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Pseudo-differential operators of class $S^m_{0,0}$ on the Herz-type spaces

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Abstract. In this paper, we show the boundedness of pseudo-differential operators of class $S^m_{0,0}$ on the Herz spaces $\dot{K}^{\alpha,p}_q$ and the Herz-type Hardy spaces $H\dot{K}^{\alpha,p}_q$.

Key words: Pseudo-differential operators, Herz spaces, Herz-type Hardy spaces.

1. Introduction

Beurling [3] introduced the space $A^q(\mathbb{R}^n) = K_q^{n(1-1/q),1}(\mathbb{R}^n)$ with $1 < q < \infty$ to study convolution algebras, which are now called Beurling algebras as a special class of the Herz spaces $K^{\alpha,p}_q$, (See Definition 2.1). The Herz spaces can be regarded as one of extensions of $L^p(\mathbb{R}^n)$ and the theory of the Herz space is developed by Feichtinger [9], Herz [15] and Flett [10].

On the other hand, Calderón and Vaillancourt [4] showed that pseudo-differential operators of class $S^0_{0,0}$ are bounded on $L^2(\mathbb{R}^n)$. And Miyachi [22] showed that those of class $S_{0,0}^{-n|1/p-1/2|}$ are bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $(0 < p < \infty)$.

The aim of this paper is to study the boundedness of pseudo-differential operators of class $S^m_{0,0}$ on the Herz spaces and the Herz-type Hardy spaces. Lu, Yabuta and Yang [21] showed that the operators having a kernel estimate are bounded from the Herz-type Hardy spaces to the Herz spaces:

Theorem A *Let $T : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ be a linear and continuous operator. Suppose that the distribution kernel of T coincides in the complement of the diagonal with a locally integrable function $k(x, y)$ satisfying*

$$|k(x,y) - k(x,0)| \le c\frac{|y|^\delta}{|x|^{n+\delta}}$$

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when $2|y| < |x|$ *for some* $\delta \in (0, 1]$. *Let* $0 < p < \infty$, $1 < q < \infty$ *and* $n(1-1/q) \le \alpha < n(1-1/q) + \delta$. *If* T *is bounded on* $L^q(\mathbb{R}^n)$, *then* T *is also bounded from* $H\dot{K}_q^{\alpha,p}$ *into* $\dot{K}_q^{\alpha,p}$.

If $m < -n-1$, pseudo-differential operators of class $S^m_{0,0}$, which is our target class, satisfies the above condition on kernels. But it is not clear that those of class $S^m_{0,0}$ with $m \ge -n-1$ satisfies the above condition or not. The main result in this paper is the following:

Theorem 3.1 *Let* $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \le \alpha < \infty$ *and* $m < -\alpha - n|1/q - 1/2|$. *Suppose that*
(i) $1 < q \le 2$, $l > n/2$, $l' > [-m] + n/2 + 1$, *or*
(ii) $2 \le q < \infty$, $l > n/q$, $l' > [-m - n(1/2 - 1/q)] + n/2 + 1$.
Then $S^m_{0,0}(l, l') \subset \mathcal{L}\ (H\dot{K}_q^{\alpha,p}, \dot{K}_q^{\alpha,p})$.

In Theorem 3.1, if q is close to 1 or 2 then we can take m close to $-n/2$. Hence we cannot use Theorem A directly in this case.

We explain more about the Herz spaces and the mapping properties of pseudo-differential operators on $L^p(\mathbb{R}^n)$ and $H^p(\mathbb{R}^n)$. Feichtinger [9] gave the different norms of Beurling algebras, which is equivalent to that in Beurling [3]. And the spaces $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$ were introduced by Herz [15]. Flett [10] gave another equivalent norms on $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $K_q^{\alpha,p}(\mathbb{R}^n)$. These spaces are useful in the analysis of mapping properties of important operators. For example, Baernstein II and Sawyer [1] showed some multiplier theorems on $H^p(\mathbb{R}^n)$ by using a norm of the Herz space as the condition. Many authors studied the boundedness of pseudo-differential operators on the Herz-type Hardy space. For example, Fan and Yang [8] studied pseudo-differential operators of class $S^0_{1,0}$ on the local Herz-type Hardy spaces $h\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$. Also there are many papers which studied several other operators on the Herz spaces and the Herz-type Hardy spaces, [16], [18], [19], [20] etc... It is well-known that Calderón and Vaillancourt [4] showed that pseudo-differential operator of class $S^0_{0,0}$ is bounded on $L^2(\mathbb{R}^n)$. Futhermore, Miyachi [22] showed that pseudo-differential operator of class $S^{-n|1/p-1/2|}_{0,0}$ is bounded from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, $(0 < p < \infty)$, by using the atomic decomposition and the analytic interpolation theory. Here we remark that the index $-n|1/p - 1/2|$ is optimal. In the proof of Theorem 3.1, we will follow the argument in [22]. However, to the best of my knowledge, there seems no literature mentioning the result of Theorem 3.1.

We also explain the theory of interpolation for families of quasi-Banach spaces. Coifman, Cwikel, Rochberg, Sagher and Weiss [6], [5] discussed the theory of interpolation for families of Banach spaces. This theory is a natural extension of the interpolation for the pair. Hernández [12] and Tabacco Vignati [25], [26] developed the theory for families of quasi-Banach spaces. In [14], Hernández and Yang characterized the intermediate spaces for families of the Herz-type Hardy spaces by using the atomic decomposition ($1 < q < \infty$) established by Lu and Yang [18].

Finally we explain the structure of this paper. In Section 2, we define the Herz spaces, the Herz-type Hardy spaces and Hörmander's symbol classes and recall tools which will be used in this paper. We use Lipschitz classes on product spaces introduced by Miyachi [22] to describe smoothness of symbols, as a substitute for Hörmander's symbol classes. In Section 3, we will prove the main theorem (Theorem 3.1) by using the tools in Section 2. Also, we state a result in the non-homogeneous case. In Section 4, first we will use the characterization of intermediate spaces for couples of the Herz-type Hardy spaces in [13] and the duality argument to show the boundedness of pseudo-differential operators on wider spaces.

2. Definitions and Tools

For $k \in \mathbb{Z}$, let $B_k = \{x \in \mathbb{R}^n \mid |x| \leq 2^k\}$, $C_k = B_k \backslash B_{k-1}$. We denote the characteristic function of E, measurable subset of $\mathbb{R}^n$, by χ_E and C_k by χ_k. We recall the definitions of the Herz spaces and the Herz-type Hardy spaces.

Definition 2.1 (Herz space) Let $0 < p$, $q \leq \infty$ and $\alpha \in \mathbb{R}$. We set

(i) $\dot{K}^{\alpha,p}_q(\mathbb{R}^n) = \{f \in L^q_{loc}(\mathbb{R}^n \setminus \{0\}) \mid \|f\|_{\dot{K}^{\alpha,p}_q} < \infty\}$: homogeneous Herz space, where

$$\|f\|_{\dot{K}^{\alpha,p}_q} = \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|f\chi_k\|^p_{L^q}\right)^{1/p}, \quad \text{and}$$

(ii) $K^{\alpha,p}_q(\mathbb{R}^n) = \{f \in L^q_{loc}(\mathbb{R}^n) \mid \|f\|_{K_q{}^{\alpha,p}} < \infty\}$: non-homogeneous Herz space, where

$$\|f\|_{K^{\alpha,p}_q} = \left(\|f\chi_{B(0,1)}\|^p_{L^q} + \sum_{k\in\mathbb{N}} 2^{k\alpha p}\|f\chi_k\|^p_{L^q}\right)^{1/p}.$$

The usual modifications in the definitions above are made when $p = \infty$.

