t+\delta}^{t+2\delta}\int_\Lambda |P_3(t, r, \omega)| r dS_\omega dr &\le C_*\varepsilon^2 \langle t\rangle^{-1/4+C_*\varepsilon^2}\\ &\le C_*\varepsilon^2\langle t\rangle^{-1/8} \qquad (6.22)\end{aligned}$$

for small ε.

Finally, (6.20), (6.21) and (6.22) yield

$$\begin{aligned}\|v_1(t)\|_E^2 &\ge \frac{\varepsilon^2}{512}\|g_1\|^2_{L^2(\Theta_0)}(1+t)^{2C_0\varepsilon^2} - C_*\varepsilon^2(1+t)^{-1/8}\\ &\ge \frac{\varepsilon^2}{1024}\|g_1\|^2_{L^2(\Theta_0)}(1+t)^{2C_0\varepsilon^2} \qquad (6.23)\end{aligned}$$

for large t. This completes the proof for the system (1.15).

We turn our attention to the system (1.14). As we have mentioned, it is equivalent to (1.15) with $f_3 = \partial_1 f_2 - \partial_2 f_1$ and $g_3 = \partial_1 g_2 - \partial_2 g_1$.

Let $f_1 = f_2 \equiv 0$. Then we have $f_3 = 0$. Let g_1 satisfy the assumption (ii), and we choose $\psi \in C_0^\infty(\mathbb{R}^2)$ satisfying $\psi \ge 0$ on Ω_Λ, and $\psi \ge 2\zeta$ on $\Omega_\Lambda \cap B_{4\delta}(0)$, like g_3 in the assumption (i).

Since g_1 and ψ are compactly supported, there exists $R_0 > 0$ such that $\operatorname{supp} g_1 \cup \operatorname{supp} \psi \subset B_{R_0}(0)$. We define

$$\widetilde{\Omega}_\Lambda = \Omega_\Lambda \cap \{(x_1, x_2) \in \mathbb{R}^2; x_1 \ge -(R_0^2 - x_2^2)^{1/2}, |x_2| \le R_0\}.$$

Then we see that $\widetilde{\Omega}_\Lambda$ is a compact set. We choose some nonnegative $C_0^\infty(\mathbb{R}^2)$ function χ satisfying $\chi \equiv 1$ on an open neighborhood of $\widetilde{\Omega}_\Lambda$. Now we define $g_2 \in C_0^\infty(\mathbb{R}^2)$ by

$$g_2(x) = \chi(x)\int_{-\infty}^{x_1}(\psi + \partial_2 g_1)(y, x_2)dy. \qquad (6.24)$$

It is easy to see

$$g_3(x) = \partial_1 g_2(x) - \partial_2 g_1(x) = \psi(x) \qquad (6.25)$$

for $x \in \Omega_\Lambda$. Hence the assumption (i) is fulfilled for this g_3. Now we find that (6.23) with $v_1 = u_1$ is valid for (1.14). This completes the proof. □

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