Michigan Math. J. 64 (2015), 169–202

Quasi-conformal Maps on Model Filiform Groups

Xiangdong Xie

Abstract. We describe all quasi-conformal maps on the higher (real and complex) model Filiform groups equipped with the Carnot metric, including nonsmooth ones. These maps have very special forms. In particular, they are all bi-Lipschitz and preserve multiple foliations. The results in this paper have implications to the large-scale geometry of nilpotent Lie groups and negatively curved solvable Lie groups.

1. Introduction

In this paper we study quasi-conformal maps on the higher real and complex model Filiform groups equipped with the Carnot metric. We identify all such maps. They are all bi-Lipschitz and preserve multiple foliations. We do not impose any regularity condition on the quasi-conformal maps. However, the group structure forces rigidity and regularity. In particular, in the case of higher complex model Filiform groups, up to taking complex conjugation, all quasi-conformal maps are biholomorphic and in fact affine.

Let K be a field. We only consider the case where K is $\mathbb{R}$ or $\mathbb{C}$. The n-step ($n \geq 2$) model Filiform algebra $\mathfrak{f}_K^n$ over K is an $(n+1)$-dimensional Lie algebra over K. It has a basis $\{e_1, e_2, \dots, e_{n+1}\}$, and the only nontrivial bracket relations are $[e_1, e_j] = e_{j+1}$ for $2 \leq j \leq n$. The Lie algebra $\mathfrak{f}_K^n$ admits a direct sum decomposition of vector subspaces $\mathfrak{f}_K^n = V_1 \oplus \cdots \oplus V_N$, where V_1 is the linear subspace spanned by e_1, e_2, and V_j $(2 \leq j \leq n)$ is the linear subspace spanned by e_{j+1}. It is easy to check that $[V_1, V_j] = V_{j+1}$ for $1 \leq j \leq n$, where $V_{n+1} = \{0\}$. Hence, $\mathfrak{f}_K^n$ is a stratified Lie algebra. For $K = \mathbb{R}$ or $\mathbb{C}$, the connected and simply connected Lie group with Lie algebra $\mathfrak{f}_K^n$ will be denoted by F_K^n and is called the n-step model Filiform group over K.

The two-dimensional subspace V_1 of $\mathfrak{f}_{\mathbb{R}}^n$ determines a left-invariant distribution (so called horizontal distribution) on $F_{\mathbb{R}}^n$. On V_1, we consider the inner product with e_1, e_2 as an orthonormal basis. This norm on V_1 then induces a Carnot metric d_c on $F_{\mathbb{R}}^n$. Similarly, the first layer V_1 of $\mathfrak{f}_{\mathbb{C}}^n$ is a four-dimensional real vector subspace spanned by e_1, ie_1, e_2, ie_2 $(i = \sqrt{-1})$, and it determines a left-invariant distribution on $F_{\mathbb{C}}^n$. On V_1 of $\mathfrak{f}_{\mathbb{C}}^n$, we consider the inner product with e_1, ie_1, e_2, ie_2 as an orthonormal basis. This norm on V_1 then induces a Carnot metric d_c on $F_{\mathbb{C}}^n$.

Recall that, for a connected and simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. We shall identify $\mathfrak{g}$ and G via the exponential map and denote the group operation by $*$.

Received February 20, 2014. Revision received October 10, 2014.

Notice that every element $x \in \mathfrak{f}_{\mathbb{R}}^n$ can be uniquely written as $x = x_1 e_1 * (x_2 e_2 + \cdots + x_{n+1} e_{n+1})$, where $x_i \in \mathbb{R}$, $1 \le i \le n+1$.

Fix $n \ge 2$. Let $h : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Set $h_2 = h$ and define $h_j : \mathbb{R} \to \mathbb{R}$, $3 \le j \le n+1$, inductively as follows:

$$h_j(x) = -\int_0^x h_{j-1}(s)\, ds.$$

Define $F_h : F_{\mathbb{R}}^n \to F_{\mathbb{R}}^n$ by

$$F_h\left(x_1 e_1 * \sum_{j=2}^{n+1} x_j e_j\right) = x_1 e_1 * \sum_{j=2}^{n+1} (x_j + h_j(x_1)) e_j.$$

THEOREM 1.1. *Let $n \ge 3$. A homeomorphism $F : (F_{\mathbb{R}}^n, d_c) \to (F_{\mathbb{R}}^n, d_c)$ is quasi-conformal if and only if it is a finite composition of left translations, graded isomorphisms, and maps of the form F_h, where $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function.*

We remark that Ben Warhurst [W] previously proved a similar statement under the assumption that the map F is smooth. We do not impose any regularity assumptions on the quasi-conformal maps. Theorem 1.1 also provides lots of examples of nonsmooth quasi-conformal maps on $F_{\mathbb{R}}^n$.

The quasi-conformal maps on the complex Filiform groups are even more rigid.

THEOREM 1.2. *Let $n \ge 3$. A homeomorphism $F : (F_{\mathbb{C}}^n, d_c) \to (F_{\mathbb{C}}^n, d_c)$ is a quasi-conformal map if and only if it is a finite composition of left translations and graded isomorphisms.*

Model Filiform groups are an important class of the so-called nonrigid Carnot groups [W; O; OW]. Recall that a Carnot group is rigid if the space of smooth contact maps is finite-dimensional and called nonrigid otherwise. Our results show that, on these nonrigid Carnot groups, quasi-conformal maps are rigid in the sense that they are bi-Lipschitz and have very special forms.

When $n = 2$, the group $F_{\mathbb{R}}^2$ is simply the first Heisenberg group. It is well known that quasi-conformal maps on the Heisenberg groups are very flexible. For instance, there exist quasi-conformal maps between Heisenberg groups that change the Hausdorff dimension of certain subsets; see [B]. There also exist bi-Lipschitz maps of the Heisenberg groups that map vertical lines to nonvertical curves [X1].

The group $F_{\mathbb{C}}^2$ is the first complex Heisenberg group. Recall that the nth complex Heisenberg algebra $\mathfrak{h}_{\mathbb{C}}^n$ is a $(2n+1)$-dimensional complex Lie algebra and has a complex vector space basis X_i, Y_i, Z $(1 \le i \le n)$ with the only nontrivial bracket relations $[X_i, Y_i] = Z$, $1 \le i \le n$. The Lie algebra $\mathfrak{h}_{\mathbb{C}}^n$ is a two-step Carnot algebra. The first layer V_1 of $\mathfrak{h}_{\mathbb{C}}^n$ is spanned by the X_i, Y_i, $1 \le i \le n$, and has complex dimension $2n$. The second layer V_2 is spanned by Z and has complex dimension 1. The nth complex Heisenberg group $H_{\mathbb{C}}^n$ is the connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{h}_{\mathbb{C}}^n$. We always equip $H_{\mathbb{C}}^n$ with a

Carnot metric associated with V_1. We identify both $H^n_{\mathbb{C}}$ and its Lie algebra $\mathfrak{h}^n_{\mathbb{C}}$ with $\mathbb{C}^{2n+1}$. So $V_1 = \mathbb{C}^{2n} \times \{0\}$ and $V_2 = \{0\} \times \mathbb{C}$.

It seems that complex Heisenberg groups are also very rigid with respect to quasi-conformal maps. In fact, using the result of Reimann and Ricci [RR], we can show that if a quasi-conformal map $F : H^n_{\mathbb{C}} \to H^n_{\mathbb{C}}$ is also a C^2 diffeomorphism, then it is complex affine after possibly taking complex conjugation; see Proposition 4.2.

Conjecture 1.3. *A homeomorphism $F : H^n_{\mathbb{C}} \to H^n_{\mathbb{C}}$ of the nth complex Heisenberg group is a quasi-conformal map if and only if it is a finite composition of left translations and graded isomorphisms.*

For general Carnot groups, we have the following.

Conjecture 1.4. *Let G be a Carnot group equipped with a Carnot metric. If G is not an Euclidean group or an Heisenberg group, then every quasi-conformal map $F : G \to G$ is bi-Lipschitz.*

The results in this paper have implications for the large-scale geometry of nilpotent groups and negatively curved homogeneous manifolds. Each Carnot group arises as (one point complement of) the ideal boundary of some negatively curved homogeneous manifold [H]. Our results imply that each quasi-isometry of the negatively curved homogeneous manifold associated to the higher model Filiform group is a rough isometry, that is, it must preserve the distance up to an additive constant. Furthermore, each quasi-isometry between finitely generated nilpotent groups descends to a bi-Lipschitz map between the asymptotic cones which are Carnot groups. Our results say that these bi-Lipschitz maps preserve multiple foliations. So they provide information about the structure of the quasi-isometries, at least after passing to the asymptotic cones.

The ideas in this paper can be used to show that quasi-conformal maps on many Carnot groups are bi-Lipschitz; see [X2; X4].

In Section 2, we recall the basics about Carnot groups and the definition of model Filiform groups. In Section 3, we study quasi-conformal maps on the real model Filiform groups and prove Theorem 1.1. In Section 4, we consider complex Heisenberg groups; in particular, we prove a special case of Conjecture 1.3, which will be used later in Section 5. In Section 5, we prove a rigidity result about quasi-conformal maps on the higher complex model Filiform groups (Theorem 1.2).

2. Preliminaries

In this section, we collect definitions and results that will be needed later. We first recall the basic definitions related to Carnot groups in Section 2.1. Then we review the definition of model Filiform groups (Section 2.2), the BCH formula (Section 2.3), the definitions of quasi-similarity and quasi-symmetric maps (Section 2.4), and the Pansu differentiability theorem (Section 2.5).

2.1. The Basics

A *Carnot Lie algebra* is a finite-dimensional Lie algebra $\mathfrak{g}$ together with a direct sum decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ of nontrivial vector subspaces such that $[V_1, V_i] = V_{i+1}$ for all $1 \le i \le r$, where we set $V_{r+1} = \{0\}$. The integer r is called the degree of nilpotency of $\mathfrak{g}$. Every Carnot algebra $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ admits a one-parameter family of automorphisms $\lambda_t : \mathfrak{g} \to \mathfrak{g}$, $t \in (0, \infty)$, where $\lambda_t(x) = t^i x$ for $x \in V_i$. Let $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ and $\mathfrak{g}' = V_1' \oplus V_2' \oplus \cdots \oplus V_s'$ be two Carnot algebras. A Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ is graded if ϕ commutes with λ_t for all $t > 0$, that is, if $\phi \circ \lambda_t = \lambda_t \circ \phi$. We observe that $\phi(V_i) \subset V_i'$ for all $1 \le i \le r$.

A connected and simply connected nilpotent Lie group is a *Carnot group* if its Lie algebra is a Carnot algebra. Let G be a Carnot group with Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$. The subspace V_1 defines a left-invariant distribution $HG \subset TG$ on G. We fix a left-invariant inner product on HG. An absolutely continuous curve γ in G whose velocity vector $\gamma'(t)$ is contained in $H_{\gamma(t)}G$ for a.e. t is called a horizontal curve. By Chow's theorem ([BR, Theorem 2.4]), any two points of G can be connected by horizontal curves. For $p, q \in G$, the *Carnot metric* $d_c(p, q)$ between them is defined as the infimum of length of horizontal curves that join p and q.

Since the inner product on HG is left-invariant, the Carnot metric on G is also left-invariant. Different choices of inner product on HG result in Carnot metrics that are bi-Lipchitz equivalent. The Hausdorff dimension of G with respect to a Carnot metric is given by $\sum_{i=1}^r i \cdot \dim(V_i)$.

Recall that, for a connected and simply connected nilpotent Lie group G with Lie algebra $\mathfrak{g}$, the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. Under this identification, the Lebesgue measure on $\mathfrak{g}$ is a Haar measure on G. Furthermore, the exponential map induces a one-to-one correspondence between Lie subalgebras of $\mathfrak{g}$ and connected Lie subgroups of G.

It is often more convenient to work with homogeneous distances defined using norms than with Carnot metrics. Let $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ be a Carnot algebra. Write $x \in \mathfrak{g}$ as $x = x_1 + \cdots + x_r$ with $x_i \in V_i$. Fix a norm $|\cdot|$ on each layer. Define the norm $\|\cdot\|$ on $\mathfrak{g}$ by

$$\|x\| = \sum_{i=1}^{r} |x_i|^{1/i}.$$

Now define a homogeneous distance on $G = \mathfrak{g}$ by $d(g, h) = \|(-g) * h\|$. An important fact is that d and d_c are bi-Lipschitz equivalent. That is, there is a constant $C \ge 1$ such that $d(p, q)/C \le d_c(p, q) \le C \cdot d(p, q)$ for all $p, q \in G$. It is often possible to calculate or estimate d by using the BCH formula (see Section 2.3). Since we are only concerned with quasi-conformal maps and bi-Lipschitz maps, it does not matter whether we use d or d_c.

Let G be a Carnot group with Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$. Since $\lambda_t : \mathfrak{g} \to \mathfrak{g}$ ($t > 0$) is a Lie algebra automorphism and G is simply connected, there is a unique Lie group automorphism $\Lambda_t : G \to G$ whose differential at the identity is λ_t. For each $t > 0$, Λ_t is a similarity with respect to the Carnot metric:

$d(\Lambda_t(p), \Lambda_t(q)) = t\,d(p, q)$ for any two points $p, q \in G$. A Lie group homomorphism $f : G \to G'$ between two Carnot groups is a graded homomorphism if it commutes with Λ_t for all $t > 0$, that is, if $f \circ \Lambda_t = \Lambda_t \circ f$. Notice that a Lie group homomorphism $f : G \to G'$ between two Carnot groups is graded if and only if the corresponding Lie algebra homomorphism is graded.

2.2. The Model Filiform Groups

Let K be a field. We only consider the case where K is $\mathbb{R}$ or $\mathbb{C}$. The n-step ($n \geq 2$) model Filiform algebra $\mathfrak{f}_K^n$ over K is an $(n+1)$-dimensional Lie algebra over K. It has a basis $\{e_1, e_2, \ldots, e_{n+1}\}$, and the only nontrivial bracket relations are $[e_1, e_j] = e_{j+1}$ for $2 \leq j \leq n$. When $n = 2$, $\mathfrak{f}_K^2$ is simply the Heisenberg algebra over K. When $K = \mathbb{C}$, $\mathfrak{f}_{\mathbb{C}}^n$ can also be viewed as a real Lie algebra. Since the brackets in $\mathfrak{f}_{\mathbb{C}}^n$ are complex linear, $\mathfrak{f}_{\mathbb{C}}^n$ (when viewed as a real Lie algebra) has the following additional nontrivial bracket relations: $[ie_1, e_j] = [e_1, ie_j] = ie_{j+1}$, $[ie_1, ie_j] = -e_{j+1}$ for $2 \leq j \leq n$.