We take a function $\varphi \in \mathscr{S}$ such that $\int \varphi \, dx = 1$ and set $\varphi_+^*(f)(x) = \sup_{t>0} |f * \varphi_t(x)|$, where $\varphi_t(x) = 1/t^n \varphi(x/t)$.

Definition 2.2 (Herz-type Hardy space) Let $0 < p, q \le \infty$ and $\alpha \in \mathbb{R}$. We set

(i) $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathscr{S}' \mid \varphi_+^*(f) \in \dot{K}_q^{\alpha,p}\}$: homogeneous Herz-type Hardy space,

$$\|f\|_{H\dot{K}_q^{\alpha,p}} = \|\varphi_+^*(f)\|_{\dot{K}_q^{\alpha,p}}, \quad \text{and}$$

(ii) $HK_q^{\alpha,p}(\mathbb{R}^n) = \{f \in \mathscr{S}' \mid \varphi_+^*(f) \in K_q^{\alpha,p}\}$: non-homogeneous Herz-type Hardy space,

$$\|f\|_{HK_q^{\alpha,p}} = \|\varphi_+^*(f)\|_{K_q^{\alpha,p}}.$$

The following basic results are well known [13], [17]: $\dot{K}_p^{0,p} = K_p^{0,p} = L^p$, if $0 < p \le \infty$, $H\dot{K}_p^{0,p} = HK_p^{0,p} = H^p$, if $0 < p < \infty$. The spaces $\dot{K}_q^{\alpha,p}$ and $K_q^{\alpha,p}$ are quasi-Banach spaces, and if $p, q \ge 1$ then $\dot{K}_q^{\alpha,p}$ and $K_q^{\alpha,p}$ are Banach spaces. The same is true for $H\dot{K}_q^{\alpha,p}$ and $HK_q^{\alpha,p}$. $H\dot{K}_q^{\alpha,p}$ and $HK_q^{\alpha,p}$ are defined independently of the choice of φ. When $1 \le p$, $q < \infty$ and $\alpha \in \mathbb{R}$, then $(\dot{K}_q^{\alpha,p})^* = \dot{K}_{q'}^{-\alpha,p'}$ where $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. In particular, $\|f\|_{\dot{K}_q^{\alpha,p}} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)dx\right| \mid g \in \dot{K}_{q'}^{-\alpha,p'} \text{ with } \|g\|_{\dot{K}_{q'}^{-\alpha,p'}} \le 1\right\}$ if $1 \le p$, $q < \infty$ and $-n/q < \alpha < n(1-1/q)$. Also $\dot{K}_q^{\alpha,p} = H\dot{K}_q^{\alpha,p}$, if $0 < p < \infty$, $1 < q < \infty$ and $-n/q < \alpha < n(1-1/q)$.

For an integer k, $\mathcal{P}_k$ denotes the set of all polynomial functions on $\mathbb{R}^n$ of degree not exceeding k. If k is a negative integer, we set $\mathcal{P}_k = 0$. We say $f \perp \mathcal{P}_k$ for $f \in L^1_{\mathrm{loc}}$, when $fP \in L^1$ and $\int f(x)P(x)dx = 0$ for all $P \in \mathcal{P}_k$. Let $[m]$ denote the integer part of real nunber m.

Proposition B ([22]) *Let $0 < p$, $q < \infty$ and $-n/q < \alpha < \infty$. Then the following are dense subspaces of $H\dot{K}_q^{\alpha,p}$:*

(i) $X_k = \{f \in C_0^\infty \mid f \perp \mathcal{P}_k\}$ *with* $k \ge [\alpha - (1 - 1/q)]$,
(ii) $\mathscr{S}_0 = \{f \in \mathscr{S} \mid \operatorname{supp} \hat{f}$ *is a compact subset of* $\mathbb{R}^n \setminus \{0\}\}$.

Next, we recall the definition of Hörmander's symbol classes. For $\xi \in$

$\mathbb{R}^n$, we set $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$.

Definition 2.3 Let $m \in \mathbb{R}$ and $0 \le \delta \le \varrho \le 1$. We set $S^m_{\varrho,\delta} = \{p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \mid |\partial^\alpha_\xi \partial^\beta_x p(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^{m-\varrho|\alpha|+\delta|\beta|}$, for all multi-indexes α and $\beta\}$.

For any $L \in \mathbb{N}\cup\{0\}$ and $p \in S^m_{\varrho,\delta}$, let $|p|^m_L = \max_{|\alpha+\beta|\le L}\sup_{x,\xi\in\mathbb{R}^n} \cdot|\partial^\alpha_\xi\partial^\beta_x p(x,\xi)|\langle\xi\rangle^{-m+\varrho|\alpha|-\delta|\beta|}$. For $p \in S^m_{\varrho,\delta}$ we define pseudo-differential operator $p(X,D)$ whose symbol is p:

$$p(X,D)f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}p(x,\xi)\hat{f}(\xi)d\xi,$$

for any $f \in \mathscr{S}$, where $\hat{f}$ denotes the Fourier transformation of f.

For $m \in \mathbb{R}$ and $L \in \mathbb{N}\cup\{0\}$, we set

$$S^m_{0,0}(L) = \left\{p \in C^L(\mathbb{R}^n\times\mathbb{R}^n) \mid |\partial^\alpha_\xi\partial^\beta_x p(x,\xi)| \le C_{\alpha,\beta}\langle\xi\rangle^m, \text{ for } |\alpha+\beta| \le L, \right\}.$$

It is trivial that $(S^m_{0,0}(L), |\cdot|^m_L)$ is a Banach space. But we adopt the next Lipschitz classes on product spaces in the main theorem, an extension of Hörmander's symbol classes, [22].

We define the Fourier transform of f, a function on $\mathbb{R}^n\times\mathbb{R}^n$, by

$$\mathscr{F}[f](\xi,\eta) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-i(x\xi+y\eta)}f(x,y)dxdy.$$

Then, the inverse Fourier transform $\mathscr{F}^{-1}$ is given by

$$\mathscr{F}^{-1}[f](x,y) = \frac{1}{(2\pi)^{2n}}\iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{i(x\xi+y\eta)}f(\xi,\eta)d\xi d\eta.$$

Let $\theta \in C^\infty_0(\mathbb{R}^n)$ with $\theta(\xi) = \begin{cases}1, & \text{if } |\xi|\le 1\\ 0, & \text{if } |\xi| \ge 2,\end{cases}$ $\theta_0 = \theta$ and $\theta_j(\xi) = \theta\left(\frac{\xi}{2^j}\right) - \theta\left(\frac{\xi}{2^{j-1}}\right)$, for $j \in \mathbb{N}$. Then, $\sum_{j=0}^\infty \theta_j \equiv 1$ and $\operatorname{supp}\theta_j \subset \{\xi \in \mathbb{R}^n \mid 2^{j-1} \le |\xi| \le 2^j\}$.

Definition 2.4 (Lipschitz classes on product spaces [22]) For $m \in \mathbb{R}$ and non-negative integers l, l', we set

$$S^m_{0,0}(l,l') = \Big\{ p \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n) \mid \|p\|_{m;l,l'} := \sup_{\substack{x,\xi\in\mathbb{R}^n\\ j,k\in\mathbb{N}\cup\{0\}}} 2^{jl}2^{kl'} \left|\mathscr{F}^{-1}[\theta_j(y)\theta_k(\eta)\mathscr{F}[p](y,\eta)](x,\xi)\langle\xi\rangle^{-m}\right| < \infty \Big\}.$$

For quasi-Banach spaces X, Y, we write $S^m_{0,0}(l,l') \subset \mathcal{L}(X,Y)$ if and only if $\|p(X,D)f\|_Y \le C\|p\|_{m;l,l'}\|f\|_X$, $(p \in \mathscr{S},\ f \in X)$. Also we write $\mathcal{L}(X) = \mathcal{L}(X,X)$. Before stating our result, we recall Miyachi's result [22].

