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Seiberg-Witten theory and the geometric structure $\mathbf{R} \times H^2$

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Abstract. The moduli space of the solutions to the monopole equations over an oriented closed 3-manifold M carrying the geometric structure $\mathbf{R} \times H^2$ is studied. Solving the parallel spinor equation, we obtain an explicit solution to the monopole equations. The moduli space consists of a single point with the Seiberg-Witten invariant ± 1. Further, the (anti-)canonical line bundle $K_M^{\pm 1}$ gives a monopole class of M.

Key words: Seiberg-Witten theory, geometric structure, monopole class, parallel spinor.

1. Introduction

Similar to the four-dimensional Seiberg-Witten theory, the study of solutions to the three-dimensional monopole equations

$$\begin{cases} c(*F_A) = \Phi \otimes \Phi^* - \dfrac{1}{2}|\Phi|^2 \mathrm{Id}_W \\ D_A \Phi = 0 \end{cases}$$

over an oriented closed 3-manifold provides a new invariant of topology, the so-called Seiberg-Witten invariant. A class $\alpha = c_1(L) \in H^2(M; \mathbf{R})$ is called a basic class if the Seiberg-Witten invariant is non-trivial. Furthermore, as a larger class, α is called a monopole class if the monopole equations associated with α have a solution for any metric h on M.

A generalization of Lichnerowicz's theorem holds also in the three-dimensional monopole equations as

$$0 = D_A D_A \Phi = \nabla_A^* \nabla_A \Phi + \frac{1}{4} s_h \Phi + \frac{1}{2} c(*F_A)\Phi$$

which leads to the well-known strong maximum principle that M with a metric of positive scalar curvature does not admit an irreducible solution. Another implication of this formula is the L^2-inequality

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$$4\int_M |F_A|^2 dv_h \leq \int_M s_h^2 dv_h \quad \text{so that} \quad \|\alpha_h\|_{(L^2,h)} \leq \frac{1}{4\pi}\|s_h\|_{(L^2,h)}$$

for the h-harmonic part α_h of the 2-form representing α. In [3] we obtained that if we assume the extremal situation above, namely, a solution satisfying

$$\|\alpha_h\|_{(L^2,h)} = \frac{1}{4\pi}\|s_h\|_{(L^2,h)}, \tag{1.1}$$

then Φ and F_A are parallel and the scalar curvature s_h of h is negative constant so that the 3-manifold (M, h) must carry the geometric structure $\mathbf{R} \times H^2$. In this article, we call (1.1) the monopole extremal condition.

The main aims of this article are to determine the monopole class α satisfying the monopole extremal condition above and to exhibit that under this condition the moduli space of solutions to the monopole equations consists of a single point, cut out transversely so that the Seiberg-Witten invariant is ± 1.

Main Theorem *Let M be an oriented closed 3-manifold carrying the geometric structure $\mathbf{R} \times H^2$ with the (anti-)canonical line bundle $K_M^{\pm 1}$. Here, $K_M^{\pm 1} \to M$ is a complex line bundle naturally induced from the (anti-) canonical line bundle $K_{H^2}^{\pm 1}$ over H^2 by the quotient map: $\mathbf{R} \times H^2 \to M$. Suppose $b_1(M) > 1$. It follows then that (1) the moduli space of solutions to the monopole equations associated with the class $\alpha = c_1(K_M^{\pm 1})$ and the metric h such that $\pi^* h = dt^2 \oplus a^2 g_H$ consists of a single point and is transversal at this point and that (2) α is a monopole class.*

Remark Proposition 5.1 in [4] is similar to the above theorem, although its proof is quite different from ours.

In Section 2, we review the three-dimensional Seiberg-Witten theory with the result of [3] and determine the monopole class $\alpha = c_1(L)$ under the monopole extremal condition as $L = K_M^{\pm 1}$. An explicit form of spinor fields $\Phi \in \Gamma(M; W)$, which are parallel with respect to the canonical metric h is given in Section 3. Making use of these parallel spinor fields which turn out to be solutions to the monopole equations, we examine in Section 4 the moduli space $\mathcal{M}(M; \alpha, h)$ of solutions associated with the metric h stated in Main Theorem (1). We can furthermore exhibit by applying the perturbation argument which is a typical device in the Seiberg-Witten theory that the moduli space $\mathcal{M}(M; \alpha, h')$ of solutions associated with an arbitrary metric h' cut out transversely so that the invariant $SW(M, K_M^{\pm 1}) = \pm 1$ and as a byproduct that $\alpha = c_1(K_M^{\pm 1})$ becomes a monopole class of M. Here, we need the topological restriction $b_1(M) > 1$ for the perturbation trick being valid.