For $K = \mathbb{R}$ or $\mathbb{C}$, the n-step ($n \geq 2$) model Filiform group F_K^n over K is the connected and simply connected Lie group whose Lie algebra is $\mathfrak{f}_K^n$. For $F_{\mathbb{R}}^n$, we use the Carnot metric corresponding to the inner product on V_1 with e_1 and e_2 as an orthonormal basis. The homogeneous distance on $F_{\mathbb{R}}^n$ is determined by the following norm:

$$\left\| \sum_{i=1}^{n+1} x_i e_i \right\| = (x_1^2 + x_2^2)^{1/2} + \sum_{i=3}^{n+1} |x_i|^{1/(i-1)}.$$

We make the obvious modifications in the case of $F_{\mathbb{C}}^n$.

2.3. The Baker–Campbell–Hausdorff Formula

Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. The exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism. One can then pull back the group operation from G to get a group structure on $\mathfrak{g}$. This group structure can be described by the Baker–Campbell–Hausdorff formula (BCH formula in short), which expresses the product $X * Y$ ($X, Y \in \mathfrak{g}$) in terms of the iterated Lie brackets of X and Y. The group operation in G will be denoted by $\cdot$. The pull-back group operation $*$ on $\mathfrak{g}$ is defined as follows. For $X, Y \in \mathfrak{g}$, define

$$X * Y = \exp^{-1}(\exp X \cdot \exp Y).$$

Then the BCH formula ([CG], page 11) says

$$X * Y = \sum_{n>0} \frac{(-1)^{n+1}}{n} \sum_{p_i+q_i>0, 1\leq i\leq n} \frac{(\sum_{i=1}^n (p_i+q_i))^{-1}}{p_1!q_1!\cdots p_n!q_n!} \times (\operatorname{ad} X)^{p_1} (\operatorname{ad} Y)^{q_1} \cdots (\operatorname{ad} X)^{p_n} (\operatorname{ad} Y)^{q_n - 1} Y,$$

where $\operatorname{ad} A(B) = [A, B]$. If $q_n = 0$, the term in the sum is $\cdots (\operatorname{ad} X)^{p_n - 1} X$; if $q_n > 1$ or if $q_n = 0$ and $p_n > 1$, then the term is zero. The first few terms are well known:

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] - \frac{1}{48}[Y, [X, [X, Y]]] - \frac{1}{48}[X, [Y, [X, Y]]] + (\text{commutators in five or more terms}).$$

LEMMA 2.1. *There are universal constants* $c_j \in \mathbb{Q}$, $2 \le j \le n-1$, *with the following property*: *for any* $X \in \mathfrak{k}^n_K$ *and any* $Y \in Ke_2 \oplus V_2 \oplus \cdots \oplus V_n$, *we have*

$$X * Y = X + Y + \frac{1}{2}[X, Y] + \sum_{j=2}^{n-1} c_j (\operatorname{ad} X)^j Y \tag{2.1}$$

and

$$Y * X = Y + X + \frac{1}{2}[Y, X] + \sum_{j=2}^{n-1} (-1)^j c_j (\operatorname{ad} X)^j Y. \tag{2.2}$$

Proof. Formula (2.1) follows from the fact that $[Z_1, Z_2] = 0$ for any two $Z_1, Z_2 \in Ke_2 \oplus V_2 \oplus \cdots \oplus V_n$. In particular, $[Y, [X, Y]] = 0$, so the only possible nonzero terms in the BCH formula are multiples of $(\operatorname{ad} X)^j Y$. We stop at $j = n-1$ since $\mathfrak{k}^n_K$ has n layers.

Formula (2.2) follows from (2.1) by taking the inverse of both sides and then replacing X with $-X$ and Y with $-Y$. □

COROLLARY 2.2. *In* $\mathfrak{k}^n_K$ *the following holds for all* $t, x_2, \dots, x_{n+1} \in K$:

$$(-te_1) * \sum_{j=2}^{n+1} x_j e_j * te_1 = \sum_{j=2}^{n+1} x'_j e_j,$$

where $x'_2 = x_2$, $x'_j = x_j - tx_{j-1} + G_j$ *for* $3 \le j \le n+1$. *Here* G_j *is a polynomial of* t *and* x_i, *and each of its terms has a factor* t^k *for some* $k \ge 2$.

Proof. We apply Lemma 2.1 to $Z := (-te_1) * \sum_{j=2}^{n+1} x_j e_j$ and obtain:

$$\begin{aligned} Z &= (-te_1) * \sum_{j=2}^{n+1} x_j e_j \\ &= -te_1 + \sum_{j=2}^{n+1} x_j e_j + \frac{1}{2}\Big[-te_1, \sum_{j=2}^{n+1} x_j e_j\Big] + \sum_{j=4}^{n+1} H_j e_j, \end{aligned}$$

where H_j is a polynomial of t and x_i, and each of its terms has a factor t^k for some $k \ge 2$. Notice that the last term in the formula in Lemma 2.1 contains only higher-degree terms in t. Now we apply the BCH formula to $Z * te_1$. The first three terms are

$$Z + te_1 + \frac{1}{2}[Z, te_1] = \sum_{j=2}^{n+1} x_j e_j + \frac{1}{2}\Big[-te_1, \sum_{j=2}^{n+1} x_j e_j\Big] + \sum_{j=4}^{n+1} H_j e_j$$
$$+ \frac{1}{2}\Big[\sum_{j=2}^{n+1} x_j e_j, te_1\Big] + \sum_{j=4}^{n+1} I_j e_j$$
$$= x_2 e_2 + \sum_{j=3}^{n+1}(x_j - t x_{j-1}) e_j + \sum_{j=4}^{n+1}(H_j + I_j) e_j,$$

where I_j is a polynomial of t and x_i, and each of its terms has a factor t^k for some $k \geq 2$. For the iterated brackets in the BCH formula for $Z * te_1$, we only need to consider the terms of the form $(\operatorname{ad} Z)^j te_1$ $(2 \leq j \leq n-1)$ since all the other terms involve te_1 at least twice and so have higher degree in t. However, a direct calculation shows that $(\operatorname{ad} Z)^2 te_1$ also has higher degree in t. So the same is true for all the terms $(\operatorname{ad} Z)^j te_1$, $2 \leq j \leq n-1$. The corollary follows. □

2.4. *Various Maps*

Here we recall the definitions of quasi-similarity and quasi-symmetric maps.

Let $K \geq 1$ and $C > 0$. A bijection $F : X \to Y$ between two metric spaces is called a (K, C)-*quasi-similarity* if

$$\frac{C}{K} d(x, y) \leq d(F(x), F(y)) \leq CK\, d(x, y)$$

for all $x, y \in X$.

Clearly, a map is a quasi-similarity if and only if it is bi-Lipschitz. The point here is that often there is control on K but not on C. In this case, the notion of quasi-similarity provides more information about the distortion. To see this, just compare the (1, 100)-quasi-similarity and 100-bi-Lipschitz properties.

Let $F : X \to Y$ be a homeomorphism between two metric spaces. For any $x \in X$ and any $r > 0$, set

$$H_F(x, r) = \frac{\sup\{d(F(y), F(x)) : \ d(y, x) \leq r\}}{\inf\{d(F(y), F(x)) : \ d(y, x) \geq r\}}.$$

The map F is called *quasi-conformal* if there is some $H < \infty$ such that

$$\limsup_{r \to 0} H_F(x, r) \leq H$$

for all $x \in X$.

Let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. A homeomorphism $F : X \to Y$ between two metric spaces is η-*quasi-symmetric* if for all distinct triples $x, y, z \in X$, we have

$$\frac{d(F(x), F(y))}{d(F(x), F(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right).$$

If $F : X \to Y$ is an η-quasi-symmetry, then $F^{-1} : Y \to X$ is an η_1-quasi-symmetry, where $\eta_1(t) = (\eta^{-1}(t^{-1}))^{-1}$. See [V], Theorem 6.3. A homeomorphism between metric spaces is quasi-symmetric if it is η-quasi-symmetric for some η.

We remark that quasi-symmetric homeomorphisms between general metric spaces are quasi-conformal. In the case of Carnot groups (and more generally Loewner spaces), a homeomorphism is quasi-symmetric if and only if it is quasi-conformal; see [HK].

Theorem 1.1 in [BKR] implies that a quasi-conformal map between two proper, locally Ahlfors Q-regular ($Q > 1$) metric spaces is absolutely continuous on almost every curve. This result applies to quasi-conformal maps $F : G \to G$ on Carnot groups.

Pansu [P2] proved that a quasisymmetric map $F : G_1 \to G_2$ between two Carnot groups is absolutely continuous. That is, a measurable set $A \subset G_1$ has measure 0 if and only if $F(A)$ has measure 0.

2.5. *Pansu Differentiability Theorem*

We begin with the definition.

DEFINITION 2.3. Let G and G' be two Carnot groups endowed with Carnot metrics, and $U \subset G$, $U' \subset G'$ open subsets. A map $F : U \to U'$ is *Pansu differentiable* at $x \in U$ if there exists a graded homomorphism $L : G \to G'$ such that

$$\lim_{y \to x} \frac{d(F(x)^{-1} \cdot F(y), L(x^{-1} \cdot y))}{d(x, y)} = 0.$$

In this case, the graded homomorphism $L : G \to G'$ is called the *Pansu differential* of F at x and is denoted by $dF(x)$.

We have the following chain rule for Pansu differentials.

LEMMA 2.3 (Lemma 3.7 in [CC]). *Suppose that $F_1 : U_1 \to U_2$ is Pansu differentiable at p and $F_2 : U_2 \to U_3$ is Pansu differentiable at $F_1(p)$. Then $F_2 \circ F_1$ is Pansu differentiable at p, and $d(F_2 \circ F_1)(p) = dF_2(F_1(p)) \circ dF_1(p)$.*

Notice that the Pansu differential of the identity map $U_1 \to U_1$ is the identity isomorphism. Hence, if $F : U_1 \to U_2$ is bijective, F is Pansu differentiable at $p \in U_1$, and F^{-1} is Pansu differentiable at $F(p)$, then $dF^{-1}(F(p)) = (dF(p))^{-1}$.

The following result (except the terminology) is due to Pansu [P2].

THEOREM 2.4. *Let G, G' be Carnot groups, and $U \subset G$, $U' \subset G'$ open subsets. Let $F : U \to U'$ be a quasi-conformal map. Then F is a.e. Pansu differentiable. Furthermore, at a.e. $x \in U$, the Pansu differential $dF(x) : G \to G'$ is a graded isomorphism.*

In Theorem 2.4 and the proofs below, "a.e." is with respect to the Lebesgue measure on $\mathfrak{g} = G$.

To simplify the exposition, we introduce the following terminology.

DEFINITION 2.4. Let $F : U \to U'$ be a quasi-conformal map between open subsets of Carnot groups. A point $p \in U$ is called a *good point* (with respect to F) if:

(1) F is Pansu differentiable at p, and $dF(p)$ is a graded isomorphism;
(2) F^{-1} is Pansu differentiable at $F(p)$, and $dF^{-1}(F(p))$ is a graded isomorphism.

It follows from the Pansu differentiability theorem and the Pansu theorem on absolute continuity of quasi-conformal maps that a.e. $p \in U$ is a good point.

Let G be a Carnot group with Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$. A *horizontal line* in G is the image of a map $\gamma : \mathbb{R} \to G$ of the form $\gamma(t) = g * tv$ for some $g \in \mathfrak{g}$ and some $v \in V_1 \setminus \{0\}$. A nondegenerate compact connected subset of a horizontal line is called a horizontal line segment.

LEMMA 2.5. *Let $F : G \to G$ be a continuous map on a Carnot group G. Suppose that F is Pansu differentiable a.e. and is absolutely continuous on almost all curves. If the Pansu differential is a.e. the identity isomorphism, then F is a left translation of G.*

Proof. After composing F with a left translation, we may assume that $F(0) = 0$. For each $v \in V_1 \setminus \{0\}$, consider the family of horizontal lines consisting of the left cosets of $\mathbb{R}v$. Then for a.e. horizontal line $L = g * \mathbb{R}v$ in this family, the Pansu differential of F is a.e. on L the identity isomorphism, and F is absolutely continuous on L. It follows that $F(L) = F(g) * \mathbb{R}v$ and $F(g * tv) = F(g) * tv$ for all $t \in \mathbb{R}$. Since this is true for a.e. L in the family and F is continuous, the same is true for all horizontal lines in the family. Since the vector $v \in V_1$ is arbitrary, the same is true for all horizontal lines.

Let now $g \in G$ be arbitrary. Then there exists a finite sequence of points $0 = p_0, p_1, \ldots, p_n = g$ and horizontal line segment α_i from p_{i-1} to p_i. The first paragraph applied to α_1 implies that F fixes all points on α_1. An induction argument shows that F fixes all points on α_i. In particular, $F(g) = g$. □

3. Quasi-conformal Maps on the Real Model Filiform Groups

In this section, we will prove Theorem 1.1. Given a quasi-conformal map $F : F^n_{\mathbb{R}} \to F^n_{\mathbb{R}}$ $(n \geq 3)$, we shall precompose and postcompose F with left translations and graded automorphisms to obtain a map of the form F_h described in the Introduction.

3.1. Graded Automorphisms of $\mathfrak{f}^n_{\mathbb{R}}$

In this subsection, we will identify all the graded automorphisms of $\mathfrak{f}^n_{\mathbb{R}}$.

For an element $x \in \mathfrak{g}$ in a Lie algebra, let $\operatorname{rank}(x)$ be the rank of the linear transformation $\operatorname{ad}(x) : \mathfrak{g} \to \mathfrak{g}$, $\operatorname{ad}(x)(y) = [x, y]$. In other words, $\operatorname{rank}(x)$ is the dimension of the image of $\operatorname{ad}(x)$.

LEMMA 3.1. *Let $\mathfrak{f}^n_{\mathbb{R}}$ be the n-step real model Filiform algebra. Assume that $n \geq 3$. Let $X = ae_1 + be_2 \in V_1$ be a nonzero element in the first layer. Then* $\operatorname{rank}(X) = 1$ *if and only if $a = 0$.*

Proof. Let $Y = y_1e_1 + \cdots + y_{n+1}e_{n+1}$. Then

$$[X, Y] = a(y_2e_3 + \cdots + y_ne_{n+1}) - by_1e_3.$$

If $a = 0$, then $[X, Y] = -by_1e_3$, and so $\operatorname{rank}(X) = 1$. If $a \neq 0$, then we can vary the y_i, and it is clear that $\operatorname{rank}(X) = n - 1 \geq 2$. The lemma follows. □

It is clear that a Lie algebra isomorphism preserves the rank of elements. A graded isomorphism also preserves the first layer V_1. Hence, we obtain the following.

