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Pretzel Knots with L-Space Surgeries

Tye Lidman & Allison H. Moore

Abstract. A rational homology sphere whose Heegaard Floer homology is the same in rank as that of a lens space is called an L-space. We classify pretzel knots with any number of tangles that admit L-space surgeries. This rests on Gabai's classification of fibered pretzel links.

1. Introduction

The Heegaard Floer homology of three-manifolds and its refinement for knots, knot Floer homology, have proved to be particularly useful for studying Dehn surgery questions in three-manifold topology. Recall that the knot Floer homology of a knot K in the three-sphere is a bigraded Abelian group,

$$\widehat{\mathrm{HFK}}(K) = \bigoplus_{m,s} \widehat{\mathrm{HFK}}_m(K,s),$$

introduced by Ozsváth and Szabó [OS04b] and independently by Rasmussen [Ra03]. The graded Euler characteristic is the symmetrized Alexander polynomial of K [OS04b],

$$\Delta_K(t) = \sum_s \chi(\widehat{\mathrm{HFK}}(K,s)) \cdot t^s.$$

These theories have been especially useful for studying knots that admit lens space surgeries, the classification of which has been an outstanding problem in low-dimensional topology for decades. For example, if $K \subset S^3$ admits a lens space surgery, then for all $s \in \mathbb{Z}$, we have $\widehat{\mathrm{HFK}}(K,s) \cong 0$ or $\mathbb{Z}$ [OS05, Thm. 1.2]. Knot Floer homology detects both the genus of K by

$$g(K) = \max\{s \mid \widehat{\mathrm{HFK}}(K,s) \neq 0\}$$

[OS04a] and the fiberedness of K by whether $\widehat{\mathrm{HFK}}(K,g(K))$ is isomorphic to $\mathbb{Z}$ [Ghi08; Ni07]. Together, these facts imply that a knot in S^3 with a lens space surgery is fibered. Indeed, this result applies more generally to knots in S^3 admitting L-space surgeries. Recall that a rational homology sphere Y is an *L-space* if $|H_1(Y;\mathbb{Z})| = \operatorname{rank}\widehat{\mathrm{HF}}(Y)$, where $\widehat{\mathrm{HF}}$ is the "hat" flavor of Heegaard Floer homology. The class of L-spaces includes all lens spaces, and more generally, three-manifolds with elliptic geometry [OS05, Prop. 2.3] (or equivalently, with finite fundamental group by the Geometrization theorem; see [KL08]). A knot admitting an L-space surgery is called an *L-space knot.*

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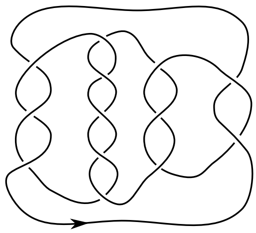

Figure 1 The pretzel knot $(3, -5, 3, -2)$

The goal of this paper is to classify L-space pretzel knots.

THEOREM 1. *Let K be a pretzel knot. Then, K admits an L-space surgery if and only if K is isotopic to the $\pm(-2, 3, q)$-pretzel knot for odd $q \geq 1$ or the $(2, 2n+1)$-torus knot for some n.*

We establish the following notation and conventions for the paper. For notation, we use $(n_1, \ldots, n_r)$ to denote the pretzel knot with r tangles, where the ith tangle consists of $n_i \in \mathbb{Z}$ half-twists. For examples of pretzel links, refer to Figures 1 and 2. We use $T(a, b)$ to denote the (a, b)-torus knot, which is positive if $a, b > 0$. It was proved by Krcatovich [Krc14] that L-space knots are prime, so we adopt the convention that $n_i \neq 0$ for all i.

We first remark that the pretzel knots $(-2, 3, 1)$, $(-2, 3, 3)$, and $(-2, 3, 5)$ are isotopic to the torus knots $T(2, 5)$, $T(3, 4)$, and $T(3, 5)$, respectively. In general, torus knots are well known to admit lens space surgeries [Mos71]; the hyperbolic pretzel knot $(-2, 3, 7)$ is also known to have two lens space surgeries [FS80]. The knot $(-2, 3, 9)$ has two finite, noncyclic surgeries [BH96]. Finally, the remaining knots, $(-2, 3, q)$ for $q \geq 11$, are known to have Seifert fibered L-space surgeries with infinite fundamental group [OS05]. Therefore, in this paper we show that no other pretzel knot admits an L-space surgery. This will be proved by appealing to Gabai's classification of fibered pretzel links [Gab86] and the state-sum formula for the Alexander polynomial [Kau83; OS03].

Using Theorem 1, we are able to easily recover the classification of pretzel knots that admit surgeries with finite fundamental group due to Ichihara and Jong.

COROLLARY 2 (Ichihara–Jong [IJ09]). *The only nontrivial pretzel knots that admit nontrivial finite surgeries are, up to mirroring,* $(-2, 3, 7)$, $(-2, 3, 9)$, $T(3, 4)$, $T(3, 5)$, *and* $T(2, 2n+1)$ *for* $n > 0$.

Proof. As discussed before, the knots in the statement of the corollary are known to admit finite surgeries. Therefore, it remains to rule out the case of $(-2, 3, q)$ for odd $q \geq 11$. Using the theory of character varieties, Mattman proved that the only knots of the form $K = (-2, 3, q)$ with $q \neq 1, 3, 5$ that admit a finite surgery are $(-2, 3, 7)$ and $(-2, 3, 9)$ [Mat02]. This completes the proof. $\square$

REMARK 3. In fact, Ichihara and Jong [IJ09] show that Corollary 2 holds more generally for Montesinos knots. Like the approach of this paper, their proof studies the Alexander polynomials of pretzel knots. However, they first appeal to an analysis of essential laminations on the exteriors of Montesinos knots by Delman. This allows them to restrict their attention to a specific family of pretzel knots before reducing to the case of the $(-2, 3, q)$-pretzel knots. Because of this, they do not need to make the graph-theoretic arguments we make here.

We observe that whereas many pretzel knots have essential Conway spheres, the pretzel knots with L-space surgeries do not. We conjecture that this holds for L-space knots in general. If true, this fact would imply that an L-space knot admits no nontrivial mutations.

CONJECTURE 4. *If K is an L-space knot, then there are no essential Conway spheres in the complement of K.*

Note that a Montesinos knot with an essential Conway sphere has at least four tangles. Indeed, after writing this note, the second author and Baker extended Theorem 1 to a classification of Montesinos knots admitting L-space surgeries (using different techniques) and confirmed Conjecture 4 for Montesinos knots. Moreover, work of Wu [Wu96] implies any surgery on an arborescent knot (a class which generalizes Montesinos knots) that is not a Montesinos knot of length at most three has infinite fundamental group. This means that amongst arborescent knots, the existence of an essential Conway sphere obstructs the knot from admitting a finite surgery. Thus, it is natural to wonder whether similar geometric obstructions exist for L-space surgeries.

In fact, for many families of pretzel knots in the arguments that follow, we are able to leverage the existence of essential Conway spheres to our advantage; we perform mutations along these surfaces so that the Kauffman state sum descriptions of the Alexander polynomials of the corresponding pretzel knots become more predictable.

Another interesting phenomenon can be observed as a result of the proof of Theorem 1. In all cases (exempting the two families of knots mentioned in Theorem 1), for each fibered knot K, we will exhibit an Alexander grading s where $\widehat{\mathrm{HFK}}(K, s)$ is neither trivial nor isomorphic to $\mathbb{Z}$. As discussed, this implies that these knots are not L-space knots. For *most* fibered pretzel knots, we will do this by showing that there is a coefficient of the Alexander polynomial with $|a_s| > 1$. Except for a few sporadic knots, we accomplish this by making repeated use of two basic arguments: either studying $a_{-g(K)+1}$ with the state-sum formula (see Section 2.3) or by analyzing the determinant of K and applying Lemma 6 (see Section 2.1). In fact, the Alexander polynomial serves as an obstruction for all but one knot, and thus the following will result directly from the proof of Theorem 1.

Table 1 The knot Floer homology groups of the knot (3, −5, 3, −2) are displayed with Maslov grading on the vertical axis and Alexander grading on the horizontal axis

	$\widehat{\mathrm{HFK}}(K = (3, -5, 3, -2))$						
	-3	-2	-1	0	1	2	3
4							$\mathbb{F}$
3						$\mathbb{F}^3$	
2					$\mathbb{F}^4$	$\mathbb{F}^2$	
1				$\mathbb{F}^3$	$\mathbb{F}^4$		
0			$\mathbb{F}^4$	$\mathbb{F}^4$			
-1		$\mathbb{F}^3$	$\mathbb{F}^4$				
-2	$\mathbb{F}$	$\mathbb{F}^2$					

PROPOSITION 5. *Up to mirroring, there is a unique fibered pretzel knot that has the Alexander polynomial of an L-space knot but does not admit an L-space surgery. This knot is* $(3, -5, 3, -2)$.

The knot $K = (3, -5, 3, -2)$ is pictured in Figure 1. Its Alexander polynomial,

$$\Delta_{(3,-5,3,-2)}(t) = t^{-3} - t^{-2} + 1 - t^2 + t^3,$$

does not obstruct it from admitting an L-space surgery. Therefore, we compute the knot Floer homology of $K = (3, -5, 3, -2)$ in Table 1 using the Python program for $\widehat{\mathrm{HFK}}$ with $\mathbb{F}_2$ coefficients by Droz [Dro] to observe directly that there exist Alexander gradings s such that $\dim \widehat{\mathrm{HFK}}(K, s; \mathbb{F}_2) \geq 2$. This implies that for these Alexander gradings, $\widehat{\mathrm{HFK}}(K, s; \mathbb{Z}) \ncong 0$ or $\mathbb{Z}$. Therefore, $K = (3, -5, 3, -2)$ is not an L-space knot. We remark that whereas the nontrivial Conway mutants of K share its Alexander polynomial, these mutants are nonfibered (this will be implied by Theorem 7 and made clear in Section 5.1). That the classical tools succeed for all but a single knot is a testament to the stringency of the conditions on L-space knots.