2. The monopole class and the (anti-)canonical line bundle

First, we will outline the three-dimensional Seiberg-Witten theory.

Let M be an oriented closed 3-manifold. Then there exists a $Spin(3)^c$ structure on M defining the principal $Spin(3)^c$-bundle P associated with the orthonormal frame bundle $SO(TM)$. Let W be the spinor bundle associated with P and $L = \det(W)$ be the determinant line bundle of W. The monopole equations are for a unitary connection A on L and a section Φ of W as follows.

$$\begin{cases} c(*F_A) = \Phi \otimes \Phi^* - \dfrac{1}{2}|\Phi|^2 \mathrm{Id}_W \\ D_A \Phi = 0 \end{cases}$$

Here, $c : T^*M \to End(W)$ denotes the Clifford multiplication and $*$ is the Hodge star operation. Further F_A is the curvature form of A and D_A is the Dirac operator twisted with A:

$$D_A : \Gamma(M; W) \xrightarrow{\nabla_A} \Gamma(M; T^*M \otimes W) \xrightarrow{c} \Gamma(M; W),$$

where ∇_A is the spin connection on W.

As is well known, the monopole equations are invariant under the gauge action

$$(A, \Phi) \mapsto (A + g^{-1}dg, g^{-1}\Phi), \quad g \in \mathcal{G} = \Gamma(M; U(1))$$

so that we can define the moduli space of solutions to the monopole equations by the gauge action, namely, $\mathcal{M} = \mathcal{S}/\mathcal{G}$. Here, $\mathcal{S}$ is the set of the solutions. It is known that $\mathcal{M}$ has 0-dimensional compact oriented manifold structure ([1]). $b_1(M) > 0$ guarantees that every solution (A, Φ) is irreducible, that is, $\Phi \neq 0$. We usually define the Seiberg-Witten invariant $SW(M, L)$ as the number of irreducible points, counted with sign in $\mathcal{M}$. Notice that $\mathcal{M}$ has irreducible points, provided $SW(M, L) \neq 0$.

In [3], for an oriented closed 3-manifold M and a monopole class α of M we obtained that if M has a smooth metric h satisfying the monopole extremal condition, then Φ and F_A are parallel and the scalar curvature s_h is negative constant. Moreover, in this article, we will get the explicit form of the monopole class α when the monopole extremal condition is fulfilled. For this, denote by $\pi : \mathbf{R} \times H^2 \to M$ the universal covering projection of M.

Proposition 2.1 *Let α be a monopole class of an oriented closed 3-manifold M. Suppose that there exists a smooth metric h on M which satisfies the monopole extremal condition. Then, the harmonic part α_h is*

$$\alpha_h = \pm \frac{1}{2\pi} d\sigma_H,$$

where $d\sigma_H$ denotes the 2-form on M whose lift $\pi^* d\sigma_H$ is the area form of (H^2, g_H).

Proof. From the argument in [3], the h-harmonic part α_h of the monopole class α is $\alpha_h = \frac{i}{2\pi} F_A$ which is parallel. Then the lift $\pi^* \alpha_h$ is parallel and symplectic over H^2 so that it is proportional to the area form of (H^2, g_H):

$$\alpha_h = \frac{i}{2\pi} F_A = \frac{c}{2\pi} d\sigma_H$$

for some real constant c. To determine c, we take the pull-back metric $\pi^* h$ described as

$$\pi^* h = dt^2 \oplus a^2 g_H,$$

where $a > 0$ and $g_H = (dx^2 + dy^2)/y^2$ is the hyperbolic metric. Regarding H^2 as the upper half plane $\{z = x + iy \mid y > 0\}$, we see that the scalar curvature s_h of h is $-2/a^2$. Since s_h is constant, we obtain

$$\|s_h\|_{(L^2,h)} = \sqrt{\int_M s_h^2 dv_h} = |s_h| \sqrt{Vol(M, h)} = \frac{2}{a^2} \sqrt{Vol(M, h)}.$$

Therefore, we get from the monopole extremal condition

$$\frac{|c|}{2\pi a^2}=\frac{1}{4\pi}\cdot\frac{2}{a^2}$$

so that $c=\pm 1$ and hence $\alpha_h=\pm\frac{1}{2\pi}d\sigma_H$. □

Conversely, if the monopole class $\alpha=c_1(L)$ satisfies $\alpha_h=\pm\frac{1}{2\pi}d\sigma_H$, then, as is easily seen, the monopole extremal condition holds.

