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THE MCSHANE INTEGRAL IN THE LIMIT

Abstract

We introduce the notion of the McShane integral in the limit for functions defined on a σ-finite outer regular quasi Radon measure space $(S, \Sigma, \mathcal{T}, \mu)$ into Banach space X and we study its relation with the generalized McShane integral introduced by D. H. Fremlin [2]. It is shown that if a function from S into X is McShane integrable in the limit on S and scalarly locally τ-upper McShane bounded for some $\tau > 0$, then it is McShane integrable on S. On the other hand, we prove that if X-valued function is McShane integrable in the limit on S, then it is McShane integrable on each member of an increasing sequence $(S_\ell)_{\ell \geq 1}$ of measurable sets of finite measure with union S. We also prove a Beppo Levi's version Theorem for this new integral.

1 Introduction

In [2], D. H. Fremlin generalized the classical McShane integral to the case of an arbitrary σ-finite outer regular quasi Radon measure space $(S, \Sigma, \mathcal{T}, \mu)$. It turns out that for any McShane integrable function taking values in Banach space $(X, \|.\|)$, the McShane integral on S can be approximated with respect to the norm $\|.\|$ by sequence consisting of McShane sums.

In a previous paper [12], we defined a new method of integrability, named *weak McShane integrability*, for functions defined on σ-finite outer regular quasi Radon measure space $(S, \Sigma, \mathcal{T}, \mu)$ into a Banach space X; "roughly speaking, a function f from S into X is weakly McShane integrable on S if all sequences

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consisting of McShane sums of f corresponding to some class of generalized McShane partitions of S converge to the same limit with respect to the weak topology.” Then we studied its relation with the Pettis integral, and we proved that a function from S into X is weakly McShane integrable on each member of Σ if and only if it's Pettis and weakly McShane integrable on S. We also proved that if a function is weakly McShane integrable on S, then it is Pettis integrable on each member of an increasing sequence of measurable sets of finite measure with union S. Moreover, it can be seen from this methods that for weakly sequentially complet spaces or spaces that do not contain a copy of c_0, a weakly McShane integrable function on S is is always Pettis integrable.

In the above cited work we also presented an example of a weakly McShane integrable function but not McShane integrable. The aim of this work is to introduce a new notion of McShane integrability named *McShane integrability in the limit*, which is situated between McShane integrability and weak McShane integrability for functions defined on σ-finite outer regular quasi Radon measure space $(S, \Sigma, \mathcal{T}, \mu)$ into a Banach space X. In this work we investigate the relation that may exist between this new integral and the McShane integral. More precisely, we seek to determine when a McShane integrable in the limit function is also McShane integrable. For this purpose, we introduced the concept of the locally upper McShane boundedness. It used to pass from McShane integrability in the limit to McShane integrability. This depends on an exhaustion-type lemma (Lemma 4.1). It is shown that if a function from S into X is McShane integrable in the limit on S and scalarly locally τ-upper McShane bounded for some $\tau > 0$, then it is McShane integrable on S (Theorem 4.1). On the other hand, we prove that if an X-valued function is McShane integrable in the limit on S, then it is McShane integrable on each member of an increasing sequence $(S_\ell)_{\ell \geq 1}$ of measurable sets of finite measure with union S (Theorem 4.2). In Section 5 we extend a Beppo Levi's theorem to the space of McShane integrable in the limit vector-valued functions (Theorem 5.1). In the case of the McShane integral this theorem is proved by C. Swartz [13], but only for the functions defined on $\mathbb{R}$. As an application of this theorem we prove that the space of McShane integrable in the limit functions equipped with the Pettis norm is not complete.

2 Preliminaries

In the sequel, X stands for a Banach space whose norm is denoted by $\|.\|$, and X^* for the topological dual of X. The closed unit ball of X^* is denoted by $\overline{B}_{X^*}$. Let (S, Σ, μ) be a positive measure space. By Σ_f we denote the collection of all measurable sets of finite measure. for each $E \in \Sigma$ with $\mu(E) > 0$, we denote $\Sigma^+(E) = \{A \subseteq E : \mu(A) > 0\}$ and denote $\Sigma^+(S)$ by just Σ^+. $L^1_{\mathbb{R}}(\mu)$ we denote the Banach space of all (equivalence classes of) Σ-measurable and μ-integrable real-valued functions on S. A function $f : S \to X$ is said to be *scalarly integrable* if for every $x^* \in X^*$, the real-valued function $\langle x^*, f\rangle$ is a member of $L^1_{\mathbb{R}}(\mu)$. We say also that f is *Dunford integrable.* If $f : S \to X$ is a scalarly integrable function, then for each $E \in \Sigma$, there is $x^{**}_E \in X^{**}$ such that

$$\langle x^*, x^{**}_E\rangle = \int_E \langle x^*, f\rangle \, d\mu.$$

The vector x^{**}_E is called the *Dunford integral* of f over E. In the case that $x^{**}_E \in X$ for all $E \in \Sigma$, then f is called *Pettis integrable* and we write $(\mathcal{P}e)$-$\int_E f \, d\mu$ instead of x^{**}_E to denote the *Pettis integral* of f over E. The spaces of (equivalence class of) all Pettis integrable functions forms a normed linear space under the Pettis (semi) norm

$$\|f\|_{\mathcal{P}e} = \sup_{x^* \in \overline{B}_{X^*}} \int_S |\langle x^*, f\rangle| \, d\mu.$$

If $f : S \to X$ is a Pettis integrable function, then the set $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is relatively weakly compact in $L^1_{\mathbb{R}}(\mu)$ (see [9], p. 162).

Definition 2.1. (Definition 246A, [4]). A subset $\mathcal{H}$ of $L^1_{\mathbb{R}}(\mu)$ is *uniformly integrable* if for every $\varepsilon > 0$ we can find a set $E \in \Sigma_f$ and an $M \geq 0$ such that

$$\int_S (|h| - M1_E)^+ \, d\mu \leq \varepsilon \text{ for every } h \in \mathcal{H},$$

where $(|h| - M1_E)^+ := \max(|h| - M1_E, 0)$.

- Let $\varphi \in L^1_{\mathbb{R}^+}(\mu)$. Then $\{h \in L^1_{\mathbb{R}}(\mu) : |h| \leq \varphi\}$ is uniformly integrable.

Definition 2.2. A subset $\mathcal{H}$ of $L^1_{\mathbb{R}}(\mu)$ is *equi-continuous* if for every $\varepsilon > 0$ there are $E \in \Sigma_f$ and a $\eta > 0$ such that $|\int_F h \, d\mu| \leq \varepsilon$ for every $h \in \mathcal{H}$ and for every $F \in \Sigma$ with $\mu(F \cap E) \leq \eta$.

Note that a subset $\mathcal{H}$ of $L^1_{\mathbb{R}}(\mu)$ is *equi-continuous* if and only if $\lim_{n\to\infty} \sup_{h\in H} |\int_{F_n} h \, d\mu| = 0$ for every non-increasing sequence $(F_n)_{n\geq 1}$ in Σ with empty intersection.

Theorem 2.1. *([4], Theorem 246G). A subset $\mathcal{H}$ of $L^1_{\mathbb{R}}(\mu)$ is uniformly integrable if and only if*

(1) $\sup_{h\in\mathcal{H}} |\int_A h\, d\mu| < \infty$ *for every μ-atom (in the measure space sense (see [10], 211I)) $A \in \Sigma$ and*

(2) $\mathcal{H}$ is equi-continuous.

Note ([4], Corollary 246I) that in case (S, Σ, μ) is a probability space, *(1)* and *(2)* may be replaced with

$$\lim_{\lambda\to\infty} \sup_{h\in\mathcal{H}} \int_{\{t\in S:|h(t)|\geq\lambda\}} |h|\, d\mu = 0.$$

Theorem 2.2. *([4], Theorem 247C). A subset $\mathcal{H}$ of $L^1_{\mathbb{R}}(\mu)$ is uniformly integrable if and only if it is relatively weakly compact in $L^1_{\mathbb{R}}(\mu)$.*

The following well known result ([7], [9]), which is the Pettis analogous of the classical Vitali convergence theorem, will play a key role in this work. An alternative proof based on the Eberlein-Smulyan-Grothendieck theorem can be found in [1].

Theorem 2.3. *Let $f : S \to X$ be a scalarly integrable function satisfying the following two conditions:*

(1) $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is relatively weakly compact in $L^1_{\mathbb{R}}(\mu)$.
(2) There exists a sequence (f_n) of Pettis integrable functions from S into X such that

$$\lim_{n\to\infty} \int_E \langle x^*, f_n\rangle\, d\mu = \int_E \langle x^*, f\rangle\, d\mu,$$

for each $x^* \in X^*$ and each $E \in \Sigma$.

Then f is Pettis integrable.