LEMMA 3.2. *Suppose $n \geq 3$. Then $h(\mathbb{R}e_2) = \mathbb{R}e_2$ for every graded isomorphism $h : \mathfrak{f}^n_{\mathbb{R}} \to \mathfrak{f}^n_{\mathbb{R}}$.*

Given $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$, define the linear map $h = h_{a_1,a_2,b} : \mathfrak{f}^n_{\mathbb{R}} \to \mathfrak{f}^n_{\mathbb{R}}$ by

$$h(e_1) = a_1e_1 + be_2,$$
$$h(e_j) = a_1^{j-2}a_2e_j \quad \text{for } 2 \leq j \leq n+1.$$

It is easy to check that h is a graded isomorphism of $\mathfrak{f}^n_{\mathbb{R}}$. The following lemma says that these are the only graded isomorphisms of $\mathfrak{f}^n_{\mathbb{R}}$.

LEMMA 3.3. *A linear map $h : \mathfrak{f}^n_{\mathbb{R}} \to \mathfrak{f}^n_{\mathbb{R}}$ is a graded isomorphism if and only if $h = h_{a_1,a_2,b}$ for some $a_1, a_2 \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$.*

Proof. Let $h : \mathfrak{f}^n_{\mathbb{R}} \to \mathfrak{f}^n_{\mathbb{R}}$ be a graded isomorphism. Then $h(V_i) = V_i$. Hence, there are constants $a_1, a_2, b, c \in \mathbb{R}$ and $a_j \in \mathbb{R}$, $3 \leq j \leq n+1$, such that $h(e_1) = a_1e_1 + be_2$, $h(e_2) = ce_1 + a_2e_2$, $h(e_j) = a_je_j$ for $3 \leq j \leq n+1$. By Lemma 3.2 we have $c = 0$. Since h is a Lie algebra isomorphism, for $2 \leq j \leq n$, we have

$$a_{j+1}e_{j+1} = h(e_{j+1}) = h([e_1, e_j]) = [h(e_1), h(e_j)]$$
$$= [a_1e_1 + be_2, a_je_j] = a_1a_je_{j+1},$$

so $a_{j+1} = a_1a_j$. It follows that $a_j = a_1^{j-2}a_2$ for $2 \leq j \leq n+1$ and $h = h_{a_1,a_2,b}$. □

LEMMA 3.4. *Let $F : F^n_{\mathbb{R}} \to F^n_{\mathbb{R}}$ be a map. Suppose that there are functions $f : \mathbb{R} \to \mathbb{R}$, $f_j : \mathbb{R}^{n+1} \to \mathbb{R}$, $2 \leq j \leq n+1$, such that*

$$F\Bigl(x_1e_1 * \sum_{j=2}^{n+1} x_je_j\Bigr) = f(x_1)e_1 * \sum_{j=2}^{n+1} f_j(x)e_j,$$

where $x = (x_1, \ldots, x_{n+1})$. If F is Pansu differentiable at $p := x_1e_1 * \sum_{j=2}^{n+1} x_je_j$ with Pansu differential $dF(p) = h_{a_1(p),a_2(p),b(p)}$, then $f'(x_1)$ and $\partial f_2(x)/\partial x_2$ exist, and

$$f'(x_1) = a_1(p) \quad \textit{and} \quad \frac{\partial f_2}{\partial x_2}(x) = a_2(p).$$

Proof. For $q \in F^n_{\mathbb{R}}$, let $y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$ be determined by $q = y_1 e_1 * \sum_{j=2}^{n+1} y_j e_j$. By the definition of the Pansu differential we have

$$\lim_{q \to p} \frac{d((-F(p)) * F(q), dF(p)((-p) * q))}{d(p,q)} = 0. \tag{3.1}$$

In particular, this limit is 0 if $q = p * (y_1 - x_1)e_1$ and $y_1 \to x_1$. Let $q = p * (y_1 - x_1)e_1$. Then $(-p) * q = (y_1 - x_1)e_1$, and so $d(p,q) = d(0, (y_1 - x_1)e_1) = |y_1 - x_1|$ and

$$dF(p)((-p) * q) = (y_1 - x_1)(a_1(p)e_1 + b(p)e_2).$$

Notice that the coefficient of e_1 in q is y_1, and so the coefficient of e_1 in $(-F(p)) * F(q)$ is $f(y_1) - f(x_1)$. The coefficient of e_1 in $[-dF(p)((-p)*q)] * (-F(p)) * F(q)$ is $f(y_1) - f(x_1) - a_1(p)(y_1 - x_1)$. Hence,

$$d((-F(p)) * F(q), dF(p)((-p) * q)) \geq |f(y_1) - f(x_1) - a_1(p)(y_1 - x_1)|.$$

Now (3.1) implies

$$\lim_{y_1 \to x_1} \frac{|f(y_1) - f(x_1) - a_1(p)(y_1 - x_1)|}{|y_1 - x_1|} = 0.$$

Hence, f is differentiable at x_1, and $f'(x_1) = a_1(p)$.

The proof of the statement about $\partial f_2(x)/\partial x_2$ is similar. Let $q = p * (y_2 - x_2)e_2$. Since $[e_i, e_j] = 0$ for $i, j \geq 2$, we can write

$$q = x_1 e_1 * \sum_{i=2}^{n+1} x_i e_i * (y_2 - x_2)e_2 = x_1 e_1 * \left(y_2 e_2 + \sum_{i=3}^{n+1} x_i e_i \right).$$

So x and y differ only in the second coordinate. Notice that we have $(-p) * q = (y_2 - x_2)e_2$, $d(p,q) = |y_2 - x_2|$ and $dF(p)((-p) * q) = a_2(p)(y_2 - x_2)e_2$. Also the coefficient of e_2 in $(-F(p)) * F(q)$ is

$$f_2(y) - f_2(x) = f_2(x_1, y_2, x_3, \ldots, x_{n+1}) - f_2(x_1, x_2, x_3, \ldots, x_{n+1}).$$

Hence,

$$d((-F(p)) * F(q), dF(p)((-p) * q)) \geq |f_2(y) - f_2(x) - a_2(p)(y_2 - x_2)|.$$

Now (3.1) implies

$$\lim_{y_2 \to x_2} \frac{|f_2(y) - f_2(x) - a_2(p)(y_2 - x_2)|}{|y_2 - x_2|} = 0.$$

Hence, $\partial f_2(x)/\partial x_2$ exists, and $\partial f_2(x)/\partial x_2 = a_2(p)$. □

3.2. Quasi-conformal Implies bi-Lipschitz

In this subsection, we show that every quasi-conformal map of $F^n_{\mathbb{R}}$ $(n \geq 3)$ is bi-Lipschitz.

Let $n \geq 3$, and let $F : F^n_{\mathbb{R}} \to F^n_{\mathbb{R}}$ be an η-quasi-symmetric map for some η. Pansu's differentiability theorem says that F is Pansu differentiable a.e. and the Pansu differential is a.e. a graded isomorphism. Notice that the Lie subalgebra generated by $\mathbb{R}e_2$ is itself. We shall abuse notation and also denote the corresponding connected subgroup of $F^n_{\mathbb{R}}$ by $\mathbb{R}e_2$. It follows from Fubini's theorem that for a.e. left coset L of $\mathbb{R}e_2$, the map F is Pansu differentiable a.e. on L and the Pansu differential is a.e. a graded isomorphism on L. By Lemma 3.2, $dF(x)(\mathbb{R}e_2) = \mathbb{R}e_2$ for a.e. $x \in F^n_{\mathbb{R}}$. Now the following result implies that F sends left cosets of $\mathbb{R}e_2$ to the left cosets of $\mathbb{R}e_2$.

PROPOSITION 3.5 (Proposition 3.4 in [X3]). *Let G and G' be two Carnot groups, and $W \subset V_1$, $W' \subset V'_1$ be subspaces. Denote by $\mathfrak{g}_W \subset \mathfrak{g}$ and $\mathfrak{g}'_{W'} \subset \mathfrak{g}'$ respectively the Lie subalgebras generated by W and W'. Let $H \subset G$ and $H' \subset G'$ respectively be the connected Lie subgroups of G and G' corresponding to $\mathfrak{g}_W$ and $\mathfrak{g}'_{W'}$. Let $F : G \to G'$ be a quasi-symmetric homeomorphism. If $dF(x)(W) \subset W'$ for a.e. $x \in G$, then F sends left cosets of H into left cosets of H'.*

To simplify the exposition, we introduce the following terminology (see Definition 2.4 for the definition of a "good point").

DEFINITION 3.2. Let $n \geq 3$, and let $F : F^n_{\mathbb{R}} \to F^n_{\mathbb{R}}$ be a quasi-symmetric map. A left coset L of $\mathbb{R}e_2$ is called a *good left coset* (with respect to F) if the following conditions hold:

(1) a.e. point $p \in L$ is a good point with respect to F;
(2) $F|_L$ is absolutely continuous;
(3) $F^{-1}|_{F(L)}$ is absolutely continuous.

LEMMA 3.6. *Let $n \geq 3$, and let $F : F^n_{\mathbb{R}} \to F^n_{\mathbb{R}}$ be a quasi-symmetric map. Then a.e. left coset of $\mathbb{R}e_2$ is a good left coset.*

Proof. Since a.e. point is a good point, Fubini's theorem implies (1) for a.e. left coset L. Theorem 1.1 of [BKR] implies that $F|_L$ is absolutely continuous on a.e. left coset L of $\mathbb{R}e_2$. Since F^{-1} is also quasi-symmetric, $F^{-1}|_{L'}$ is absolutely continuous on a.e. left coset L' of $\mathbb{R}e_2$. Since quasi-symmetric maps preserve the conformal modulus of curve families and F maps left cosets of $\mathbb{R}e_2$ to left cosets of $\mathbb{R}e_2$, we see that (2) and (3) hold for a.e. left coset L. □

For a good point $x \in F^n_{\mathbb{R}}$, let $b(x), a_1(x), a_2(x) \in \mathbb{R}$ be such that $dF(x) = h_{a_1(x),a_2(x),b(x)}$; see Lemma 3.3. Then $dF(x)(e_1) = a_1(x)e_1 + b(x)e_2$, $dF(x)(e_j) = (a_1(x))^{j-2}a_2(x)e_j$ for $2 \leq j \leq n+1$. Set $a_{n+1}(x) = (a_1(x))^{n-1} \times a_2(x)$. Notice that $dF(x)(v) = a_{n+1}(x)v$ for any $v \in V_n = \mathbb{R}e_{n+1}$.

LEMMA 3.7. *Let L be a good left coset of $\mathbb{R}e_2$. Then there is a constant $a_L \in \mathbb{R} \setminus \{0\}$ such that $a_{n+1}(x) = a_L$ for a.e. $x \in L$.*

Proof. Suppose that there exist good points $p, q \in L$ such that $a_{n+1}(p) \neq a_{n+1}(q)$. We shall show that this implies $a_{n+1}(z) \to +\infty$ as $z \to \infty$ along L. This provides a contradiction since the same claim applied to F^{-1} implies $a_{n+1}(z) \to 0$ as $z \to \infty$ along L; see the remark after Lemma 2.3 about the relation between the Pansu differential of F and that of F^{-1}.

We may assume that $|a_{n+1}(p)| \geq |a_{n+1}(q)|$. Notice that $L = p * \mathbb{R}e_2$ and $F(L) = F(p) * \mathbb{R}e_2$. In particular, there are $y, y_0 \in \mathbb{R}$ such that $q = p * ye_2$ and $F(q) = F(p) * y_0e_2$. After composing with the inverse of $dF(p)$, we may assume that $dF(p) = \mathrm{Id}$ is the identity isomorphism. Then $a_{n+1}(p) = 1$ and either $|a_{n+1}(q)| < 1$ or $a_{n+1}(q) = -1$. We shall consider the image of the left coset $r^n e_{n+1} * L$ under the map F as $r \to 0$.

Notice that $d(0, r^n e_{n+1}) = r$. Since $dF(p) = \mathrm{Id}$, the definition of the Pansu differential implies that there exists some $\tilde{x} = \sum_i \tilde{x}_i e_i$ with $\tilde{x}_i = \tilde{x}_i(r)$ such that $d(0, \tilde{x}) = o(r)$ and $F(r^n e_{n+1} * p) = F(p) * r^n e_{n+1} * \tilde{x}$, where $o(r)$ is the little-o notation. Here we used the fact that $r^n e_{n+1}$ is in the center. Similarly, there is some $\tilde{y} = \sum_i \tilde{y}_i e_i$ with $\tilde{y}_i = \tilde{y}_i(r, y)$ such that $d(0, \tilde{y}) = o(r)$ and $F(r^n e_{n+1} * q) = F(q) * a_{n+1}(q) r^n e_{n+1} * \tilde{y}$. For later use, we notice here that $\tilde{x}_1, \tilde{x}_2, \tilde{y}_2 = o(r)$ and $\tilde{x}_{n+1}, \tilde{y}_{n+1} = o(r^n)$. Since F sends left cosets of $\mathbb{R}e_2$ to left cosets of $\mathbb{R}e_2$, we have

$$L_r := F(r^n e_{n+1} * L) = F(r^n e_{n+1} * p) * \mathbb{R}e_2 = F(p) * r^n e_{n+1} * \tilde{x} * \mathbb{R}e_2.$$

In particular, $F(r^n e_{n+1} * q) = F(p) * r^n e_{n+1} * \tilde{x} * \tilde{s}_1 e_2$ for some $\tilde{s}_1 \in \mathbb{R}$. Using Lemma 2.1, we find two expressions for $F(r^n e_{n+1} * q)$:

$$\begin{aligned} &F(r^n e_{n+1} * q) \\ &\quad = F(q) * a_{n+1}(q) r^n e_{n+1} * \tilde{y} \\ &\quad = F(p) * y_0 e_2 * a_{n+1}(q) r^n e_{n+1} * \tilde{y} \\ &\quad = F(p) * (\tilde{y}_1 e_1 + \{\tilde{y}_2 + y_0\} e_2 + S_1 \\ &\qquad + \{\tilde{y}_{n+1} + a_{n+1}(q) r^n + (-1)^{n-1} c_{n-1} \tilde{y}_1^{n-1} y_0\} e_{n+1}) \end{aligned}$$

and

$$\begin{aligned} F(r^n e_{n+1} * q) &= F(p) * r^n e_{n+1} * \tilde{x} * \tilde{s}_1 e_2 \\ &= F(p) * (\tilde{x}_1 e_1 + \{\tilde{x}_2 + \tilde{s}_1\} e_2 + S_2 \\ &\quad + \{\tilde{x}_{n+1} + r^n + c_{n-1} \tilde{x}_1^{n-1} \tilde{s}_1\} e_{n+1}), \end{aligned}$$

where S_1 and S_2 are linear combinations of the e_i with $3 \leq i \leq n$. Comparing the coefficients, we obtain:

$$\tilde{y}_1 = \tilde{x}_1, \tag{3.3}$$

$$\tilde{s}_1 = y_0 + (\tilde{y}_2 - \tilde{x}_2), \tag{3.4}$$

$$\begin{aligned} &a_{n+1}(q) r^n + \tilde{y}_{n+1} + (-1)^{n-1} c_{n-1} \tilde{y}_1^{n-1} y_0 \\ &\quad = r^n + \tilde{x}_{n+1} + c_{n-1} \tilde{x}_1^{n-1} \tilde{s}_1. \end{aligned} \tag{3.5}$$