Assume that M admits the geometric structure $\mathbf{R}\times H^2$. The (anti-) canonical line bundle $K_{H^2}^{\pm 1}$ over H^2 induce a complex line bundle denoted by $K_M^{\pm 1}$. This is because M is a Γ-quotient of $\mathbf{R}\times H^2$, where Γ is the discrete subgroup of $Isom^+(\mathbf{R}\times H^2)$, and the frame field $\frac{1}{y}dz$ of K_{H^2} or $\frac{1}{y}d\bar{z}$ of $K_{H^2}^{-1}$ is invariant respectively under the action

$$z\mapsto\frac{az+b}{cz+d},\quad a,b,c,d\in\mathbf{R},\quad ad-bc=1$$

so that $K_{H^2}^{\pm 1}$ over $\mathbf{R}\times H^2$ well descends to the bundle $K_M^{\pm 1}$ over M, which we call the (anti-)canonical line bundle over M.

As a corollary of Main Theorem in [3], we can determine the complex line bundle L under the monopole extremal condition.

Corollary 2.2 *Let L be a complex line bundle over an oriented closed 3-manifold M. Assume that the first Chern class of L is a monopole class α of M and satisfies the monopole extremal condition. Then, L must be bundle-isomorphic to $F\otimes K_M^{\pm 1}$, where F is a complex line bundle with a flat connection and $K_M^{\pm 1}$ is the (anti-)canonical line bundle over M.*

Proof. For simplicity, we write $K_M^{\pm 1}=K$. It suffices to show that

$$c_1(L\otimes K^{-1})=c_1(L)-c_1(K)=0\ (=c_1(F)),$$

since over M the multiplicative group $H^1(M;\mathcal{D}^\times)$, the space of all equivalence classes of complex line bundle over M, is isomorphic to $H^2(M;\mathbf{Z})$ via the map assigning a complex line bundle to its first Chern class (see [2]).

For this purpose, let D be the Hermitian holomorphic connection on K_{H^2} induced from the Levi-Civita connection ∇. Its connection form A is easily computed as $A=-\frac{i}{y}dz$ and $F_A=dA=-i(dx\wedge dy)/y^2$ so that $c_1(K_{H^2})$ coincides with $\frac{1}{2\pi}[d\sigma_H]$. This completes the proof. □

3. Parallel spinor solutions to the monopole equations

From now on, we take $L = K_M^{\pm 1}$ and investigate an explicit form of the solutions to the monopole equations. The spinor bundle W is described as $W = W_0 \otimes L_1$, where W_0 is the product bundle $W_0 = M \times \mathbf{C}^2$ and L_1 is some complex line bundle. Taking care that $L = \det(W)$, we obtain $K_M^{\pm 1} = L_1^2$ so that $L_1 = K_M^{\pm 1/2}$. Hence we can take spinor fields

$$\Phi_0 = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \otimes \sqrt{dz}, \quad \Phi_0^- = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \otimes \sqrt{d\bar{z}} \in \Gamma(M; W), \quad W = W_0 \otimes K_M^{\pm 1/2},$$

where dz and $d\bar{z}$ are regarded as sections of $K_M^{\pm 1}$. Under these conditions, we can show the following proposition.

Proposition 3.1 *If $\nabla_{A_0}\Phi_0 = \nabla_{A_0^-}\Phi_0^- = 0$, where A_0 and A_0^- are the connections of $K_M^{\pm 1}$ associated with the Levi-Civita connection of (M, h), then*

$$\Phi_0 = \begin{pmatrix} C/\sqrt{y} \\ 0 \end{pmatrix} \otimes \sqrt{dz}, \quad \Phi_0^- = \begin{pmatrix} 0 \\ C/\sqrt{y} \end{pmatrix} \otimes \sqrt{d\bar{z}}, \quad C = \pm\sqrt{-s_h}.$$

Proof. We consider the case for (A_0, Φ_0) with $L = K_M$. (The case for (A_0^-, Φ_0^-) is similar.) First, we see

$$(\nabla_{A_0})_X \Phi_0 = \left(\nabla_X \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \right) \otimes \sqrt{dz} + \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \otimes \left((\nabla_{A_0})_X \sqrt{dz} \right), \tag{3.1.1}$$

where X is any tangent vector to M. By the definition of the spin connection, the first term is computed as follows.