3 The McShane integral in the limit

In this section, we introduce the concept of the McShane integral in the limit and we investigate some of its properties. For this purpose, we need to introduce some terminology. Assume that (S, Σ, μ) is a σ-finite positive measure space and $\mathcal{T} \subset \Sigma$ a topology on S making $(S, \mathcal{T}, \Sigma, \mu)$ a *quasi-Radon* measure space which is *outer regular*, that is, such that

$$\mu(E) = \inf\{\mu(G) : E \subset G,\, G \in \mathcal{T}\} \ (E \in \Sigma).$$

Recall that if $(S, \mathcal{T}, \Sigma, \mu)$ is any σ-finite outer regular quasi Radon measure space, and $A \subset S$ is any set (not necessarily measurable), so is $(A, A \cap \mathcal{T}, A \cap \Sigma, \mu_{|A})$. For an extensive study of quasi-Radon measure spaces, the reader is referred to ([6], Chapter 41). A *partial McShane partition* is a countable (may be finite) collection $\{(E_i, t_i)\}_{i\in I}$, where the E_i's are pairwise disjoint measurable subsets of S with finite measure and t_i is a point of S for each $i \in I$. A *generalized McShane partition* of S is an infinite partial McShane partition $\{(E_i, t_i)\}_{i\geq 1}$ such that $\mu(S \setminus \cup_{i=1}^{i=\infty} E_i) = 0$. A *gauge* on S is a function $\Delta : S \to \mathcal{T}$ such that $t \in \Delta(t)$ for every $t \in S$. For a given Δ on S, we say that a partial McShane partition $\{(E_i, t_i)\}_{i\in I}$ is *subordinate* to Δ if $E_i \subset \Delta(t_i)$ for every $i \in I$. Let $f : S \to X$ be a function. We set

$$\sigma_n(f, \mathcal{P}_\infty) := \sum_{i=1}^{i=n} \mu(E_i) f(t_i),$$

for each infinite partial McShane partition $\mathcal{P}_\infty = \{(E_i, t_i)\}_{i\geq 1}$.

From now on $(S, \mathcal{T}, \Sigma, \mu)$ is a σ-finite outer regular quasi-Radon measure space.

Definition 3.1. ([**2**]).
(1) A function $f : S \to X$ is *McShane integrable* (*$\mathcal{M}$-integrable* for short), with McShane integral ϖ, if for every $\varepsilon > 0$ there is a gauge $\Delta : S \to \mathcal{T}$ such that

$$\limsup_{n\to\infty} \|\sigma_n(f, \mathcal{P}_\infty) - \varpi\| \leq \varepsilon,$$

for every generalized McShane partition $\mathcal{P}_\infty$ of S subordinate to Δ. We set $\varpi := (\mathcal{M})\text{-} \int_S f \, d\mu$.

(2) f is $\mathcal{M}$-integrable on a measurable subset E of S, if the function $1_E f$ is $\mathcal{M}$-integrable on S. We set $(\mathcal{M})\text{-} \int_E f \, d\mu := (\mathcal{M})\text{-} \int_S 1_E f \, d\mu$.

Remark 3.1. A function $f : S \to X$ is $\mathcal{M}$-integrable, with McShane integral ϖ, if and only if there is a sequence of gauges (Δ_m) from S into $\mathcal{T}$ such that

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{n\to\infty} \|\sigma_n(f, \mathcal{P}_\infty) - \varpi\| = 0,$$

where $\Pi_\infty(\Delta_m)$ denotes the collection of all generalized McShane partitions of S subordinate to Δ_m.

Before proceeding further, we list below some basic properties of the McShane integral that will be needed in this work. They are borrowed from [2].

Theorem 3.1 (*[2]*). *Let $f : S \to X$ be a function.*
(1) If f is $\mathcal{M}$-integrable on S, then the restriction $f_{|A}$ is $\mathcal{M}$-integrable on S (with respect to the σ-finite outer regular quasi-Radon measure space $(A, A \cap \mathcal{T}, A \cap \Sigma, \mu_{|A})$), for every $A \subset S$.

(2) Let $E \in \Sigma$. Then f is $\mathcal{M}$-integrable on E if and only if $f_{|E}$ is $\mathcal{M}$-integrable on E, and in this case the integrals are equal.

(3) Suppose $X = \mathbb{R}$. Then f is $\mathcal{M}$-integrable, if and only if it is integrable in the ordinary sense, and the two integrals are equal.

Recall that for compact Radon measure space $(S, \mathcal{T}, \Sigma, \mu)$, generalized McShane partitions can be replaced by *finite strict generalized McShane partitions* of S (that is, finite partial McShane partitions $\{(E_i, t_i)\}_{1 \le i \le p}$ such that $\cup_{i=1}^{i=p} E_i = S$):

Proposition 3.1 (*Proposition 1E, [2]*). *Suppose that $(S, \mathcal{T}, \Sigma, \mu)$ is a compact Radon measure space and let $f : S \to X$ be a function. Then f is $\mathcal{M}$-integrable on S, with McShane integral ϖ, if and only if for every $\varepsilon > 0$ there is a gauge $\Delta : S \longrightarrow \mathcal{T}$ such that*

$$\| \sum_{i=1}^{i=p} \mu(E_i) f(t_i) - \varpi \| \le \varepsilon,$$

for all finite strict generalized McShane partitions $(E_i, t_i)_{1 \le i \le p}$ of S subordinate to Δ.

Lemma 3.1 (*The strong Saks-Henstock Lemma*). ***(Lemma 3B, [2])***.
Let $f : S \to X$ be a function $\mathcal{M}$-integrable on S and $\varepsilon > 0$. Then there exists a gauge $\Delta : S \longrightarrow \mathcal{T}$ such that

$$\sup_{\{(E_i,t_i)\}_{1 \le i \le p} \in P\Pi_f(\Delta)} \| \sum_{i=1}^{i=p} \mu(E_i) f(t_i) - (\mathcal{M})\text{-} \int_{\cup_{i=1}^{i=p} E_i} f \, d\mu \| \le \varepsilon,$$

where $P\Pi_f(\Delta)$ denotes the collection of all finite partial McShane partitions of S subordinate to Δ.

Definition 3.2. (**Definition 3.2, [12]**). (1) A function $f : S \to X$ is said to be *weakly McShane integrable* ($\mathcal{WM}$-integrable for short) on S, with weak McShane integral ϖ, if there is a sequence of gauges (Δ_m) from S into $\mathcal{T}$ such that the following property holds

$$(*) \quad \lim_{m \to \infty} \limsup_{n \to \infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty^m) \rangle - \langle x^*, \varpi \rangle| = 0,$$

for every $x^* \in X^*$ and for every sequence $(\mathcal{P}_\infty^m)$ of generalized McShane partitions of S adapted to (Δ_m) (i.e. $\mathcal{P}_\infty^m$ is subordinate to Δ_m for each $m \geq 1$).

We set $\varpi = (\mathcal{WM})\text{-}\int_S f\, d\mu$.

(2) f is $\mathcal{WM}$-integrable on a measurable subset E of S, if the function $1_E f$ is $\mathcal{WM}$-integrable on S. We set $(\mathcal{WM})\text{-}\int_E f\, d\mu := (\mathcal{WM})\text{-}\int_S 1_E f\, d\mu$.

(3) f is $\mathcal{WM}$-integrable on Σ, if it is $\mathcal{WM}$-integrable on every measurable subset of S.

According to (Proposition 3.2, [12]), $(*)$ may be replaced with:

$$\lim_{m\to\infty} \sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle| = 0 \text{ for all } x^* \in X^*,$$

where $\Pi_\infty(\Delta_m)$ denotes the collection of all generalized McShane partitions of S subordinate to Δ_m.

Lemma 3.2 (*The weak Saks-Henstock lemma*)**.** ***(Lemma 3.2, [12]).*** *Let $f : S \to X$ be a $\mathcal{WM}$-integrable function. Then there exists a sequence (Δ_m) of gauges from S into $\mathcal{T}$ such that*

$$\lim_{m\to\infty} \sup_{\{(E_i,t_i)\}_{1\leq i\leq p} \in P\Pi_f(\Delta_m)} |\langle x^*, \sum_{i=1}^{i=p} \mu(E_i) f(t_i)\rangle - \int_{\cup_{i=1}^{i=p} E_i} \langle x^*, f\rangle\, d\mu| = 0$$

for all $x^* \in X^*$,

where $P\Pi_f(\Delta_m)$ denotes the collection of all finite partial McShane partitions of S subordinate to Δ_m.

Now we define our new notion of McShane integrability namely McShane integrability in the limit:

Definition 3.3. (1) A function $f : S \to X$ is said to be *McShane intgrable in the limit* ($\mathcal{M}$-integrable in the limit for short) on S, with McShane integral in the limit ϖ, if for every $\varepsilon > 0$ there is a gauge $\Delta : S \to \mathcal{T}$ such that

$$\limsup_{n\to+\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle| \leq \varepsilon,$$

for all $x^* \in \overline{B}_{X^*}$ and for every generalized McShane partition $\mathcal{P}_\infty$ of S subordinate to Δ. We set $\varpi := (\mathcal{ML})\text{-}\int_S f\,d\mu$.

(2) f is $\mathcal{M}$-integrable in the limit on a measurable subset E of S, if the function $1_E f$ is $\mathcal{M}$-integrable in the limit on S. We set $(\mathcal{ML})\text{-}\int_E f\,d\mu = (\mathcal{ML})\text{-}\int_S 1_E.f\,d\mu$.

(3) f is $\mathcal{M}$-integrable in the limit on Σ, if it is $\mathcal{M}$-integrable in the limit on every measurable subset of S.