Pluging (3.3) and (3.4) into (3.2), we get:

$$\begin{aligned} &(1 - a_{n+1}(q)) r^n + (1 - (-1)^{n-1}) c_{n-1} \tilde{x}_1^{n-1} y_0 \\ &\quad = (\tilde{y}_{n+1} - \tilde{x}_{n+1}) - c_{n-1} \tilde{x}_1^{n-1} (\tilde{y}_2 - \tilde{x}_2). \end{aligned} \tag{3.6}$$

Since $\tilde{x}_1, \tilde{x}_2, \tilde{y}_2 = o(r)$, $\tilde{x}_{n+1}, \tilde{y}_{n+1} = o(r^n)$, we see that we must have $a_{n+1}(q) = 1$ if $c_{n-1} = 0$ or $n - 1$ is even. In these cases, the lemma holds. So from now on, we shall assume that $n-1$ is odd and $c_{n-1} \neq 0$. Then (3.6) implies that, for all sufficiently small r, we have

$$-\frac{11}{10}(1-a_{n+1}(q))r^n \le 2c_{n-1}\tilde{x}_1^{n-1}y_0 \le -\frac{9}{10}(1-a_{n+1}(q))r^n. \tag{3.7}$$

Our next goal is to bound $|a_{n+1}(z)|$ from below for $z \in L$. For this, we shall bound from below the distance from a point on $F(L)$ to L_r. So we fix $q_2 = F(p) * s_2e_2 \in F(L)$ and let $q_1 \in L_r$ vary. Then $q_1 = F(r^n e_{n+1} * p) * s_1e_2 = F(p) * r^n e_{n+1} * \tilde{x} * s_1 e_2$ with $s_1 \in \mathbb{R}$, and $d(q_2, q_1) = d(0, (-s_2e_2) * r^n e_{n+1} * \tilde{x} * s_1e_2)$. Using Lemma 2.1 twice, we obtain

$$\begin{aligned}(-s_2e_2) &* r^n e_{n+1} * \tilde{x} * s_1 e_2\\ &= \tilde{x}_1 e_1 + \{\tilde{x}_2 - s_2 + s_1\}e_2 + S\\ &\quad + \{r^n + \tilde{x}_{n+1} + c_{n-1}\tilde{x}_1^{n-1}s_2 + c_{n-1}\tilde{x}_1^{n-1}s_1\}e_{n+1},\end{aligned}$$

where S is a linear combination of the e_i with $3 \le i \le n$. Write $s_2 = ty_0$ and $s_1 - s_2 = u$. Fix any t with $|t| \ge 100/(1-a_{n+1}(q))$. Notice that $100/(1-a_{n+1}(q)) \ge 50$ since $|a_{n+1}(q)| \le 1$. Let r be sufficiently small so that $|\tilde{x}_2| \le r/10$ and (3.7) holds. If s_1 is such that $|u| \ge \sqrt{|t|}r$, then $d(q_2, q_1) \ge |(\tilde{x}_2 - s_2 + s_1)| \ge \sqrt{|t|}r/2 \ge \frac{1}{2}|t|^{1/n} \cdot r$. If $|u| \le \sqrt{|t|}r$, then (3.7) and the assumption on $|t|$ imply that the following holds for sufficiently small r:

$$\begin{aligned}d(q_2, q_1) &\ge |r^n + \tilde{x}_{n+1} + c_{n-1}\tilde{x}_1^{n-1}s_2 + c_{n-1}\tilde{x}_1^{n-1}s_1|^{1/n}\\ &= |r^n + \tilde{x}_{n+1} + 2c_{n-1}\tilde{x}_1^{n-1}s_2 + c_{n-1}\tilde{x}_1^{n-1}u|^{1/n}\\ &= |r^n + \tilde{x}_{n+1} + t\cdot 2c_{n-1}\tilde{x}_1^{n-1}y_0 + c_{n-1}\tilde{x}_1^{n-1}u|^{1/n}\\ &\ge \left|\frac{1}{2}\cdot t\cdot 2c_{n-1}\tilde{x}_1^{n-1}y_0\right|^{1/n}\\ &\ge \left|\frac{9(1-a_{n+1}(q))}{20}|t|r^n\right|^{1/n}\\ &= \left|\frac{9(1-a_{n+1}(q))}{20}\right|^{1/n}\cdot |t|^{1/n}\cdot r.\end{aligned}$$

It follows that $d(q_2, L_r) \ge c|t|^{1/n}r$, where $c = \min\{\frac{1}{2}, |9(1-a_{n+1}(q))/20|^{1/n}\}$. This implies that $a_{n+1}(F^{-1}(q_2)) \ge c^n|t|$, which goes to $+\infty$ as $|t| \to \infty$, finishing the proof of our claim. □

Recall that F is η-quasi-symmetric.

LEMMA 3.8. *For every good left coset L of $\mathbb{R}e_2$, the restriction $F|_L : L \to F(L)$ is a $(\eta(1), |a_L|^{1/n})$-quasi-similarity.*

Proof. Notice that (L, d) is isometric to the real line and under this identification the usual derivative of the map $F|_L : L \to F(L)$ is simply $a_2(p)$. So we only need to control $a_2(p)$ for $p \in L$. Since F is η-quasi-symmetric, its Pansu differentials are also η-quasi-symmetric. Notice that $dF(p)(e_2) = a_2(p)e_2$, $dF(p)(e_{n+1}) =$

$a_{n+1}(p)e_{n+1}$. So $d(0, dF(p)(e_2)) = |a_2(p)|$ and $d(0, dF(p)(e_{n+1})) = |a_{n+1}(p)|^{1/n}$. Since $d(0, e_2) = d(0, e_3) = 1$, the quasi-symmetric condition now implies

$$\frac{1}{\eta(1)} \cdot |a_{n+1}(p)|^{1/n} \le |a_2(p)| \le \eta(1) \cdot |a_{n+1}(p)|^{1/n}.$$

By Lemma 3.7 there is a constant a_L such that $a_{n+1}(p) = a_L$ for a.e. $p \in L$. Since $F|_L : L \to F(L)$ is a homeomorphism between two lines, either $a_2(p) \ge 0$ for a.e. $p \in L$ or $a_2(p) \le 0$ for a.e. $p \in L$. The lemma now follows from the fact that $F|_L$ is absolutely continuous. □

LEMMA 3.9. *For every two good left cosets L, L' of $\mathbb{R}e_2$, we have $|a_L|/(2^n\eta(1)^{2n}) \le |a_{L'}| \le 2^n\eta(1)^{2n}|a_L|$.*

Proof. Let L, L' be two good left cosets of $\mathbb{R}e_2$. We assume that $|a_L| \ge 2^n\eta(1)^{2n}|a_{L'}|$, and will get a contradiction. Fix $g_1 \in L$ and $g_2 \in L'$. By Lemma 3.8,

$$d(F(g_1 * te_2), F(g_1)) \ge |t||a_L|^{1/n}/\eta(1) \ge 2|t|\eta(1)|a_{L'}|^{1/n}$$

and $d(F(g_2 * te_2), F(g_2)) \le |t|\eta(1)|a_{L'}|^{1/n}$. It follows that

$$\frac{d(F(g_1 * te_2), F(g_1)) - d(F(g_2 * te_2), F(g_2))}{d(F(g_2 * te_2), F(g_2))} \ge 1. \tag{3.8}$$

It is an easy calculation to show that either $d(g_1 * te_2, g_2 * te_2)$ is a constant function of t if the coefficients of e_1 in g_1 and g_2 are the same, or

$$d(g_1 * te_2, g_2 * te_2) \sim \sqrt{t} \quad \text{as } t \to \infty.$$

Hence,

$$\frac{d(g_1 * te_2, g_2 * te_2)}{d(g_2, g_2 * te_2)} \to 0 \quad \text{as } t \to \infty. \tag{3.9}$$

The quasi-symmetry condition of F implies

$$\frac{d(F(g_1 * te_2), F(g_2 * te_2))}{d(F(g_2), F(g_2 * te_2))} \to 0 \quad \text{as } t \to \infty. \tag{3.10}$$

Let p_t be the point on $F(L)$ between $F(g_1)$ and $F(g_1 * te_2)$ such that $d(F(g_1), p_t) = d(F(g_2), F(g_2 * te_2))$. Similarly to (3.9), we have

$$\frac{d(p_t, F(g_2 * te_2))}{d(F(g_2), F(g_2 * te_2))} \to 0 \quad \text{as } t \to \infty. \tag{3.11}$$

Now (3.10) and (3.11) imply

$$\frac{d(F(g_1 * te_2), F(g_1)) - d(F(g_2 * te_2), F(g_2))}{d(F(g_2 * te_2), F(g_2))} = \frac{d(p_t, F(g_1 * te_2))}{d(F(g_2), F(g_2 * te_2))} \to 0$$

as $t \to \infty$, contradicting (3.8). Similarly, we get a contradiction if $|a_{L'}| \ge 2^n\eta(1)^{2n}|a_L|$. □

LEMMA 3.10. *Suppose $n \ge 3$. Then every η-quasi-symmetric map $F : F^n_{\mathbb{R}} \to F^n_{\mathbb{R}}$ is a $(2\eta(1)^4, C)$-quasi-similarity for some constant C.*

Proof. Fix a good left coset L_0. Then $F|_{L_0}$ is an $(\eta(1), |a_{L_0}|^{1/n})$-quasi-similarity. Lemma 3.8 and Lemma 3.9 now imply that $F|_L$ is a $(2\eta(1)^3, |a_{L_0}|^{1/n})$-quasi-similarity for every good left coset L of $\mathbb{R}e_2$. Since a.e. left coset of $\mathbb{R}e_2$ is a good left coset (Lemma 3.6) and F is continuous, $F|_L$ is a $(2\eta(1)^3, |a_{L_0}|^{1/n})$-quasi-similarity for every left coset L of $\mathbb{R}e_2$. Now let $p, q \in F^n_{\mathbb{R}}$ be two arbitrary points. If p, q lie on the same left coset, then

$$\frac{|a_{L_0}|^{1/n}}{2\eta(1)^3} \cdot d(p,q) \le d(F(p), F(q)) \le 2\eta(1)^3 |a_{L_0}|^{1/n} \cdot d(p,q).$$

Suppose that p, q do not lie on the same left coset. Pick a point q' such that p, q' lie on the same left coset and $d(p,q') = d(p,q)$. Then the quasi-symmetric condition implies

$$\begin{aligned} d(F(p), F(q)) &\le \eta(1) \cdot d(F(p), F(q')) \le \eta(1) \cdot 2\eta(1)^3 |a_{L_0}|^{1/n} \cdot d(p,q') \\ &= 2\eta(1)^4 |a_{L_0}|^{1/n} \cdot d(p,q). \end{aligned}$$

Now the same argument applied to F^{-1} finishes the proof. □

REMARK 3.12. The arguments in this section can be modified to show a local version of Lemma 3.10: if $F : U \to V$ is a quasi-symmetric map between two open subsets of $F^n_{\mathbb{R}}$ with $n \ge 3$, then F is locally bi-Lipschitz, that is, every point $x \in U$ has a neighborhood U' such that $F|_{U'}$ is bi-Lipschitz. In the proofs of the above lemmas, it is not necessary to let $|t| \to \infty$. One can get around this by choosing a sufficiently small neighborhood of x. For instance, for any fixed $t > 0$, the quotient in (3.9) becomes very small if g_2 is sufficiently close to g_1.

3.3. Quasi-conformal Maps Have Special Forms

In this subsection, we show that quasi-conformal maps on $F^n_{\mathbb{R}}$ ($n \ge 3$) have very special forms.

Let $n \ge 3$, and let $F : F^n_{\mathbb{R}} \to F^n_{\mathbb{R}}$ be an η-quasi-symmetric map for some homeomorphism $\eta : [0, \infty) \to [0, \infty)$. By Lemma 3.10, F is M-bi-Lipschitz for some $M \ge 1$. Let $G = \mathbb{R}e_2 \oplus V_2 \oplus \cdots \oplus V_n \subset F^n_{\mathbb{R}}$. Notice that G is a subgroup of $F^n_{\mathbb{R}}$.

LEMMA 3.11. *F sends left cosets of G to left cosets of G.*

Proof. An easy calculation using the BCH formula shows that two left cosets of $\mathbb{R}e_2$ lie in the same left coset of G if and only if the Hausdorff distance between them is finite. Now the lemma follows since by Lemma 3.10 F is bi-Lipschitz. □

LEMMA 3.12. *There is an M-bi-Lipschitz homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $F(x_1 e_1 * G) = f(x_1) e_1 * G$.*

Proof. Lemma 3.11 implies that there is a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $F(x_1 e_1 * G) = f(x_1) e_1 * G$. We need to prove that f is bi-Lipschitz. Let

$x_1, x_1' \in \mathbb{R}$. Notice that $d(x_1e_1 * G, x_1'e_1 * G) = |x_1 - x_1'|$. Pick $p \in x_1e_1 * G$ and $q \in x_1'e_1 * G$ with $d(p, q) = |x_1 - x_1'|$. Then

$$\begin{aligned}|f(x_1) - f(x_1')| &= d(f(x_1)e_1 * G, f(x_1')e_1 * G) \leq d(F(p), F(q)) \\ &\leq M \cdot d(p, q) = M \cdot |x_1 - x_1'|.\end{aligned}$$

A similar argument shows that f^{-1} is also M-Lipschitz. □

LEMMA 3.13. *For every good left coset L of $\mathbb{R}e_2$, there exists a constant $a_{2,L} \in \mathbb{R} \setminus \{0\}$ such that $a_2(p) = a_{2,L}$ for a.e. $p \in L$. Furthermore, for any $g \in L$, we have $F(g * te_2) = F(g) * a_{2,L}te_2$ for all $t \in \mathbb{R}$.*

Proof. For $p \in F_{\mathbb{R}}^n$, let $x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ be determined by $p = x_1e_1 * \sum_{i=2}^{n+1} x_ie_i$. By Lemma 3.7 we have $a_1(p)^{n-1}a_2(p) = a_{n+1}(p) = a_L$ for a.e. $p \in L$. On the other hand, there is a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $F(x_1e_1 * G) = f(x_1)e_1 * G$. Lemma 3.4 implies that $a_1(p) = f'(x_1)$. Since all the points on L have the same e_1 coefficient, $a_1(p)$ is a.e. constant along L. Therefore, $a_2(p)$ is a.e. a constant along L. By Lemma 3.4 again, $a_2(p) = \partial f_2(x)/\partial x_2$. Since $F|_L$ is absolutely continuous (being bi-Lipschitz), f_2 is an affine function of x_2 (whereas the other variables are fixed), and the last statement in the lemma holds. □

LEMMA 3.14. *There exists a constant $a_2 \in \mathbb{R} \setminus \{0\}$ such that $F(g * te_2) = F(g) * a_2te_2$ for every $g \in F_{\mathbb{R}}^n$.*

Proof. Since F is continuous, it suffices to show $a_{2,L} = a_{2,L'}$ for any two good left cosets L, L' of $\mathbb{R}e_2$. Fix $g_1 \in L$ and $g_2 \in L'$. Then $F(g_1 * te_2) = F(g_1) * a_{2,L}te_2$ and $F(g_2 * te_2) = F(g_2) * a_{2,L'}te_2$. Now the equality $a_{2,L} = a_{2,L'}$ follows from the following fact that was derived in the proof of Lemma 3.9:

$$\frac{d(F(g_1 * te_2), F(g_1)) - d(F(g_2 * te_2), F(g_2))}{d(F(g_2 * te_2), F(g_2))} \to 0 \quad \text{as } t \to \infty. \qquad \square$$

By replacing F with $h_{1,a_2^{-1},0} \circ F$ we may assume that $a_2 = 1$.