$$\begin{aligned} \nabla_X \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} &= \begin{pmatrix} X\phi_1 \\ X\phi_2 \end{pmatrix} - \frac{1}{2}\sum_{i<j}^{3} \omega_{ij}(X) c(e_i) c(e_j) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \\ &= \begin{pmatrix} X\phi_1 \\ X\phi_2 \end{pmatrix} - \frac{i}{2}\omega_{23}(X) \begin{pmatrix} \phi_1 \\ -\phi_2 \end{pmatrix}, \end{aligned} \tag{3.1.2}$$

where ω_{ij} are the connection forms of (M, h) with respect to the orthonormal frame $\{dt, \ \frac{1}{ay}dx, \ \frac{1}{ay}dy\}$. Here, the lift of h is $dt^2 \oplus a^2 g_H$. On the other hand, since dz is regarded as a section of K_M, the second term of (3.1.1) is computed for $X = \frac{\partial}{\partial z}$ as

$$(\nabla_{A_0})_{\frac{\partial}{\partial z}} \sqrt{dz} = \left(\frac{\partial}{\partial z} \log y \right) \sqrt{dz}, \tag{3.1.3}$$

where $z = x + iy$. Moreover from the local product structure of M, we see

$$(\nabla_{A_0})_{\frac{\partial}{\partial t}} \sqrt{dz} = 0, \quad (\nabla_{A_0})_{\frac{\partial}{\partial \bar{z}}} \sqrt{dz} = 0.$$

Substituting (3.1.2) and (3.1.3) into (3.1.1), from $(\nabla_{A_0})_X \Phi_0 = 0$ for $X = \frac{\partial}{\partial z}$, we obtain

$$X\phi_1 - \frac{i}{2}\omega_{23}(X)\phi_1 + \left(\frac{\partial}{\partial z} \log y \right) \phi_1 = 0,$$

$$X\phi_2 + \frac{i}{2}\omega_{23}(X)\phi_2 + \left(\frac{\partial}{\partial z} \log y \right) \phi_2 = 0.$$

Using $\omega_{23} = -\frac{1}{y} dx$, we get

$$\frac{\partial \phi_1}{\partial z} - \frac{i}{4y}\phi_1 = 0, \quad \frac{\partial \phi_2}{\partial z} - \frac{3i}{4y}\phi_2 = 0.$$

Similarly, for $X = \frac{\partial}{\partial \bar{z}}$ we get

$$\frac{\partial \phi_1}{\partial \bar{z}} + \frac{i}{4y}\phi_1 = 0, \quad \frac{\partial \phi_2}{\partial \bar{z}} - \frac{i}{4y}\phi_2 = 0.$$

Solving the simultaneous equations for ϕ_1 and ϕ_2, we get $\phi_1 = \frac{C}{\sqrt{y}}$ and $\phi_2 = 0$.

Now we obtain

$$\Phi_0 = \begin{pmatrix} C/\sqrt{y} \\ 0 \end{pmatrix} \otimes \sqrt{dz}.$$

On the other hand, using $\nabla_{A_0}\Phi_0 = 0$ and Lichnerowicz's formula, we get

$$s_h = -|\Phi_0|^2 \quad \text{so that} \quad C = \pm\sqrt{-s_h}. \qquad \square$$

Moreover, we can show the following corollary.

Corollary 3.2 *For the monopole class α whose h-harmonic part is*

$$\alpha_h = \frac{i}{2\pi}F_{A_0} = \frac{1}{2\pi}d\sigma_H \quad \text{or} \quad \alpha_h = \frac{i}{2\pi}F_{A_0^-} = -\frac{1}{2\pi}d\sigma_H,$$

(A_0, Φ_0) or (A_0^-, Φ_0^-) with $|\Phi_0| = |\Phi_0^-| = \sqrt{2}$ is a solution to the monopole equations for $L = K_M$ or $L = K_M^{-1}$, respectively.

Proof. We consider the case for (A_0, Φ_0) with $L = K_M$. (The case for (A_0^-, Φ_0^-) is similar.) Since $D_A = c \circ \nabla_A$, it is clear that $D_{A_0}\Phi_0 = 0$. On the other hand, the curvature form of A_0 is described as

$$F_{A_0} = \pm i d\sigma_H = \pm i e^2 \wedge e^3,$$

where e^1, e^2, e^3 are the dual orthonormal frame of (M, h). Therefore we obtain

$$c(*F_{A_0}) = \pm\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In general for $\Phi = \binom{\varphi_1}{\varphi_2}$, we get

$$\Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \mathrm{Id}_W = \begin{pmatrix} \frac{1}{2}(|\varphi_1|^2 - |\varphi_2|^2) & \varphi_1\overline{\varphi_2} \\ \overline{\varphi_1}\varphi_2 & \frac{1}{2}(|\varphi_2|^2 - |\varphi_1|^2) \end{pmatrix}.$$