Theorem C ([22]) *Let $m(p) = -n|1/p - 1/2|$. Suppose that*
(i) $0 < p \le 2$, $l > n/2$, $l' > n/p$ *or*
(ii) $2 < p < \infty$, $l > n/p$, $l' > n/2$.
Then $S^{m(p)}_{0,0}(l,l') \subset \mathcal{L}\ (H^p, L^p)$.

Lu and Yang [18] showed the atomic decomposition of Herz-type Hardy spaces, whose statement is similar to that of Hardy spaces.

Theorem D ([18]) *Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \le \alpha < \infty$, and $s \ge [\alpha - n(1-1/q)]$, s is a integer. Then $f \in H\dot{K}^{\alpha,p}_q$ if and only if there exist $a_j \in L^q$ and complex numbers λ_j such that $f = \sum_{j\in\mathbb{Z}} \lambda_j a_j$ in the sence $\mathscr{S}'$,* $\operatorname{supp} a_j \subset B_j$, $\|a_j\|_{L^q} \le |B_j|^{-\alpha/n}$, $\int a_j(x)x^\beta dx = 0$, $(0 \le |\beta| \le s)$ *and $\{\lambda_j\}_{j\in\mathbb{Z}} \in l^p$. Moreover $\|f\|_{H\dot{K}^{\alpha,p}_q} \sim \inf \left(\sum_{j\in\mathbb{Z}} |\lambda_j|^p\right)^{1/p}$.*

Later, we will use theorems C and D to prove the main result in the next section.

3. Pseudo-differential operators on the Herz-type spaces

Here we state the main result.

Theorem 3.1 *Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \le \alpha < \infty$ and $m < -\alpha - n|1/q - 1/2|$. Suppose that*
(i) $1 < q \le 2$, $l > n/2$, $l' > [-m] + n/2 + 1$, *or*
(ii) $2 \le q < \infty$, $l > n/q$, $l' > [-m - n(1/2 - 1/q)] + n/2 + 1$.
Then $S^m_{0,0}(l,l') \subset \mathcal{L}\ (H\dot{K}^{\alpha,p}_q, \dot{K}^{\alpha,p}_q)$.

Proof. The proof follows the idea of Miyachi [22]. We take $p \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, $f = \sum_{j\in\mathbb{Z}} \lambda_j a_j \in H\dot{K}^{\alpha,p}_q \cap \mathscr{S}_0$. First we prove the case $0 < p \le 1$. Let s be an integer sufficiently large. Then we have

$$\|p(X,D)f\|^p_{\dot K^{\alpha,p}_q} \le \sum_{j\in\mathbb{Z}} |\lambda_j|^p \|p(X,D)a_j\|^p_{\dot K^{\alpha,p}_q}.$$

On the other hand, we have

$$\begin{aligned}
\|p(X,D)a_j\|^p_{\dot K^{\alpha,p}_q} &= \sum_{k\in\mathbb{Z}} 2^{k\alpha p}\|p(X,D)a_j\chi_k\|^p_{L^q} \\
&= \sum_{k=-\infty}^{j+1} 2^{k\alpha p}\|p(X,D)a_j\chi_k\|^p_{L^q} + \sum_{k=j+2}^{\infty} 2^{k\alpha p}\|p(X,D)a_j\chi_k\|^p_{L^q} \\
&=: A_1 + A_2.
\end{aligned}$$

By Theorem C, $A_1 \le \sum_{k=-\infty}^{j+1} 2^{k\alpha p}\|p(X,D)a_j\|^p_{L^q} \lesssim \|p\|^p_{m;l,l'} \sum_{k=-\infty}^{j+1} \cdot 2^{(k-j)\alpha p} \lesssim \|p\|^p_{m;l,l'}$. It suffices to prove that $A_2 \lesssim \|p\|^p_{m;l,l'}$. To estimate A_2 we decompose the symbol p by using the above partition of unity $\{\theta_t\}_{t=0}^{\infty}$ in ξ-space:

$$p(x,\xi) = \sum_{t=0}^{\infty} p(x,\xi)\theta_t(\xi) = \sum_{t=0}^{\infty} p_t(x,\xi) \text{ where } p_t(x,\xi) := p(x,\xi)\theta_t(\xi).$$

Also, let $K(x,y)$ ($K_t(x,y)$, resp.) be the kernel of the pseudo-differential operator $p(X,D)$ ($p_t(X,D)$, resp.): $K(x,y) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{iy\xi}p(x,\xi)d\xi$ ($K_t(x,y) := (2\pi)^{-n}\int_{\mathbb{R}^n} e^{iy\xi}p_t(x,\xi)d\xi$, resp.).

• In the case (i): Let $k \ge j+2$.
We consider the case $j \le 0$. Let γ be a multi-index such that $|\gamma| = [-m]+1$ if $k \ge 0$, $= [-m]$ if $k < 0$. By using vanishing moments of order s we have

$$\begin{aligned}
&\|p_t(x,\xi)a_j\chi_k\|_{L^q} \\
&= \left(\int_{C_k}\left|\int_{B_j} K_t(x,x-y)a_j(y)dy\right|^q dx\right)^{1/q} \\
&= \left(\int_{C_k}\left|\int_{B_j} \sum_{|\beta|=s+1} \frac{(-1)^{s+1}}{\beta!}\partial_2^\beta K_t(x,x-\theta y)y^\beta a_j(y)dy\right|^q dx\right)^{1/q}
\end{aligned}$$

$$\lesssim 2^{j(s+1)} \sum_{|\beta|=s+1} \left(\int_{C_k} \left(\int_{B_j} \left|\partial_2^\beta K_t(x, x-\theta y)\right| \left|a_j(y)\right| dy \right)^q dx \right)^{1/q}.$$

We use the notation $\langle x \rangle = (1+|x|^2)^{1/2}$. By Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\|p_t(x,\xi) a_j \chi_k\|_{L^q} \\
&\quad \lesssim 2^{-j(\alpha-s-1)} \sum_{|\beta|=s+1} \left(\int_{B_j} \left(\int_{C_k} \left|\partial_2^\beta K_t(x, x-\theta y)\right|^q dx \right)^{q'/q} dy \right)^{1/q'} \\
&\quad \lesssim 2^{-j(\alpha-s-1)} 2^{kn(1/q-1/2)} 2^{-k|\gamma|} \\
&\qquad \times \sum_{|\beta|=s+1} \left(\int_{B_j} \left(\int_{C_k} \langle x \rangle^{2|\gamma|} \left|\partial_2^\beta K_t(x, x-\theta y)\right|^2 dx \right)^{q'/2} dy \right)^{1/q'}.
\end{aligned}$$

To estimate the integral, we write

$$\begin{aligned}
p_{\gamma'}(x,\xi) &= \langle \xi \rangle^{-m} \partial_\xi^{\gamma'} p(x,\xi) \in S^0_{0,0}(l, l'-|\gamma'|), \\
\psi_{y,\gamma',t}(\xi) &= e^{-i\theta y \xi} \langle \xi \rangle^m \partial_\xi^{\gamma-\gamma'} \big((i\xi)^\beta \theta_t(\xi) \big), \\
g_{y,\gamma',t}(x) &= \mathscr{F}^{-1}[\psi_{y,\gamma',t}](x).
\end{aligned}$$

Integration by parts gives

$$(-i(x-\theta y))^\gamma \partial_2^\beta K_t(x, x-\theta y) = \sum_{\gamma' \le \gamma} \binom{\gamma}{\gamma'} p_{\gamma'}(X,D) g_{y,\gamma',t}(x).$$

By Plancherel's theorem, we get $\|g_{y,\gamma',t}\|_{L^2} \lesssim 2^{t(m+s+1+n/2)}$. Therefore an easy computation and Theorem C yield

$$\begin{aligned}
&\left(\int_{B_j} \left(\int_{C_k} \left|(-i(x-\theta y))^\gamma\right|^2 \left|\partial_2^\beta K_t(x, x-\theta y)\right|^2 dx \right)^{q'/2} dy \right)^{1/q'} \\
&\quad \lesssim \|p\|_{m;l,l'} 2^{t(m+s+1+n/2)} 2^{jn(1-1/q)}.
\end{aligned}$$

Now we remark that $k \ge j+2$, $x \in C_k$ and $y \in B_j$ implies $\langle x-\theta y \rangle \sim \langle x \rangle$.