2. Background

Throughout, K (resp. L) is an oriented knot (resp. link) in S^3. Let $g(K)$ denote the genus of K. Let $L = (n_1, \dots, n_r)$ be a pretzel link. We will sometimes use the integer n_i to refer to this specific tangle in the pretzel projection. To set the sign conventions for tangles, we declare the first row of crossings in Figure 3 to be $+1$ and -1, respectively. The *length* of the tangle n_i refers to $|n_i|$. Notice that tangles of length one can be permuted to any spot in a pretzel link by flype moves. Furthermore, if there exist indices i and j such that $n_i = +1$ and $n_j = -1$ in L, then n_i and n_j can be pairwise removed by flyping followed by an isotopy.

Similarly, if there exist indices i and j such that $n_i = \pm 1$ and $n_j = \mp 2$, these tangles may be made consecutive via flypes and then consolidated with the isotopy $(\pm 1, \mp 2) \simeq (\pm 2)$. Unless otherwise stated, we assume that a diagram of a pretzel link L is in pretzel form and that r is the minimal possible number of strands to present L as a pretzel projection; in particular, we assume that there are no pairs of indices i and j such that $n_i = \pm 1$ and $n_j = \mp 1$ or ∓ 2. Throughout, we will assume the classification of pretzel knots due to Kawauchi [Kaw85].

2.1. Determinants of Pretzel Knots

Since $\chi(\widehat{\mathrm{HFK}}(K, s))$ is equal to a_s, the coefficient of t^s in the symmetrized Alexander polynomial of K, this will give us an easy way to approach Theorem 1 in many cases; whenever there exists a coefficient a_s of $\Delta_K(t)$ with $|a_s| > 1$, K is not an L-space knot [OS05]. We therefore establish the following lemma.

LEMMA 6. *If* $\det(K) > 2g(K) + 1$, *then* $\Delta_K(t)$ *contains some coefficient* a_s *with* $|a_s| > 1$.

Proof. If the coefficients of $\Delta_K(t)$ are at most one in absolute value, then

$$\det(K) = |\Delta_K(-1)| \leq \sum_s |a_s| \leq 2g(K) + 1.$$

□

Suppose that Y is a Seifert fibered rational homology sphere with base orbifold S^2 and Seifert invariants $(b; (a_1, b_1), \ldots, (a_r, b_r))$. Then

$$|H_1(Y; \mathbb{Z})| = \left| a_1 \cdots a_r \cdot \left(b + \sum_{i=1}^{r} \frac{b_i}{a_i} \right) \right|$$

(see, e.g., [Sav02]). The branched double covers of Montesinos knots (and consequently, pretzel knots) are such Seifert fibered spaces. If

$$K = (\underbrace{1, \ldots, 1}_{d}, n_1, \ldots, n_k),$$

where $|n_i| > 1$ for $1 \leq i \leq k$, then the branched double cover of K, denoted $\Sigma_2(K)$, has Seifert invariants $(d; (n_1, 1), \ldots, (n_k, 1))$. Therefore,

$$\det(K) = |H_1(\Sigma_2(K))| = \left| n_1 \cdots n_k \cdot \left(d + \sum_{i=1}^{k} \frac{1}{n_i} \right) \right|. \tag{1}$$

As permuting tangles in a pretzel knot corresponds with doing a series of Conway mutations, $\Delta_K(t)$, and consequently $\det(K)$, are unchanged. Invariance of the determinant under permutations is also evident from equation (1). Since the symmetrized Alexander polynomial of a fibered knot is monic of degree $g(K)$, when K is fibered and the mutation preserves fiberedness, the genus of K is also unchanged.

2.2. *Fibered Pretzel Links*

As mentioned earlier, if K is an L-space knot, then K is fibered. Theorem 1 is therefore automatic for any nonfibered knot. Thus, for the proof of Theorem 1, we will only be interested in fibered pretzel knots. In [Gab86, Thm. 6.7], Gabai classified oriented fibered pretzel links together with their fibers; we recall this in Theorem 7. An oriented pretzel link L may be written

$$L = (m_1, m_{11}, m_{12}, \dots, m_{1\ell_1}, m_2, m_{21}, \dots, m_{2\ell_2}, \dots, m_R, m_{R1}, \dots, m_{R\ell_R}),$$

where m_i denotes a tangle in which the two strands are oriented consistently (i.e., both up or both down), and m_{ij} denotes a tangle where the two strands are oriented inconsistently (i.e., one up and one down). An oriented pretzel link falls into one of three types, depending on the surface obtained by applying the Seifert algorithm to the oriented pretzel presentation of L (see Gabai [Gab86]). Moreover, the type of surface associated to an oriented pretzel link can be ascertained from its diagram; for ease of exposition, we restate the classification in [Gab86] by defining the types in terms of link diagrams. A type 1 link contains no m_i, a type 2 link contains both an m_i and an m_{ij}, and a type 3 link contains no m_{ij}. With this, we call a surface resulting from performing the Seifert algorithm to an oriented pretzel presentation type 1, 2, or 3 based on whether it arises from a type 1, 2, or 3 link diagram, respectively.

Associated with a type 2 or type 3 link L will be an auxiliary oriented pretzel link L',

$$\begin{aligned} L' = \Bigg(&\frac{-2m_1}{|m_1|}, m_{11}, m_{12}, \dots, m_{1\ell_1}, \frac{-2m_2}{|m_2|}, \\ &m_{21}, \dots, m_{2\ell_2}, \dots, \frac{-2m_R}{|m_R|}, m_{R1}, \dots, m_{R\ell_R}\Bigg), \end{aligned} \tag{2}$$

where the term $-2m_i/|m_i|$ is omitted if $|m_i| = 1$. The link L' is oriented so that the surface obtained by applying the Seifert algorithm is of type 1. See Figure 2. The auxiliary link L' is derived from a procedure of Gabai in which a minimal genus Seifert surface is desummed and its sutured manifold hierarchy is analyzed to determine whether L fibers [Gab86].

THEOREM 7 (Gabai, Thm. 6.7 in [Gab86]). *The algorithm that follows determines whether an oriented pretzel link fibers.*

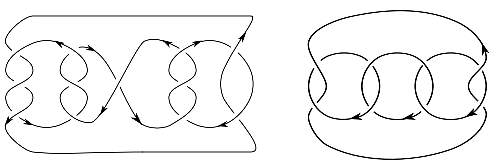

Figure 2 The pretzel knot $(3, -3, 1, 3, 2)$ and its associated auxiliary link $(-2, 2, -2, 2)$

(I) *A type 1 link L fibers if and only if one of the following holds*:
  (a) *Each* $n_i = \pm 1$ *or* ∓ 3 *and some* $n_i = \pm 1$.
  (b) $(n_1, \dots, n_r) = \pm(2, -2, 2, -2, \dots, 2, -2, n)$, $n \in \mathbb{Z}$ (*here*, r *is odd*).
  (c) $(n_1, \dots, n_r) = \pm(2, -2, 2, -2, \dots, -2, 2, -4)$ (*here*, r *is even*).

(II) *A fibered type 2 link falls into the following two subcases*:
  (a) *The numbers of positive and negative* m_i *differ by two. Then L fibers if and only if* $|m_{ij}| = 2$ *for all indices* ij.
  (b) *The numbers of positive and negative* m_i *in L are equal*, *and* $L' \neq \pm(2, -2, \dots, 2, -2)$. *Then L fibers if and only if* L' *fibers.*

(III) *For type 3 links*, *if either the numbers of positive and negative tangles are unequal or if* $L' \neq \pm(2, -2, \dots, 2, -2)$, *then treat L as if it was type 2(a) or 2(b). Otherwise*, *L is fibered if and only if there is a unique* m_i *of minimal absolute value.*

Finally, if L is a fibered pretzel link of type 1, type 2(a), or the type 2(a) subcase of type 3, then the fiber surface is necessarily isotopic to the surface obtained by applying the Seifert algorithm to the pretzel diagram of L.

The original formulation describes the fiber surfaces for all types; we include this information only when it is relevant to our calculations. In our case analysis, we denote the three subcases of type 3 by type 3-2(a), type 3-2(b), and type 3-min accordingly.

Remark 8. In Gabai's classification of oriented fibered pretzel links, there is a third subcase of fibered type 2 links, called type 2(c). For these links, the numbers of positive and negative m_i are equal and $L' = \pm(2, -2, \dots, 2, -2)$. However, these links are not minimally presented and can be isotoped to be in type 3.

Remark 9. If a pretzel knot K (as opposed to a link) is type 1, then there is an odd number of tangles m_{ij}, all of which are odd. If K is type 2, there is exactly one m_{ij}, which we denote by $\bar{m}$, and this unique $\bar{m}$ must also be the unique even tangle. Moreover, there is an even number of m_i. If K is type 3, there is an even number of m_i, exactly one of which is even. Finally, note that adjacent tangles of type m_i have alternating orientations.