The McShane integral in the limit is the concept intermediate between to the McShane integral and the weak McShane integral:

Proposition 3.2. *Let $f : S \to X$ be a function. Then, f is $\mathcal{M}$-integrable on $S \Longrightarrow f$ is $\mathcal{M}$-integrable in the limit on $S \Longrightarrow f$ is $\mathcal{WM}$-integrable on S.*

PROOF. The first implication is obvious. For a proof of the second implication. Assume that f is $\mathcal{M}$-integrable in the limit on S. Then for each $m \geq 1$ there is a gauge $\Delta_m : S \longrightarrow \mathcal{T}$ such that

$$\sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta_m)} \limsup_{n\to+\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, (\mathcal{ML})\text{-}\int_S f\,d\mu\rangle| \leq \frac{1}{m},$$

for all $x^* \in \overline{B}_{X^*}$, where $\Pi_\infty(\Delta_m)$ denotes the collection of all generalized McShane partitions of S subordinate to Δ_m, by letting $m \to +\infty$, we conclude that f is $\mathcal{WM}$-integrable on S. $\square$

The next theorem provides the linearity properties of the McShane integral in the limit.

Theorem 3.2. *Let f, g: $S \to X$ be two functions.*
(1) *If f and g are $\mathcal{M}$-integrable in the limit on S and $\alpha \in \mathbb{R}$, then $\alpha f + g$ is $\mathcal{M}$-integrable in the limit on S, and we have*

$$(\mathcal{ML})\text{-}\int_S \alpha f + g\,d\mu = \alpha(\mathcal{ML})\text{-}\int_S f\,d\mu + (\mathcal{ML})\text{-}\int_S g\,d\mu$$

(2) *if f is $\mathcal{M}$-integrable in the limit on S and if $f = g$ μ-p.p, then g is $\mathcal{M}$-integrable in the limit on S, and we have*

$$(\mathcal{ML})\text{-}\int_S g\,d\mu = (\mathcal{ML})\text{-}\int_S f\,d\mu.$$

(3) *If f is $\mathcal{M}$-integrable in the limit on E, then it is scalarly integrable on E (that is, $\langle x^*, f\rangle$ is Lebesgue integrable on E for all $x^* \in X^*$), and we have*

$$\int_E \langle x^*, f\rangle \, d\mu = \langle x^*, (\mathcal{ML})\text{-}\int_E f \, d\mu\rangle \text{ for all } x^* \in X^*.$$

(4) *If f is $\mathcal{M}$-integrable in the limit on Σ, then is Pettis integrable, and we have*

$$(\mathcal{ML})\text{-}\int_E f \, d\mu = (\mathcal{P}e)\text{-}\int_E f \, d\mu \text{ for all } E \in \Sigma.$$

PROOF. We will prove (2) only; the rest of the proof is straightforward. Set $\theta := f - g$. Since $\theta := 0$ μ-a.e., by ([2], Corollary 2G), θ is $\mathcal{M}$-integrable on S, therefore $\mathcal{M}$-integrable in the limit on S. In turn, by (1), $g = f + \theta$ is $\mathcal{M}$-integrable in the limit on S. □

Proposition 3.3. *Let $f : S \to X$ be a function. If f is $\mathcal{M}$-integrable in the limit on S, then for every $\varepsilon > 0$, there is a gauge $\Delta : S \longrightarrow \mathcal{T}$ such that*

$$\limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, f\rangle \, d\mu| \leq \varepsilon$$

for all $E \in \Sigma$, for all $x^* \in \overline{B}_{X^*}$, and for every generalized McShane partition $\mathcal{P}_\infty$ of E subordinate to Δ.

PROOF. Let $\varepsilon > 0$ By theorem 3.2(3) and theorem 3.1(3), there is a gauge Δ from S to $\mathcal{T}$ such that

$$\sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta)} \limsup_{n\to+\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_S \langle x^*, f\rangle \, d\mu| \leq \frac{\varepsilon}{2},$$

for all $x^* \in \overline{B}_{X^*}$. Let $E \in \Sigma$. We can then repeat mutatis mutandis the arguments used in the proof of [2], Theorem 1N) for each function $\langle x^*, f\rangle$, $x^* \in \overline{B}_{X^*}$ to obtain

$$\sup_{\mathcal{P}_\infty, \mathcal{Q}_\infty \in \Pi_{\infty|E}(\Delta)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle| \leq \frac{\varepsilon}{2}. \tag{3.3.1}$$

On the other hand, as $\langle x^*, f\rangle_{|E}$ is $\mathcal{M}$-integrable (by Theorem 3.2(4) and Theorem 3.1*(2)-(3)*) we may select a gauge Λ from S to $\mathcal{T}$ (which may depend on x^*) with $\Lambda(t) \subset \Delta(t)$ for all $t \in S$ such that

$$\sup_{\mathcal{Q}_\infty \in \Pi_{\infty|E}(\Lambda)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle - \int_E \langle x^*, f\rangle \, d\mu| \leq \frac{\varepsilon}{2}. \tag{3.3.2}$$

Now, by the triangle inequality and the fact that $\Lambda(t) \subset \Delta(t)$ for all $t \in S$, we have

$$\begin{aligned} & \sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, f\rangle \, d\mu| \\ \leq & \sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta), \mathcal{Q}_\infty \in \Pi_{\infty|E}(\Lambda)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle| \\ + & \sup_{\mathcal{Q}_\infty \in \Pi_{\infty|E}(\Lambda)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle - \int_E \langle x^*, f\rangle \, d\mu| \\ \leq & \sup_{\mathcal{P}_\infty, \mathcal{Q}_\infty \in \Pi_{\infty|E}(\Delta)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle| \\ + & \sup_{\mathcal{Q}_\infty \in \Pi_{\infty|E}(\Lambda)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{Q}_\infty)\rangle - \int_E \langle x^*, f\rangle \, d\mu|. \end{aligned}$$

Hence, by (3.3.1) and (3.3.2)

$$\sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, f\rangle \, d\mu| \leq \varepsilon.$$

□

As a consequence of Proposition 3.3, we have:

Corollary 3.1. *Let $f : S \to X$ be a function and let $F \in \Sigma$. If f is $\mathcal{M}$-integrable in the limit on S and Pettis integrable on F (that is, $1_F f$ is Pettis integrable), then $f_{|E\cap F}$ is $\mathcal{M}$-integrable in the limit on $E \cap F$ for every $E \in \Sigma$, and we have*

$$(\mathcal{ML})\text{-}\int_{E\cap F} f_{|E\cap F} \, d\mu = (\mathcal{P}e)\text{-}\int_E 1_F f \, d\mu.$$

Corollary 3.2. *A function $f : S \to X$ is $\mathcal{M}$-integrable in the limit on Σ if and only if it is $\mathcal{M}$-integrable in the limit on S and Pettis integrable, and the corresponding integrals are equal.*

PROOF. The only "if part" is proved by Theorem 3.2(4). Whereas the "if part" is a direct consequence of Proposition 3.3 and Corollary 3.1. □

Lemma 3.3 (*The Saks-Henstock Lemma in the limit*)**.** *Let $f : S \to X$ be a function $\mathcal{M}$-integrable in the limit on S and $\varepsilon > 0$. Then there exists a gauge $\Delta : S \longrightarrow \mathcal{T}$ such that*

$$\sup_{\{(E_i,t_i)\}_{1\leq i\leq p} \in P\Pi_f(\Delta)} |\langle x^*, \sum_{i=1}^{i=p} \mu(E_i) f(t_i)\rangle - \int_{\cup_{i=1}^{i=p} E_i} \langle x^*, f\rangle \, d\mu| \leq \varepsilon,$$

for all $x^* \in \overline{B}_{X^*}$*, where* $P\Pi_f(\Delta)$ *denotes the collection of all finite partial McShane partitions of* S *subordinate to* Δ*.*

PROOF. We will follow the same line of reasoning as in the proof of ([2], Lemma 2B) with suitable modifications. Let $x^* \in \overline{B}_{X^*}$ and $\varepsilon > 0$. By the hypothesis there exists a gauge $\Delta : S \longrightarrow \mathcal{T}$ such that

$$\sup_{\mathcal{P}_\infty \in \Pi_\infty(\Delta)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, (\mathcal{ML})\text{-}\int_S f\, d\mu\rangle| \leq \frac{\varepsilon}{2}.$$

Let $\{(E_i, t_i)\}_{1\leq i\leq p}$ be a member of $P\Pi_f(\Delta)$. Let $E := \cup_{i=1}^{i=p} E_i$. As $\langle x^*, f\rangle_{|S\setminus E}$ is $\mathcal{M}$-integrable (by Theorem 3.1 *(3)*), we may select a generalized McShane partition $\{(F_i, u_i)\}_{i\geq 1}$ of $S \setminus E$ (which may depends on x^*) subordinate to Δ such that

$$\limsup_{n\to\infty} |\langle x^*, \sum_{i=1}^{i=n} \mu(F_i) f(u_i)\rangle - \int_{S\setminus E} \langle x^*, f\rangle\, d\mu| \leq \frac{\varepsilon}{2}.$$

Set

$$E_{p+i} := F_i \text{ and } t_{p+i} := u_i \quad i \geq 1.$$

Then $\{(E_i, t_i)\}_{i\geq 1}$ is a generalized McShane partition of S that is subordinate to Δ and

$$\begin{aligned}
|\langle x^*, \sum_{i=1}^{i=p} \mu(E_i) f(t_i) - \int_E \langle x^*, f\rangle\, d\mu| &= |\langle x^*, \sum_{i=1}^{i=p+n} \mu(E_i) f(t_i)\rangle - \int_S \langle x^*, f\rangle\, d\mu \\
&- \langle x^*, \sum_{i=1}^{i=n} \mu(F_i) f(u_i)\rangle + \int_{S\setminus E} \langle x^*, f\rangle\, d\mu| \\
&\leq |\langle x^*, \sum_{i=1}^{i=p+n} \mu(E_i) f(t_i)\rangle - \int_S \langle x^*, f\rangle\, d\mu| \\
&+ |\langle x^*, \sum_{i=1}^{i=n} \mu(F_i) f(u_i)\rangle - \int_{S\setminus E} \langle x^*, f\rangle\, d\mu|.
\end{aligned}$$

