A Note on Cabled Slice Knots and Reducible Surgeries

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ABSTRACT. We consider the question of when a slice knot admits a reducible Dehn surgery. By analyzing the correction terms associated to such a surgery we show that slice knots cannot admit surgeries with more than two summands. We also give a necessary Heegaard Floer theoretic condition for a positive cable of a knot to be slice.

1. Introduction

Dehn surgery is one of the simplest and most important operations in three-manifold topology, and understanding which three-manifolds can result from Dehn surgery on a knot in S^3 has been one of the major goals of modern low-dimensional topology. We refer the reader to [6] for an introduction and comprehensive overview of the history and scope of this endeavor.

Perhaps the most basic question related to this goal is when the result of Dehn surgery can be a reducible manifold (a manifold containing an essential two-sphere). Moser's classification of Dehn surgeries on torus knots [20] gave the first nontrivial examples of this phenomenon:

$$S^3_{pq}(T_{p,q}) \cong L(p,q) \# L(q,p).$$

Of course, the fact that $S^3_0(U) \cong S^1 \times S^2$ can be thought of as a degenerate case of this. Gabai [3] showed that no other knot in S^3 admits a surgery to $S^1 \times S^2$, so we may assume that any nontrivial reducible surgery decomposes as a connected sum.

Other interesting examples of reducible surgeries come from considering cabled knots [7]. Let $J_{p,q}$ denote the (p,q)-cable of J for some pair $p,q \in \mathbb{Z}$ with $p \geq 2$. (Throughout, p is the longitudinal winding number.) Then we have

$$S^3_{pq}(J_{p,q}) \cong L(p,q) \# S^3_{q/p}(J).$$

Note that this generalizes the case of torus knots, which can be thought of as cables of the unknot. Cabled knots represent the only known examples of knots admitting reducible surgeries, and we have the following conjecture of Gonzales Acuña and Short.

CABLING CONJECTURE ([4]). *If K is a nontrivial knot in S^3 and $S^3_r(K)$ is reducible, then $K = J_{p,q}$ for some knot J, and $r = pq$.*

There has been much progress made toward a positive resolution of this conjecture. For example, it is known that any reducible surgery slope must be integral [5] and that one summand must be a lens space [8]. Moreover, if a reducible surgery on K yields a connected sum of lens spaces, then K is either a torus knot or a cable thereof [9]. Many classes of knots are known to satisfy the conjecture, including knots with symmetries [2; 11; 12], alternating and arborescent knots [18; 27], and knots with low bridge number [10; 14; 17; 25; 29].

Since any reducible surgery slope must be integral, it follows that every reducible surgery on a cable knot yields a connected sum of a lens space with an *irreducible* rational homology sphere (since $p \geq 2$, the $S^3_{q/p}(J)$ summand cannot itself be reducible). In particular, all such surgeries have only two summands. This motivates the following conjecture.

TWO-SUMMANDS CONJECTURE. *If K is a nontrivial knot in S^3 and $S^3_r(K)$ is reducible, then $S^3_r(K) \cong Y_1 \# Y_2$ with irreducible Y_1 and Y_2.*

In a reducible surgery, there can be at most one summand that is *not* a lens space [26], and there can be at most two summands that are *not* homology spheres [16]. It follows that a reducible surgery with three summands (the highest possible number) must consist of two lens spaces summands and an integer homology sphere summand. The two-summands conjecture is true for knots with bridge number at most five and positive braid closures [29].

In the present paper, we restrict our attention to slice knots. A knot $K \subset S^3$ is called *slice* if there exists a smooth, properly embedded disk $D \subset B^4$ with $\partial D = K$. Our first result verifies the two-summands conjecture for slice knots.

THEOREM 1.1. *A slice knot in the three-sphere cannot admit a reducible surgery with three irreducible summands.*

The proof is straightforward. If K is a slice knot, then $S^3_{pq}(K)$ is integer homology cobordant to $L(pq, 1)$. Knowing this, we can compare the correction terms of $L(pq, 1)$ to those of $L(p, a) \# L(q, b)$. A simple lemma shows that these collections never match up. In fact, Theorem 1.1 holds for any knot K with $V_0(K) = V_0(\overline{K}) = 0$, where $V_0(K)$ is a Heegaard Floer theoretic knot invariant coming from the knot Floer complex that determines the correction terms of surgeries on K [23; 24], and $\overline{K}$ denotes the mirror of K. The condition that $V_0(K) = 0$ suffices in the case that the surgery is positive. Along these same lines, we have the following observation.