2.3. A State Sum for the Alexander Polynomial

The Alexander polynomial of K admits a state sum expression in terms of the set of Kauffman states $\mathcal{S}$ of a decorated projection of the knot [Kau83]. We will use a reformulation of the Kauffman state sum that appears in [OS03]. By a decorated knot projection we mean a knot projection with a distinguished edge. When using decorated knot projections, we will always choose the bottom-most edge in a standard projection of a pretzel knot to be the distinguished edge. Each state $\mathbf{x}$ is equipped with a bigrading $(A(\mathbf{x}), M(\mathbf{x})) \in \mathbb{Z} \oplus \mathbb{Z}$ such that the symmetrized

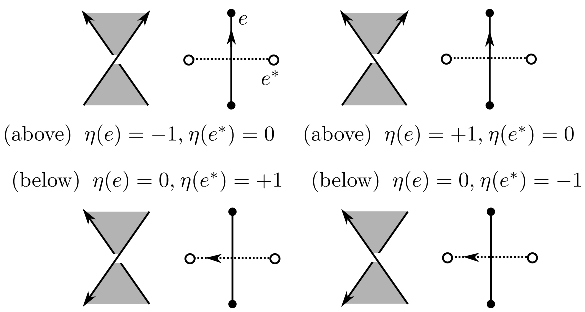

Figure 3 The labels $\eta(e)$ and $\eta(e^*)$ for the edges $e \in G_B$ and $e^* \in G_W$. The edge orientations pictured are those induced by K on G_B or G_W

Alexander polynomial of K is given by the state sum

$$\Delta_K(t) = \sum_{\mathbf{x} \in \mathcal{S}} (-1)^{M(\mathbf{x})} t^{A(\mathbf{x})}. \tag{3}$$

Let G_B and G_W denote the black and white graphs associated with a checkerboard coloring of a decorated knot projection, colored so that the unbounded region is always white. The decorated edge of K determines a decorated vertex, the *root,* in each of G_B and G_W. For a pretzel diagram, there is also clearly a top-most vertex of G_B, referred to as the *top vertex.* The set of states $\mathcal{S}$ is in a one-to-one correspondence with the set of maximal trees of G_B. Each maximal tree $T \subset G_B$ uniquely determines a maximal tree $T^* \subset G_W$. Fix a state $\mathbf{x} \in \mathcal{S}$ and let $\mathscr{T}_{\mathbf{x}} = T_{\mathbf{x}} \cup T^*_{\mathbf{x}}$ denote the black and white maximal trees that correspond to $\mathbf{x}$. By an abuse of notation, we will not always distinguish between the state $\mathbf{x}$ and the trees $\mathscr{T}_{\mathbf{x}}$. We now describe $A(\mathbf{x})$ and $M(\mathbf{x})$ in this framework, following [OS03]. Label each edge e of G_B and G_W with $\eta(e) \in \{-1, 0, 1\}$ according to Figure 3. We describe two partial orientations on the edges of $T_{\mathbf{x}}$ and $T^*_{\mathbf{x}}$. The first orientation is a total orientation that flows away from the root. The second partial orientation is induced by the orientation on the knot as in Figure 3; note that at each crossing exactly one of the edges of $T_{\mathbf{x}}$ or $T^*_{\mathbf{x}}$ is oriented. Then, $A(\mathbf{x})$ is defined by

$$A(\mathbf{x}) = \frac{1}{2} \sum_{e \in \mathscr{T}_{\mathbf{x}}} \sigma(e)\eta(e), \tag{4}$$

where

$$\sigma(e) = \begin{cases} 0 & \text{if } e \text{ is not oriented by } K, \\ +1 & \text{if the two induced orientations on } e \text{ agree,} \\ -1 & \text{if the two induced orientations on } e \text{ disagree.} \end{cases}$$

Note that though it is not indicated in the notation, $\sigma(e)$ depends on $\mathbf{x}$; which $\mathbf{x}$ will be clear from the context. The label $\eta(e)$ does not depend on $\mathbf{x}$. Next, $M(\mathbf{x})$

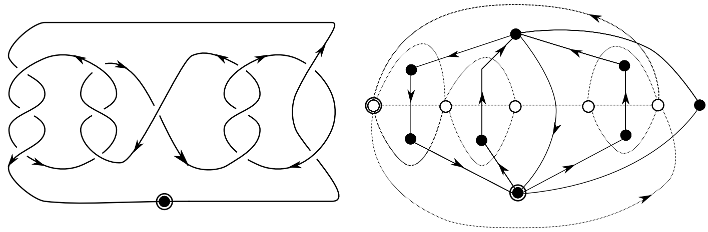

Figure 4 An example of the type 2(a) pretzel knot $K = (3, -3, 1, 3, 2)$ together with its corresponding graphs G_B and G_W, with orientations induced by K and black and white roots indicated

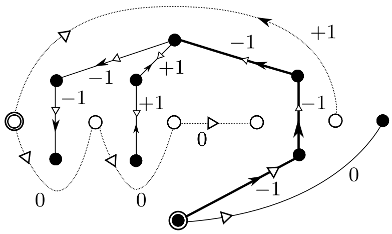

Figure 5 A state $\mathbf{x}$ of the knot $K = (3, -3, 1, 3, 2)$ in bigrading $(A(\mathbf{x}), M(\mathbf{x})) = (-4, -5)$. White arrows indicate the orientations that point away from roots, and black arrows indicate the orientations induced by K. Edges are labeled by η. The trunk is in bold

is defined by summing only over edges on which the two orientations agree,

$$M(\mathbf{x}) = \sum_{\substack{e \in \mathscr{T}_{\mathbf{x}} \\ \sigma(e)=+1}} \eta(e). \tag{5}$$

An example of a state and its bigrading is given in Figures 4 and 5.

2.4. *Counting Lemmas*

The state-sum formula, equation (3), provides an elementary way to determine the coefficients of the Alexander polynomial. Suppose that the state-sum decomposition of a diagram of a fibered knot K admits a unique state $\tilde{\mathbf{x}}$ with minimal A-grading $A(\tilde{\mathbf{x}})$. Since the symmetrized Alexander polynomial is monic of degree $g(K)$, then $A(\tilde{\mathbf{x}}) = -g(K)$ by equation (3). When such a unique minimal element $\tilde{\mathbf{x}}$ exists, it is convenient to use $\tilde{\mathbf{x}}$ to count the states in A-grading $-g(K) + 1$. We will often exploit this to show that $|a_{-g(K)+1}| > 1$, demonstrating that many pretzel knots are not L-space knots.

Definition 10. Let K be a pretzel knot with a decorated diagram, and let $\mathscr{T}_{\mathbf{x}}$ be the trees corresponding to some state $\mathbf{x}$. The *trunk* of $T_{\mathbf{x}}$ is the unique path in $T_{\mathbf{x}}$

that connects the root of G_B to the top vertex of G_B (see in Figure 5). We may also refer to the trunk of $T_{\mathbf{x}}$ as the *trunk* of $\mathbf{x}$.

Each tangle n_i determines a path in G_B from the root to the top vertex; let $T(n_i)$ denote this path. We collect the following facts to use freely throughout without reference.

FACT 11. Let $\mathbf{x}$ be any state, and let $\tilde{\mathbf{x}}$ be the unique minimally A-graded state if it exists.

- The trunk of $T_{\mathbf{x}}$ is necessarily $T(n_k)$ for some k. If $i \neq k$, then $T(n_i) \cap T_{\mathbf{x}} \neq T(n_i)$.
- If $|n_i| = 1$ and $T(n_i)$ is not the trunk of $T_{\mathbf{x}}$, then $T(n_i) \cap T_{\mathbf{x}} = \emptyset$.
- For any i, η is constant along the edges in $T(n_i)$.
- Suppose n_i is either $m_{ij} = \pm 2$ or an m_i. When $T(n_i)$ is not the trunk of the unique minimally A-graded state $\tilde{\mathbf{x}}$, there is only one terminal edge in $T(n_i) \cap T_{\tilde{\mathbf{x}}}$. In particular, $T(n_i) \cap T_{\tilde{\mathbf{x}}}$ is connected and cannot have edges incident to both the top vertex and the root.

The first three items are straightforward. For the last, note that if the subgraph $T(n_i) \cap T_{\tilde{\mathbf{x}}}$ is disconnected (i.e., there are two terminal edges in $T(n_i) \cap T_{\tilde{\mathbf{x}}}$), then it is possible to replace $\tilde{\mathbf{x}}$ with a state $\mathbf{x}'$ that agrees with $T_{\tilde{\mathbf{x}}}$ outside of $T(n_i)$ and has $T(n_i) \cap T_{\mathbf{x}'}$ connected. But then $A(\mathbf{x}') < A(\tilde{\mathbf{x}})$, and so $A(\tilde{\mathbf{x}})$ is not minimal.

DEFINITION 12. Let K be a pretzel knot with a decorated diagram and suppose that there exists a unique state $\tilde{\mathbf{x}}$ with minimal A-grading. Fix a tangle $n_i \neq \pm 1$ that does not correspond to the trunk. A *trade* is a state $\mathbf{y}$ (or $\mathscr{T}_{\mathbf{y}}$) whose corresponding black tree is obtained by replacing the terminal edge of $T_{\tilde{\mathbf{x}}}$ contained in $T(n_i)$ with the unique edge in $T(n_i) \setminus T_{\tilde{\mathbf{x}}}$. See Figure 6.

In a trade, $T_{\mathbf{y}}$ (resp. $T^*_{\mathbf{y}}$) along with its orientations and labels differs from $T_{\tilde{\mathbf{x}}}$ (resp. $T^*_{\tilde{\mathbf{x}}}$) in exactly one edge, and furthermore, $T_{\mathbf{y}}$ and $T_{\tilde{\mathbf{x}}}$ share the same trunk.

LEMMA 13. *Suppose that $K = (n_1, \dots, n_r)$ and $\tilde{\mathbf{x}}$ are as in Definition 12 and that $T(n_k)$ is the trunk of $\tilde{\mathbf{x}}$. Let ℓ be the number of tangles with $n_i = \pm 1$ and $i \neq k$. Then, there are $r - \ell - 1$ trades, all of which are supported in bigrading $(A(\tilde{\mathbf{x}}) + 1, M(\tilde{\mathbf{x}}) + 1)$.*

Proof. Let $\mathbf{y}$ be a trade. By definition there is exactly one trade corresponding with each tangle of length greater than one that is not the trunk (see Figure 6), and so there are $r - \ell - 1$ trades.