Hence we have

$$\left(\int_{B_j}\left(\int_{C_k}\langle x\rangle^{2|\gamma|}\left|\partial_2^\beta K_t(x,x-\theta y)\right|^2 dx\right)^{2/q'} dy\right)^{1/q'}$$
$$\lesssim \|p\|_{m;l,l'}2^{t(m+s+1+n/2)}2^{jn(1-1/q)},$$

and

$$\|p_t(x,\xi)a_j\chi_k\|_{L^q}$$
$$\lesssim \|p\|_{m;l,l'}2^{-j(\alpha-s-1-n(1-1/q))}2^{kn(1/q-1/2)}2^{-k|\gamma|}2^{t(m+s+1+n/2)}. \qquad (1)$$

We also have the following estimate by repeating the above argument with $\beta=0$:

$$\|p_t(x,\xi)a_j\chi_k\|_{L^q}$$
$$\lesssim \|p\|_{m;l,l'}2^{-j(\alpha-n(1-1/q))}2^{kn(1/q-1/2)}2^{-k|\gamma|}2^{t(m+n/2)}. \qquad (2)$$

For each $j\le 0$, there exists $t_j\in\mathbb{N}$ such that $2^{t_j-1}2^j\le 1<2^{t_j}2^j$. Using estimates (1), (2), we deduce the desired estimate in the following way:

$$\|p(X,D)a_j\chi_k\|_{L^q}$$
$$\le \sum_{t=0}^{t_j-1}\|p_t(X,D)a_j\chi_k\|_{L^q}+\sum_{t=t_j}^{\infty}\|p_t(X,D)a_j\chi_k\|_{L^q}$$
$$\lesssim \|p\|_{m;l,l'}2^{-j(\alpha-s-1-n(1-1/q))}2^{kn(1/q-1/2)}2^{-k|\gamma|}\sum_{t=0}^{t_j-1}2^{t(m+s+1+n/2)}$$
$$+\|p\|_{m;l,l'}2^{-j(\alpha-n(1-1/q))}2^{kn(1/q-1/2)}2^{-k|\gamma|}\sum_{t=t_j}^{\infty}2^{t(m+n/2)}$$
$$\lesssim \|p\|_{m;l,l'}2^{-j(\alpha+m-n(1/2-1/q))}2^{kn(1/q-1/2)}2^{-k|\gamma|}$$
$$\lesssim \|p\|_{m;l,l'}2^{-j(\alpha+m-n(1/2-1/q))}2^{kn(1/q-1/2)}2^{-km}.$$

Thus, for $j\le 0$, $A_2=\sum_{k=j+2}^{\infty}2^{k\alpha p}\|p(X,D)a_j\chi_k\|_{L^q}^p\lesssim\|p\|_{m;l,l'}^p\sum_{k=j+2}^{\infty}\cdot 2^{p(k-j)(\alpha+m-n(1/2-1/q))}\lesssim\|p\|_{m;l,l'}^p$.

Next we consider the case $j > 0$. In this case, we do not use the vanishing moment condition or docomposition of symbol. Let γ be a multi-index such that $|\gamma| > \alpha + n(1/q - 1/2) + 1$.

$$\|p(X,D)a_j\chi_k\|_{L^q}$$
$$\leq 2^{-j\alpha}2^{kn(1/q-1/2)}2^{-k|\gamma|}\left(\int_{B_j}\left(\int_{C_k}\langle x\rangle^{2|\gamma|}\big|K(x,x-y)\big|^2dx\right)^{1/q'}dy\right)^{1/q'}.$$

Going throught a similar argument as above, we obtain,

$$\left(\int_{B_j}\left(\int_{C_k}\big|(-i(x-y))^\gamma\big|^2\big|K(x,x-y)\big|^2dx\right)^{q'/2}dy\right)^{1/q'} \lesssim \|p\|_{m;l,l'}2^{jn(1-1/q)}.$$

Since $\langle x-y\rangle \sim \langle x\rangle$, we have

$$\left(\int_{B_j}\left(\int_{C_k}\langle x\rangle^{2|\gamma|}\big|K(x,x-y)\big|^2dx\right)^{q'/2}dy\right)^{1/q'} \lesssim \|p\|_{m;l,l'}2^{jn(1-1/q)}.$$

Hence we have

$$\|p(X,D)a_j\chi_k\|_{L^q} \lesssim \|p\|_{m;l,l'}2^{-j(\alpha-n(1-1/q))}2^{kn(1/q-1/2)}2^{-k|\gamma|}. \tag{3}$$

Now we write $|\gamma| = \alpha + n(1/q-1/2) + \varepsilon$. Then, for $j > 0$,

$$\begin{aligned} A_2 &= \sum_{k=j+2}^{\infty} 2^{k\alpha p}\|p(X,D)a_j\chi_k\|_{L^q}^p \\ &\lesssim \|p\|_{m;l,l'}^p \sum_{k=j+2}^{\infty} 2^{-jp(\alpha-n(1-1/q))}2^{kp(\alpha+n(1/q-1/2)-|\gamma|)} \\ &= \|p\|_{m;l,l'}^p \sum_{k=j+2}^{\infty} 2^{-p(k-j)\varepsilon}2^{-jp(\alpha-n(1-1/q)+\varepsilon)} \lesssim \|p\|_{m;l,l'}^p. \end{aligned}$$

We remark that $[-m] + n/2 + 1$ is larger than $\alpha + n/q$.

• In the case (ii): Let $k \geq j+2$.
We consider the case $j \leq 0$. Let γ be a multi-index such that $|\gamma| = [-m + n(1/q-1/2)] + 1$ if $k \geq 0$, $= [-m + n(1/q-1/2)]$ if $k < 0$.

$$
\begin{aligned}
&\|p(X,D)a_j\chi_k\|_{L^q}\\
&= \left(\int_{C_k}\left|\int_{B_j} K_t(x,x-y)a_j(y)dy\right|^q dx\right)^{1/q}\\
&\lesssim 2^{-j(\alpha-s-1)}\sum_{|\beta|=s+1}\left(\int_{B_j}\left(\int_{C_k}\left|\partial_2^\beta K_t(x,x-\theta y)\right|^q dx\right)^{q'/q} dy\right)^{1/q'}\\
&\lesssim 2^{-j(\alpha-s-1)}2^{-k|\gamma|}\sum_{|\beta|=s+1}\left(\int_{B_j}\left(\int_{C_k}\langle x\rangle^{q|\gamma|}\left|\partial_2^\beta K_t(x,x-\theta y)\right|^q dx\right)^{q'/q} dy\right)^{1/q'}.
\end{aligned}
$$