Letting $n \to \infty$, we get

$$|\langle x^*, \sum_{i=1}^{i=p} \mu(E_i) f(t_i)\rangle - \int_E \langle x^*, f\rangle\, d\mu| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By taking the supremum over $\{(E_i,t_i)\}_{1\leq i\leq p} \in P\Pi_f(\Delta)$ in this inequality yields

$$\sup_{\{(E_i,t_i)\}_{1\leq i\leq p}\in P\Pi_f(\Delta)} |\langle x^*, \sum_{i=1}^{i=p} \mu(E_i) f(t_i)\rangle - \int_{\cup_{i=1}^{i=p} E_i} \langle x^*, f\rangle \, d\mu| \leq \varepsilon.$$

□

If S is compact the concept of the McShane integral in the limit and the McShane integral are equivalent:

Corollary 3.3. *Suppose that $(S, \mathcal{T}, \Sigma, \mu)$ is a compact Radon measure space and let $f : S \to X$ be a function. Then f is $\mathcal{M}$-integrable on S if and only if it is $\mathcal{M}$-integrable in the limit on S and the two integrals are equal.*

PROOF. As consequence of Lemma 3.3 and Proposition 3.1. □

Corollary 3.4. *Let $f : S \longrightarrow X$ be a function and Suppose that the set $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is equi-continuous. Then f is $\mathcal{M}$-integrable on S if and only if it is $\mathcal{M}$-integrable in the limit on S and the two integrals are equal.*

PROOF. Let $\varepsilon > 0$. By Saks-Henstock Lemma in the limit (Lemma 3.3), there is a gauge $\Delta : S \longrightarrow \mathcal{T}$ such that

$$\sup_{\{(F_i,u_i)\}_{1\leq i\leq p}\in P\Pi_f(\Delta)} |\langle x^*, \sum_{i=1}^{i=p} \mu(F_i) f(u_i)\rangle - \int_{\cup_{i=1}^{i=p} F_i} \langle x^*, f\rangle \, d\mu| \leq \frac{\varepsilon}{2},$$

for all $x^* \in \overline{B}_{X^*}$. Let $E \in \Sigma_f$, by hypothesis, there exists $\eta > 0$ such that

$$\sup_{x^*\in\overline{B}_{X^*}} |\int_F \langle x^*, f\rangle \, d\mu| \leq \frac{\varepsilon}{2},$$

for all $F \in \Sigma$ with $\mu(E \cap F) \leq \eta$ (Definition 2.2). Let $\mathcal{P}_\infty := \{(E_i, t_i)\}_{i\geq 1}$ be fixed generalized McShane partition of S subordinate to Δ and $x^* \in \overline{B}_{X^*}$ and choose an integer $n_0 \geq 1$ such that $\mu(E \cap \cup_{i=n}^{i=\infty} E_i) \leq \eta$ for all $n \geq n_0$. Then

$$\sup_{x^*\in\overline{B}_{X^*}} |\int_{\cup_{i=n}^{i=\infty} E_i} \langle x^*, f\rangle \, d\mu| \leq \frac{\varepsilon}{2} \text{ for all } n \geq n_0.$$

According, using Theorem 3.2 (3), we obtain

$$\begin{aligned}
&|\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \langle x^*, (\mathcal{ML})\text{-}\int_S f\, d\mu\rangle| \\
\leq\ & |\langle x^*, \sigma_n(f, \mathcal{P}_\infty)\rangle - \int_{\cup_{i=1}^{i=n} E_i} \langle x^*, f\rangle\, d\mu| + |\int_{\cup_{i=n}^{i=\infty} E_i} \langle x^*, f\rangle\, d\mu| \\
\leq\ & \sup_{\{(F_i,u_i)\}_{1\leq i\leq p}\in P\Pi_f(\Delta)} |\langle x^*, \sum_{i=1}^{i=p} \mu(F_i) f(u_i)\rangle - \int_{\cup_{i=1}^{i=p} F_i} \langle x^*, f\rangle\, d\mu| \\
+\ & |\int_{\cup_{i=n}^{i=\infty} E_i} \langle x^*, f\rangle\, d\mu| \\
\leq\ & \frac{\varepsilon}{2} + |\int_{\cup_{i=n}^{i=\infty} E_i} \langle x^*, f\rangle\, d\mu|.
\end{aligned}$$

for all $n \geq n_0$. Taking the supremum over $\overline{B}_{X^*}$ in the above estimation yields

$$\|\sigma_n(f, \mathcal{P}_\infty) - (\mathcal{ML})\text{-}\int_S f\, d\mu\| \leq \frac{\varepsilon}{2} + \sup_{x^*\in\overline{B}_{X^*}} |\int_{\cup_{i=n}^{i=\infty} E_i} |\langle x^*, f\rangle\, d\mu| \leq \varepsilon,$$

for all $n \geq n_0$. Consequently,

$$\limsup_{n\to+\infty} \|\sigma_n(f, \mathcal{P}_\infty) - (\mathcal{ML})\text{-}\int_S f\, d\mu\| \leq \varepsilon.$$

Thus f is $\mathcal{M}$-integrable on S. □

Corollary 3.5. *Let $f : S \longrightarrow X$ a function. On suppose that there is $h \in L^1_{\mathbb{R}^+}(\mu)$ such that*

$$(**)\ \|f(t)\| \leq h(t)\ a.e.$$

Then f is $\mathcal{M}$-integrable on S if and only if it is $\mathcal{M}$-integrable in the limit on S and the two integrals are equal.

PROOF. Assume that f is $\mathcal{M}$-integrable in the limit on S. By the inequality $(**)$, the collection $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is equi-continuous. Then f is $\mathcal{M}$-integrable on S. □

Corollary 3.6. *Let $f : S \to X$ be a function. Then f is $\mathcal{M}$-integrable on S if and only if it is Pettis integrable and $\mathcal{M}$-integrable in the limit on S, and in this case the integrals are equal.*

PROOF. Assume that f is $\mathcal{M}$-integrable in the limit on S and Pettis integrable. The Pettis integrability of f yields that the set $\{\langle x^*, f\rangle, x^* \in \overline{B}_{X^*}\}$ is relatively compact in $L^1_{\mathbb{R}}(\mu)$ ([9], p. 162). Then it is equi-continuous (Theorem 2.2). Consequently f is $\mathcal{M}$-integrable on S. □

Corollary 3.7. *A function $f : S \to X$ is $\mathcal{M}$-integrable on S if and only if it is $\mathcal{M}$-integrable in the limit on Σ.*

PROOF. The proof is consequence of Corollaries 3.1 and 3.6. $\square$

In order to pass from McShane integrability in the limit to McShane integrability, we introduce the following new concept of local upper McShane boundedness.

Definition 3.4. Let $\tau > 0$. A function $f : S \longrightarrow \mathbb{R}$ is said to be *locally τ-upper McShane bounded* if for each gauge $\Delta : S \longrightarrow \mathcal{T}$ and for each $E \in \Sigma^+$, there is an $A \in \Sigma^+(E)$ and generalized McShane partition $\mathcal{P}_\infty = \{(A_i, t_i)\}_{i\geq 1}$ of A subordinate to Δ such that

$$\frac{1}{\mu(A)} \limsup_{n\to+\infty} \sigma_n(f, \mathcal{P}_\infty) \leq \tau.$$

A function $f : S \longrightarrow X$ is said to be *scalarly locally τ-upper McShane bounded* if, for each $x^* \in \overline{B}_{X^*}$, $\langle x^*, f\rangle$ is locally τ-upper McShane bounded.

The following technical lemma it used to proof the lemma 4.1 with an exhaustion-type argument.

Lemma 3.4. *Let $(a_i)_{i\geq 1}$ be a sequence in $\mathbb{R}$ and $(\lambda_i)_{i\geq 1}$ be a sequence in $\mathbb{R}^+$ such that $\sum_{i=1}^{i=\infty} \lambda_i = 1$. Then for each $\varepsilon > 0$, there is $i_0 \geq 1$ such that*

$$a_{i_0} \leq r_\varepsilon := \limsup_{n\to+\infty} \sum_{i=1}^{i=n} \lambda_i a_i + \varepsilon.$$

PROOF. **Case. 1** If $\limsup_{n\to+\infty} \sum_{i=1}^{i=n} \lambda_i a_i = +\infty$, there is nothing to prove.
Case. 2 If $\limsup_{n\to+\infty} \sum_{i=1}^{i=n} \lambda_i a_i < +\infty$. Assume that there exists $\varepsilon > 0$ such that

$$a_i > r_\varepsilon \text{ for all } i \geq 1.$$

Then, by taking n large enough, we obtain

$$\sum_{i=1}^{i=n} \lambda_i a_i > r_\varepsilon(\sum_{i=1}^{i=n} \lambda_i).$$

Finally, passing to the $\limsup_{n\to+\infty}$ in the previous inequality, we get

$$r_\varepsilon - \varepsilon = \limsup_{n\to+\infty} \sum_{i=1}^{i=n} \lambda_i a_i \geq r_\varepsilon(\lim_{n\to+\infty} \sum_{i=1}^{i=n} \lambda_i) = r_\varepsilon,$$

for which is absurd. $\square$

4 McShane integrability in the limit versus McShane integrability

In this section we attempt to determine a relation that may exist between the McShane integral in the limit and the McShane integral. The smallest class of functions for which the two integrals are equivalent is the class of scalarly locally τ-upper McShane bounded functions:

Theorem 4.1. *Let $f : S \longrightarrow X$ be a function. If the following two conditions hold,*

(i) f is $\mathcal{M}$-integrable in the limit on S

(ii) f is scalarly locally τ-upper McShane bounded for some $\tau > 0$,

then f is $\mathcal{M}$-integrable on S and the two integrals are equal.