Let $e_{\tilde{\mathbf{x}}} \in T_{\tilde{\mathbf{x}}}$ and $e_{\mathbf{y}} \in T_{\mathbf{y}}$ ($e^*_{\tilde{\mathbf{x}}}$ and $e^*_{\mathbf{y}}$, respectively) be the edges along which $T_{\tilde{\mathbf{x}}}$ and $T_{\mathbf{y}}$ ($T^*_{\tilde{\mathbf{x}}}$ and $T^*_{\mathbf{y}}$ respectively) differ. The edges $e_{\tilde{\mathbf{x}}}$ and $e_{\mathbf{y}}$ are contained in $T(n_i)$ for some $i \neq k$ and therefore share the same value for η. Assume first that $\eta(e_{\tilde{\mathbf{x}}}) = \eta(e_{\mathbf{y}}) = \pm 1$ and $\eta(e^*_{\tilde{\mathbf{x}}}) = \eta(e^*_{\mathbf{y}}) = 0$. Because $A(\tilde{\mathbf{x}})$ is minimal and $\tilde{\mathbf{x}}$ is unique, $\sigma(e_{\tilde{\mathbf{x}}})\eta(e_{\tilde{\mathbf{x}}}) = -1$, or else $A(\mathbf{y}) \leq A(\tilde{\mathbf{x}})$. This implies $\sigma(e_{\tilde{\mathbf{x}}}) = -\eta(e_{\tilde{\mathbf{x}}})$. In the trade, $e_{\tilde{\mathbf{x}}}$ is replaced with $e_{\mathbf{y}}$, and the orientations induced by the root on $T_{\tilde{\mathbf{x}}}$

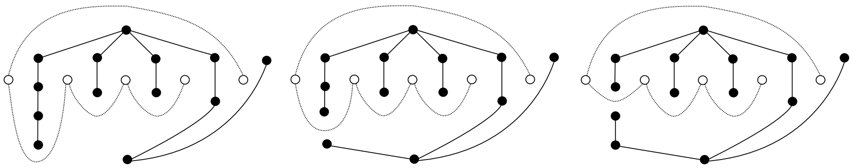

Figure 6 Three states for the pretzel knot $(5, -3, 3, 3, 2)$. If the knot is oriented so that the strands of the first tangle point downward, then the first state is the unique state with minimal A-grading, the middle state is a trade, and the last state is neither

and $T_{\mathbf{y}}$ switch from pointing down on $e_{\tilde{\mathbf{x}}}$ to pointing up on $e_{\mathbf{y}}$ (or vice versa). Hence, $\sigma(e_{\mathbf{y}}) = -\sigma(e_{\tilde{\mathbf{x}}})$. This implies $\sigma(e_{\mathbf{y}})\eta(e_{\mathbf{y}}) = +1$, and therefore both $M(\mathbf{y}) = M(\tilde{\mathbf{x}}) + 1$ and $A(\mathbf{y}) = A(\tilde{\mathbf{x}}) + 1$. Assume next that $\eta(e^*_{\tilde{\mathbf{x}}}) = \eta(e^*_{\mathbf{y}}) = \pm 1$ and $\eta(e_{\tilde{\mathbf{x}}}) = \eta(e_{\mathbf{y}}) = 0$. The trade induces a change in $T^*_{\tilde{\mathbf{x}}}$ wherein the edge $e^*_{\tilde{\mathbf{x}}}$ is replaced with an edge $e^*_{\mathbf{y}}$ that is vertically adjacent in G_W (see Figure 6). Similarly, since $A(\tilde{\mathbf{x}})$ is minimal, $\sigma(e^*_{\tilde{\mathbf{x}}})\eta(e^*_{\tilde{\mathbf{x}}}) = -1$. The same argument as for G_B applies, and we obtain $M(\mathbf{y}) = M(\tilde{\mathbf{x}}) + 1$ and $A(\mathbf{y}) = A(\tilde{\mathbf{x}}) + 1$. □

For the remainder of the paper, we proceed through the cases of Theorem 7 to prove Theorem 1. Before proceeding, we point out that pretzel knots with one strand are unknotted and that the two stranded pretzel $(a, b) \simeq T(2, a+b)$. In all of the cases that follow, K is a minimally presented fibered pretzel knot with three or more tangles, unless otherwise stated.

3. Type 1 Knots

We will only need Lemma 6 to determine which type 1 pretzel knots are L-space knots.

Lemma 14. *The only L-space pretzel knots of type 1 are those isotopic to the $T(2, 2n+1)$ torus knots. Any other fibered pretzel knot K of type 1 satisfies* $\det(K) > 2g(K) + 1$.

Proof. In our case analysis, we disregard the subcases (b) and (c) of type 1 because these are links with at least two components. Thus up to mirroring,

$$K = (\underbrace{1, \ldots, 1}_{c}, \underbrace{-3, \ldots, -3}_{d}),$$

where $c > 0$ and $d \geq 0$. When $d = 0$, K is the torus knot $T(2, c)$. Thus, assume that $d > 0$. If K has three strands, then K is isotopic to either $(1, -3, -3)$ or $(1, 1, -3)$, which are $T(2, 3)$ and the figure eight knot, respectively. The figure eight knot has $\det(K) = 5 > 2g(K) + 1$. Therefore, we may assume that K has at least four strands (in fact, five since if K is a type 1 knot, then it must have an odd number of strands).

The genus of the pretzel spanning surface, and in this case, the genus of K by Theorem 7 is given by

$$g(K) = \frac{1}{2}(d + c - 1).$$

By equation (1),

$$\det(K) = \left| 3^d \left(-c + \sum_{i=1}^{d} \frac{1}{3} \right) \right| = |3^{d-1}(d - 3c)|.$$

We will verify the inequality in two cases, $d > 3c$ and $d < 3c$, where $c, d > 0$ and $d + c \geq 5$. (When $d = 3c$, $d + c$ is even, and so K is not a knot.) If $d > 3c$, then

$$\det(K) = |3^{d-1}(d - 3c)| \geq |3^{d-1}| > \frac{4d}{3} > d + c = 2g(K) + 1.$$

Consider $d < 3c$. If $d < 3$, the inequality is easily checked by hand. If $3 \leq d < 3c$, then we have

$$\begin{aligned} 3^{d-1} - 1 > 2d \quad &\Rightarrow \quad (3^{d-1} - 1)(3c - d) > 2d - 2c \\ &\Rightarrow \quad (3^{d-1} - 1)(3c - d) + (3c - d) > d + c \\ &\Rightarrow \quad \det(K) = 3^{d-1}(3c - d) > d + c = 2g(K) + 1. \qquad \square \end{aligned}$$

4. Type 2 Knots

We remind the reader that a type 2 knot has an odd number of tangles and contains exactly one m_{ij}, which is even and denoted $\bar{m}$.

4.1. Type 2(a)

After mirroring, we may assume that a type 2(a) fibered knot has $p + 2$ positive odd tangles, p negative odd tangles, and $\bar{m} = \pm 2$. The proof of Theorem 1 for type 2(a) knots is addressed via Lemmas 15, 16, and 19.

LEMMA 15. *Up to mirroring, the only L-space pretzel knots of type 2(a) with three tangles are those isotopic to $(-2, 3, q)$ for $q \geq 1$ odd. Otherwise, there exists a coefficient a_s of $\Delta_K(t)$ such that $|a_s| \geq 2$.*

Proof. Here $p = 0$, and thus let $K = (\pm 2, r, q)$ be minimally presented, where r and q are positive odd integers. For $K = (+2, r, q)$, K is alternating and hyperbolic, hence not an L-space knot [OS05]. Therefore, we may assume that $K = (-2, r, q)$ with $r > 1$. When $r = 3$ and q is any positive odd integer, this is the family of L-space knots exempted in the assumptions of the lemma.

Without loss of generality, we may further assume that $5 \leq r \leq q$. The genus of the surface F obtained by applying the Seifert algorithm to the pretzel presentation for $K = (-2, r, q)$ is $g(F) = \frac{1}{2}(r + q)$, which is equal to $g(K)$ by Theorem 7. Thus, whenever $r > 5$ and $q > 5$ or whenever $r = 5$ and $q > 7$,

$$\det(K) = \left| 2rq \left(\frac{1}{r} + \frac{1}{q} - \frac{1}{2} \right) \right| = |2(r + q) - rq| > r + q + 1 = 2g(K) + 1.$$

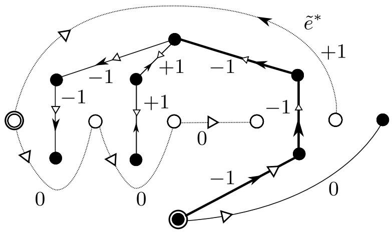

Figure 7 The trees $\mathscr{T}_{\tilde{\mathbf{x}}}$ corresponding with the unique minimal state $\tilde{\mathbf{x}}$ of the type 2(a) fibered knot $K = (3, -3, 1, 3, 2)$. Edges in the diagram are labeled by η, and $\tilde{e}^*$ is indicated

It remains to check that $r = 5$ and $q = 5$ or 7. We obtain the desired result by computing the Alexander polynomials. All Alexander polynomials in this paper are computed using the Mathematica package KnotTheory [BNM+].

$$\Delta_{P(-2,5,5)}(t) = t^{-5} - t^{-4} + t^{-2} - 2t^{-1} + 3 - 2t + t^2 - t^4 + t^5,$$
$$\Delta_{P(-2,5,7)}(t) = t^{-6} - t^{-5} + t^{-3} - 2t^{-2} + 3t^{-1} - 3 + 3t - 2t^2 + t^3 - t^5 + t^6. \qquad \square$$