Since

$$
(-i(x-\theta y))^\gamma \partial_2^\beta K_t(x,x-\theta y) = \sum_{\gamma'\le\gamma}\binom{\gamma}{\gamma'} p_{\gamma'}(X,D)g_{y,\gamma',t}(x),
$$

where

$$
\begin{aligned}
&p_{\gamma'}(x,\xi) = \langle\xi\rangle^{-m-n(1/2-1/q)}\partial_\xi^{\gamma'}p(x,\xi)\in S_{0,0}^{-n(1/2-1/q)}(l,l'-\gamma')\subset\mathcal{L}(L^q),\\
&\psi_{y,\gamma',t}(\xi) = e^{-i\theta y\xi}\langle\xi\rangle^{m+n(1/2-1/q)}\partial_\xi^{\gamma-\gamma'}\left((i\xi)^\beta\theta_t(\xi)\right),\\
&g_{y,\gamma',t}(x) = \mathscr{F}^{-1}[\psi_{y,\gamma',t}](x)
\end{aligned}
$$

and

$$
\|g_{y,\gamma',t}\|_{L^q}\lesssim 2^{t(m+s+1+n(3/2-2/q))},
$$

as a consequence, we have

$$
\begin{aligned}
&\left(\int_{B_j}\left(\int_{C_k}\langle x\rangle^{q|\gamma|}\left|\partial_2^\beta K_t(x,x-\theta y)\right|^q dx\right)^{q'/q} dy\right)^{1/q'}\\
&\qquad\lesssim \|p\|_{m;l,l'}2^{jn(1-1/q)}2^{t(m+s+1+n(3/2-2/q))}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&\|p_t(X,D)a_j\chi_k\|_{L^q}\\
&\quad\lesssim \|p\|_{m;l,l'}2^{-j(\alpha-s-1-n(1-1/q))}2^{-k|\gamma|}2^{t(m+n(3/2-2/q))}. \qquad (4)
\end{aligned}
$$

We also have the following estimate by repeating the above argument with $\beta = 0$;

$$\|p_t(X,D)a_j\chi_k\|_{L^q} \lesssim \|p\|_{m;l,l'} 2^{-j(\alpha-n(1-1/q))} 2^{-k|\gamma|} 2^{t(m+n(3/2-2/q))}. \quad (5)$$

Since the above two estimates give $\|p(X,D)a_j\chi_k\|_{L^q} \lesssim \|p\|_{m;l,l'} \cdot 2^{-j(m+\alpha-n(1/q-1/2))}2^{-k|\gamma|}$, we conclude $A_2 \lesssim \|p\|_{m;l,l'}^p$, for $j \le 0$.

We consider the last case, $j > 0$. Let $|\gamma| > \alpha$ then we set $|\gamma| = \alpha + \varepsilon$. It is easy to see

$$\begin{aligned}&\|p(X,D)a_j\chi_k\|_{L^q}\\ &\quad\lesssim 2^{-j\alpha}2^{-k|\gamma|}\left(\int_{B_j}\left(\int_{C_k}\langle x\rangle^{q|\gamma|}\big|K(x,x-y)\big|^q dx\right)^{q'/q dy}\right)^{1/q'}\end{aligned}$$

and

$$(-i(x-y))^\gamma K(x,x-y) = p_\gamma(X,D)g_y(x),$$

where

$$\begin{aligned}p_\gamma(x,\xi) &= \langle\xi\rangle^{-m-n(1/2-1/q)}\partial_\xi^\gamma p(x,\xi) \in S_{0,0}^{-n(1/2-1/q)}(l,l'-|\gamma|),\\ \psi_y(\xi) &= e^{-iy\xi}\langle\xi\rangle^{m+n(1/2-1/q)},\\ g_y(x) &= \mathscr{F}^{-1}[\psi_y](x).\end{aligned}$$

We have

$$\left(\int_{B_j}\left(\int_{C_k}\langle x\rangle^{q|\gamma|}\big|K(x,x-y)\big|^q dx\right)^{q'/q} dy\right)^{1/q'} \lesssim \|p\|_{m;l,l'}2^{jn(1-1/q)},$$

that is

$$\|p(X,D)a_j\chi_k\|_{L^q} \lesssim \|p\|_{m;l,l'}2^{-j(\alpha-n(1-1/q))}2^{-k|\gamma|}. \quad (6)$$

We obtain the following estimates of A_2 without difficulty, $A_2 \lesssim \|p\|_{m;l,l'}^p$. As a result, when $0 < p \le 1$, we get $\|p(X,D)f\|_{\dot{K}_q^{\alpha,p}} \lesssim \|p\|_{m;l,l'}\|f\|_{H\dot{K}_q^{\alpha,p}}$.

Finally we consider the case $1 < p < \infty$. In this case, we use the following decomposition, and each term can be easily estimated by (1), (4) and (6).

$$
\begin{aligned}
\|p(X,D)f\|_{\dot{K}^{\alpha,p}_q} \lesssim & \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha p}\left(\sum_{j=k-1}^{\infty} |\lambda_j| \|p(X,D)a_j\chi_k\|_{L^q}\right)^p\right)^{1/p} \\
& + \left(\sum_{k=-\infty}^{2} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|p(X,D)a_j\chi_k\|_{L^q}\right)^p\right)^{1/p} \\
& + \left(\sum_{k=3}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{-1} |\lambda_j| \|p(X,D)a_j\chi_k\|_{L^q}\right)^p\right)^{1/p} \\
& + \left(\sum_{k=3}^{\infty} 2^{k\alpha p}\left(\sum_{j=0}^{k-2} |\lambda_j| \|p(X,D)a_j\chi_k\|_{L^q}\right)^p\right)^{1/p}.
\end{aligned}
$$

This completes the proof of theorem. □

We show some results in the non-homogeneous case. We remak that we can take m to be larger than that of Theorem 3.1 and $HK^{\alpha,p}_q \subsetneq K^{\alpha,p}_q$ if $0 < p \le \infty$, $1 < q < \infty$, $-\infty < \alpha \le -n/q$ or $n(1-1/q) \le \alpha < \infty$ [13].

Theorem 3.2 *Let $0 < p \le \infty$, $1 < q < \infty$ and $n(1-1/q) \le \alpha < \infty$. Suppose that*

(i) $1 < q \le 2$, $m < -n/2$, $l > n/2$, $l' > \alpha + n/q$ or

(ii) $2 < q < \infty$, $m < -n(3/2 - 2/q)$, $l > n/q$, $l' > \alpha + n/2$.

Then $S^m_{0,0}(l,l') \subset \mathcal{L}\ (K^{\alpha,p}_q)$.

Proof. Theorem 3.2 has been already proved in the course of the proof of Theorem 3.1. When we consider non-homogeneous case, we do not need estimates of the case $j < 0$ in the proof of Theorem 3.1. We check the case (i) with $0 < p \le 1$ only.

We write

$$
f(x) = f(x)\chi_{B(0,1)}(x) + \sum_{j\in\mathbb{N}} f(x)\chi_j(x) = \sum_{j\ge 0} f_j(x) = \sum_{j\ge 0} \lambda_j a_j(x),
$$

where $f_0(x) = f(x)\chi_{B(0,1)}(x)$, $f_j(x) = f(x)\chi_j(x)$, $(j \ge 1)$, $\lambda_j = |B_j|^{\alpha/n}\|f_j\|_{L^q}$ and $a_j(x) = \frac{f_j(x)}{|B_j|^{\alpha/n}\|f_j\|_{L^q}}$.