Therefore for $\Phi_0 = \binom{\varphi}{0}$, we obtain

$$\begin{aligned} \Phi_0 \otimes \Phi_0^* - \frac{1}{2}|\Phi_0|^2 \mathrm{Id}_W &= \begin{pmatrix} \frac{1}{2}|\varphi|^2 & 0 \\ 0 & -\frac{1}{2}|\varphi|^2 \end{pmatrix} \\ &= -\frac{1}{2}|\varphi|^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{2}|\Phi_0|^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

Hence in order for (A_0, Φ_0) to satisfy the first monopole equation, we may take

$$F_{A_0} = -id\sigma_H, \quad |\Phi_0| = \sqrt{2}. \qquad \square$$

4. The moduli space

Proof of Main Theorem (1). We consider the case for $\alpha = c_1(K_M)$. (The case for $\alpha = c_1(K_M^{-1})$ is similar.) In this case, $\alpha_h = \frac{1}{2\pi} d\sigma_H$ so that by the proof of Proposition 2.1, we obtain the monopole extremal condition

$$\|\alpha_h\|_{(L^2,h)} = \frac{1}{4\pi}\|s_h\|_{(L^2,h)}$$

for the metric h whose lift $\pi^* h$ has the form $dt^2 \oplus a^2 g_H$. Let (A, Φ) be an arbitrary solution associated with the class α and the metric h. In this case, recall that $\nabla_A \Phi = 0$ holds. Now we take (A_0, Φ_0) which satisfies

$$\alpha = \frac{1}{2\pi} d\sigma_H = \frac{i}{2\pi} F_{A_0} \quad \text{and} \quad \Phi_0 = \begin{pmatrix} \sqrt{2/y} \\ 0 \end{pmatrix} \otimes \sqrt{dz}.$$

From Corollary 3.2, (A_0, Φ_0) is a solution. We can show that any solution (A, Φ) is gauge equivalent to (A_0, Φ_0).

For this, we take $A = A_0 + ia$, $a \in \Omega^1(M)$ so that $F_A = F_{A_0} + ida$. Since F_A and F_{A_0} are harmonic, we obtain $da = 0$ and $F_A = F_{A_0}$. Moreover, by the first monopole equation, we get $|\Phi_0|^2 = 2|F_{A_0}| = 2|F_A| = |\Phi|^2$ and

$$\Phi_0 \otimes \Phi_0^* - \frac{1}{2}|\Phi_0|^2 \mathrm{Id}_W = c(*F_{A_0}) = c(*F_A) = \Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \mathrm{Id}_W$$

so that $\Phi_0 \otimes \Phi_0^* = \Phi \otimes \Phi^*$. Taking $\Phi_0 = \binom{\varphi}{0}$ and $\Phi = \binom{\varphi_1}{\varphi_2}$, we get $|\varphi|^2 = |\varphi_1|^2$ and $|\varphi_2|^2 = 0$ so that there exists $g \in \mathcal{G}$ such that $\Phi = g^{-1}\Phi_0$. Therefore

$$\begin{aligned} 0 = \nabla_A \Phi &= dg^{-1} \otimes \Phi_0 + g^{-1} \nabla_A \Phi_0 \\ &= dg^{-1} \otimes \Phi_0 + g^{-1} \nabla_{A_0} \Phi_0 + g^{-1} ia \otimes \Phi_0 \\ &= dg^{-1} \otimes \Phi_0 + g^{-1} ia \otimes \Phi_0 \end{aligned}$$

and hence

$$ia \otimes \Phi_0 = -g dg^{-1} \otimes \Phi_0 = g^{-1} dg \otimes \Phi_0, \quad \text{namely,} \quad (A, \Phi) = (A_0 + g^{-1} dg, g^{-1}\Phi_0),$$

which implies that (A, Φ) is gauge equivalent to (A_0, Φ_0).