Lemma 16. *Let $K = (n_1, \ldots, n_{2p+3})$ be a fibered pretzel knot of type 2(a) with $p \geq 1$. Further, suppose there exists some tangle with $n_i < -2$. Then $|a_{-g(K)+1}| \geq 2$.*

Proof. The condition of being a type 2(a) fibered knot is preserved under permutation of tangles. As mentioned in Section 2.1, the genus and $\Delta_K(t)$ are also preserved. Therefore, we may apply mutations to assume that n_i is positive when i is odd and n_i is negative when i is even, except for $n_{2p+2} > 0$ and $n_{2p+3} = \bar{m} = \pm 2$. Thus, for all edges $e \in T(n_i) \subset G_B$,

$$\eta(e) = \begin{cases} 0 & \text{if } i = 2p+3, \\ -1 & \text{if } i < 2p+3 \text{ is odd or } i = 2p+2, \\ +1 & \text{if } i \neq 2p+2 \text{ is even.} \end{cases}$$

Claim 17. *Orient K so that the strands of the first tangle point downward. Then K admits a unique state $\tilde{\mathbf{x}}$ with minimal A-grading.*

Proof. Let $\tilde{\mathbf{x}}$ be the state defined as follows and illustrated by the example in Figure 7. The trunk of $\tilde{\mathbf{x}}$ is $T(n_{2p+2})$. The intersections $T_{\tilde{\mathbf{x}}}$ with $T(n_i)$ for $i = 1, \ldots, 2p+1$ are incident to the top vertex and therefore are not incident to the black root. There is a single edge in $T_{\tilde{\mathbf{x}}} \cap T(n_{2p+3})$ that is incident to the root if $\bar{m} = 2$ or incident to the top vertex if $\bar{m} = -2$.

By choice of the orientation on K, $T(n_i)$ is oriented downward for i odd and upward for i even, except for $T(n_{2p+3})$, where instead the corresponding edges of $T^*_{\tilde{\mathbf{x}}}$ are oriented. In $T_{\tilde{\mathbf{x}}}$, the orientation induced by the root points downward along all $T(n_i)$, $i < 2p+2$, and points upward along the trunk.

Hence, for all $e \in T_{\tilde{\mathbf{x}}}$,

$$\sigma(e) = \begin{cases} 0 & \text{if } e \in T(n_{2p+3}), \\ +1 & \text{if } e \in T(n_i) \text{for } i \text{ odd and } i \neq 2p+3 \text{ or } i = 2p+2, \\ -1 & \text{if } e \in T(n_i) \text{for } i \text{ even and } i \neq 2p+2. \end{cases}$$

As for edges in the white tree $T^*_{\tilde{\mathbf{x}}}$, all are labeled $\eta(e) = \sigma(e) = 0$ except for the one edge $\tilde{e}^*$ corresponding with $n_{2p+3} = \bar{m}$, which is labeled $\eta(\tilde{e}^*) = \pm 1$ when $\bar{m} = \pm 2$. In particular, the maximal tree with minimal A-grading is constructed so that $\sigma(\tilde{e}^*)\eta(\tilde{e}^*) = -1$ regardless of the sign of $\bar{m}$. See Figure 7. Thus, every edge of $\mathscr{T}_{\tilde{\mathbf{x}}}$ with $\eta(e) \neq 0$ contributes $\sigma(e)\eta(e) = -1$ to the sum for $A(\tilde{\mathbf{x}})$, so clearly $A(\tilde{\mathbf{x}})$ is minimal.

To see that $\tilde{\mathbf{x}}$ is unique, fix an arbitrary state $\mathbf{x}$. Because there is exactly one edge $e^* \in T^*_{\mathbf{x}}$ labeled $\eta(e^*) \neq 0$, we have

$$A(\mathbf{x}) = \frac{1}{2}\left(\sigma(e^*)\eta(e^*) + \sum_{e \in T_{\mathbf{x}}} \sigma(e)\eta(e)\right).$$

In particular, for $\tilde{\mathbf{x}}$,

$$A(\tilde{\mathbf{x}}) = \frac{1}{2}\left(-1 + \sum_{e \in T_{\tilde{\mathbf{x}}}} \sigma(e)\eta(e)\right).$$

Suppose that $\mathbf{x}$ is a state with the same trunk $T(n_{2p+2})$ as $\tilde{\mathbf{x}}$ but for which $T_{\mathbf{x}}$ differs from $T_{\tilde{\mathbf{x}}}$ along any set of edges of $T(n_i)$, $i = 1, \ldots, n_{2p+1}$. Then there exists some edge of $T_{\mathbf{x}}$ that is incident to the root and this edge will contribute $\sigma(e)\eta(e) = +1$ to the sum for $A(\mathbf{x})$. Since the contribution of the white tree $T^*_{\mathbf{x}}$ is not impacted, $A(\mathbf{x}) > A(\tilde{\mathbf{x}})$. If, instead, $\mathbf{x}$ shares the same trunk as $\tilde{\mathbf{x}}$ but $T_{\mathbf{x}}$ differs from $T_{\tilde{\mathbf{x}}}$ along $T(n_{2p+3})$, then the edge $e^* \in T^*_{\mathbf{x}}$ will contribute $\sigma(e^*)\eta(e^*) = +1$ to the sum for $A(\mathbf{x})$, and again $A(\mathbf{x}) > A(\tilde{\mathbf{x}})$. Now, suppose that $\mathbf{x}$ has a different trunk from $\tilde{\mathbf{x}}$. If the trunk of $\mathbf{x}$ is $T(n_{2p+3})$ and $T(n_i) \cap T_{\mathbf{x}}$ agrees with $T(n_i) \cap T_{\tilde{\mathbf{x}}}$ for $i = 1, \ldots, 2p+1$, then $A(\mathbf{x}) = A(\tilde{\mathbf{x}}) + 1$. Note that we exclude $i = 2p+2$ since if $T(n_{2p+2}) \cap T_{\mathbf{x}} = T(n_{2p+2}) \cap T_{\tilde{\mathbf{x}}} = T(n_{2p+2})$, then $T_{\mathbf{x}}$ would contain a cycle. If the trunk of $\mathbf{x}$ is $T(n_{2p+3})$ and $T(n_i) \cap T_{\mathbf{x}}$ does not agree with $T(n_i) \cap T_{\tilde{\mathbf{x}}}$ for some $i = 1, \ldots, 2p+1$, then $A(\mathbf{x}) > A(\tilde{\mathbf{x}}) + 1$. If, instead, the trunk of $\mathbf{x}$ is $T(n_i)$ for some $i = 1, \ldots, 2p+1$, then certainly $A(\mathbf{x}) \geq A(\tilde{\mathbf{x}}) + 1$. Hence, $\tilde{\mathbf{x}}$ is unique. $\square$

Let ℓ be the number of length one tangles excluding the trunk. By Lemma 13 there are $2p+2-\ell$ trades, all supported in bigradings $(-g(K)+1, M(\tilde{\mathbf{x}})+1)$. To determine that $|a_{-g(K)+1}| \geq 2$, we need to count the other states in A-grading $-g(K)+1$ and compute their M-gradings.

Because $\bar{m} = \pm 2$, all of the trees that share the same trunk as $T_{\tilde{\mathbf{x}}}$ that are not trades represent states that have an A-grading greater than $-g(K)+1$. Thus, the states in A-grading $-g(K)+1$ that are not trades are states with different trunks. One of these states is denoted $\mathbf{x}'$, where $T_{\mathbf{x}'}$ differs from $T_{\tilde{\mathbf{x}}}$ only as follows. The trunk of $\mathbf{x}'$ is $T(n_{2p+3})$, and $T_{\mathbf{x}'} \cap T(n_{2p+2})$ is connected and incident to the root. If $\bar{m} = -2$, then $\mathbf{x}'$ is supported in bigrading $(-g(K)+1, M(\tilde{\mathbf{x}})+2)$, and if $\bar{m} = +2$, then $\mathbf{x}'$ is supported in bigrading $(-g(K)+1, M(\tilde{\mathbf{x}})+1)$. Each remaining state in A-grading $-g(K)+1$ corresponds to a state denoted $\mathbf{x}_j$, where $T_{\mathbf{x}_j}$ differs from $T_{\tilde{\mathbf{x}}}$ only as follows. The trunk of $\mathbf{x}_j$ is $T(n_j)$ for some $n_j = \pm 1$, $j \neq 2p+2$, and $T(\mathbf{x}_j) \cap T(n_{2p+2})$ is connected and incident to the root. The trunk of $T_{\mathbf{x}_j}$ is necessarily of length one because otherwise $A(\mathbf{x}_j) > -g(K)+1$ due to the contribution of at least two edges in $T(n_j)$ with $\sigma(e)\eta(e) = +1$.