Notice that the left cosets of G with the metric induced from d are isometric to $\mathbb{R}^n$ with the metric $D((x_j), (y_j)) = \sum_j |x_j - y_j|^{1/j}$. It was proved in Section 15 of [T] that each quasi-symmetric map $(\mathbb{R}^n, D) \to (\mathbb{R}^n, D)$ preserves the foliation consisting of affine subspaces parallel to $\mathbb{R}^i \times \{0\}$ for each $1 \leq i \leq n-1$. This implies that there exist continuous functions $f_j := f_j(x_1, x_j, x_{j+1}, \dots, x_{n+1})$, $2 \leq j \leq n+1$, such that F has the following form:

$$F\left(x_1e_1 * \left(\sum_{j=2}^{n+1} x_je_j\right)\right) = f(x_1)e_1 * \sum_{j=2}^{n+1} f_je_j. \tag{3.13}$$

Let $E \subset \mathbb{R}$ be the subset consisting of all $x_1 \in \mathbb{R}$ with the following properties:

(1) f is differentiable at x_1;
(2) almost every point p in the left coset $x_1e_1 * G$ is a good point with respect to F.

By Fubini's theorem and the fact that f is bi-Lipschitz, we see that E has full measure in $\mathbb{R}$.

LEMMA 3.15. *For $x_1 \in E$, there exist functions $h_j := h_j(x_1, x_{j+1}, \dots, x_{n+1})$, $2 \leq j \leq n+1$, such that the following holds when $x_1 \in E$:*

$$F\left(x_1 e_1 * \left(\sum_{j=2}^{n+1} x_j e_j\right)\right) = f(x_1) e_1 * \sum_{j=2}^{n+1} \{(f'(x_1))^{j-2} x_j + h_j\} e_j.$$

Proof. We shall first show that $f_j = f_j(x_1, x_j, x_{j+1}, \dots, x_{n+1})$, $2 \leq j \leq n+1$, is Lipschitz in x_j. Let

$$p = x_1 e_1 * (x_2 e_2 + \cdots + x_{n+1} e_{n+1})$$

and

$$q = x_1 e_1 * (x_2 e_2 + \cdots + x_{j-1} e_{j-1} + y_j e_j + x_{j+1} e_{j+1} + \cdots + x_{n+1} e_{n+1}).$$

Notice that the only difference between p and q is in the coefficient of e_j. We have $(-p) * q = (y_j - x_j) e_j$ and $d(p, q) = |y_j - x_j|^{1/(j-1)}$. Hence,

$$d(F(p), F(q)) \leq M |y_j - x_j|^{1/(j-1)}.$$

On the other hand, by (3.13) the coefficient of e_j in $(-F(p)) * F(q)$ is

$$f_j(x_1, y_j, x_{j+1}, \dots, x_{n+1}) - f_j(x_1, x_j, x_{j+1}, \dots, x_{n+1}). \tag{3.14}$$

Hence,

$$\begin{aligned} |f_j(x_1, y_j, x_{j+1}, \dots, x_{n+1}) - f_j(x_1, x_j, x_{j+1}, \dots, x_{n+1})|^{1/(j-1)} \\ \leq d(F(p), F(q)) \leq M \cdot |y_j - x_j|^{1/(j-1)}, \end{aligned}$$

and so

$$|f_j(x_1, y_j, x_{j+1}, \dots, x_{n+1}) - f_j(x_1, x_j, x_{j+1}, \dots, x_{n+1})| \leq M^{j-1} |y_j - x_j|.$$

Hence, f_j is Lipschitz in x_j.

Set $G_j = \mathbb{R}e_2 \oplus \cdots \oplus \widehat{(\mathbb{R}e_j)} \oplus \cdots \oplus \mathbb{R}e_{n+1}$, where $\widehat{(\mathbb{R}e_j)}$ means that the term $\mathbb{R}e_j$ is absent. For $x_1 \in E$, define the subset $E_{x_1} \subset G_j$ as follows:

$$w = x_2 e_2 + \cdots + x_{j-1} e_{j-1} + x_{j+1} e_{j+1} + \cdots + x_{n+1} e_{n+1} \in E_{x_1}$$

if and only if the point

$$\begin{aligned} p &= p(x_1, w, x_j) \\ &:= x_1 e_1 * (x_2 e_2 + \cdots + x_{j-1} e_{j-1} + x_j e_j + x_{j+1} e_{j+1} + \cdots + x_{n+1} e_{n+1}) \end{aligned}$$

is a good point for a.e. $x_j \in \mathbb{R}$. Since a.e. point in $x_1 e_1 * G$ is a good point, Fubini's theorem implies that E_{x_1} has full measure in G_j. Fix $x_1 \in E$ and $w \in E_{x_1}$. Let x_j be such that $p = p(x_1, w, x_j)$ is a good point. By Lemma 3.4, $a_1(p) = f'(x_1)$. By our normalization $a_2 = 1$, we obtain $a_j(p) = (f'(x_1))^{j-2}$. Let q be as defined at the beginning of the proof. Then $dF(p)((-p)*q) = (f'(x_1))^{j-2}(y_j - x_j)e_j$. By the definition of the Pansu differential we have

$$\frac{d(0, [-dF(p)((-p)*q)]*(-F(p))*F(q))}{d(p,q)} \to 0 \tag{3.15}$$

as $y_j \to x_j$. Notice that $(-F(p)) * F(q) \in G$ and the coefficient of e_j in $(-F(p)) * F(q)$ is given by (3.14). It follows that the coefficient of e_j in $[-dF(p)((-p)*q)]*(-F(p))*F(q)$ is

$$\begin{aligned} A_j := {} & f_j(x_1, y_j, x_{j+1}, \dots, x_{n+1}) - f_j(x_1, x_j, x_{j+1}, \dots, x_{n+1}) \\ & - (f'(x_1))^{j-2}(y_j - x_j). \end{aligned}$$

Now (3.15) implies

$$\begin{aligned} \frac{|A_j|^{1/(j-1)}}{|y_j - x_j|^{1/(j-1)}} &= \frac{|A_j|^{1/(j-1)}}{d(p,q)} \\ &\le \frac{d(o, [-dF(p)((-p)*q)]*(-F(p))*F(q))}{d(p,q)} \to 0. \end{aligned}$$

It follows that $\partial f_j(x)/\partial x_j = (f'(x_1))^{j-2}$. Since this is true for a.e. x_j and f_j is Lipschitz in x_j, we see that f_j is an affine function of x_j (when the other variables are fixed). Hence, there is a real number $H_j := H_j(x_1, x_{j+1}, \dots, x_{n+1})$ such that $f_j = (f'(x_1))^{j-2}x_j + H_j$. So far, $H_j(x_1, x_{j+1}, \dots, x_{n+1})$ is defined only for those $(x_{j+1}, \dots, x_{n+1})$ such that there are $(x_2, \dots, x_{j-1})$ with

$$x_2e_2 + \dots + x_{j-1}e_{j-1} + x_{j+1}e_{j+1} + \dots + x_{n+1}e_{n+1} \in E_{x_1}.$$

Since E_{x_1} has full measure, Fubini's theorem implies that H_j is defined for a.e. $(x_{j+1}, \dots, x_{n+1})$.

Let $x_1 \in E$ be fixed. Set

$$h_j(x_1, x_j, x_{j+1}, \dots, x_{n+1}) = f_j(x_1, x_j, \dots, x_{n+1}) - (f'(x_1))^{j-2}x_j.$$

To complete the proof of the lemma, it suffices to show that h_j is independent of x_j. We have proved in the preceding paragraph that for a.e. $(x_{j+1}, \dots, x_{n+1})$, the function $h_j(x_1, x_j, x_{j+1}, \dots, x_{n+1})$ is independent of x_j. Since h_j is continuous in $(x_j, x_{j+1}, \dots, x_{n+1})$, it is independent of x_j for all $(x_{j+1}, \dots, x_{n+1})$. □

Next, we show that f is an affine function and that $h_i(x_1, x_{i+1}, \dots, x_{n+1})$ depends only on x_1.

Let $x_1 \in E$. Let $p = x_1e_1 * \sum_{i=2}^{n+1} x_ie_i$ and $q = x_1e_1 * (\sum_{i=2}^{n} x_ie_i + \tilde{x}_{n+1}e_{n+1})$. Then $d(p,q) = |x_{n+1} - \tilde{x}_{n+1}|^{1/n}$. Since F is M-bi-Lipschitz, we have

$$d(F(p), F(q)) \le M|x_{n+1} - \tilde{x}_{n+1}|^{1/n}.$$

On the other hand,

$$\begin{aligned} (-F(q)) * F(p) = {} & \sum_{i=2}^{n} (h_i(x_1, x_{i+1}, \dots, x_{n+1}) - h_i(x_1, x_{i+1}, \dots, \tilde{x}_{n+1}))e_i \\ & + f'(x_1)^{n-1}(x_{n+1} - \tilde{x}_{n+1})e_{n+1}. \end{aligned}$$

It follows that for $2 \le i \le n$,

$$|h_i(x_1, x_{i+1}, \dots, x_{n+1}) - h_i(x_1, x_{i+1}, \dots, \tilde{x}_{n+1})| \\ \le M^{i-1} \cdot |x_{n+1} - \tilde{x}_{n+1}|^{(i-1)/n}. \tag{3.16}$$

LEMMA 3.16. *The function f is affine.*

Proof. Let $x_1, \tilde{x}_1 \in E$. We prove that $f'(x_1) = f'(\tilde{x}_1)$. The lemma then follows since f is bi-Lipschitz and E has full measure in $\mathbb{R}$. Let $p = x_1 e_1 * x_{n+1} e_{n+1}$ and $q = \tilde{x}_1 e_1 * x_{n+1} e_{n+1}$. Eventually, we will let $x_{n+1} \to \infty$. Set $s_{n+1} = f'(x_1)^{n-1} x_{n+1} + h_{n+1}(x_1)$ and $\tilde{s}_{n+1} = f'(\tilde{x}_1)^{n-1} x_{n+1} + h_{n+1}(\tilde{x}_1)$. By Lemma 3.15,

$$F(p) = f(x_1)e_1 * \left\{ \left(\sum_{i=2}^{n} h_i e_i \right) + s_{n+1} e_{n+1} \right\}$$

and

$$F(q) = f(\tilde{x}_1)e_1 * \left\{ \left(\sum_{i=2}^{n} \tilde{h}_i e_i \right) + \tilde{s}_{n+1} e_{n+1} \right\},$$

where $h_i = h_i(x_1, 0, \dots, 0, x_{n+1})$ and $\tilde{h}_i = h_i(\tilde{x}_1, 0, \dots, 0, x_{n+1})$. Denote $a := f(x_1) - f(\tilde{x}_1)$. Now

$$(-F(q)) * F(p) = \left\{ -\left(\sum_{i=2}^{n} \tilde{h}_i e_i \right) - \tilde{s}_{n+1} e_{n+1} \right\} * a e_1 * \left\{ \left(\sum_{i=2}^{n} h_i e_i \right) + s_{n+1} e_{n+1} \right\}.$$

Notice that $d(p, q) = |x_1 - \tilde{x}_1|$. Hence, $d(F(p), F(q)) \le M \cdot |x_1 - \tilde{x}_1|$ is bounded from above by a constant independent of x_{n+1}. By using Lemma 2.1 twice we find that the coefficient of e_{n+1} in $(-F(q)) * F(p)$ is given by

$$\begin{aligned} s_{n+1} - \tilde{s}_{n+1} &+ \frac{1}{2} a(h_n + \tilde{h}_n) + \sum_{j=2}^{n-1} c_j a^j h_{n-j+1} + \sum_{j=2}^{n-1} (-1)^{j+1} c_j a^j \tilde{h}_{n-j+1} \\ &= [f'(x_1)^{n-1} - f'(\tilde{x}_1)^{n-1}] x_{n+1} + h_{n+1} - \tilde{h}_{n+1} \\ &\quad + \frac{1}{2} a(h_n + \tilde{h}_n) + \sum_{j=2}^{n-1} c_j a^j h_{n-j+1} + \sum_{j=2}^{n-1} (-1)^{j+1} c_j a^j \tilde{h}_{n-j+1}, \end{aligned}$$

which is bounded from above by a constant independent of x_{n+1}. By (3.16),

$$|h_i(x_1, 0, \dots, 0, x_{n+1}) - h_i(x_1, 0, \dots, 0)| \le M^{i-1} \cdot |x_{n+1}|^{(i-1)/n};$$

we see that $|h_i|$ (for $2 \le i \le n$) is bounded from above by a sublinear function of x_{n+1} as $x_{n+1} \to \infty$. The same is true for $\tilde{h}_i$. It follows that $[f'(x_1)^{n-1} - f'(\tilde{x}_1)^{n-1}] x_{n+1}$ is also bounded above by a sublinear function of x_{n+1}. This can happen only when $(f'(x_1))^{n-1} = (f'(\tilde{x}_1))^{n-1}$. Since $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, $f'(x_1)$ and $f'(\tilde{x}_1)$ have the same sign, and hence $f'(x_1) = f'(\tilde{x}_1)$. □

By Lemma 3.16 there are constants $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}$ such that $f(x_1) = ax_1 + b$. After replacing F with $h_{a^{-1},1,0} \circ L_{-F(0)} \circ F$, we may assume that $f(x_1) = x_1$ is the identity map. So, for $x_1 \in E$, we have

$$f_i(x_1, x_i, \ldots, x_{n+1}) = x_i + h_i(x_1, x_{i+1}, \ldots, x_{n+1}). \tag{3.17}$$

Now we can extend the definition of h_i to the case where $x_1 \notin E$. For any $(x_1, x_i, \ldots, x_{n+1})$, set

$$H_i(x_1, x_i, x_{i+1}, \ldots, x_{n+1}) = f_i(x_1, x_i, \ldots, x_{n+1}) - x_i.$$

Notice that H_i is continuous in all variables since f_i is. Equality (3.17) implies that $H_i(x_1, x_i, \ldots, x_{n+1})$ is independent of x_i when $x_1 \in E$. Since E has full measure and H_i is continuous, we conclude that H_i is independent of x_i for all $x_1 \in \mathbb{R}$. Hence, $H_i(x_1, x_i, \ldots, x_{n+1})$ is a function of $x_1, x_{i+1}, \ldots, x_{n+1}$ only. So we can define

$$h_i(x_1, x_{i+1}, \ldots, x_{n+1}) = H_i(x_1, x_i, \ldots, x_{n+1})$$

for any $(x_1, x_{i+1}, \ldots, x_{n+1})$. Now the following holds for all points in $F^n_{\mathbb{R}}$:

$$F\left(x_1e_1 * \left(\sum_{i=2}^{n+1} x_ie_i\right)\right) = x_1e_1 * \sum_{i=2}^{n+1}\{x_i + h_i(x_1, x_{i+1}, \ldots, x_{n+1})\}e_i.$$

Next, we shall show that $h_i(x_1, x_{i+1}, \ldots, x_{n+1})$ depends only on x_1.