From now on, we will show the transversality of the moduli space $\mathcal{M}$. To show this, we consider the following complex which turns out to be elliptic by the subsequent lemma, Lemma 4.1.

$$\mathcal{C} \ : \ 0 \to \Omega^0(M) \xrightarrow{G} \Omega^1(M) \oplus \Gamma(W) \xrightarrow{T} \Omega^1(M) \oplus \Gamma(W) \xrightarrow{S} \Omega^0(M) \to 0,$$

where

$$G_{(A_0,\Phi_0)}(u) = (du, -iu\Phi_0), \quad S_{(A_0,\Phi_0)}(a, \varphi) = \delta a + i\mathrm{Im}\langle \Phi_0, \varphi \rangle$$

and $T_{(A_0,\Phi_0)}(a, \varphi) = (b, \psi)$, where

$$b = c(i * da) - \Phi_0 \otimes \varphi^* - \varphi \otimes \Phi_0^* + \frac{1}{2}(\langle \Phi_0, \varphi \rangle + \langle \varphi, \Phi_0 \rangle)\mathrm{Id}_W,$$

$$\psi = D_{A_0}\varphi + ic(a)\Phi_0.$$

Lemma 4.1 (1) $T \circ G = 0$, (2) $S \circ T = 0$, (3) *Index*$(\mathcal{C}) = 0$.

Proof of (1). By definition,

$$(b, \psi) = T \circ G_{(A_0,\Phi_0)}(u) = T_{(A_0,\Phi_0)}(du, -iu\Phi_0),$$

where

$$b = c(i * d(du)) - iu|\Phi_0|^2 + iu|\Phi_0|^2 + \frac{1}{2}(iu|\Phi_0|^2 - iu|\Phi_0|^2)\mathrm{Id}_W,$$

$$\psi = D_{A_0}(-iu\Phi_0) + ic(du)\Phi_0 = -i(c(du)\Phi_0 + uD_{A_0}\Phi_0) + ic(du)\Phi_0.$$

Obviously b and ψ vanish. □

Proof of (2). Let $S \circ T_{(A_0,\Phi_0)}(a, \varphi) = S_{(A_0,\Phi_0)}(b, \psi)$. It is sufficient to show

$$\int_M \langle S(b, \psi), u \rangle dv_h = 0 \quad \text{for any} \quad u \in \Omega^0(M).$$

By definition,

$$b = *da + ic^{-1}\left(\Phi_0 \otimes \varphi^* + \varphi \otimes \Phi_0^* - \frac{1}{2}(\langle \Phi_0, \varphi \rangle + \langle \varphi, \Phi_0 \rangle)\mathrm{Id}_W\right),$$

$$\psi = D_{A_0}\varphi + ic(a)\Phi_0$$

so that

$$\begin{aligned} S(b,\psi) &= \delta\Big(*\,da + ic^{-1}\Big(\Phi_0\otimes\varphi^* + \varphi\otimes\Phi_0^* - \frac{1}{2}(\langle\Phi_0,\varphi\rangle + \langle\varphi,\Phi_0\rangle)\mathrm{Id}_W\Big)\Big) \\ &\quad + i\mathrm{Im}\langle\Phi_0, D_{A_0}\varphi + ic(a)\Phi_0\rangle \\ &= i\delta\Big(c^{-1}\Big(\Phi_0\otimes\varphi^* + \varphi\otimes\Phi_0^* - \frac{1}{2}(\langle\Phi_0,\varphi\rangle + \langle\varphi,\Phi_0\rangle)\mathrm{Id}_W\Big)\Big) \\ &\quad + i\mathrm{Im}\langle\Phi_0, D_{A_0}\varphi\rangle. \end{aligned}$$

Therefore

$$\begin{aligned} &\int_M \langle S(b,\psi), u\rangle dv_h \\ &\quad = \int_M \Big\langle i\delta\Big(c^{-1}\Big(\Phi_0\otimes\varphi^* + \varphi\otimes\Phi_0^* - \frac{1}{2}(\langle\Phi_0,\varphi\rangle + \langle\varphi,\Phi_0\rangle)\mathrm{Id}_W\Big)\Big) \\ &\qquad\qquad + i\mathrm{Im}\langle\Phi_0, D_{A_0}\varphi\rangle, u\Big\rangle dv_h \\ &\quad = i\int_M \big\langle\Phi_0\otimes\varphi^* + \varphi\otimes\Phi_0^*, c(du)\big\rangle dv_h \\ &\qquad + i\int_M \frac{1}{2}u(\langle\Phi_0, D_{A_0}\varphi\rangle - \langle D_{A_0}\varphi,\Phi_0\rangle)dv_h \\ &\quad = i\int_M \big\langle\Phi_0\otimes\varphi^* + \varphi\otimes\Phi_0^*, c(du)\big\rangle dv_h \\ &\qquad + \frac{i}{2}\int_M \big(\langle c(du)\Phi_0,\varphi\rangle - \langle\varphi, c(du)\Phi_0\rangle\big)dv_h \\ &\quad = i\int_M \frac{1}{2}\mathrm{tr}\begin{pmatrix}\varphi_{01}\overline{\varphi_1} + \varphi_1\overline{\varphi_{01}} & \varphi_{01}\overline{\varphi_2} \\ \varphi_2\overline{\varphi_{01}} & 0\end{pmatrix}\begin{pmatrix}-ia_1 & -a_2 - ia_3 \\ a_2 - ia_3 & ia_1\end{pmatrix} dv_h \\ &\qquad + \frac{i}{2}\int_M \Bigg(\Bigg\langle \begin{pmatrix} ia_1\varphi_{01} \\ -a_2\varphi_{01} + ia_3\varphi_{01}\end{pmatrix}, \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}\Bigg\rangle \\ &\qquad - \Bigg\langle \begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}, \begin{pmatrix} ia_1\varphi_{01} \\ -a_2\varphi_{01} + ia_3\varphi_{01}\end{pmatrix}\Bigg\rangle\Bigg)dv_h = 0. \end{aligned}$$