Claim 18. *Let $\mathbf{x}_j$ be as before. Then,*

$$M(\mathbf{x}_j) = \begin{cases} M(\tilde{\mathbf{x}})+1 & j \text{ odd and } j \neq 2p+3, \\ M(\tilde{\mathbf{x}})+2 & j \text{ even and } j \neq 2p+2. \end{cases}$$

Proof. In $T(n_{2p+2})$, all edges are labeled $\eta(e) = -1$. For all $e \in T_{\tilde{\mathbf{x}}} \cap T(n_{2p+2})$, $\sigma(e) = +1$. Because $n_j = \pm 1$, $T_{\tilde{\mathbf{x}}} \cap T(n_j) = \emptyset$. Now $T_{\mathbf{x}_j} \cap T(n_{2p+2})$ contains $n_{2p+2} - 1$ edges, all with $\sigma(e) = +1$. For the single edge $e \in T(n_j) \cap T_{\mathbf{x}_j}$, $\sigma(e) = \eta(e) = -1$ if j is odd and $\sigma(e) = \eta(e) = +1$ if j is even. All other edges and labels of $T_{\mathbf{x}_j}$ and $T_{\tilde{\mathbf{x}}}$ agree, and the changes in the white graphs do not affect the M-grading. The net change to the M-grading from $\tilde{\mathbf{x}}$ to $\mathbf{x}_j$ is $+1$ or $+2$, respectively. □

By equation (3) the coefficient $|a_{-g(K)+1}|$ is given by the absolute value of the difference in the numbers of states in M-gradings $M(\tilde{\mathbf{x}})+1$ and $M(\tilde{\mathbf{x}})+2$. Suppose first that $\bar{m} = 2$. Since K is minimally presented, there are no j with $n_j = -1$. Thus, we may assume that any tangle of length one is positive, and therefore all states in A-grading $-g(K)+1$ are supported in M-grading $M(\tilde{\mathbf{x}})+1$ by Lemma 13 and Claim 18. This implies $|a_{-g(K)+1}| > 1$ since clearly there is more than one such state. Suppose now that $\bar{m} = -2$. We may similarly assume that each length one tangle is negative. Since $n_{2p+2} > 0$, the trunk is not of length one, and therefore ℓ is the number of length one tangles. Again by Lemma 13 and Claim 18,

$$|a_{-g(K)+1}| = (2p+2-\ell) - (\ell+1) = 2p - 2\ell + 1,$$

and so $|a_{-g(K)+1}| > 1$ whenever $p > \ell$. When $p = \ell$, every negative tangle other than $\bar{m}$ is of length one. In other words, if there exists some tangle with $n_i < -2$, then $|a_{-g(K)+1}| \geq 2$. This verifies the statement of the lemma. □

In light of Lemmas 15 and 16, after isotopy and our assumptions on mirroring,

$$K = (-2, \underbrace{-1, \ldots, -1}_{p}, w_1, w_2, \ldots, w_{p+2}), \tag{6}$$

where $w_i \geq 3$ is odd for $1 \leq i \leq p+2$ and $p \geq 1$.

Lemma 19. *Let K be as in equation (6). Then* $\det(K) > 2g(K)+1$.

Proof. Since K is a type 2(a) fibered knot, then by Theorem 7 the minimal genus Seifert surface that is the fiber for K is obtained by applying the Seifert algorithm to the standard projection. This gives

$$g(K) = \frac{1}{2}\left(\sum_{i=1}^{p+2}(w_i - 1) + 2\right).$$

Let $W = w_1 \cdots w_{p+2}$. By equation (1) and the fact that $w_i \geq 3$ is odd for $1 \leq i \leq p+2$,

$$\begin{aligned}
\det(K) &= \left|-2W\left(-p - \frac{1}{2} + \sum_{i=1}^{p+2}\frac{1}{w_i}\right)\right| \\
&= \left|W + 2W\left(p - \sum_{i=1}^{p+2}\frac{1}{w_i}\right)\right| \\
&\geq \left|W + 2W\left(\frac{2p-2}{3}\right)\right|.
\end{aligned}$$

Since $p \geq 1$,

$$\begin{aligned}
\left|W + 2W\left(\frac{2p-2}{3}\right)\right| &\geq W \\
&> \left(\sum_{i=1}^{p+2} w_i\right) + 1 \\
&\geq \sum_{i=1}^{p+2}(w_i - 1) + 3 \\
&= 2g(K) + 1. \qquad \square
\end{aligned}$$

4.2. *Type 2(b)*

A type 2(b) fibered pretzel knot K has exactly one even tangle $\bar{m}$, which is the unique m_{ij}, p positive odd tangles, and p negative odd tangles, where $p \geq 1$. The auxiliary link $L' \neq \pm(2, -2, \ldots, 2, -2)$, and K fibers if and only if L' fibers (see equation (2) for the construction of L').

LEMMA 20. *For all minimally presented fibered pretzel knots of type 2(b),* $|a_{-g(K)+1}| \geq 2$.

Proof. Suppose first that K has $c > 0$ length one tangles. Recall that length one tangles do not factor into L'. If $c \geq 3$, then L' is not a type 1 fibered link (see Theorem 7). If $c = 2$, then the fiberedness of L' implies $\bar{m} = \pm 2$ when the length one tangles are ∓ 1, and this is not allowed because K is then not minimally presented.

Suppose $c = 1$. Since L' has an even number of tangles, $L' = \pm(2, -2, \ldots, 2, -4)$. Thus, up to mirroring,

$$K = (1, m_1, \ldots, m_{2p-1}, -4),$$

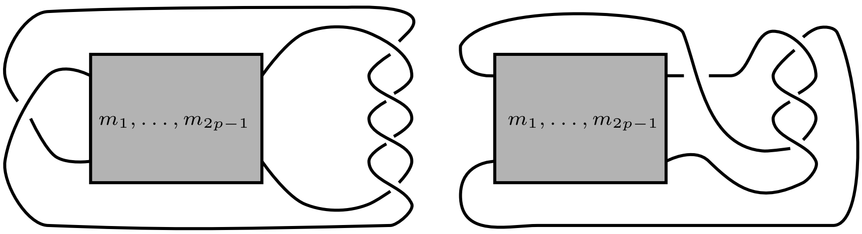

Figure 8 The isotopy performed on the type 2(b) knot $K = (1, m_1, \ldots, m_{2p-1}, -4)$

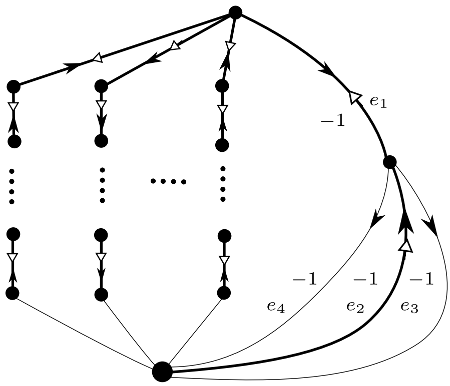

Figure 9 The black graph after isotoping the type 2(b) knot $(1, m_1, \ldots, m_{2p-1}, -4)$ with $T_{\tilde{\mathbf{x}}}$ in bold

where $m_i < -2$ for $1 \le i \le 2p-1$, i odd, and $m_i > 2$ for $1 < i < 2p-1$, i even. Isotope K according to Figure 8. After this isotopy, the knot diagram admits a black graph whose edges are all labeled $\eta(e) = \pm 1$ and a white graph where all of the edges are labeled 0. Thus, we only need to consider maximal trees of the black graph to compute $\Delta_K(t)$. This is no longer a pretzel presentation, but as can be seen in Figure 9, we can make sense of the terms trunk, top vertex, trade, and so on and may apply the content of Section 2.4 in an analogous manner.

Claim 21. *After the isotopy specified in Figure 8, orient K so that the strands of the first tangle point upward. Then there is a unique state $\tilde{\mathbf{x}}$ with minimal A-grading.*

Proof. Refer to Figures 8 and 9. Since $m_i < -2$ for i odd and $m_i > 2$ for i even, we have that for all edges $e \in T(m_i) \subset G_B$, $i = 1, \ldots, 2p-1$,

$$\eta(e) = \begin{cases} +1 & e \in T(m_i), i \text{ odd}, \\ -1 & e \in T(m_i), i \text{ even}. \end{cases}$$

There are four additional edges in G_B, and each is labeled $\eta(e) = -1$. Let $\tilde{\mathbf{x}}$ be the state with trunk $e_1 \cup e_2$ and with no other edges incident to the black root. For all $e \in T_{\tilde{\mathbf{x}}}$ with $e \neq e_1$, $\sigma(e)\eta(e) = -1$, and for e_1, $\sigma(e_1)\eta(e_1) = +1$. Because $|m_i| > 2$ for $i = 1, \ldots, 2p-1$, we have that for any other state, the corresponding A-grading is strictly greater than $A(\tilde{\mathbf{x}})$. Hence, $\tilde{\mathbf{x}}$ is the unique state with minimal A-grading. □

It is easy to verify that there are exactly $2p + 1$ states in A-grading $-g(K) + 1$, all of which are obtained by trades along any of $m_1, \ldots, m_{2p-1}$ or by replacing e_2 with e_3 or e_4. Each of these $2p + 1$ states is supported in the same M-grading by an argument similar to Lemma 13. Hence, $|a_{-g(K)+1}| = 2p + 1$ whenever K contains any tangle of length one, thus completing the proof of Lemma 20 in this case (i.e., when $c > 0$).

Suppose now that there are no tangles of length one in K. In particular, we are no longer working with the isotopy of Figure 8. Since L' is a fibered type 1 link, L' is isotopic to $\pm(2, -2, 2, -2, \ldots, 2, -2, n)$ for some $n \in \mathbb{Z}$. Since K is type 2(b), up to mirroring, there exists a permutation of the tangles such that the resulting knot, denoted K^τ, is of the form

$$K^\tau = (m_1, \ldots, m_{2p}, \bar{m}),$$

where $m_i > 0$ when i is odd, $m_i < 0$ is negative when i is even, and $\bar{m}$ is even. Since K^τ has no tangles of length one, the auxiliary link for K^τ is isotopic to $\pm(2, -2, 2, -2, \ldots, 2, -2, n^\tau)$ for some $n^\tau \in \mathbb{Z}$, and therefore K^τ is a type 2(b) fibered pretzel knot. Because K^τ is a fibered mutant of the fibered knot K, it shares the same Alexander polynomial and genus. Therefore, it suffices to work with K^τ.