THEOREM 1.2. *Suppose that K is a positive cable of a knot J. If K is slice, then $V_0(J) = 0$.*

As an application, let D denote the untwisted Whitehead double of the right-handed trefoil knot. By Proposition 6.1 of [13] we know that $V_0(D) = 1$. It follows that no positive cable of D is smoothly slice, even though the $(p, 1)$-cable of D is topologically slice for all p. (Note that this particular fact can also be

deduced from work of Hom [15].) Theorem 1.2 complements the main result of [28]; here, the proof is simplified by the following proposition.

PROPOSITION 1.3. *If $J_{p,q}$ is algebraically slice, then $q = 1$.*

There is a simple geometric argument that any $(p, 1)$-cable of a slice knot is also slice. Thus, we have the following conjecture.

CONJECTURE 1.4. *The cabled knot $J_{p,q}$ is slice if and only if J is slice and $q = 1$.*

Theorem 1.2 and [15] give evidence that the conjecture is true at the level of Heegaard Floer homology. Note that Conjecture 1.4 is true for fibered knots in the homotopy-ribbon setting by Theorem 8.5 of [19].

2. Heegaard Floer Correction Terms

The main tool used in proving Theorem 1.1 is the correction terms coming from Heegaard Floer homology. This theory was first formulated for closed three-manifolds [22] before being shown to give invariants of four-manifolds [21]. For our purposes, the most important aspects of the theory will be the Heegaard Floer correction terms defined in [21].

Let Y be an oriented, closed three-manifold, and let $\mathrm{Spin}^c(Y)$ denote the collection of Spin^c structures associated to Y. For each $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, let $d(Y, \mathfrak{s})$ denote the *correction term* associated to $(Y, \mathfrak{s})$. A detailed development of this invariant can be found in [21], where it was shown to have the following properties.

(1) Let $-Y$ denote the opposite orientation of Y. Then $d(-Y, \mathfrak{s}) = -d(Y, \mathfrak{s})$.
(2) Let $\bar{\mathfrak{s}}$ be the image of $\mathfrak{s}$ under conjugation. Then $d(Y, \bar{\mathfrak{s}}) = d(Y, \mathfrak{s})$.
(3) For any pairs $(Y_1, \mathfrak{s}_1)$ and $(Y_2, \mathfrak{s}_2)$,

$$d(Y_1 \# Y_2, \mathfrak{s}_1 \# \mathfrak{s}_2) = d(Y_1, \mathfrak{s}_1) + d(Y_2, \mathfrak{s}_2).$$

Let $\mathcal{D}(Y)$ denote the collection of correction terms associated to Y. (Note that elements of this set can appear with multiplicity greater than one.) Recall that there is an affine identification $\mathrm{Spin}^c(Y) \approx H^2(Y; \mathbb{Z})$. For our purposes, the most important aspect of the correction terms is that they are preserved under integer homology cobordism.

PROPOSITION 2.1. *If Y_1 and Y_2 are integer homology cobordant, then $\mathcal{D}(Y_1) = \mathcal{D}(Y_2)$.*

The correspondence between the collections of correction terms in the proposition can be strengthened; see [1] for a proof and more detail. Here, we will not be too concerned with fixing identifications and labelings of spin^c structures.

Let $L(p, q)$ denote the lens space obtained by p/q-surgery on the unknot in S^3. In this case, Ozváth and Szabó [21] gave a canonical ordering on $\mathrm{Spin}^c(L(p, q))$ by elements $i \in \mathbb{Z}_p$ and a recursive formula for the correction terms.

PROPOSITION 2.2. *For any positive, relatively prime integers $p > q$ and any integer $0 \leq i < p + q$, we have*

$$d(-L(p,q),i) = \frac{pq - (2i + 1 - p - q)^2}{4pq} - d(-L(q,r),j), \qquad (1)$$

where r and j are the reductions modulo q of p and i, respectively.

This formula allows for the calculation the correction terms of any lens space since $L(1,0) \cong S^3$ and $d(S^3) = 0$.