The proof of Theorem 4.1 involves the following exhaustion type Lemma.

Lemma 4.1. *Let $f : S \longrightarrow \mathbb{R}$ a function locally τ-upper McShane bounded for some $\tau > 0$ and L be a member of Σ^+. Then, given any gauge $\Delta : S \longrightarrow \mathcal{T}$, there exists a generalized McShane partition $\mathcal{P}_\infty := \{E_i, t_i)\}_{i\geq 1}$ of L subordinate to Δ such that*

$$f(t_i) \leq \tau + 1 \textit{ for all } i \geq 1$$

PROOF. *1) Case μ is finite.* The proof is an exhaustion-type argument in the spirit of [10]. Fixed a gauge $\Delta : S \longrightarrow \mathcal{T}$.
Let $\mathcal{A}_1$ denote the collection of subsets $E \in \Sigma^+(L)$ such that there is $t \in L$ for which

$$E \subset \Delta(t) \text{ and } f(t) \leq \tau + 1.$$

Since f is locally τ-upper McShane bounded, there is $A \in \Sigma^+(L)$ and generalized McShane partition $\mathcal{P}_\infty = \{(A_i, t_i)\}_{i\geq 1}$ of A subordinate to Δ such that

$$\frac{1}{\mu(A)} \limsup_{n\to+\infty} \sigma_n(f, \mathcal{P}_\infty) \leq \tau.$$

Using lemma 3.4, there is $(A, t_A) \in \mathcal{P}_\infty$ such that

$$A \subset \Delta(t_A) \text{ and } f(t_A) \leq \tau + 1.$$

Thus the collection $\mathcal{A}_1$ is not empty. If there is a set $E \in \mathcal{A}_1$ with $\mu(S\setminus E) = 0$, then we are finished. Otherwise, let l_1 be the smallest positive integer for which there is a set $E_1 \in \mathcal{A}_1$ with $\frac{1}{l_1} \leq \mu(E_1) < \mu(L)$. According there is $t_1 \in L$ such that

$$E_1 \subset \Delta(t_1) \text{ and } f(t_1) \leq \tau + 1.$$

Let $\mathcal{A}_2$ denote the collection of subsets $E \in \Sigma^+(L \setminus E_1)$ such that there is $t \in L$ such that

$$E \subset \Delta(t) \text{ and } f(t) \leq \tau + 1.$$

Since f is locally τ-upper McShane bounded, there is $A \in \Sigma^+(L \setminus E_1)$ and generalized McShane partition $\mathcal{P}_\infty = \{(A_i, t_i)\}_{i \geq 1}$ of A subordinate to Δ such that

$$\frac{1}{\mu(A)} \limsup_{n \to +\infty} \sigma_n(f, \mathcal{P}_\infty) \leq \tau.$$

Applying again Lemma 3.4, there is $(A, t_A) \in \mathcal{P}_\infty$ such that

$$A \subset \Delta(t_A) \text{ and } f(t_A) \leq \tau + 1.$$

Thus the collection $\mathcal{A}_2$ is not empty. If there is a set $E \in \mathcal{A}_2$ with $\mu(L \setminus E_1 \cup E) = 0$, then we are finished. Otherwise, let l_2 be the smallest positive integer for which there is a set $E_2 \in \mathcal{A}_2$ such that $\frac{1}{l_2} \leq \mu(E_2) < \mu(L)$. Thus there exists $t_2 \in L$ with

$$E_2 \subset \Delta(t_2) \text{ and } f(t_2) \leq \tau + 1.$$

Continue in this way. If the process stops in finite numbers of steps then we are finished. If the process does not stop, then we obtain a countable family (E_i) of pairwise disjoint measurable subsets of L and a sequence (t_i) in L such that

$$\frac{1}{l_i} \leq \mu(E_i) < \mu(L), \; E_i \subset \Delta(t_i) \text{ and}$$

$$f(t_i) \leq \tau + 1 \text{ for all } i \geq 1$$

(l_i being the smallest positive integer for which there is $E \in \mathcal{A}_i$ with $\frac{1}{l_i} \leq \mu(E) < \mu(L)$).
Set $E_\infty := \cup_{i=1}^{i=\infty} E_i$. We claim that $\mu(L \setminus E_\infty) = 0$. Indeed, if $\mu(L \setminus E_\infty) > 0$, then the local τ-upper McShane boundedness ensures the existence of $A \in \Sigma^+$ contained in $L \setminus E_\infty$ and $t \in L$ such that

$$A \subset \Delta(t) \text{ and } f(t) \leq \tau + 1.$$

Since for each positive integer $n \geq 1$

$$\sum_{i=1}^{i=n} \frac{1}{l_i} \leq \mu(\cup_{i=1}^{i=n} E_i) < \mu(L)$$

and $l_n > 1$, we can choose an integer $n \geq 1$ such that

$$\frac{1}{l_n - 1} < \mu(A).$$

As

$$A \subset L \setminus E_\infty \subset L \setminus \cup_{i=1}^{i=n-1} E_i,$$

we conclude that A is a member of $\mathcal{A}_n$. This contradicts the definition of l_n. Thus $\mu(L \setminus E_\infty) = 0$ as claimed.

2) General case. Let $L \in \Sigma^+$ and (A_k) be sequence of pairwise disjoint measurable subsets of L such that $L = \cup_{k=1}^\infty A_k$ and $\mu_{|L}(A_k) < +\infty$, for all $k \geq 1$. As f is locally τ-upper McShane bounded, by the first case, for each $k \geq 1$, there is a generalized McShane partition $\{E_{k,i}, t_{k,i})\}_{i \geq 1}$ of A_k subordinate to Δ such that

$$f(t_{k,i}) \leq \tau + 1 \text{ for all } i \geq 1.$$

It suffices to verify that

$$\mu(L \setminus \cup_{k=1}^{k=\infty} \cup_{i=1}^{i=\infty} E_{k,i}) = 0.$$

By remarking that

$$\cup_{k=1}^{k=\infty} A_k \setminus (\cup_{k=1}^{k=\infty} \cup_{i=1}^{k=\infty} E_{k,i}) \subset \cup_{k=1}^{k=\infty} (A_k \setminus \cup_{i=1}^{i=\infty} E_{k,i}),$$

we get

$$\begin{aligned} \mu(L \setminus \cup_{k=1}^{k=\infty} \cup_{i=1}^{i=\infty} E_{k,i}) &= \mu(\cup_{k=1}^{k=\infty} A_k \setminus \cup_{k=1}^{k=\infty} \cup_{i=1}^{i=\infty} E_{k,i}) \\ &\leq \mu(\cup_{k=1}^{k=\infty} (A_k \setminus \cup_{i=1}^{i=\infty} E_{k,i})) \\ &= \sum_{k=1}^{k=\infty} \mu(A_k \setminus \cup_{i=1}^{i=\infty} E_{k,i}) = 0, \end{aligned}$$

since

$$\mu(A_k \setminus \cup_{i=1}^{i=\infty} E_{k,i}) = \mu_{|A_k}(A_k \setminus \cup_{i=1}^{i=\infty} E_{k,i}) = 0 \text{ for all } k \geq 1.$$

$\square$

Proof of Theorem 4.1. By virtue of Corollary 3.4 it suffices to prove that condition (ii) implies that the set $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is equi-continuous. Indeed, let $x^* \in \overline{B}_{X^*}$ and a fixed $E \in \Sigma^+$. As $\langle x^*, 1_E f\rangle$ is integrable, by