When the pretzel diagram for K^τ is oriented so that the strands of m_1 point downward, K^τ admits a unique state $\tilde{\mathbf{x}}$ with minimal A-grading $-g(K^\tau)$. This state has trunk $T(\bar{m})$, and no other edges of $T_{\tilde{\mathbf{x}}}$ are incident to the root. See Figure 10. Because the tangles alternate sign, every edge of $\mathscr{T}_{\tilde{\mathbf{x}}}$ contributes $\sigma(e)\eta(e) = -1$ or 0 to the sum for $A(\tilde{\mathbf{x}})$. Because there are no tangles of length one, any other state will have a strictly greater A-grading. Hence, $\tilde{\mathbf{x}}$ is unique and minimally A-graded. Moreover, every state supported in A-grading $-g(K^\tau) + 1$ is a trade because there is a unique $\bar{m}$ and there are no tangles of length one. By Lemma 13, there are $2p$ trades, each supported in M-grading $M(\tilde{\mathbf{x}}) + 1$. Hence, $|a_{-g(K^\tau)+1}| = 2p \geq 2$, and this implies $|a_{-g(K)+1}| = 2p \geq 2$. □

5. Type 3 Knots

Each tangle in a type 3 knot is an m_i, and therefore all edges $e \in G_B$ and $e^* \in G_W$ are labeled $\eta(e) = \pm 1$ and $\eta(e^*) = 0$, respectively (see Figure 3). In particular, the Alexander polynomials of type 3 knots can be computed solely using the black graph G_B. Moreover, in this case, K is a pretzel knot of even length, so we will assume that K has at least four tangles.

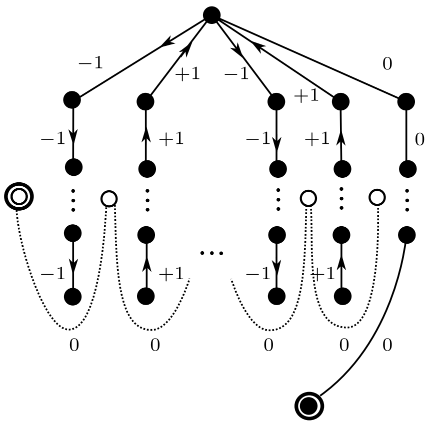

Figure 10 The unique minimal state for a type 2(b) knot K^τ with no tangles of length one. Labels $\eta(e)$ are indicated in the diagram

5.1. Type 3-Min

A type 3-min knot K has p positive tangles and p negative tangles. Of these there is a unique tangle of minimal length and an even tangle, which are possibly the same tangle. By assumption, since K is fibered, $L' = \pm(2, -2, \ldots, 2, -2)$ also has an even number of tangles, and thus by uniqueness of the minimal tangle, there are no tangles of length one.

LEMMA 22. *For all fibered pretzel knots of type* 3*-min not isotopic to* $\pm(3, -5, 3, -2)$, *there exists a coefficient of the Alexander polynomial such that* $|a_s| \geq 2$.

Proof. By the conditions on L', the tangles of K alternate sign. After mirroring and cyclic permutation, we may assume that n_i is positive when i is odd, n_i is negative when i is even, and $|n_{2p}|$ is minimal. For all $e \in T(n_i)$, $\eta(e) = -1$ when i is odd and $\eta(e) = +1$ when i is even. Orient the pretzel diagram so that the first tangle points downward. Let $\tilde{\mathbf{x}}$ be the state with trunk $T(n_{2p})$ and no other edges incident to the root (see the example in Figure 11). Because the tangles alternate sign, $\eta(e)\sigma(e) = -1$ for all $e \in T(n_i)$ for $i = 1, \ldots, 2p-1$, and for $e \in T(n_{2p})$, $\eta(e)\sigma(e) = +1$. Since n_{2p} is the unique minimal length tangle, $A(\tilde{\mathbf{x}})$ is minimal, and $\tilde{\mathbf{x}}$ is the unique state with minimal A-grading.

By Lemma 13 there are $2p - 1$ trades in bigrading $(-g(K) + 1, M(\tilde{\mathbf{x}}) + 1)$. Since there is no m_{ij}, all states in A-grading $-g(K) + 1$ that are not trades have a corresponding black tree with trunk $T(n_j)$ such that $|n_j| = |n_{2p}| + 1$ by an argument similar to that in Lemma 16. Denote such a state by $\mathbf{x}_j$. First, suppose n_{2p} is odd. Since there is exactly one even tangle, there is at most one state $\mathbf{x}_j$. If no such $\mathbf{x}_j$ exists, then $|a_{-g(K)+1}| = 2p - 1$. Otherwise, $|a_{-g(K)+1}| = (2p-1) \pm 1$, depending on $M(\mathbf{x}_j)$. Since $2p \geq 4$, we have $|a_{-g(K)+1}| \geq 2$. Now suppose n_{2p} is even.

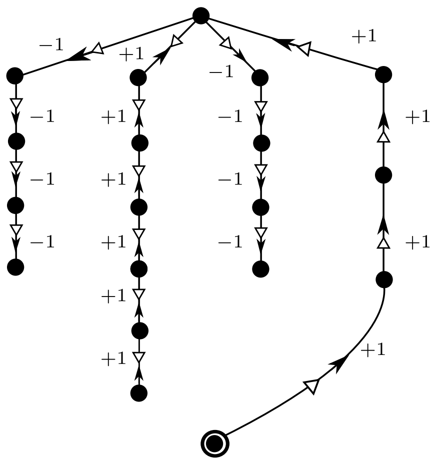

Figure 11 An example of the unique minimal state for the type 3-min knot $(5, -7, 5, -4)$ with trunk along the unique tangle of minimum length

CLAIM 23. *Let n_j be a tangle of length $|n_{2p}| + 1$. Then,*

$$M(\mathbf{x}_j) = \begin{cases} M(\tilde{\mathbf{x}}) & j \text{ odd}, \\ M(\tilde{\mathbf{x}}) + 1 & j \text{ even}. \end{cases}$$

Proof. Fix j such that $|n_j| = |n_{2p}| + 1$. Recall that the trunk of $T_{\tilde{\mathbf{x}}}$ is $T(n_{2p})$, the trunk of $T_{\mathbf{x}_j}$ is $T(n_j)$, and for all $e \in T(n_i)$, $\eta(e) = -1$ when i is odd and $\eta(e) = +1$ when i is even. Additionally, outside of $T(n_{2p})$ and $T(n_j)$, $T_{\tilde{\mathbf{x}}}$ and $T_{\mathbf{x}_j}$ agree. For $\tilde{\mathbf{x}}$, the values of σ are given by

$$\sigma(e) = \begin{cases} +1 & e \in T_{\tilde{\mathbf{x}}} \cap T(n_{2p}), \\ +1 & e \in T_{\tilde{\mathbf{x}}} \cap T(n_j),\ j \text{ odd}, \\ -1 & e \in T_{\tilde{\mathbf{x}}} \cap T(n_j),\ j \text{ even}, \end{cases}$$

and for $\mathbf{x}_j$, the values of σ are given by

$$\sigma(e) = \begin{cases} -1 & e \in T_{\mathbf{x}_j} \cap T(n_{2p}),\ j \text{ odd or even}, \\ -1 & e \in T_{\mathbf{x}_j} \cap T(n_j),\ j \text{ odd}, \\ +1 & e \in T_{\mathbf{x}_j} \cap T(n_j),\ j \text{ even}. \end{cases}$$

Suppose j is odd. Then because $|n_j| = |n_{2p}| + 1$,

$$M(\mathbf{x}_j) - M(\tilde{\mathbf{x}}) = \sum_{\substack{e \in T_{\mathbf{x}_j} \\ \sigma(e)=1}} \eta(e) - \sum_{\substack{e \in T_{\tilde{\mathbf{x}}} \\ \sigma(e)=1}} \eta(e) = -(|n_{2p}| - (|n_j| - 1)) = 0.$$

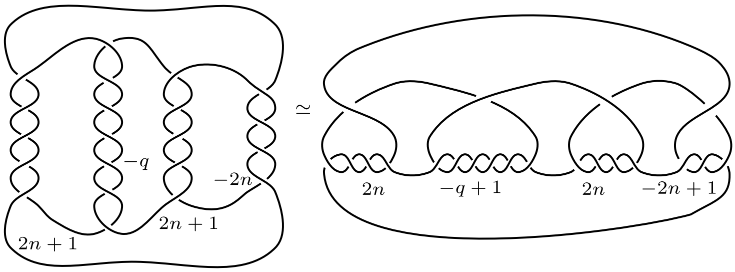

Figure 12 The isotopy performed on $(2n+1, -q, 2n+1, -2n)$ to obtain a Seifert surface with reduced genus

Suppose j is even. Then

$$M(\mathbf{x}_j) - M(\tilde{\mathbf{x}}) = \sum_{\substack{e \in T_{\mathbf{x}_j} \\ \sigma(e)=1}} \eta(e) - \sum_{\substack{e \in T_{\tilde{\mathbf{x}}} \\ \sigma(e)=1}} \eta(e) = |n_j| - |n_{2p}| = 1. \qquad \square$$

Let E and O be the numbers of states $\mathbf{x}_j$ with j even and odd, respectively. Since there are no tangles of length one, there are $2p-1$ trades supported in M-grading $M(\tilde{\mathbf{x}})+1$. By Claim 23,

$$|a_{-g(K)+1}| = |(2p-1) + E - O|.$$

Therefore, $|a_{-g(K)+1}| \geq p-1$, so whenever $p > 2$, we are done.

The case $p = 2$ remains. In particular, $|a_{-g(K)+1}| = |(2p-1)+E-O| \leq 1$ only when $E = 0$ and $O = 2$. Thus, it suffices to consider

$$K = (2n+1, -q, 2n+1, -2n),$$

where $n \geq 1$ and $q \geq 2n+3$ is odd.