LEMMA 3.17. *For each $2 \le i \le n$, the function $h_i(x_1, x_{i+1}, \ldots, x_{n+1})$ depends only on x_1.*

Proof. The idea is very simple: F sends horizontal vectors to horizontal vectors. Notice that for any $g \in F^n_{\mathbb{R}}$, the tangent vectors of $g * te_1$ ($t \in \mathbb{R}$) are horizontal. Since F is Pansu differentiable a.e., we see that for a.e. $g \in F^n_{\mathbb{R}}$, the tangent vector of the curve $F(g * te_1)$ at $t = 0$ exists and is horizontal. We shall calculate this tangent vector.

Let $g = x_1e_1 * \sum_{i=2}^{n+1} x_ie_i$ be a point where F is Pansu differentiable. By Corollary 2.2 we have:

$$\begin{aligned} g * te_1 &= x_1e_1 * \sum_{i=2}^{n+1} x_ie_i * te_1 = (x_1 + t)e_1 * (-te_1) * \sum_{i=2}^{n+1} x_ie_i * te_1 \\ &= (x_1 + t)e_1 * \sum_{i=2}^{n+1} x'_ie_i, \end{aligned}$$

where $x'_2 = x_2$ and $x'_i = x_i - tx_{i-1} + G_i$ for $3 \le i \le n+1$. Here G_i is a polynomial of t and the x_j and each of its terms has a factor t^k for some $k \ge 2$. Denote $x'_1 = x_1 + t$, $h_i = h_i(x_1, x_{i+1}, \ldots, x_{n+1})$ and $\tilde{h}_i = h_i(x'_1, x'_{i+1}, \ldots, x'_{n+1})$. Now

$$F(g * te_1) = F\left(x'_1e_1 * \left(\sum_{i=2}^{n+1} x'_ie_i\right)\right) = x'_1e_1 * \sum_{i=2}^{n+1}(x'_i + \tilde{h}_i)e_i.$$

Since

$$F(g) = F\left(x_1e_1 * \left(\sum_{i=2}^{n+1} x_ie_i\right)\right) = x_1e_1 * \sum_{i=2}^{n+1}(x_i + h_i)e_i,$$

by Corollary 2.2 we obtain

$$\begin{aligned}
&(-F(g)) * F(g * te_1)\\
&\quad= \sum_{i=2}^{n+1}(-x_i - h_i)e_i * (te_1) * \sum_{i=2}^{n+1}(x_i' + \tilde{h}_i)e_i\\
&\quad= (te_1) * (-te_1) * \sum_{i=2}^{n+1}(-x_i - h_i)e_i * (te_1) * \sum_{i=2}^{n+1}(x_i' + \tilde{h}_i)e_i\\
&\quad= (te_1) * \sum_{i=2}^{n+1} x_i''e_i * \sum_{i=2}^{n+1}(x_i' + \tilde{h}_i)e_i\\
&\quad= (te_1) * \sum_{i=2}^{n+1}(x_i'' + x_i' + \tilde{h}_i)e_i,
\end{aligned}$$

where $x_2'' = -[x_2 + h_2]$, $x_j'' = -[x_j + h_j] + t[x_{j-1} + h_{j-1}] + H_j$ for $3 \le j \le n+1$. Here H_j is a polynomial, and all its terms have degree at least 2 in t. Set $\tilde{x}_i = x_i'' + x_i' + \tilde{h}_i$ for $2 \le i \le n+1$. Observe that $\tilde{x}_2 = -[x_2 + h_2] + x_2 + \tilde{h}_2 = \tilde{h}_2 - h_2$ and for $3 \le i \le n+1$,

$$\begin{aligned}
\tilde{x}_i = x_i'' + x_i' + \tilde{h}_i &= -[x_i + h_i] + t[x_{i-1} + h_{i-1}] + H_i + x_i - tx_{i-1} + G_i + \tilde{h}_i\\
&= \tilde{h}_i - h_i + th_{i-1} + G_i + H_i.
\end{aligned}$$

We continue the calculation by using Lemma 2.1:

$$\begin{aligned}
&(-F(g)) * F(g * te_1)\\
&\qquad= (te_1) * \sum_{i=2}^{n+1} \tilde{x}_ie_i = te_1 + \tilde{x}_2e_2 + \sum_{i=3}^{n+1}\left(\tilde{x}_i + \frac{1}{2}t\tilde{x}_{i-1} + I_i\right)e_i,
\end{aligned}$$

where I_i is a polynomial in t, and the $\tilde{x}_j$ and all its terms have degree at least 2 in t. So for $3 \le i \le n+1$, the coefficient of e_i is

$$\begin{aligned}
&\tilde{x}_i + \frac{1}{2}t\tilde{x}_{i-1} + I_i\\
&\quad= \tilde{h}_i - h_i + th_{i-1} + G_i + H_i\\
&\qquad+ \frac{1}{2}t[\tilde{h}_{i-1} - h_{i-1} + th_{i-2} + G_{i-1} + H_{i-1}] + I_i\\
&\quad= \tilde{h}_i - h_i + th_{i-1} + \frac{1}{2}t(\tilde{h}_{i-1} - h_{i-1}) + J_i,
\end{aligned}$$

where J_i is a polynomial, and all its terms have degree at least 2 in t. Now the fact that the tangent vector of the curve $F(g * te_1)$ at $t = 0$ is horizontal implies that for all $3 \leq i \leq n+1$,

$$\lim_{t\to 0} \frac{\tilde{x}_i + t\tilde{x}_{i-1}/2 + I_i}{t} = 0.$$

Hence,

$$\lim_{t\to 0} \frac{\tilde{h}_i - h_i + th_{i-1} + t(\tilde{h}_{i-1} - h_{i-1})/2 + J_i}{t} = 0.$$

Clearly, $\lim_{t\to 0} J_i/t = 0$. Since $\lim_{t\to 0} \tilde{h}_{i-1} = h_{i-1}$, we have $\lim_{t\to 0} t(\tilde{h}_{i-1} - h_{i-1})/t = 0$. Hence, we have

$$\begin{aligned} -h_{i-1} &= \lim_{t\to 0} \frac{\tilde{h}_i - h_i}{t} \\ &= \lim_{t\to 0} \frac{h_i(x_1', x_{i+1}', \ldots, x_{n+1}') - h_i(x_1, x_{i+1}, \ldots, x_{n+1})}{t}. \end{aligned} \tag{3.18}$$

For $i = n+1$, we have

$$\begin{aligned} -h_n(x_1, x_{n+1}) &= \lim_{t\to 0} \frac{h_{n+1}(x_1') - h_{n+1}(x_1)}{t} \\ &= \lim_{t\to 0} \frac{h_{n+1}(x_1 + t) - h_{n+1}(x_1)}{t} = h'_{n+1}(x_1). \end{aligned}$$

We have shown that $-h_n(x_1, x_{n+1}) = h'_{n+1}(x_1)$ at every point $x_1 e_1 * \sum_{i=2}^{n+1} x_i e_i$, where F is Pansu differentiable. Since F is Pansu differentiable a.e., Fubini's theorem implies that for a.e. $x_1 \in \mathbb{R}$, the equality $-h_n(x_1, x_{n+1}) = h'_{n+1}(x_1)$ holds for a.e. x_{n+1}. The continuity of h_n implies that $-h_n(x_1, x_{n+1}) = h'_{n+1}(x_1)$ for all x_{n+1}. In particular, for a.e. $x_1 \in \mathbb{R}$, $h_n(x_1, x_{n+1})$ is independent of x_{n+1}. Now the continuity of h_n implies that for all x_1, $h_n(x_1, x_{n+1})$ is independent of x_{n+1}. This shows that $h_n(x_1, x_{n+1})$ is a function of x_1 only. Now an induction argument on i using (3.18) implies that for all $2 \leq i \leq n+1$, h_i is a function of x_1 only. □

3.4. *Completing the Proof of Theorem 1.1*

Here we finish the proof of Theorem 1.1.

We shall first show that every map of the form F_h is bi-Lipschitz and hence quasi-conformal. For this part, we shall use the Carnot metric d_c. Recall that d_c and d are bi-Lipschitz equivalent. Let h be M-Lipschitz, and $F := F_h$ be defined as in the Introduction. Clearly, F isometrically maps each left coset of G to itself. Since $h_j(x) = -\int_0^x h_{j-1}(s)\,ds$, the calculation in the proof of Lemma 3.17 shows that the curve $F(g * te_1)$ is horizontal. Furthermore, the coefficients of e_1 and e_2 in $-F(g) * F(g * te_1)$ are t and $\tilde{x}_2 = h_2(x_1 + t) - h_2(t) = h(x_1 + t) - h(x_1)$. So the tangent vector of $F(g * te_1)$ is $e_1 + h'(x_1)e_2$ and has the length $\leq \sqrt{1 + M^2}$. It follows that for each horizontal line segment S contained in some left coset of $\mathbb{R}e_1$, its image $F(S)$ has the length at most $\sqrt{1 + M^2} \cdot \text{length}(S)$. Now let

$p, q \in F^n_{\mathbb{R}}$ be arbitrary. Then $p \in x_1e_1 * G$ and $q \in x_1'e_1 * G$ for some $x_1, x_1' \in \mathbb{R}$. If

$$d_c(q, p * G) \geq \frac{1}{10\sqrt{1+M^2}} d_c(p, q),$$

then

$$\begin{aligned} d_c(F(q), F(p)) &\geq d_c(F(q*G), F(p*G)) = d_c(q*G, p*G) = d_c(q, p*G) \\ &\geq \frac{1}{10\sqrt{1+M^2}} d_c(p, q). \end{aligned}$$

Now suppose that

$$d_c(q, p * G) \leq \frac{1}{10\sqrt{1+M^2}} d_c(p, q).$$

We may assume that $x_1 \geq x_1'$. In this case, the horizontal line segment $S = \{q * te_1 | t \in [0, x_1 - x_1']\}$ has the length $|x_1 - x_1'|$ and connects q and $q' = q * (x_1 - x_1')e_1 \in p * G$. It follows that $F(S)$ has the length $\leq \sqrt{1+M^2} \cdot |x_1 - x_1'|$. Hence,

$$d_c(F(q), F(q')) \leq \sqrt{1+M^2} \cdot |x_1 - x_1'| = \sqrt{1+M^2} \cdot d_c(q, p*G) \leq d_c(p, q)/10.$$

On the other hand, $d_c(F(p), F(q')) = d_c(p, q') \geq (1 - 1/(10\sqrt{1+M^2}))d_c(p, q)$. By the triangle inequality we have

$$\begin{aligned} d_c(F(p), F(q)) &\geq d_c(F(p), F(q')) - d_c(F(q'), F(q)) \\ &\geq \left(1 - \frac{1}{10\sqrt{1+M^2}} - \frac{1}{10}\right) d_c(p, q). \end{aligned}$$

Hence, $d_c(F(p), F(q))$ is bounded from below in terms of $d_c(p, q)$. Since $F_h^{-1} = F_{-h}$, the same argument applied to F_h^{-1} shows that $d_c(p, q)$ is bounded from below in terms of $d_c(F(p), F(q))$. Hence, F is bi-Lipschitz.

Conversely, let $n \geq 3$, and let $F : F^n_{\mathbb{R}} \to F^n_{\mathbb{R}}$ be a quasi-conformal map. We have shown that after composing F with graded automorphisms and left translations, F has the following form:

$$F\left(x_1e_1 * \left(\sum_{i=2}^{n+1} x_ie_i\right)\right) = x_1e_1 * \sum_{i=2}^{n+1} \{x_i + h_i(x_1)\}e_i.$$

We next show that h_2 is Lipschitz. Given $x_1, x_1' \in \mathbb{R}$, let $p = x_1e_1$, $q = x_1'e_1$. Then $d(p, q) = |x_1 - x_1'|$. Since we have shown that F is bi-Lipschitz, we have $d(F(p), F(q)) \leq M \cdot d(p, q) = M \cdot |x_1 - x_1'|$. On the other hand, the coefficient of e_2 in $(-F(q)) * F(p)$ is $h_2(x_1) - h_2(x_1')$. It follows that

$$|h_2(x_1) - h_2(x_1')| \leq d(0, (-F(q)) * F(p)) = d(F(q), F(p)) \leq M \cdot |x_1 - x_1'|.$$

Hence, h_2 is Lipschitz. Now consider the quasi-conformal map $F_0 := F_{h_2}^{-1} \circ F = F_{-h_2} \circ F$. Its projection on the first layer is the identity map. That is, if $\pi_1 : F^n_{\mathbb{R}} \to V_1$ denotes the projection onto the first layer, and if $g \in F^n_{\mathbb{R}}$ is such that $\pi_1(g) = x_1e_1 + x_2e_2$, then $\pi_1(F_0(g)) = x_1e_1 + x_2e_2$. It follows that the Pansu differential of F_0 is the identity isomorphism whenever it exists. Now Lemma 2.5 implies that F_0 is a left translation. Hence, the original map F is a composition of left translations, graded isomorphisms, and a map of the form F_h.

The proof of Theorem 1.1 is now complete.