Theorem 3.1 (3) and the strong Saks-Henstock lemma (Lemma 3.1), we may select a gauge $\Delta : S \longrightarrow \mathcal{T}$ such that

$$\sup_{\{(F_i,u_i)\}_{1\le i\le p}\in P\Pi_{f_{|E}}(\Delta)} |\langle x^*, \sum_{i=1}^{i=p} \mu(F_i) 1_E f(u_i)\rangle - \int_{\cup_{i=1}^{i=p} F_i} \langle x^*, 1_E f\rangle \, d\mu| \le \mu(E),$$

where $P\Pi_{f_{|E}}(\Delta)$ denotes the collection of all finite partial McShane partitions of E subordinate to Δ. Now, Lemma 4.1 ensures the existence of a generalized McShane partition $\{(A_i, t_i)\}_{i\ge 1}$ of E subordinate to Δ such that

$$\langle x^*, f(t_i)\rangle \le \tau + 1 \text{ for all } i \ge 1.$$

Next, because $\mathcal{P}_m := \{(E \cap A_i, t_i)\}_{1\le i\le m} \in P\Pi_{f_{|E}}(\Delta)$ for all $m \ge 1$, we get

$$\begin{aligned} & |\langle x^*, \sigma(1_E f, \mathcal{P}_m)\rangle - \int_{\cup_{i=1}^{i=m} E\cap A_i} \langle x^*, f\rangle \, d\mu| \\ \le & \sup_{\{(F_i,u_i)\}_{1\le i\le p}\in P\Pi_{f_{|E}}(\Delta)} |\langle x^*, \sum_{i=1}^{i=p} \mu(F_i) 1_E f(u_i)\rangle - \int_{\cup_{i=1}^{i=p} F_i} \langle x^*, 1_E f\rangle \, d\mu| \le \mu(E) \end{aligned}$$

for all $m \ge 1$. Whence

$$\begin{aligned} \int_{E\cap\cup_{i=1}^{i=m} A_i} \langle x^*, f\rangle \, d\mu &\le \langle x^*, \sigma(1_E f, \mathcal{P}_m)\rangle + \mu(E) \\ &\le \sup_{i\ge 1} \langle x^*, 1_E f(t_i)\rangle \sum_{i=1}^{i=m} \mu(E\cap A_i) + \mu(E) \\ &\le (\tau+1)\mu(E \cap \cup_{i=1}^{i=m} A_i) + \mu(E) \\ &\le (\tau+2)\mu(E), \end{aligned}$$

for every $m \ge 1$. As

$$\int_E \langle x^*, f\rangle \, d\mu = \lim_{m\to\infty} \int_{E\cap\cup_{i=1}^{i=m} A_i} \langle x^*, f\rangle \, d\mu,$$

(since $(\cup_{i=1}^{i=m} A_i)_m$ is an increasing sequence and $\mu(E \setminus \cup_{i=1}^{i=\infty} A_i) = 0$), the above estimation yields

$$\int_E \langle x^*, f\rangle \, d\mu \le (\tau+2)\mu(E).$$

By arbitrariness of x^* and E , we get

$$|\int_E \langle x^*, f\rangle \, d\mu| \le (\tau+2)\mu(E).$$

Since this holds for all $x^* \in \overline{B}_{X^*}$ and for every $E \in \Sigma^+$, by virtue of the remark 2.1 of [12], we conclude that $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is equi-continuous. The following Theorem presents a decomposition relation between the McShane integral in the limit and the McShane integral:

Theorem 4.2. *If a function $f : S \to X$ is $\mathcal{M}$-integrable in the limit on S, then there exists an increasing sequence $(S_\ell)_{\ell\geq 1}$ in Σ_f with union S such that $1_{S_\ell} f$ is $\mathcal{M}$-integrable on S for each $\ell \geq 1$.*

The proof of Theorem 4.2 involves the following Lemmas.

Lemma 4.2. *If $f : S \to X$ is a scalarly integrable function, then there exists an increasing sequence $(S_\ell)_{\ell\geq 1}$ in Σ_f with union S such that $\{\langle x^*, 1_{S_\ell} f\rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable for each $\ell \geq 1$.*

PROOF. Since μ is σ-finite, there is an increasing sequence $(R_k)_{k\geq 1}$ in Σ_f such that $S = \cup_{k\geq 1} R_k$. For each $k \geq 1$, set

$$C_k := \{t \in R_k : \|f(t)\| \leq k\}.$$

Then $(C_k)_{k\geq 1}$ is an increasing sequence with union S and $\mu^*(C_k) < \infty$ for all $k \geq 1$, where μ^* stands for the outer measure induced by μ. Let $D_k \in \Sigma_f$ be such that $C_k \subset D_k$ and $\mu(D_k) = \mu^*(C_k)$. Since $\langle x^*, f\rangle$ is uniformly bounded on C_k and $\mu^*(C_k) = \mu(D_k)$, $\langle x^*, f\rangle$ is uniformly bounded almost everywhere on D_k. Set

$$S_\ell := \cup_{k=1}^{k=\ell} D_k \qquad \ell \geq 1.$$

Clearly, $(S_\ell)_{\ell\geq 1}$ is a non-decreasing sequence in Σ_f with union S. Further, the function $\langle x^*, 1_{S_\ell} f\rangle$ is uniformly bounded almost everywhere for each $\ell \geq 1$, in turn $\{\langle x^*, 1_{S_\ell} f\rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable. $\square$

Lemma 4.3. *Let $f : S \to X$ be a function and let $E \in \Sigma$. Then $1_E f$ is $\mathcal{M}$-integrable in the limit on S if and only if the restriction $f_{|E}$ is $\mathcal{M}$-integrable in the limit on E, and the two integrals are equal.*

PROOF. Set $g := 1_E f$. Let $\varepsilon > 0$. If g is $\mathcal{M}$-integrable in the limit on S, then by Proposition 3.3, there exists a gauge $\Delta : S \longrightarrow \mathcal{T}$ such that

$$\sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(g, \mathcal{P}_\infty)\rangle - \int_E \langle x^*, g\rangle \, d\mu| \leq \varepsilon$$

for all $x^* \in \overline{B}_{X^*}$ with

$$\int_E \langle x^*, g\rangle \, d\mu = \int_S 1_E \langle x^*, g\rangle \, d\mu = \int_S \langle x^*, g\rangle \, d\mu = \langle x^*, (\mathcal{ML})\text{-}\int_S g \, d\mu\rangle,$$

where the last equality follows from Theorem 3.2(3). As $g_{|E} = f_{|E}$, we obtain

$$\sup_{\mathcal{P}_\infty \in \Pi_{\infty|E}(\Delta)} \limsup_{n\to\infty} |\langle x^*, \sigma_n(f,\mathcal{P}_\infty)\rangle - \langle x^*, (\mathcal{ML})\text{-}\int_S g\, d\mu\rangle| \leq \varepsilon$$

for all $x^* \in \overline{B}_{X^*}$. Thus $f_{|E}$ is $\mathcal{M}$-integrable in the limit on E, with integral $(\mathcal{ML})$-$\int_E f_{|E}\, d\mu = (\mathcal{ML})$-$\int_S g\, d\mu$. Conversely, suppose that $f_{|E}$ is $\mathcal{M}$-integrable in the limit on E and set $\varpi_E := (\mathcal{ML})$-$\int_E f_{|E}\, d\mu$. We prove that g is $\mathcal{M}$-integrable in the limit on S. Applying the Saks-Henstock Lemma in the limit (Lemma 3.3) to $f_{|E}$, we way select a gauge (Δ_E) from E into $\mathcal{T}$ such that

$$\text{(4.3.1).} \qquad \sup_{\{(F_i,u_i)\}_{1\leq i\leq p} \in P\Pi_f(\Delta_E)} |\langle x^*, \sum_{i=1}^{i=p} \mu(F_i) f(u_i)\rangle - \int_{\cup_{i=1}^{i=p} F_i} \langle x^*, f\rangle\, d\mu| \leq \frac{\varepsilon}{2}$$

for all $x^* \in \overline{B}_{X^*}$. Now for each $n \geq 1$, choose a closed set F_n and an open set O_n with $F_n \subset E \subset O_n$ such that

$$\text{(4.3.2)} \qquad \mu(E \setminus F_n) \leq \frac{1}{n} \text{ and}$$

$$\text{(4.3.3)} \qquad \mu(O_n \setminus E) \leq \frac{2^{-(n+1)}\varepsilon}{n+1}$$

and define the gauge (Δ) of from S into $\mathcal{T}$ by

$$\begin{aligned} \Delta(t) &:= \Delta_E(t) \cap O_n \text{ if } t \in E \text{ and } n \leq \|f(t)\| < n+1 \\ &:= S \setminus F_n \qquad \text{if } t \in S \setminus E. \end{aligned}$$

Let $(\{(E_i,t_i)\}_{i\geq 1}$ be a generalized McShane partition of S subordinate to Δ and for each $i \geq 1$ set

$$\begin{aligned} H_i &:= E_i \cap E \text{ if } t_i \in E \\ &:= \emptyset \qquad \text{otherwise.} \end{aligned}$$

Since $\{(H_i,t_i)\}_{i\geq 1}$ is a partial McShane partition of E subordinate to (Δ_E), (4.3.1) gives

$$\begin{aligned} &|\langle x^*, \sum_{i=1}^{i=n} \mu(H_i) f(t_i)\rangle - \int_{\cup_{i=1}^{i=n} H_i} \langle x^*, f\rangle\, d\mu| \\ \leq &\sup_{\{(F_i,u_i)\}_{1\leq i\leq p} \in P\Pi_f(\Delta_E)} |\langle x^*, \sum_{i=1}^{i=p} \mu(F_i) f(u_i)\rangle - \int_{\cup_{i=1}^{i=p} F_i} \langle x^*, f\rangle\, d\mu| \leq \frac{\varepsilon}{2} \end{aligned}$$

for every $n \geq 1$. Therefore, by the triangle inequality and the definition of H_i, we find that

$$\begin{aligned}
|\langle x^*, \sum_{i=1}^{i=n} \mu(E_i) g(t_i)\rangle \quad &- \quad \int_{\cup_{i=1}^{i=n} H_i} \langle x^*, f\rangle \, d\mu| \\
&\leq \quad |\langle x^*, \sum_{i=1}^{i=n} \mu(E_i) g(t_i)\rangle - \langle x^*, \sum_{i=1}^{i=n} \mu(H_i) f(t_i)\rangle| \\
&+ \quad |\langle x^*, \sum_{i=1}^{i=n} \mu(H_i) f(t_i)\rangle - \int_{\cup_{i=1}^{i=n} H_i} \langle x^*, f\rangle \, d\mu| \\
&\leq \quad \sum_{\{i=1,\ldots,n/t_i \in E\}} \mu(E_i \setminus E) \|f(t_i)\| + \frac{\varepsilon}{2} \\
&= \quad \sum_{k=1}^{k=\infty} \sum_{\{i=1,\ldots,n/t_i \in E,\, k \leq \|f(t_i)\| < k+1\}} \mu(E_i \setminus E) \|f(t_i)\| + \frac{\varepsilon}{2}
\end{aligned}$$