Hence,

$$\|p(X,D)f\|^p_{K^{\alpha,p}_q} = \|p(X,D)f\chi_{B(0,1)}\|^p_{L^q} + 2^{\alpha p}\|p(X,D)f\chi_1\|^p_{L^q} + \sum_{k=2}^{\infty} 2^{k\alpha p}\|p(x,D)f\chi_k\|^p_{L^q} := A + B + C.$$

The term A is easily estimated as: $A \le \sum_{j=0}^{\infty}|\lambda_j|^p\|p(X,D)a_j\chi_{B(0,1)}\|^p_{L^q} \le \sum_{j=0}^{\infty}|\lambda_j|^p 2^{-jp} \lesssim \|f\|^p_{K^{\alpha,p}_q}$.

Similarly, $B \lesssim \|f\|^p_{K^{\alpha,p}_q}$.

Finally, to estimate C we decompose it into two parts.

$$\begin{aligned} C &= \sum_{k=2}^{\infty} 2^{k\alpha p}\|p(X,D)f\chi_k\|^p_{L^q} \le \sum_{k=2}^{\infty} 2^{k\alpha p}\sum_{j=0}^{\infty}|\lambda_j|^p\|p(X,D)a_j\chi_k\|^p_{L^q} \\ &\lesssim \sum_{k=2}^{\infty} 2^{k\alpha p}\sum_{j=0}^{k-2}|\lambda_j|^p\|p(X,D)a_j\chi_k\|^p_{L^q} + \sum_{k=2}^{\infty} 2^{k\alpha p}\sum_{j=k-1}^{\infty}|\lambda_j|^p\|p(X,D)a_j\chi_k\|^p_{L^q} \\ &:= C_1 + C_2. \end{aligned}$$

By the L^q-boundedness of $p(X,D)$, $C_2 \lesssim \sum_{k=2}^{\infty}\sum_{j=k-1}^{\infty}|\lambda_j|^p 2^{p(k-j)\alpha} \lesssim \|f\|^p_{K^{\alpha,p}_q}$. The estimates (3) gives $C_1 \lesssim \|f\|^p_{K^{\alpha,p}_q}$. Therefore we have $\|p(X,D)f\|_{K^{\alpha,p}_q} \lesssim \|f\|_{K^{\alpha,p}_q}$. □

Remark 3.1 If $n(1-1/q) \lneq \alpha$, then we can take $-\alpha - n|1/q - 1/2|$ as the order of the symbol in Theorem 3.2.

By using the following Proposition 3.1, the symbol $S^m_{0,0}(l,l')$ can be written as $S^m_{0,0}(L)$ in the statements of Theorem 3.1 and Theorem 3.2.

Proposition 3.1 *For any $m \in \mathbb{R}$ and nonnegative integers l, l', there exist nonnegative integers P, Q such that $\|p\|_{m;l,l'} \lesssim |p|^m_{P+Q}$, $(p \in \mathscr{S})$.*

Proof. We define $\langle D_y\rangle^2 = 1 + \sum_{i=1}^{n} D^2_{y_i} = 1 - \triangle$. And let $N = [n+1-m/2]+1$, $M = N+1$, $P = (n+l)/2$ if $n+l$ is even, $= [(n+l)/2]+1$ if $n+l$ is odd, $Q = n+l'$ if $n+l'$ is even, $= [(n+l')/2]+1$ if $n+l'$ is odd. Then we have

$$
\begin{aligned}
&\left|\mathscr{F}^{-1}[\theta_j(y)\theta_k(\eta)\hat{p}(y,\eta)](x,\xi)\right| = \left|\mathscr{F}^{-1}[\theta_j(y)\theta_k(\eta)] * p(x,\xi)\right| \\
&\quad = \left| \iint \left(\frac{1}{(2\pi)^{2n}} \iint e^{i(yu+\eta v)} \langle D_y\rangle^{2M}\theta_j(y)\langle D_\eta\rangle^{2N}\theta_k(\eta)dyd\eta \right)\right. \\
&\qquad \left. \times \langle u\rangle^{-2M}\langle v\rangle^{-2N} p(x-u,\xi-v)dudv \right| \\
&\quad \le \iint \left(\frac{1}{(2\pi)^{2n}} \iint \langle y\rangle^{-2P}\langle D_y\rangle^{2M}\theta_j(y)\langle \eta\rangle^{-2Q}\langle D_\eta\rangle^{2N}\theta_k(\eta)dyd\eta \right) \\
&\qquad \times C_{P,Q,M,N}|p|^m_{P+Q}\langle\xi\rangle^m\langle v\rangle^{-m}\langle u\rangle^{-2M}\langle v\rangle^{-2N}dudv \\
&\quad \le C_{P,Q,M,N}|p|^m_{P+Q} 2^{-jl}2^{-kl'}\langle\xi\rangle^m, \quad (j,k\in\mathbb{Z}_+,\ x,\xi\in\mathbb{R}^n). \qquad \square
\end{aligned}
$$

4. Interpolations

In this section, by using the interpolation theory for bilinear operators, we get rid of the condition of $\alpha : n(1-1/q) \le \alpha$ in Theorem 3.1. Furthermore, the duality argument allows us to take negative index α.

First of all, we recall the defintions of interpolation for families of quasi-Banach spaces, [14]. Let $\triangle$ be the open unit disc in $\mathbb{C}$, and T the boundary of $\triangle$. We put a quasi-Banach space on for each $\theta \in T : (B(\theta), \|\cdot\|_{B(\theta)})$, and denote by $c(\theta)$ the constants in the quasi-triangle inequalities. We say that family $\{B(\theta)\}_{\theta\in T}$ is an interpolation family of quasi-Banach spaces if each $B(\theta)$ is cotinuously embedded in a Hausdorff topological vecter space $\mathcal{U}$, the function $\theta \to \|b\|_{B(\theta)}$ is measurable for each $b \in \bigcap_{\theta\in T} B(\theta)$, and log $c(\theta) \in L^1(T)$;$\mathcal{U}$ is called the containing space of the given family $\{B(\theta)\}_{\theta\in T}$. We define

$$
\beta = \left\{ b \in \bigcap_{\theta\in T} B(\theta) \,\middle|\, \int_T \log^+ \|b\|_{B(\theta)} d\theta < \infty \right\},
$$

called the log-intersection space of the given family $\{B(\theta)\}_{\theta\in T}$. Let $\mathcal{G} = \mathcal{G}$ $(\triangle, B(\cdot))$ be the space of all the β-valued analytic function of the form

$$
g(z) = \sum_{j=1}^{m} \psi_j(z) b_j
$$

for which $\|g\|_{\mathcal{G}} = \sup_{\theta \in T} \|g(\theta)\|_{B(\theta)} < \infty$, where $m \in \mathbb{N}$, $\psi_j \in N^+(\triangle)$, the positive Nevalinna class for $\triangle$ ([7]), and $b_j \in \beta$, $j = 1, \dots, m$. For any $a \in \beta$ and $z \in \triangle$, we define

$$\|a\|_z = \inf \left\{ \|g\|_{\mathcal{G}} \mid g \in \mathcal{G},\ g(z) = a \right\}.$$

If N_z denotes the set of functions of β such that $\|a\|_z = 0$, the completion $B(z)$ of $(\beta/N_z, \|\cdot\|_z)$ will be called the interpolation space at z of the family $\{B(\theta)\}_{\theta \in T}$. We also denote $B(z)$ by $[B(\theta)]_z$.