Now, if K is slice, then there exists a concordance from K to the unknot U. It follows that $S^3_{p/q}(K)$ is integer homology cobordant to $S^3_{p/q}(U) \cong L(p,q)$. (The Dehn surgery manifolds cobound a four-manifold that is obtained by performing "Dehn surgery cross I" on the concordance.) Thus, we obtain the following corollary to Proposition 2.1.

COROLLARY 2.3. *If K is slice, then*

$$\mathcal{D}(S^3_{p/q}(K)) = \mathcal{D}(L(p,q)).$$

In fact, this corollary holds for any K with $V_0(K) = 0$. This follows from the integer and rational surgery formulae developed in [23; 24].

3. Proof of Theorem 1.1

In this section, we prove the main theorem. First, we present a simple lemma that gives an upper bound on the range of the correction terms for a given lens space. Let $\delta(p,q) = \max \mathcal{D}(L(p,q)) - \min \mathcal{D}(L(p,q))$ denote this range.

LEMMA 3.1. $\delta(p,q) \leq p/4$.

Proof. Note that $\delta(p,q) = \delta(p, p - q)$ since $L(p,q) \cong -L(p, p - q)$. Therefore, we may assume without loss of generality that $p > q/2 > 0$. Furthermore, the recursive formula in Proposition 2.2 is easy to understand for small values of q, and it can be checked that $\delta(p,q) \leq p/4$ whenever $q < 8$. (The calculations breaks down into a finite number of cases based on the values of q and r.)

Thus, we assume that $q \geq 8$. From equation (1) we have

$$\delta(p,q) \leq \max_{i,i'} \left\{ \frac{(2i - p - q + 1)^2 - pq}{4pq} - \frac{(2i' - p - q + 1)^2 - pq}{4pq} \right\} + \delta(q,r).$$

The bracketed term is maximized when $i = 0$ and $i' = (p + q - 1)/2$ or $i' = (p + q)/2$, depending on whether p and q have opposite parity or not. The former choice of i' yields a more extremal value; so, in either case, we have

$$\delta(p,q) \leq \frac{(p + q - 1)^2 - pq}{4pq} - \frac{-pq}{4pq} + \delta(q,r)$$

$$= \frac{(p + q - 1)^2}{4pq} + \delta(q,r).$$

Since $q < p$, we have $p + q - 1 < 2p$, so

$$\delta(p, q) \leq \frac{4p^2}{4pq} + \delta(q, r)$$
$$= \frac{p}{q} + \delta(q, r).$$

Now, having shown that the claim holds for small values of p, we can proceed by induction and assume that $\delta(q, r) \leq q/4$. Since $q \geq 8$ and $p > q/2$, we have

$$\delta(p, q) \leq \frac{p}{8} + \frac{q}{4} \leq \frac{p}{8} + \frac{p}{8} = \frac{p}{4}. \qquad \square$$

We remark that this bound is sharp since $\delta(p, 1)$ is $p/4$ when p is even and $(p^2 - 1)/4p$ when p is odd. However, this bound appears to be far from sharp once $q > 1$.

THEOREM 1.1. *A slice knot in the three-sphere cannot admit a reducible surgery with three irreducible summands.*

Proof. Suppose that K is a slice knot such that $S_r^3(K)$ has three irreducible summands. Since $S_{-r}^3(\overline{K}) \cong -S_r^3(K)$, there will be no loss of generality in assuming that $r > 0$. Recall that at most one summand is *not* a lens space [26], whereas at most two summands are *not* integer homology spheres [16]. It follows that precisely two summands are lens spaces and the third is a homology sphere:

$$S_{pq}^3(K) = L(p, a)\#L(q, b)\#Y.$$

It follows that $r = pq$ with p and q coprime. Since Y is an integer homology sphere, it has only one correction term, $d(Y)$. By Corollary 2.3 we know that $\mathcal{D}(S_r^3(K)) = \mathcal{D}(L(r, 1))$. These two facts give us that

$$\mathcal{D}(L(pq, 1)) = \mathcal{D}(L(p, a)\#L(q, b)) + d(Y). \qquad (2)$$

It follows that the ranges of values on the left and the right must match. In particular, since $d(Y)$ is constant, we must have

$$\delta(pq, 1) \leq \delta(p, a) + \delta(q, b).$$

By Lemma 3.1 we have that $\delta(p, a) \leq p/4$ and $\delta(q, b) \leq q/4$, so we have