for every $n \geq 1$. As $E_i \subset \Delta(t_i) \subset O_k$ for all $i \geq 1$ such that $t_i \in E$ and $k \leq \|f(t_i)\| < k+1$, we obtain

$$\begin{aligned}
(4.3.4) \quad |\langle x^*, \sum_{i=1}^{i=n} \mu(E_i) g(t_i)\rangle \quad &- \quad \int_{\cup_{i=1}^{i=n} H_i} \langle x^*, f\rangle \, d\mu| \\
&\leq \quad \sum_{k=1}^{k=\infty} (k+1)\mu(O_k \setminus E) + \varepsilon \leq \sum_{k=1}^{k=\infty} 2^{-k} \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

for every $n \geq 1$. On the other hand, we have

$$(4.3.5), \qquad |\int_E \langle x^*, f\rangle \, d\mu - \int_{\cup_{i=1}^{i=\infty} H_i} \langle x^*, f\rangle \, d\mu| \leq \int_{E \setminus F_n} |\langle x^*, f\rangle| \, d\mu$$

because, for every $n \geq 1$

$$\begin{aligned}
E \quad &= \quad [\cup_{i \geq 1, t_i \in E} (E \cap E_i)] \cup [\cup_{i \geq 1, t_i \in S \setminus E} (E \cap E_i)] \\
&\subset \quad (\cup_{i=1}^{i=\infty} H_i) \cup [\cup_{i \geq 1, t_i \in S \setminus E} (E \cap \Delta(t_i))] = (\cup_{i=1}^{i=\infty} H_i) \cup (E \setminus F_n) \subset E,
\end{aligned}$$

in view of the definition of H_i and Δ. Remarking that $\langle x^*, \varpi_E\rangle = \int_E \langle x^*, f_{|E}\rangle \, d\mu =$

$\int_E \langle x^*, f\rangle\, d\mu$ and putting (4.3.4) and (4.3.5) together, we get

$$\begin{aligned} |\langle x^*, \sum_{i=1}^{i=n} \mu(E_i) g(t_i)\rangle - \langle x^*, \varpi_E\rangle| &\leq |\langle x^*, \sum_{i=1}^{i=n} \mu(E_i) g(t_i)\rangle - \int_{\cup_{i=1}^{i=n} H_i} \langle x^*, f\rangle\, d\mu| \\ &+ |\int_E \langle x^*, f\rangle\, d\mu - \int_{\cup_{i=1}^{i=\infty} H_i} \langle x^*, f\rangle\, d\mu| \\ &\leq \varepsilon + \int_{E\setminus F_n} |\langle x^*, f\rangle|\, d\mu \end{aligned}$$

for every $n \geq 1$. As (4.3.1) and the integrability of $\langle x^*, f\rangle$ on E ensure

$$\lim_{n\to\infty} \int_{E\setminus F_n} |\langle x^*, f\rangle|\, d\mu = 0,$$

we obtain

$$\limsup_{n\to\infty} |\langle x^*, \sum_{i=1}^{i=n} \mu(E_i) g(t_i)\rangle - \langle x^*, \varpi_E\rangle| \leq \varepsilon.$$

Thus g is $\mathcal{M}$-integrable in the limit on S with integral ϖ_E. □

Proof of Theorem 4.2. Let $(S_\ell)_{\ell\geq 1}$ be the sequence given in Lemma 4.2. By proposition 3.2 and Lemma 3.3 [12] , we can select a sequence (Δ_m) of gauges, and a fixed sequence of generalized McShane partitions of S adapted to (Δ_m) $(\{(E_i^m, t_i^m)\}_{i\geq 1})_{m\geq 1}$ such that for any fixed $\ell \geq 1$, there exists a strictly increasing sequence $(p_m)_{m\geq 1}$ of positive integers (possibly depending on ℓ) such that

$$\lim_{m\to\infty} \langle x^*, \sum_{i=1}^{i=p_m} \mu(S_\ell \cap E_i^m \cap E) f(t_i^m)\rangle = \int_{S_\ell\cap E} \langle x^*, f\rangle\, d\mu$$

for all $x^* \in X^*$ and for all $E \in \Sigma$. In other words, this equality becomes

$$\lim_{m\to\infty} \int_{S_\ell\cap E} \langle x^*, \sum_{i=1}^{i=p_m} 1_{E_i^m} f(t_i^m)\rangle\, d\mu = \int_{S_\ell\cap E} \langle x^*, f\rangle\, d\mu$$

for all $x^* \in X^*$ and for all $E \in \Sigma$. As the functions $\sum_{i=1}^{i=p_m} 1_{E_i^m} f(t_i^m)$ $(m \geq 1)$ are obviously Pettis integrable and, by Lemma 4.2, the set $\{\langle x^*, 1_{S_\ell} f\rangle : x^* \in \overline{B}_{X^*}\}$ is uniformly integrable, it follows from Theorems 2.2 and 2.1 that $1_{S_\ell} f$ is Pettis integrable. Therefore, by Corollary 3.1, $f_{|S_\ell}$ is $\mathcal{M}$-integrable in the limit on S_ℓ. Equivalently $1_{S_\ell} f$ is $\mathcal{M}$-integrable in the limit on S, in view of Lemma 4.3. The desired conclusion then follows from Corollary 3.6.

5 Beppo Levi's Theorem for the McShane integral in the limit

In this section we state our Beppo Levi's version convergence theorem for the McShane integral in the limit:

Theorem 5.1. *Suppose that μ is finite. Let (f_n) be a sequence of $\mathcal{M}$-integrable in the limit functions from S to X such that $f = \sum_{n=1}^{n=\infty} f_n$ pointwise on S and the series $\sum_{n\geq 1} \|f_n\|_{\mathcal{P}e}$ is convergent.*

Then the series $\sum_{n\geq 1}(\mathcal{ML})\text{-}\int_S f_n\, d\mu$ is convergent, f is $\mathcal{M}$-integrable in the limit on S and

$$(\mathcal{ML})\text{-}\int_S f\, d\mu = \sum_{n=1}^{n=\infty} (\mathcal{ML})\text{-}\int_S f_n\, d\mu$$

Proof. Let $\varepsilon > 0$, $x^* \in \overline{B}_{X^*}$ and $F_n = \sum_{i=1}^{i=n} f_i$. Since

$$\sum_{n=1}^{n=\infty} \|(\mathcal{ML})\text{-}\int_S f_n\, d\mu\| \leq \sum_{n=1}^{n=\infty} \|f_n\|_{\mathcal{P}e} < \infty \text{ (Theorem 3.2 (3))},$$

the series $\sum_{n\geq 1}(\mathcal{ML})\text{-}\int_S f_n\, d\mu$ is (absolutely) convergent by completeness of X. For convenience, set $\varpi = \sum_{n=1}^{n=\infty}(\mathcal{ML})\text{-}\int_S f_n\, d\mu$. By Theorem 3.2 (1) and Saks Henstock Lemma in the limit (Lemma 3.3), for each $n \geq 1$ we may select a gauge Δ_n from S to $\mathcal{T}$ such that

$$\text{(5.1)} \quad \sup_{\{(E_i,t_i)\}_{1\leq i\leq p} \in P\Pi_f(\Delta_n)} |\langle x^*, \sum_{i=1}^{i=p} \mu(E_i) F_n(t_i)\rangle - \int_{\cup_{i=1}^{i=p} E_i} \langle x^*, F_n\rangle\, d\mu| \leq \frac{\varepsilon}{3}\frac{1}{2^n}$$

where $P\Pi_f(\Delta_n)$ denotes the collection of all finite partial McShane partitions of S subordinate to Δ_n. Pick $n_0 \geq 1$ such that $\sum_{n=n_0}^{n=\infty} \|f_n\|_{\mathcal{P}e} < \frac{\varepsilon}{3}$. For every $t \in S$ there exists $n(t) \geq n_0$ such that $n \geq n(t)$ implies $\|F_n(t) - f(t)\| < \frac{\varepsilon}{3}\varphi(t)$, where φ is the function in Lemma 7 [13]. Define a gauge Δ from S to $\mathcal{T}$ by setting $\Delta(t) = \Delta_{n(t)}(t) \cap \Delta_\varphi(t)$, $t \in S$ and let $\mathcal{P}_\infty = \{(E_i, t_i)\}_{i\geq 1}$ be a generalized McShane partition of S subordinate to Δ and $m \geq 1$, then