4. The Complex Heisenberg Groups

In this section, we will provide evidence that quasi-conformal maps on the complex Heisenberg groups are very special (Section 4.3). For this purpose, we need to introduce differential forms associated with two-step Carnot groups (Section 4.1) and discuss their relations with horizontal liftings (Section 4.2).

4.1. Differential Forms Associated with Two-Step Carnot Groups

Here we introduce differential forms associated with two-step Carnot groups.

Let $\mathfrak{g} = V_1 \oplus V_2$ be a two-step Carnot group. We identify the Lie group with its Lie algebra via the exponential map. The Lie bracket restricted to the first layer V_1 gives rise to a skew symmetric bilinear map

$$\omega : V_1 \times V_1 \to V_2,$$
$$\omega(X, Y) = [X, Y].$$

We view ω as a (constant) V_2-valued differential 2-form on V_1.

We next define a V_2-valued differential 1-form α on V_1 as follows. For each $X \in V_1$, we need to define a linear map $\alpha_X : T_X V_1 \to V_2$. We identify $T_X V_1$ with V_1. Let $\alpha_X : V_1 \to V_2$ be given by

$$\alpha_X(Y) = [X, Y] \quad \text{for } Y \in V_1.$$

It is convenient to write the differential forms α and ω in coordinates. Fix a vector space basis $\{e_1, \dots, e_m\}$ for V_1 and a vector space basis $\{\eta_1, \dots, \eta_n\}$ for V_2. Then a point $X \in V_1$ can be written as $X = x_1 e_1 + x_2 e_2 + \cdots + x_m e_m$. For $X = x_1 e_1 + x_2 e_2 + \cdots + x_m e_m$ and $Y = y_1 e_1 + y_2 e_2 + \cdots + y_m e_m$, we obtain:

$$\omega(X, Y) = [X, Y] = \sum_{i,j} x_i y_j [e_i, e_j].$$

It follows that

$$\omega = \sum_{i,j} [e_i, e_j]\, dx_i \wedge dx_j = 2 \sum_{i<j} [e_i, e_j]\, dx_i \wedge dx_j$$

and

$$\alpha = \sum_{i,j} x_i [e_i, e_j]\, dx_j.$$

We notice that $d\alpha = \omega$.

Now we work out the differential forms associated to the complex Heisenberg group $H^1_{\mathbb{C}} = \mathbb{C}^3$. Let X, Y, Z be the basis for the complex Lie algebra with bracket relation $[X, Y] = Z$. We choose a basis for the real Lie algebra $\{e_1 = X, e_2 = iX, e_3 = Y, e_4 = iY, \eta_1 = Z, \eta_2 = iZ\}$. The nontrivial bracket relations for the real Lie algebra are $[e_1, e_3] = \eta_1$, $[e_1, e_4] = \eta_2$, $[e_2, e_3] = \eta_2$, and $[e_2, e_4] = -\eta_1$.

A point with coordinates (x_1, x_2, x_3, x_4) with respect to the real vector space basis $\{e_1, e_2, e_3, e_4\}$ has the coordinates (w_1, w_2) with respect to the complex vector space basis $\{X, Y\}$, where $w_1 = x_1 + i x_2$, $w_2 = x_3 + i x_4$. Then $dw_1 = dx_1 + i\, dx_2$ and $dw_2 = dx_3 + i\, dx_4$. We obtain:

$$\begin{aligned}\omega &= 2[e_1, e_3]\, dx_1 \wedge dx_3 + 2[e_1, e_4]\, dx_1 \wedge dx_4 \\ &\quad + 2[e_2, e_3]\, dx_2 \wedge dx_3 + 2[e_2, e_4]\, dx_2 \wedge dx_4 \\ &= 2(dx_1 \wedge dx_3 - dx_2 \wedge dx_4)\eta_1 + 2(dx_1 \wedge dx_4 + dx_2 \wedge dx_3)\eta_2 \\ &= 2[(dx_1 \wedge dx_3 - dx_2 \wedge dx_4) + i(dx_1 \wedge dx_4 + dx_2 \wedge dx_3)]Z \\ &= 2(dw_1 \wedge dw_2)Z.\end{aligned}$$

Similarly,

$$\begin{aligned}\alpha &= [e_1, e_3]x_1\, dx_3 + [e_3, e_1]x_3\, dx_1 + [e_1, e_4]x_1\, dx_4 + [e_4, e_1]x_4\, dx_1 \\ &\quad + [e_2, e_3]x_2\, dx_3 + [e_3, e_2]x_3\, dx_2 + [e_2, e_4]x_2\, dx_4 + [e_4, e_2]x_4\, dx_2 \\ &= (x_1\, dx_3 - x_3\, dx_1 + x_4\, dx_2 - x_2\, dx_4)\eta_1 \\ &\quad + (x_1\, dx_4 - x_4\, dx_1 + x_2\, dx_3 - x_3\, dx_2)\eta_2 \\ &= [(x_1\, dx_3 - x_3\, dx_1 + x_4\, dx_2 - x_2\, dx_4) \\ &\quad + i(x_1\, dx_4 - x_4\, dx_1 + x_2\, dx_3 - x_3\, dx_2)]Z \\ &= (w_1\, dw_2 - w_2\, dw_1)Z.\end{aligned}$$

4.2. *Horizontal Lifts in Two-Step Carnot Groups*

Here we give a criterion for a closed curve in V_1 whose horizontal lifts to $\mathfrak{g}$ are also closed curves.

We first recall that a horizontal curve is completely determined by its initial point and its first layer component. This result is well known.

Let $c(t) = (c_1(t), c_2(t)) \in \mathfrak{g} = V_1 \oplus V_2$ be an absolutely continuous curve in $\mathfrak{g}$. For each t_0, let $\gamma(t) = L_{-c(t_0)}c(t)$ be the translated curve. Notice that $\gamma(t_0) = 0$. Using the BCH formula, we find that the tangent vector of γ at $t = t_0$ is

$$\left(c_1'(t_0), c_2'(t_0) - \frac{1}{2}[c_1(t_0), c_1'(t_0)]\right).$$

It follows that the curve $c(t)$ is horizontal if and only if

$$c_2'(t) = \frac{1}{2}[c_1(t), c_1'(t)] \quad \text{for a.e. } t. \tag{4.1}$$

Here is a criterion for a closed curve in V_1 to have closed horizontal lifts to $\mathfrak{g}$.

LEMMA 4.1. *Let $c_1 : [0, 1] \to V_1$ be a closed Lipschitz curve. Then the following conditions are equivalent*:

(1) *The horizontal lifts of c_1 to $\mathfrak{g}$ are closed curves*;
(2) $\int_{c_1} \alpha = 0$;
(3) $\int_D \omega = 0$ *for any Lipschitz 2-disk D with boundary curve c_1*.

Proof. (1) $\iff$ (2) Let $c(t) = (c_1(t), c_2(t))$ be a horizontal lift of c_1. Then c is closed if and only if $c_2(1) = c_2(0)$. By (4.1) we have $c_2'(t) = \frac{1}{2}[c_1(t), c_1'(t)]$ for a.e. $t \in [0, 1]$. Write $c_1(t) = \sum_i x_i(t)e_i$. Then $c_1'(t) = \sum_i x_i'(t)e_i$ and $[c_1(t), c_1'(t)] = \sum_{i,j} x_i(t)x_j'(t)[e_i, e_j]$. Now the fundamental theorem of calculus gives

$$\begin{aligned} c_2(1) - c_2(0) &= \int_0^1 c_2'(t)\,dt = \frac{1}{2}\int_0^1 [c_1(t), c_1'(t)]\,dt \\ &= \frac{1}{2}\int_0^1 \sum_{i,j} x_i(t)x_j'(t)[e_i, e_j]\,dt = \frac{1}{2}\int_{c_1} \alpha. \end{aligned}$$

Hence, (1) and (2) are equivalent.

Conditions (2) and (3) are equivalent due to Stokes' theorem and the fact that $d\alpha = \omega$. □

4.3. *Quasi-conformal Maps on the Complex Heisenberg Groups*

Here we provide evidence that quasi-conformal maps on the complex Heisenberg groups are affine.

Recall that the nth complex Heisenberg group $H^n_{\mathbb{C}}$ is the simply connected Lie group whose Lie algebra $\mathfrak{h}^n_{\mathbb{C}}$ is a complex Lie algebra and has a complex vector space basis X_i, Y_i, Z $(1 \le i \le n)$ with the only nontrivial bracket relations $[X_i, Y_i] = Z$, $1 \le i \le n$. Of course, it has more bracket relations as a real Lie algebra coming from the fact that the bracket is complex linear: $[X_j, iY_j] = [iX_j, Y_j] = iZ$, and $[iX_j, iY_j] = -Z$. The first layer V_1 of $\mathfrak{h}^n_{\mathbb{C}}$ is spanned by the X_i, Y_i, $1 \le i \le n$, and has complex dimension $2n$. The second layer V_2 is spanned by Z and has complex dimension 1. We identify both $H^n_{\mathbb{C}}$ and its Lie algebra $\mathfrak{h}^n_{\mathbb{C}}$ with $\mathbb{C}^{2n+1}$. So $V_1 = \mathbb{C}^{2n} \times \{0\}$ and $V_2 = \{0\} \times \mathbb{C}$. Let $\pi_1 : H^n_{\mathbb{C}} = \mathbb{C}^{2n+1} \to \mathbb{C}^{2n}$ be the projection onto V_1. A map $F : \mathbb{C}^{2n+1} \to \mathbb{C}^{2n+1}$ is called a lifting of a map $f : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ if $F(\pi_1^{-1}(p)) = \pi_1^{-1}(f(p))$ for all $p \in \mathbb{C}^{2n}$.

Let $\tau : H^n_{\mathbb{C}} \to H^n_{\mathbb{C}}$ be defined by $\tau(w_1, \ldots, z) = (\bar{w}_1, \ldots, \bar{z})$. It is easy to see that τ is a graded isomorphism of $H^n_{\mathbb{C}}$.

Here is the first evidence for Conjecture 1.3.

PROPOSITION 4.2. *Let $F : H^n_{\mathbb{C}} \to H^n_{\mathbb{C}}$ be a homeomorphism of the complex Heisenberg group. If F is both a quasi-conformal map and a C^2 diffeomorhism, then, after possibly composing with τ, F is the lifting of a complex affine map. Furthermore, F or $F \circ \tau$ is a complex affine map.*

Proof. Write F as $F(w_1, \ldots, w_{2n}, z) = (F_1, \ldots, F_{2n}, F_{2n+1})$, where $F_i = F_i(w_1, \ldots, w_{2n}, z)$ is the ith component function of F. Since F is quasi-conformal and $H^n_{\mathbb{C}}$ is connected, F is power quasi-symmetric. This implies that the F_i have polynomial growth. On the other hand, by the main result in [RR], F or $F \circ \tau$ is biholomorphic. We shall assume that F is biholomorphic. By a Liouville type theorem we conclude that the F_i are actually polynomials. By symmetry, the component functions of F^{-1} are also polynomials. It follows that the horizontal Jacobian $J_H(F,x)$ of F is a polynomial (recall that the horizontal Jacobian $J_H(F,x)$ is the determinant of the linear map $dF(x)|_{V_1} : V_1 \to V_1$). The same is true for the horizontal Jacobian $J_H(F^{-1}, y)$ of F^{-1}. However, the horizontal Jacobian of $\mathrm{Id} = F^{-1} \circ F$ is $1 = J_H(F,x) \cdot J_H(F^{-1}, F(x))$. So the product of the polynomials $J_H(F,x)$ and $J_H(F^{-1}, F(x))$ is 1. This happens only when both polynomials are constants. Therefore, $J_H(F,x)$ is a constant function. This implies that F is bi-Lipschitz. It follows that there is a constant $L > 0$ such that $d(F(p), F(q)) \le L \cdot d(p,q)$ for all $p, q \in H^n_{\mathbb{C}}$. Let $p = 0$ and $q = (w_1, \dots, w_{2n}, z)$. We see that for each $1 \le i \le 2n$,

$$|F_i(w_1, \dots, w_{2n}, z) - F_i(0, \dots, 0)| \le L \cdot \left\{ \sum_i |w_i| + |z|^{1/2} \right\}.$$

Since F_i is a polynomial, we conclude that F_i is independent of z and is affine in $w_1, \dots, w_{2n}$. Hence, F is the lifting of a complex affine map $f := (F_1, \dots, F_{2n}) : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$.

Let g be the linear part of f. Notice that $g = dF(p)|_{V_1}$ for any point p. In other words, g lifts to the graded isomorphism $dF(p)$ of $H^n_{\mathbb{C}}$. The translational part of f of course lifts to a left translation L_q (for some $q \in H^n_{\mathbb{C}}$) in $H^n_{\mathbb{C}}$. Let $F' = L_q \circ dF(p)$. Notice that $dF(x)|_{V_1} = g = dF'(x)|_{V_1}$ for all $x \in H^n_{\mathbb{C}}$. It follows that $dF(x) = dF'(x)$ for all $x \in H^n_{\mathbb{C}}$. By Lemma 2.5 there is some $q' \in H^n_{\mathbb{C}}$ such that $F = L_{q'} \circ F' = L_{q'} \circ L_q \circ dF(p)$, which is a complex affine map. □

Here is more evidence for Conjecture 1.3.

PROPOSITION 4.3. *Let $F : H^1_{\mathbb{C}} \to H^1_{\mathbb{C}}$ be a quasi-conformal map. Suppose that F is the lifting of a map $f : \mathbb{C}^2 \to \mathbb{C}^2$ of the form $f(w_1, w_2) = (w_1, w_2 + g(w_1))$, where $g : \mathbb{C} \to \mathbb{C}$ is a map. Then there are constants $a, b \in \mathbb{C}$ such that $g(w_1) = aw_1 + b$.*

Proof. Notice that for each fixed $w_1 \in \mathbb{C}$, we have $f(\{w_1\} \times \mathbb{C}) = \{w_1\} \times \mathbb{C}$. Since F is a lifting of the map f, it follows that F preserves each left coset of $\{0\} \times \mathbb{C}^2$ in $H^1_{\mathbb{C}}$. Now the arguments in [SX] show that F is bi-Lipschitz.