□

Proof of (3). Split $\mathcal{C}$ into the direct sum of the following complexes;

$$\mathcal{C}_1 \; : \; 0 \to \Omega^0(M) \overset{d}{\to} \Omega^1(M) \overset{*d}{\to} \Omega^1(M) \overset{\delta}{\to} \Omega^0(M) \to 0$$

$$\mathcal{C}_2 \; : \; 0 \to \Gamma(W) \overset{D_{A_0}}{\to} \Gamma(W) \to 0.$$

The first one is equivalent to the de Rham complex so that $\mathrm{Index}(\mathcal{C}_1) = \chi(M) = 0$. The second one is the spin complex and so $\mathrm{Index}(\mathcal{C}_2) = \mathrm{Index}D_{A_0} = 0$. Therefore

$$\mathrm{Index}(\mathcal{C}) = \mathrm{Index}(\mathcal{C}_1) + \mathrm{Index}(\mathcal{C}_2) = 0. \qquad \square$$

Using Lemma 4.1 (3), by definition,

$$\mathrm{Index}(\mathcal{C}) = \dim H^0(\mathcal{C}) - \dim H^1(\mathcal{C}) + \dim H^2(\mathcal{C}) - \dim H^3(\mathcal{C}) = 0.$$

Since the solution is irreducible, if $u \in \Omega^0(M)$ satisfies

$$G_{(A_0,\Phi_0)}(u) = (du, -iu\Phi_0) = (0,0),$$

then $u = 0$ so that $H^0(\mathcal{C}) = \mathrm{Ker}G = \{0\}$. Moreover since $S^*_{(A_0,\Phi_0)}(u) = (du, iu\Phi_0)$, we have $S^*_{(A_0,\Phi_0)} = G_{(A_0,-\Phi_0)}$ so that $\mathrm{Ker}S^* = \{0\}$. Therefore $H^3(\mathcal{C}) = \Omega^0(M)/\mathrm{Im}S$ is isomorphic to $\mathrm{Ker}S^* = \{0\}$ and hence $H^3(\mathcal{C}) = \{0\}$. Consequently, $H^1(\mathcal{C}) \cong H^2(\mathcal{C})$. Therefore the surjectivity of T is equivalent to $\mathrm{Ker}S/\mathrm{Im}T = \{0\}$ which is equivalent to $\mathrm{Ker}T/\mathrm{Im}G = \{0\}$. This is also equivalent to

$$\left\{(a,\varphi) \mid T_{(A_0,\Phi_0)}(a,\varphi) = (0,0),\ G^*_{(A_0,\Phi_0)}(a,\varphi) = 0\right\} = \{(0,0)\},$$

where

$$G^*_{(A_0,\Phi_0)}(a,\varphi) = \delta a - i\mathrm{Im}\langle \Phi_0, \varphi\rangle.$$

It is clear that $T_{(A_0,\Phi_0)}(a,\varphi) = 0$ implies

$$c(da) = -i\left(\Phi_0 \otimes \varphi^* + \varphi \otimes \Phi_0^* - \frac{1}{2}(\langle \Phi_0, \varphi\rangle + \langle \varphi, \Phi_0\rangle)\mathrm{Id}_W\right),$$

$$D_{A_0}\varphi = -ic(a)\Phi_0$$

and that $G^*_{(A_0,\Phi_0)}(a,\varphi) = 0$ implies

$$\delta a = i\mathrm{Im}\langle \Phi_0, \varphi\rangle = \frac{i}{2}(\langle \Phi_0, \varphi\rangle - \langle \varphi, \Phi_0\rangle).$$