$$\begin{aligned}
& |\langle x^*, \sigma_m(f, \mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle| \\
= \; & |\langle x^*, \sum_{n=1}^{n=\infty} \sum_{i=1}^{i=m} \mu(E_i) f_n(t_i)\rangle - \langle x^*, \sum_{n=1}^{n=\infty} (\mathcal{ML})\text{-}\int_S f_n \, d\mu\rangle| \\
= \; & |\sum_{n=1}^{n=\infty} \sum_{i=1}^{i=m} \mu(E_i)\langle x^*, f_n(t_i)\rangle - \sum_{n=1}^{n=\infty} \int_S \langle x^*, f_n\rangle \, d\mu| \\
\leq \; & |\sum_{i=1}^{i=m} \{ \sum_{n=1}^{n=\infty} \mu(E_i)\langle x^*, f_n(t_i)\rangle - \sum_{n=1}^{n=\infty} \int_{\cup_{i=1}^{i=m} E_i} \langle x^*, f_n\rangle \, d\mu\}| \\
+ \; & |\sum_{n=1}^{n=\infty} \int_{S \setminus \cup_{i=1}^{i=m} E_i} \langle x^*, f_n\rangle \, d\mu| \\
\leq \; & |\sum_{i=1}^{i=m} \sum_{n=n(t_i)+1}^{n=\infty} \langle x^*, f_n(t_i)\rangle \mu(E_i)| \\
+ \; & |\sum_{i=1}^{i=m} \{ \sum_{n=1}^{n=n(t_i)} \langle x^*, f_n(t_i)\rangle - \sum_{n=1}^{n=n(t_i)} \int_{E_i} \langle x^*, f_n\rangle \, d\mu\}| \\
+ \; & |\sum_{i=1}^{i=m} \sum_{n=n(t_i)+1}^{n=\infty} \int_{E_i} \langle x^*, f_n\rangle \, d\mu| + \sum_{n=1}^{n=\infty} \int_{S \setminus \cup_{i=1}^{i=m} E_i} |\langle x^*, f_n\rangle| \, d\mu \\
= \; & T_1 + T_2 + T_3 + \sum_{n=1}^{n=\infty} \int_{S \setminus \cup_{i=1}^{i=m} E_i} |\langle x^*, f_n\rangle| \, d\mu,
\end{aligned}$$

with obvious definitions for the T_i. First

$$\begin{aligned}
T_1 \; & \leq \; \sum_{i=1}^{i=m} |\sum_{n=n(t_i)+1}^{n=\infty} \langle x^*, f_n(t_i)\rangle| \mu(E_i) \\
& \leq \; \sum_{i=1}^{i=m} \| \sum_{n=n(t_i)+1}^{n=\infty} f_n(t_i)\| \mu(E_i) \leq \frac{\varepsilon}{3} \sum_{i=1}^{i=m} \varphi(t_i)\mu(E_i) \leq \frac{\varepsilon}{3}
\end{aligned}$$

by Lemma 7 [13]. Next for estimating T_2, let $p_m = \max\{n(t_1), ..., n(t_m)\}$.

Then, by inequality (5.1),

$$\begin{aligned}
T_2 &= \Big|\sum_{i=1}^{i=m}\{\langle x^*, F_{n(t_i)}(t_i)\rangle - \int_{E_i}\langle x^*, F_{n(t_i)}\rangle\,d\mu\}\Big| \\
&= \Big|\sum_{k=1}^{k=p_m}\sum_{i,n(t_i)=k}\{\langle x^*, F_{n(t_i)}(t_i)\rangle - \int_{E_i}\langle x^*, F_{n(t_i)}\rangle\,d\mu\}\Big| \\
&\leq \sum_{k=1}^{k=p_m}\Big|\sum_{i,n(t_i)=k}\{\langle x^*, F_{n(t_i)}(t_i)\rangle - \int_{E_i}\langle x^*, F_{n(t_i)}\rangle\,d\mu\}\Big| \leq \sum_{k=1}^{k=p_m}\frac{\varepsilon}{3}\frac{1}{2^k} < \frac{\varepsilon}{3}.
\end{aligned}$$

Note that the series in T_3 converges (absolutely) by the observation above. Then

$$\begin{aligned}
T_3 = \sum_{i=1}^{i=m}\sum_{n=n(t_i)+1}^{n=\infty}\int_{E_i}\langle x^*, f_n\rangle\,d\mu\Big| &\leq \sup_{x^*\in\overline{B}_{X^*}}\sum_{i=1}^{i=m}\sum_{n=n(t_i)+1}^{n=\infty}\int_{E_i}\langle x^*, f_n\rangle\,d\mu\Big| \\
&\leq \sup_{x^*\in\overline{B}_{X^*}}\sum_{i=1}^{i=m}\sum_{n=n_0+1}^{n=\infty}\int_{E_i}|\langle x^*, f_n\rangle|\,d\mu \\
&\leq \sup_{x^*\in\overline{B}_{X^*}}\sum_{n=n_0+1}^{n=\infty}\int_{S}|\langle x^*, f_n\rangle|\,d\mu \\
&\leq \sum_{n=n_0+1}^{n=\infty}\|f_n\|_{\mathcal{P}e} < \frac{\varepsilon}{3}.
\end{aligned}$$

Consequently, we have

$$\limsup_{m\to\infty}|\langle x^*, \sigma_m(f,\mathcal{P}_\infty)\rangle - \langle x^*, \varpi\rangle| \leq \varepsilon + \limsup_{m\to\infty}\sum_{n=1}^{n=\infty}\int_{S\setminus\cup_{i=1}^{i=m}E_i}|\langle x^*, f_n\rangle|\,d\mu.$$

Since the series $\sum_{n\geq 1}\int_{S\setminus\cup_{i=1}^{i=m}E_i}|\langle x^*, f_n\rangle|\,d\mu$ is dominated term by term by convergent series $\sum_{n\geq 1}\|f_n\|_{\mathcal{P}e}$ and by the integrability of $\langle x^*, f_n\rangle$ on S we have

$$\lim_{m\to\infty}\int_{S\setminus\cup_{i=1}^{i=m}E_i}|\langle x^*, f_n\rangle|\,d\mu = 0,$$

the dominated convergence theorem for series gives

$$\limsup_{m\to\infty}\sum_{n=1}^{n=\infty}\int_{S\setminus\cup_{i=1}^{i=m}E_i}|\langle x^*, f_n\rangle|\,d\mu = \sum_{n=1}^{n=\infty}\limsup_{m\to\infty}\int_{S\setminus\cup_{i=1}^{i=m}E_i}|\langle x^*, f_n\rangle|\,d\mu = 0.$$

Thus

$$\limsup_{m\to\infty} |\langle x^*, \sigma_m(f, \mathcal{P}_\infty) - \langle x^*, \varpi\rangle| \le \varepsilon,$$

that is f is $\mathcal{M}$-integrable in the limit on S and

$$(\mathcal{ML})\text{-}\int_S f\, d\mu = \sum_{n=1}^{n=\infty} (\mathcal{ML})\text{-}\int_S f_n\, d\mu.$$

□

Problem . Let (f_n) be a sequence of $\mathcal{M}$-integrable in the limit functions from S to X such that $f(t) = \lim_{n\to\infty} f_n(t)$ exists in X, for the weak (resp. norm) topology of X, for almost every $t \in S$ and the set $\{\langle x^*, f\rangle : x^* \in \overline{B}_{X^*}\}$ is not equi-continuous, does it have f to be $\mathcal{M}$-integrable in the limit on S? If the answer is no, what are the conditions so that f becomes $\mathcal{M}$-integrable in the limit on S?
As consequence of Theorem 5.1, we have

Corollary 5.1. *Suppose that μ is finite. Let (f_n) be a sequence of $\mathcal{M}$-integrable in the limit functions from S to X and suppose that $f = \lim_{n\to\infty} f_n$ pointwise on S. If (f_n) is $\|.\|_{\mathcal{P}e}$-Cauchy, then f is $\mathcal{M}$-integrable in the limit on S and* $\lim_{n\to\infty} \|f_n - f\|_{\mathcal{P}e} = 0$.

PROOF. Pick a subsequence (n_k) satisfying $\|f_{n_k+1} - f_{n_k}\|_{\mathcal{P}e} \le \frac{1}{2^k}$ and set $g_k = f_{n_k+1} - f_{n_k}$. Then

$$\lim_{k\to\infty} \sum_{j=1}^{j=k} g_j = \lim_{k\to\infty} (f_{n_k+1} - f_1) = f - f_{n_1}$$

pointwise and and the series $\sum_{k\ge 1} \|g_k\|_{\mathcal{P}e}$ is convergent so Theorem 5.1 implies that $f - f_{n_1}$ is $\mathcal{M}$-integrable in the limit on S and

$$\lim_{k\to\infty} \|f_{n_k} - f\|_{\mathcal{P}e} = \lim_{k\to\infty} \|\sum_{j=1}^{j=k} g_j - (f - f_{n_1})\|_{\mathcal{P}e} = 0.$$

Since the same argument can be applied to any subsequence of (f_n), it follows that

$$\lim_{n\to\infty} \|f_n - f\|_{\mathcal{P}e} = 0.$$

□

Corollary 5.2. *Suppose that μ is finite. Let $(E_n)_{n\geq 1}$ be a sequence of disjoint subsets of Σ_f, let $(x_n)_{n\geq 1}$ be a sequence in X, and let $f : S \to X$ be the function defined by*

$$f(t) := \sum_{n=1}^{n=\infty} x_n 1_{E_n}(t) \quad (t \in S).$$

If the series $\sum_{n\geq 1} \mu(E_n)x_n$ is absolutely convergent, then f is $\mathcal{M}$-integrable in the limit on S, and

$$(\mathcal{ML})\text{-}\int_S f\, d\mu = \sum_{n=1}^{n=\infty} \mu(E_n)x_n.$$

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