Let $0 < p_0 < \infty$, $1 < q_0$, $q_1 < \infty$, $n(1 - 1/q_0) \le \alpha_0 < \infty$, $m_0 < -\alpha_0 - n|1/q_0 - 1/2|$, $m_1 = -n|1/q_1 - 1/2|$, and $0 < \theta < 1$. Then, we define $1/p(\theta) = (1-\theta)/p_0 + \theta/q_1$, $1/q(\theta) = (1-\theta)/q_0 + \theta/q_1$, $\alpha(\theta) = (1-\theta)\alpha_0$, and $m(\theta) = (1-\theta)m_0 + \theta m_1$. Let L be an integer sufficiently large.

Following three equalities, which characterize the intermediate spaces obtained by the complex method of interpolation for the couples or families, are well known.

$$\left[S^{m_0}_{0,0,}(L), S^{m_1}_{0,0}(L)\right]_\theta = S^{m(\theta)}_{0,0}(L) : \text{ Páivárinta and Sommersaro, [24]},$$

$$\left[H\dot{K}^{\alpha_0,p_0}_{q_0}, H\dot{K}^{0,q_1}_{q_1}\right]_\theta = H\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)} : \text{ Hernández and Yang, [14]},$$

$$\left[\dot{K}^{\alpha_0,p_0}_{q_0}, \dot{K}^{0,q_1}_{q_1}\right]_\theta = \dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)} : \text{ Hernández and Yang, [13]}.$$

Next we consider the following bilinear operator:

$$\begin{aligned}\mathscr{T} :\ & S^{m_0}_{0,0}(L) \times H\dot{K}^{\alpha_0,p_0}_{q_0} \to \dot{K}^{\alpha_0,p_0}_{q_0} \\ \text{or } & S^{m_1}_{0,0}(L) \times H\dot{K}^{0,q_1}_{q_1} \to \dot{K}^{0,q_1}_{q_1}; (p, f) \longmapsto p(X, D)f.\end{aligned}$$

Theorem 4.1 *In the above situation, if L is sufficiently large, then $\|\mathscr{T}(p,f)\|_{\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)}} \lesssim |p|^{m(\theta)}_L \|f\|_{H\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)}}$, $(p \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$, $f \in H\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)})$.*

Proof. We follow the argument in Theorem 4.4.1 in [2]. For the sake of convenience, we write

$$A^0_\theta = S^{m(\theta)}_{0,0}(L), \quad A^1_\theta = H\dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)}, \quad A^2_\theta = \dot{K}^{\alpha(\theta),p(\theta)}_{q(\theta)},$$

$(B^0(\tau), B^1(\tau), B^2(\tau)) = (S^{m_0}_{0,0}(L), H\dot{K}^{\alpha_0,p_0}_{q_0}, \dot{K}^{\alpha_0,p_0}_{q_0})$ if $\tau \in T_0$, and $(B^0(\tau), B^1(\tau), B^2(\tau)) = (S^{m_1}_{0,0}(L), H\dot{K}^{0,p_1}_{q_1}, \dot{K}^{0,p_1}_{q_1})$ if $\tau \in T_1$, where T_0 and T_1 are subsets of T so that $\frac{1}{p(\theta)} = \int_{T_0} \frac{1}{p_0} P_\theta(\tau)d\tau + \int_{T_1} \frac{1}{p_1} P_\theta(\tau)d\tau$, $\frac{1}{q(\theta)} = \int_{T_0} \frac{1}{q_0} P_\theta(\tau)d\tau + \int_{T_1} \frac{1}{q_1} P_\theta(\tau)d\tau$, $\alpha(\theta) = \int_{T_0} \alpha_0 P_\theta(\tau)d\tau + \int_{T_1} \alpha_1 P_\theta(\tau)d\tau$, $m(\theta) = \int_{T_0} m_0 P_\theta(\tau)d\tau + \int_{T_1} m_1 P_\theta(\tau)d\tau$ and $P_\theta(\tau)$ is the Poisson kernel for evaluation at θ.

Let us be reminded that the space $B(\theta)$ defines β, $\mathcal{G}$ and N_z as was explained above. We use the notations β^k, $\mathcal{G}^k$ and N^k_z, if β, $\mathcal{G}$ and N_z is defined by $B(\theta) = A^k_\theta$ $(k = 0, 1, 2)$.

It is not hard to see that $N^k_\theta = \{0\}$ for each k. Let $\varepsilon > 0$, $a_0 \in \beta^0$ and $a_1 \in \beta^1$. Then there exist $f_0 \in \mathcal{G}^0$ and $f_1 \in \mathcal{G}^1$: $f_0 = \sum_{i=1}^{k_0} \varphi_i b_i$, $f_1 = \sum_{j=1}^{k_1} \psi_j c_j$ such that $f_0(\theta) = a_0$, $f_1(\theta) = a_1$, $\|f_0\|_{\mathcal{G}^0} \le \|a_0\|_\theta + \varepsilon$ and $\|f_1\|_{\mathcal{G}^1} \le \|a_1\|_\theta + \varepsilon$, where φ_i, $\psi_j \in N^+(\triangle)$ and $b_i \in \beta^0$, $c_j \in \beta^1$. Let C_0 and C_1 be constants in the inequalities $\|\mathscr{T}(p,f)\|_{\dot{K}^{\alpha_0,p_0}_{q_0}} \le C_0 |p|^{m_0}_L \|f\|_{H\dot{K}^{\alpha_0,p_0}_{q_0}}$, $\|\mathscr{T}(p,f)\|_{\dot{K}^{\alpha_1,p_1}_{q_1}} \le C_1 |p|^{m_1}_L \|f\|_{H\dot{K}^{\alpha_1,p_1}_{q_1}}$, and we set $g(z) = (C_0 + C_1)^{-1} \sum_{i=1}^{k_0} \sum_{j=1}^{k_1} \varphi_i(z)\psi_j(z)\mathscr{T}(b_i, c_j)$. Then $g(\theta) = (C_0 + C_1)^{-1}\mathscr{T}(a_0, a_1)$, $g \in \mathcal{G}^2$ and

$$\begin{aligned}\|g\|_{\mathcal{G}^2} &= \sup_{\tau\in T} \|g(\tau)\|_{B^2(\tau)} = \sup_{\tau\in T} \frac{1}{C_0 + C_1} \left\| \mathscr{T}\left(\sum_{i=1}^{k_0} \varphi_i(\tau) b_i, \sum_{j=1}^{k_1} \psi_j(\tau) c_j \right) \right\|_{B^2(\tau)} \\ &\le \sup_{\tau\in T} \left\| \sum_{i=1}^{k_0} \varphi_i(\tau) b_i \right\|_{B^0(\tau)} \sup_{\tau\in T} \left\| \sum_{j=1}^{k_1} \psi_j(\tau) c_j \right\|_{B^1(\tau)} = \|f_0\|_{\mathcal{G}^0} \|f_1\|_{\mathcal{G}^1}.\end{aligned}$$

Hence we have

$$\begin{aligned}\|\mathscr{T}(a_0, a_1)\|_\theta &\le (C_0 + C_1)\|g\|_{\mathcal{G}^2} \le (C_0 + C_1)\|f_0\|_{\mathcal{G}^0}\|f_1\|_{\mathcal{G}^1} \\ &\le (C_0 + C_1)(\|a_0\|_\theta + \varepsilon)(\|a_1\|_\theta + \varepsilon),\end{aligned}$$

which implies the conclusion $\|\mathscr{T}(p,f)\|_{A^2_\theta} \lesssim |p|^{m(\theta)}_L \|f\|_{A^1_\theta}$. □

In particular, we consider the case $q_1 = 2$ in Theorem 4.1. By elementary calculation, we obtain

Corollary 4.1 *Let $0 < p < \infty$, $1 < q < \infty$, $0 < \alpha$, $1 - 1/q - \alpha/n < \min(1/p, 1/q, 1/2)$ and L be an integer sufficiently large. If $m < -\alpha - n|1/q - 1/2|$, then $S^m_{0,0}(L) \subset \mathcal{L}\ (H\dot{K}^{\alpha,p}_q, \dot{K}^{\alpha,p}_q)$.*

Proof. The condition $1 - 1/q - \alpha/n < \min(1/p, 1/q, 1/2)$ guarantees that there exists $0 < \theta < 1$, $0 < p_0 < \infty$ and $1 < q_0 < \infty$ such that

$$1/p = (1-\theta)/p_0 + \theta/2, \quad 1/q = (1-\theta)/q_0 + \theta/2$$
$$\text{and} \quad n(1 - 1/q_0) \leq \alpha/(1-\theta).$$

This and Theorem 4.1 complete the proof of Corollary 4.1. □

Remark 4.1 If $0 < p, q \leq 2$ and $1 < q$, then the condition $1 - 1/q - \alpha/n < \min(1/p, 1/q, 1/2)$ is always satisfied. The range of α in Corollary 4.1 is wider than that of Theorem 4.1.