By the direct computation together with the fact that $\nabla_{A_0}\Phi_0 = D_{A_0}\Phi_0 = 0$, we get

$$\begin{aligned} D_{A_0}D_{A_0}\varphi &= -iD_{A_0}(c(a)\Phi_0) \\ &= -i\big((\delta a)\Phi_0 - 2(\nabla_{A_0})_{a^\sharp}\Phi_0 + c(da)\Phi_0 - c(a)D_{A_0}\Phi_0\big) \\ &= \frac{1}{2}(\langle \Phi_0, \varphi\rangle - \langle \varphi, \Phi_0\rangle)\Phi_0 - ic(da)\Phi_0. \end{aligned}$$

Here we made use of the formula:

$$D_A(c(a)\Phi) = (\delta a)\Phi - 2(\nabla_A)_X\Phi + c(da)\Phi,$$

$a \in \Omega^1(M)$ and $X = a^\sharp \in \mathcal{X}(M)$. Now we have

$$\begin{aligned} c(da)\Phi_0 &= -i\left(\Phi_0 \otimes \varphi^* + \varphi \otimes \Phi_0^* - \frac{1}{2}(\langle \Phi_0, \varphi\rangle + \langle \varphi, \Phi_0\rangle)\mathrm{Id}_W\right)\Phi_0 \\ &= -i\left(\langle \Phi_0, \varphi\rangle\Phi_0 + |\Phi_0|^2\varphi - \frac{1}{2}(\langle \Phi_0, \varphi\rangle + \langle \varphi, \Phi_0\rangle)\Phi_0\right) \\ &= -i\left(|\Phi_0|^2\varphi + \frac{1}{2}(\langle \Phi_0, \varphi\rangle - \langle \varphi, \Phi_0\rangle)\Phi_0\right), \end{aligned}$$

so that the term $D_{A_0}D_{A_0}\varphi$ becomes

$$\begin{aligned} D_{A_0}D_{A_0}\varphi &= \frac{1}{2}(\langle \Phi_0, \varphi\rangle - \langle \varphi, \Phi_0\rangle)\Phi_0 - ic(da)\Phi_0 \\ &= \frac{1}{2}(\langle \Phi_0, \varphi\rangle - \langle \varphi, \Phi_0\rangle)\Phi_0 - |\Phi_0|^2\varphi - \frac{1}{2}(\langle \Phi_0, \varphi\rangle - \langle \varphi, \Phi_0\rangle)\Phi_0 \\ &= -|\Phi_0|^2\varphi. \end{aligned}$$

Therefore

$$\int_M \langle D_{A_0} D_{A_0} \varphi, \varphi \rangle dv_h = -\int_M |\Phi_0|^2 |\varphi|^2 dv_h \quad \text{or}$$
$$\int_M |D_{A_0} \varphi|^2 dv_h = -\int_M |\Phi_0|^2 |\varphi|^2 dv_h.$$

$|\Phi_0|$ is positive constant because the solution (A_0, Φ_0) is irreducible and Φ_0 is parallel. Hence we conclude $\varphi = 0$ so that $a = 0$ by $-ic(a)\Phi_0 = D_{A_0}\varphi = 0$. From the above arguments, the transversality of $\mathcal{M}$ is completely derived.

□

Proof of Main Theorem (2). In order to see that the class $\alpha = c_1(K_M)$ is a monopole class, we show that the Seiberg-Witten invariant does not vanish with respect to an arbitrary metric on M. We consider the case where a given metric h is arbitrary. In this case, we cannot always make use of the condition $\nabla_A \Phi = 0$. We usually think of the perturbed monopole equations as follows.

$$\begin{cases} c(*F_A + i\rho) = \Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \mathrm{Id}_W \\ D_A \Phi = 0 \end{cases}$$

Here, ρ is a co-closed 1-form. With respect to these perturbed equations, it is known that the Seiberg-Witten invariant is independent of metrics g and perturbations ρ ([1]). More precisely, given a generic path (g_t, ρ_t), $t \in [0,1]$ connecting (g_0, ρ_0) and (g_1, ρ_1), it is known that

$$SW_{(g_0,\rho_0)}(M, L) = SW_{(g_1,\rho_1)}(M, L).$$

To apply the perturbed argument to our case, we take $L = K_M$ and $(g_0, \rho_0) = (h, 0)$. Main Theorem (1) together with the definition of the Seiberg-Witten invariant implies $SW_{(h,0)}(M, K_M) = \pm 1$ so that

$$SW(M, K_M) = \pm 1 \ (\neq 0).$$

This implies that the monopole equations associated with $\alpha = c_1(K_M)$ has solutions which are irreducible by $b_1(M) > 1$. Hence α is a monopole class.

□

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