$$\delta(pq, 1) \leq \frac{p + q}{4}. \qquad (3)$$

Now, if pq is even, we have $\delta(pq, 1) = pq/4$; so $pq \leq p + q$. If pq is odd, then we have $\delta(pq, 1) = (pq - 1)(pq + 1)/4pq$; so $pq - 1 \leq p + q$. It follows that either $p = 1$, $q = 1$, or $p = q = 2$. However, we have assumed that p and q are coprime. Therefore, either $L(p, a)$ or $L(q, b)$ is S^3, and the proof is complete. $\square$

We conclude this section by remarking that the proof of Theorem 1.1 also shows that surgery on a slice knot cannot produce a connected sum of two lens spaces; thus, we recover a special case of a much stronger theorem of Greene [9].

4. Proof of Theorem 1.2

We are grateful to Chuck Livingston for pointing out that the following result should hold in the algebraically slice setting. Not only is this fact interesting in its own right, but it greatly simplifies the proof of the ensuing theorem, which was originally entirely Heegaard Floer theoretic. Let $\Delta_K(t)$ denote the Alexander polynomial of a knot K in S^3.

PROPOSITION 1.3. *Let $K = J_{p,q}$ denote the (p, q)-cable of a knot J. If K is algebraically slice, then $q = 1$.*

Proof. Suppose that $K = J_{p,q}$ is algebraically slice. Since K is algebraically slice, we have that $\Delta_K(t) = f(t)f(t^{-1})$ for some $f(t) \in \mathbb{Z}[t]$. Furthermore, since K is a cable, we can write $\Delta_K(t) = \Delta_J(t^p)\Delta_{T_{p,q}}(t)$. It follows that

$$\Delta_J(t^p)\Delta_{T_{p,q}}(t) = f(t)f(t^{-1}).$$

Let ξ be a root of $\Delta_{T_{p,q}}(t)$. This means that ξ is a pqth root of unity that is neither a pth root of unity nor a qth root of unity. It follows that ξ is a root of $f(t)f(t^{-1})$. Without loss of generality, we can assume that $f(\xi) = 0$. It follows that $f(\bar{\xi}) = 0$ as well, and we note that $\bar{\xi} = \xi^{-1}$. It follows that ξ is a root of both $f(t)$ and $f(t^{-1})$. Since ξ has multiplicity one as a root of $\Delta_{T_{p,q}}(t)$ and since we have observed that it has multiplicity two as a root of $f(t)f(t^{-1})$, it follows that ξ is a root of $\Delta_J(t^p)$. Thus, ξ^p is a root of $\Delta_J(t)$. This is true of any root ξ of $\Delta_{T_{p,q}}$.

Now, let q_1 be a prime factor of q, and let $\xi = e^{2\pi i/pq_1}$. It follows that $\xi^{pq} = 1$, but $\xi^q \neq 1$ and $\xi^p \neq 1$, so ξ is a root of $\Delta_{T_{p,q}}(t)$. Therefore, ξ^p is a root of $\Delta_J(t)$. However, ξ^p is a primitive q_1th root of unity, and q_1 is prime. This implies that the cyclotomic polynomial $\Phi_{q_1}(t)$ divides $\Delta_J(t)$. This implies that $\Phi_{q_1}(1) = q_1$ divides $\Delta_J(1)$, which is one, since $\Delta_J(t)$ is an Alexander polynomial. It follows that $q_1 = 1$ and that this conclusion must hold for all prime factors of q. Therefore, we must have $q = 1$, as desired. $\square$

THEOREM 1.2. *Suppose that K is a positive cable of a knot J. If K is slice, then $V_0(J) = 0$.*

Proof. If K is the (p, q)-cable of J, and K is algebraically slice, then $q = 1$ by Proposition 1.3. It follows that

$$S_p^3(K) = L(p, 1) \# S_{1/p}^3(J).$$

If K is slice, then Corollary 2.3 tells us that

$$\mathcal{D}(L(p, 1)) = \mathcal{D}(L(p, 1)) + d(S_{1/p}^3(J)).$$

It follows that $d(S_{1/p}^3(J)) = 0$, which is equivalent to $V_0(J) = 0$ [23; 24]. $\square$

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