We will reduce the genus of the surface obtained by the Seifert algorithm by performing a particular isotopy of K, which is described in [Gab86] and pictured in Figure 12. Applying the Seifert algorithm to the new diagram gives a lower genus Seifert surface F for K, suitable to apply Lemma 6, but not necessarily a genus minimizing Seifert surface. We obtain

$$g(F) = \frac{1}{2}(6n+q-3).$$

By equation (1),

$$\det(K) = |4n(2n+1)q - (2n+1)^2 q - 2n(2n+1)^2|.$$

In general, $\det(K) > 2g(F)+1 \geq 2g(K)+1$ is satisfied whenever

$$(4n(2n+1) - (2n+1)^2 - 1)q > 2n(2n+1)^2 + 6n - 2, \tag{7}$$

and since $q \geq 2n+3$, this inequality holds for all $n > 3$. Moreover, if $n = 1$, $n = 2$, or $n = 3$, then $\det(K) > 2g(F)+1 \geq 2g(K)+1$ whenever $q \geq 13$, $q \geq 9$, or $q \geq 11$, respectively. The only pairs (n, q) not satisfying inequality (7) are:

$(1,5)$, $(1,7)$, $(1,9)$, $(1,11)$, $(2,7)$, and $(3,9)$. The Alexander polynomials for the knots corresponding to the last five pairs are:

$$\begin{aligned}
\Delta_{(3,-7,3,-2)} &= t^{-4} - t^{-3} + 2t^{-1} - 3 + 2t - t^3 + t^4, \\
\Delta_{(3,-9,3,-2)} &= t^{-5} - t^{-4} + 2t^{-2} - 3t^{-1} + 3 - 3t + 2t^2 - t^4 + t^5, \\
\Delta_{(3,-11,3,-2)} &= t^{-6} - t^{-5} + 2t^{-3} - 3t^{-2} + 3t^{-1} \\
&\quad - 3 + 3t - 3t^2 + 2t^3 - t^5 + t^6, \\
\Delta_{(5,-7,5,-4)} &= t^{-5} - t^{-4} + t^{-2} - 2t^{-1} + 3 - 2t + t^2 - t^4 + t^5, \\
\Delta_{(7,-9,7,-6)} &= t^{-7} - t^{-6} + t^{-4} - 2t^{-3} + 3t^{-2} - 4t^{-1} \\
&\quad + 5 - 4t + 3t^2 - 2t^3 + t^4 - t^6 + t^7.
\end{aligned}$$

Clearly, each polynomial has some coefficient with $|a_s| > 1$. The first pair of integers corresponds to $(3, -5, 3, -2)$, the knot exempted in the statement of the lemma. □

As discussed in the introduction, the Alexander polynomial of the knot $(3, -5, 3, -2)$ does not obstruct it from admitting an L-space surgery. However, its knot Floer homology (see Table 1) readily provides an obstruction. This completes the proof of Theorem 1 for type 3-min pretzel knots.

5.2. *Type 3-2(a)*

After mirroring, we may assume that for pretzel knots of Type 3-2(a), there are $p+2$ positive tangles and p negative tangles, and that of these $2p+2$ tangles, there is exactly one even tangle. Note that the property of being a type 3-2(a) fibered pretzel knot does not change under mutation.

LEMMA 24. *Let K be as before. If K does not have exactly p negative tangles of length one, then $|a_{-g(K)+1}| \geq 2$.*

Proof. Up to mutation, we may assume that n_i is positive when i is odd and that n_i is negative when i is even, except n_{2p+2}, which is positive. In G_B, $e \in T(n_i)$ is labeled $\eta(e) = -1$ for i odd or $i = 2p+2$ and $\eta(e) = +1$ for i even, $i \neq 2p+2$. Orient K so that the strands of the first tangle point downward. Then there is a unique state $\tilde{\mathbf{x}}$ with minimal A-grading represented by a black tree with trunk $T(n_{2p+2})$ as in Lemma 16. In particular, for all $e \in T_{\tilde{\mathbf{x}}}$, $\sigma(e) = +1$ if $e \in T(n_i)$ for i odd or $i = 2p+2$ and $\sigma(e) = -1$ if i even, $i \neq 2p+2$. Every edge in $T_{\tilde{\mathbf{x}}}$ contributes $\eta(e)\sigma(e) = -1$ to the sum for $A(\tilde{\mathbf{x}})$, so $\tilde{\mathbf{x}}$ is clearly minimally graded. It is unique because in any other tree there will be an edge contributing $\sigma(e)\eta(e) = +1$ to the A-grading.

There are $2p - \ell + 1$ trades in bigrading $(-g(K)+1, M(\tilde{\mathbf{x}})+1)$ by Lemma 13, where ℓ is the number of tangles of length one not counting the trunk. There are precisely ℓ other states in A-grading $-g(K)+1$. Each of these additional states, denoted $\mathbf{x}_j$, corresponds to a tangle n_j of length one, as obtained in Lemma 16.

Then,

$$M(\mathbf{x}_j) = \begin{cases} M(\tilde{\mathbf{x}}) + 1 & j \text{ is odd}, \\ M(\tilde{\mathbf{x}}) + 2 & j \neq 2p+2 \text{ is even}, \end{cases}$$

as in Claim 18. If the length one tangles are positive (i.e., each j is odd), then

$$|a_{-g(K)+1}| = (2p - \ell + 1) + \ell = 2p + 1 > 2,$$

and we are done. If the length one tangles are negative, then

$$|a_{-g(K)+1}| = (2p - \ell + 1) - \ell > 1 \quad \Longleftrightarrow \quad \ell < p.$$

This verifies the statement of Lemma 24. □

The next lemma will complete the proof of Theorem 1 for type 3-2(a) pretzel knots.

LEMMA 25. *Let K be a type 3-2(a) knot with exactly p negative length one tangles and $p+2$ positive tangles. Then there exists some coefficient a_s of $\Delta_K(t)$ with $|a_s| > 1$.*

Proof. After reindexing the tangles,

$$K = (\underbrace{-1, \ldots, -1}_{p}, w_1, \ldots, w_{p+2}),$$

where there exists some i such that $w_i \geq 4$ is even (since K is minimally presented, $w_i \neq 2$ for any i) and for all other i, $w_i \geq 3$ is odd. By Theorem 7 the genus of K is obtained by applying the Seifert algorithm to the standard projection,

$$g(K) = \frac{1}{2}\left(\sum_{i=1}^{p+2}(w_i - 1) + 1\right).$$

Let $W = w_1 \cdots w_{p+2}$. Using equation (1), we have

$$\begin{aligned}
\det(K) &= \left| W\left(-p + \sum_{i=1}^{p+2} \frac{1}{w_i}\right)\right| \\
&\geq \left| W\left(p - \frac{1}{4} - \sum_{i=1}^{p+1} \frac{1}{3}\right)\right| \\
&\geq W \cdot \frac{8p-7}{12}.
\end{aligned}$$

Whenever $p \geq 2$, we have

$$\det(K) \geq W \cdot \frac{8p-7}{12} > \left(\sum_{i=1}^{p+2} w_i\right) - p = 2g(K) + 1.$$

Now apply Lemma 6.

If $p = 1$, then $K = (-1, w_1, w_2, w_3)$. Now suppose that one of the w_i is at least five. Then,

$$\begin{aligned}\det(K) &= \left| W\left(1 - \sum_{i=1}^{3} \frac{1}{w_i}\right)\right| \\ &\geq W \cdot \frac{13}{60} \\ &> \left(\sum_{i=1}^{3} w_i\right) - 1 \\ &= 2g(K) + 1.\end{aligned}$$

The only type 3-2(a) fibered pretzel knot with four or more strands that has not been addressed is $K = (-1, 3, 3, 4)$, which has the Alexander polynomial

$$\Delta_{(-1,3,3,4)} = t^{-4} - t^{-3} + 2t^{-1} - 3 + 2t - t^3 + t^4.$$

Clearly, there exist coefficients with $|a_s| > 1$. □

5.3. *Type 3-2(b)*

Let K be a fibered type 3-2(b) pretzel knot. There are p positive tangles and p negative tangles. By assumption the auxiliary link L' is not isotopic to $\pm(2, -2, \dots, 2, -2)$, and K is fibered if and only if L' is fibered. There are no tangles of L' equal to ± 1, and therefore L' cannot be of type 1(a). Since there is no m_{ij}, there are no tangles equal to ± 4, and so we may also rule out type 1(c). Therefore, L' must fall into the type 1(b) subcase of type 1 knots, which are of the form $\pm(2, -2, \dots, 2, -2, n)$, where $n \in \mathbb{Z}$. This can only happen if $n = \pm 2$ and K contains a unique tangle of length one.

Up to mirroring and isotopy, $K = (n_1, \dots, n_{2p})$, where n_i is positive for i odd, negative for i even, and $n_{2p} = -1$. Orient K so that the strands of the first tangle point downward. Then $\eta(e) = -1$ when $e \in T(n_i)$ for i odd and $\eta(e) = +1$ when $e \in T(n_i)$ for i even. As in the proof of Lemma 22, there exists a unique state $\tilde{\mathbf{x}}$ with minimal A-grading with trunk $T(n_{2p})$ and with the property that $\sigma(e) = +1$ when $e \in T(n_i)$ for i odd and $\sigma(e) = -1$ when $e \in T(n_i)$ for i even. The only possible states that are not trades must occur along tangles of length two. Since there is a single even tangle, there is at most one such state. By equation (3) and Lemma 13 this implies that $|a_{-g(K)+1}|$ is at least $2p - 2$, and hence $|a_{-g(K)+1}| \geq 2$.

This completes the case analysis required to prove Theorem 1.

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