Remark 4.2 We remark that the conclusion of Corollary 4.1 holds if the index L is larger than at least $\left[\frac{3n}{4}\right] + \left[\frac{3n}{4} + \frac{1}{2}\left[\frac{\alpha+n/q}{1-min(1/p,1/2)} - \frac{n^2}{2(\alpha+n/q)}\right] + 1\right] + 4$ if $1 < q \leq 2$, $\left[\frac{3n}{4}\right] + \left[\frac{3n}{4} + \frac{1}{2}\left[\frac{\alpha}{1-min(1/p,1/q)}\right] + 1\right] + 4$ if $2 < q < \infty$.

The duality argument gives us the boundedness of $S^m_{0,0}(L)$ on the Herz spaces with $\alpha < 0$,

Corollary 4.2 *Let* $1 < p, q < \infty$, $0 < \alpha < n(1 - 1/q)$, $1 - 1/q - \alpha/n < \min(1/p, 1/q, 1/2)$, *and* L *be an integer sufficiently large. If* $m < -\alpha - n|1/q - 1/2|$ *then* $S^m_{0,0}(L) \subset \mathcal{L}\ (\dot{K}^{-\alpha,p'}_{q'})$, *where* $1/p + 1/p' = 1$ *and* $1/q + 1/q' = 1$.

Remark 4.3 Let p be in $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and p^* be the adjoint operator of p. Note that the inequality $|p^*|^m_L \leq |p|^m_{L+2([(n+1)/2]+1)}$ holds. Hence, the index L in Corollary 4.2 must be larger than $L_0 + 2([(n+1)/2]+1)$ where L_0 is the minimal requirement for L in Corollary 4.1.

Remark 4.4 The author believes that the complex interpolation theorem for Herz-type Hardy spaces with $q \leq 1$ holds. If the interpolation theorem holds, we will be able to obtain the boundedness of pseudo-differential operators of class $S^m_{0,0}$ on the Herz-type Hardy spaces with $q \leq 1$.

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References

[1] Baernstein II A. and Sawyer E.T., *Embedding and multiplier theorems for $H^p(\mathbb{R}^n)$*. Mem. Amer. Math. Soc., vol. 59, no. 318, Amer. Math. Soc., Providence, R.I., 1985.

[2] Bergh J. and Löfström J., *Interpolation spaces*. Springer Verlag, Berlin-Heidelberg-New York, 1976.

[3] Beurling A., *Construction and analysis of some convolution algebras*. Ann. Inst. Fourier Grenoble **14** (1964), 1–32.

[4] Calderón A.P. and Vaillancourt R., *On the boundedness of pseudo-differential operators*. J. Math. Soc. Japan **23** (1971), 374–378.

[5] Coifman R.R., Cwikel M., Rochberg R., Sagher Y. and Weiss G., *Complex interpolation for families of Banach spaces*. Proceedings, Symposia in Pure Mathematics, Vol. 35, Part 2, Amer. Math. Soc., Providence, R.I., (1979), 269–282

[6] Coifman R.R., Cwikel M., Rochberg R., Sagher Y. and Weiss G., *A Theory of Complex Interpolation for Families of Banach Spaces*. Adv. in Math. **43** (1982), 203–229.

[7] Duren P.L., *Theory of H^p spaces*. Academic Press, New York, 1970.

[8] Fan D. and Yang D., *The Weighted Herz-type Hardy spaces $h\dot{K}_q^{\alpha,p}(w_1, w_2)$*. Approx. Theory & its Appl. **13** (1997), 19–41.

[9] Feightinger H.G., *An elementary approach to Wiener's third Tauberian theorem for the Euclidian n-spaces*. Proceedings of Conference at Cortona 1984, Symposia Mathematica, vol. 29, Academic Press, New York, 1987, pp. 264–301.

[10] Flett T.M., *Some elementary inequalities for integrals with applications to Fourier transforms*. Proc. London Math. Soc. **(3)29** (1974), 538–556.

[11] García-Cuerva J. and Herrero M.J.L., *A Theory of Hardy spaces associated to the Herz spaces*. Proc.London Math. Soc. **69** (1994), 605–628.

[12] Hernández E., *Intermediate spaces and the complex method of interpolation for families of Banach spaces*. Ann. Scoula Norm. Sup. Pisa Cl. Sci. **Serie iv xiii** (1986), 245–266.

[13] Hernández E. and Yang D., *Interpolation of Herz-type Hardy spaces*. Illinous J. Math. **42** (1998), 564–581.

[14] Hernández E. and Yang D., *Interpolation of Herz spaces and Applications*. Math. Nachr. **205** (1999), 69–87.

[15] Herz C., *Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms*. J. Math. Mech. **18** (1968), 283–324.

[16] Komori Y., *Weak type estimates for Calderón-Zygmund operators on Herz spaces at critical indexes*. Math. Nachr. **259** (2003), 42–50.

[17] Li X. and Yang D., *Boundedness of some sublinear operators on Herz spaces*. Illinois J. Math. **40** (1996), 484–501.

[18] Lu S. and Yang D., *The Weighted Herz-type Hardy spaces and its applications*. Science in China (Ser. A). **38** (1995), 662–673.

[19] Lu S. and Yang D., *Hardy-Littlewood-Sobolev Theorems of fractional integration on Herz-type spaces and its applications*. Canad. J. Math. **48** (1996), 363–380.

[20] Lu S. and Yang D., *The Continuity of commutators on Herz-Type Spaces*. Michigan Math. J. **44** (1997), 255–281.

[21] Lu S., Yabuta K. and Yang D., *Boundedness of some sublinear operators in Weighted Herz-type Hardy spaces*. Kodai Math. J. **23** (2000), 391–410.

[22] Miyachi A., *Estimates for Pseudo-Differential operators of Class* $S_{0,0}$. Math. Nachr. **133** (1987), 135–154.

[23] Miyachi A., *Remarks on Herz-type Hardy spaces*. Acta Math. Sinica. English Series **17** (2001), 339–360.

[24] Páivárinta L. and Sommersalo E., *A Generalization of the Calderón-Vaillancourt Theorem to* L^p *and* h^p. Math. Nachr. **138** (1988), 145–156.

[25] Tabacco Vignati A., *Complex interpolation for families of quasi-Banach spaces*. Indiana Univ. Math. J. **37** (1988), 1–21.

[26] Tabacco Vignati A., *Some techniques for the characterization of intermediate spaces*. Ann. Scoula Norm. Sup. Pisa Cl. Sci. Serie iv **17** (1990), 323–341.

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