Fix $w_2, z \in \mathbb{C}$ and let $c : [0,1] \to \mathbb{C} \times \{w_2\} \times \{z\} \subset H^1_{\mathbb{C}}$ be a closed C^1 curve. Then c is a horizontal curve in $H^1_{\mathbb{C}}$. Since F is bi-Lipschitz, $F \circ c$ is also a closed horizontal curve in $H^1_{\mathbb{C}}$. The projection $\pi_1 \circ F \circ c$ of $F \circ c$ under π_1 is a closed Lipschitz curve in $\mathbb{C}^2$ that admits a closed horizontal lift. By Lemma 4.1 we have

$$\int_{\pi_1 \circ F \circ c} w_1\, dw_2 - w_2\, dw_1 = 0.$$

Since $\int_\gamma w_1\, dw_2 + w_2\, dw_1 = 0$ for any closed curve γ, we have $\int_{\pi_1 \circ F \circ c} w_2\, dw_1 = 0$. Notice that $\pi_1 \circ F \circ c(t) = (c(t), w_2 + g(c(t)))$. So

$$0 = \int_{\pi_1 \circ F \circ c} w_2\, dw_1 = \int_c [w_2 + g(w_1)]\, dw_1 = \int_c g(w_1)\, dw_1.$$

Hence, $\int_c g(w_1)\,dw_1 = 0$ for any closed C^1 curve in the complex plane. Since g is continuous (actually Lipschitz; see the next paragraph), Morera's theorem implies that $g(w_1)$ is holomorphic.

We next show that g is Lipschitz. Let $w_1, w_1' \in \mathbb{C}$ be arbitrary. Fix any $w_2, z \in \mathbb{C}$ and let $p = (w_1, w_2, z)$, $q = (w_1', w_2, z + \frac{1}{2}(w_1 - w_1')w_2)$. Then $d(p,q) = |w_1 - w_1'|$. Notice that $\pi_1 \circ F(p) = f(w_1, w_2) = (w_1, w_2 + g(w_1))$ and $\pi_1 \circ F(q) = f(w_1', w_2) = (w_1', w_2 + g(w_1'))$. It follows that $d(F(p), F(q)) \geq |(w_2 + g(w_1)) - (w_2 + g(w_1'))| = |g(w_1) - g(w_1')|$. By the first paragraph, F is M-bi-Lipschitz for some $M > 0$. Hence,

$$|g(w_1) - g(w_1')| \leq d(F(p), F(q)) \leq M \cdot d(p,q) = M \cdot |w_1 - w_1'|.$$

So g is Lipschitz. Since g is also holomorphic, it has to be affine. □

5. Quasi-conformal Maps on the Complex Model Filiform Groups

In this section, we show that quasi-conformal maps on the higher complex model Filiform groups are affine. The proof is mostly similar to the real case. We will only indicate the difference in the proofs.

5.1. Graded Automorphisms of $\mathfrak{f}_{\mathbb{C}}^n$

In this subsection, we identify the graded automorphisms of $\mathfrak{f}_{\mathbb{C}}^n$.

Let $F_{\mathbb{C}}^n$ be the n-step complex model Filiform group. Recall that its Lie algebra $\mathfrak{f}_{\mathbb{C}}^n$ is a complex Lie algebra with basis $\{e_1, e_2, \ldots, e_{n+1}\}$ and the only nontrivial bracket relations are $[e_1, e_j] = e_{j+1}$ for $2 \leq j \leq n$. Viewed as a real Lie algebra, $\mathfrak{f}_{\mathbb{C}}^n$ has the additional bracket relations $[ie_1, e_j] = ie_{j+1} = [e_1, ie_j]$, $[ie_1, ie_j] = -e_{j+1}$. The Lie algebra $\mathfrak{f}_{\mathbb{C}}^n$ decomposes as $\mathfrak{f}_{\mathbb{C}}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_n$, where $V_1 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ is the first layer, and $V_j = \mathbb{C}e_{j+1}$, $2 \leq j \leq n$, is the jth layer.

The proof of the following lemma is similar to that of Lemma 3.1.

LEMMA 5.1. *Let $\mathfrak{f}_{\mathbb{C}}^n$ be the n-step complex model Filiform algebra. Assume that $n \geq 3$. Let $X = ae_1 + be_2 \in V_1$ be a nonzero element in the first layer. Then* $\operatorname{rank}(X) = 2$ *if $a = 0$ and* $\operatorname{rank}(X) > 2$ *otherwise.*

Hence, we have the following.

LEMMA 5.2. *Suppose $n \geq 3$. Then $h(\mathbb{C}e_2) = \mathbb{C}e_2$ for every graded isomorphism $h : \mathfrak{f}_{\mathbb{C}}^n \to \mathfrak{f}_{\mathbb{C}}^n$.*

A graded isomorphism $h : \mathfrak{f}_{\mathbb{C}}^n \to \mathfrak{f}_{\mathbb{C}}^n$ is in particular an isomorphism between real vector spaces. In general, h needs not to be an isomorphism of the complex vector spaces. Here is an example. Define $\tau : \mathfrak{f}_{\mathbb{C}}^n \to \mathfrak{f}_{\mathbb{C}}^n$ by

$$\tau\left(\sum z_j e_j\right) = \sum \bar{z}_j e_j.$$

It is easy to check that τ is a graded isomorphism of $\mathfrak{f}_{\mathbb{C}}^n$. Notice that τ is complex antilinear, not complex linear.

LEMMA 5.3. *Let $h : \mathfrak{f}^n_{\mathbb{C}} \to \mathfrak{f}^n_{\mathbb{C}}$ be a graded isomorphism. Then h is either complex linear or complex antilinear.*

Proof. The map h satisfies $[hv, hw] = h[v, w]$ for all $v, w \in V_1$. In particular, we have $[h(ie_1), h(ie_2)] = h([ie_1, ie_2]) = -h([e_1, e_2])$, $[h(ie_1), h(e_2)] = h([ie_1, e_2]) = h([e_1, ie_2]) = [h(e_1), h(ie_2)]$, and $[h(e_1), h(ie_1)] = h([e_1, ie_1]) = 0$.

By Lemma 5.2 we know that $h(\mathbb{C}e_2) = \mathbb{C}e_2$. Hence, there are $0 \neq w_1, w_2 \in \mathbb{C}$ such that

$$h(e_2) = w_1 e_2,$$
$$h(ie_2) = w_2 e_2.$$

There are also constants $a, b, c, d \in \mathbb{C}$ such that

$$h(e_1) = ae_1 + be_2,$$
$$h(ie_1) = ce_1 + de_2.$$

The equation $[h(ie_1), h(e_2)] = [h(e_1), h(ie_2)]$ yields $cw_1 = aw_2$. Similarly, $[h(ie_1), h(ie_2)] = -[h(e_1), h(e_2)]$ yields $cw_2 = -aw_1$, and $[h(e_1), h(ie_1)] = 0$ yields $ad - bc = 0$. It follows that $c(w_1^2 + w_2^2) = 0$. Assume that $c = 0$; then $ad = 0$. Since h is an isomorphism, $d \neq 0$. So $a = 0$, which implies that h maps V_1 into $\mathbb{C}e_2$, contradicting the fact that h is an isomorphism. Hence, $c \neq 0$. It follows that $w_2 = iw_1$ or $w_2 = -iw_1$.

If $w_2 = iw_1$, then $c = ia$ and $d = ib$. In this case, $h|_{V_1}$ is complex linear. Since the bracket is also complex linear, an induction argument shows that $h|_{V_j}$ is complex linear for all j. If $w_2 = -iw_1$, then $c = -ia$ and $d = -ib$. In this case, $h|_{V_1}$ is complex antilinear. A similar argument also shows that $h|_{V_j}$ is complex antilinear for all j. □

LEMMA 5.4. *A real linear map $h : \mathfrak{f}^n_{\mathbb{C}} \to \mathfrak{f}^n_{\mathbb{C}}$ is a graded isomorphism if and only if h has one of the following two forms*:

(1) *$h = h_{a_1,a_2,b}$ for some $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$; in this case, h is complex linear*;
(2) *$h = \tau \circ h_{a_1,a_2,b}$ for some $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$; in this case, h is complex antilinear.*

Proof. One direction is clear. So we start with a graded isomorphism $h : \mathfrak{f}^n_{\mathbb{C}} \to \mathfrak{f}^n_{\mathbb{C}}$. By Lemma 5.3 h is complex linear or complex antilinear. If h is complex linear, then the proof of Lemma 3.3 shows that $h = h_{a_1,a_2,b}$ for some $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. If h is complex antilinear, then $\tau \circ h$ is a graded isomorphism and is complex linear. Hence, $\tau \circ h = h_{a_1,a_2,b}$ for some $a_1, a_2 \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. It follows that $h = \tau \circ h_{a_1,a_2,b}$. □

5.2. *Proof of Theorem 1.2*

In this subsection, we show that each quasi-conformal map of $F^n_{\mathbb{C}}$ $(n \geq 3)$ is a composition of left translations and graded automorphisms. Hence, by

Lemma 5.4, after possibly taking complex conjugation (that is, composing with τ), each quasi-conformal map of $F_{\mathbb{C}}^n$ ($n \geq 3$) is biholomorphic.

Let $n \geq 3$, and let $F : F_{\mathbb{C}}^n \to F_{\mathbb{C}}^n$ be a quasi-conformal map. Then F is a quasi-symmetric map and sends left cosets of $\mathbb{C}e_2$ to left cosets of $\mathbb{C}e_2$. For a.e. left coset L of $\mathbb{C}e_2$, the map F is Pansu differentiable a.e. on L. For any point $x \in F_{\mathbb{C}}^n$ where F is Pansu differentiable, let $a_j(x), b(x) \in \mathbb{C}$, $1 \leq j \leq n+1$, be such that $dF(x)(e_1) = a_1(x)e_1 + b(x)e_2$, $dF(x)(e_j) = a_j(x)e_j$ ($2 \leq j \leq n+1$). Then $a_j(x) = (a_1(x))^{j-2}a_2(x)$ for $2 \leq j \leq n+1$. If $dF(x)$ is complex linear, then $dF(x)(ze_j) = z \cdot dF(x)(e_j)$ for any $z \in \mathbb{C}$. If $dF(x)$ is complex antilinear, then $dF(x)(ze_j) = \bar{z} \cdot dF(x)(e_j)$ for any $z \in \mathbb{C}$.

LEMMA 5.5. *Let L be a good left coset of $\mathbb{C}e_2$. Then there is a constant $a_L \in \mathbb{C} \setminus \{0\}$ such that $a_{n+1}(x) = a_L$ for a.e. $x \in L$. Furthermore, if $x, y \in L$ are two good points, then either both $dF(x)$ and $dF(y)$ are complex linear, or both are complex antilinear.*

Proof. The proof of the first statement is the same as in the proof of Lemma 3.7 with $\mathbb{R}$ replaced by $\mathbb{C}$. For the second statement, if $dF(x)$ is complex linear and $dF(y)$ is complex antilinear, then use the left coset $ir^n e_{n+1} * L$ instead of $r^n e_{n+1} * L$ and repeat the argument to get a contradiction. For this, one uses the fact that $dF(x)(ir^n e_{n+1}) = ir^n a_{n+1}(x)e_{n+1}$ and $dF(y)(ir^n e_{n+1}) = -ir^n a_{n+1}(y)e_{n+1} = -ir^n a_{n+1}(x)e_{n+1}$. □

Now statements similar to Lemma 3.8 through Lemma 3.13 all hold, and the proofs are the same. In particular, F is bi-Lipschitz, and if we denote by $G = \mathbb{C}e_2 \oplus \cdots \oplus \mathbb{C}e_{n+1}$, then F sends left cosets of G to left cosets of G. Hence, there is a homeomorphism $f : \mathbb{C} \to \mathbb{C}$ such that $F(x_1e_1 * G) = f(x_1)e_1 * G$. For a.e. left coset L of $\mathbb{C}e_2$, there exists a constant $a_{2,L} \in \mathbb{C} \setminus \{0\}$ such that $a_2(x) = a_{2,L}$ for a.e. $x \in L$. Furthermore, for any $g \in L$, either $F(g * te_2) = F(g) * a_{2,L}te_2$ for all $t \in \mathbb{C}$, or $F(g * te_2) = F(g) * a_{2,L}\bar{t}e_2$ for all $t \in \mathbb{C}$.

LEMMA 5.6. *There is a constant $a_2 \in \mathbb{C} \setminus \{0\}$ such that either $F(g * te_2) = F(g) * a_2te_2$ for all $g \in F_{\mathbb{C}}^n$ or $F(g * te_2) = F(g) * a_2\bar{t}e_2$ for all $g \in F_{\mathbb{C}}^n$.*

Proof. Let L, L' be two good left cosets of $\mathbb{C}e_2$. We run the argument in the proof of Lemma 3.14 for $t \in \mathbb{R}$ to show $a_{2,L} = a_{2,L'}$. If $F(g * te_2) = F(g) * a_{2,L}te_2$ on L and $F(g * te_2) = F(g) * a_{2,L'}\bar{t}e_2$ on L', then we get a contradiction by using $t \in i\mathbb{R}$. Since F is a homeomorphism and a.e. left coset L of $\mathbb{C}e_2$ is a good left coset, the lemma follows by continuity. □

After composing F with a graded isomorphism, we may assume that $a_2 = 1$ and $F(g * te_2) = F(g) * te_2$ for all $g \in F_{\mathbb{C}}^n$. So the Pansu differential is complex linear whenever it exists.

When the Pansu differential is complex linear, the derivatives that appear in the proofs in Section 3 can be taken to be complex derivatives:

(1) proof of Lemmas 3.4 and 3.13, $a_1(p) = f'(x_1)$ and $a_2(p) = \partial f_2(x)/\partial x_2$;

(2) proof of Lemma 3.15, $\partial f_j(x)/\partial x_j = (f'(x_1))^{j-2}$; here both are complex derivatives;
(3) proof of Lemma 3.17, $-h_n(x_1, x_{n+1}) = h'_{n+1}(x_1)$.

Notice that the left cosets of G are isometric to $\mathbb{C}^n$ with the metric $D((z_i), (w_i)) = \sum_i |z_i - w_i|^{1/i}$. By the proof in Section 15 of [T] each quasi-symmetric map $h : (\mathbb{C}^n, D) \to (\mathbb{C}^n, D)$ preserves the foliation consisting of affine subspaces parallel to $\mathbb{C}^i \times \{0\}$ for each $1 \le i \le n-1$. An analogue of Lemma 3.15 holds, and so for a.e. $x_1 \in \mathbb{C}$, F has the following form:

$$F\left(x_1e_1 * \left(\sum_{i=2}^{n+1} x_ie_i\right)\right) = f(x_1)e_1 * \sum_{i=2}^{n+1}\{(f'(x_1))^{i-2}x_i + h_i(x_1, x_{i+1}, \dots, x_{n+1})\}e_i.$$

We next show the following.

LEMMA 5.7. *The function f is complex affine.*

Proof. Recall that $f : \mathbb{C} \to \mathbb{C}$ is a homeomorphism such that $F(x_1e_1 * G) = f(x_1)e_1 * G$. This implies that at any point $p \in F^n_{\mathbb{C}}$ where F is Pansu differentiable, $a_1(p) = f'(x_1)$. Here x_1 is the coefficient of e_1 in the expression for p, and $f'(x_1)$ is the complex derivative. Hence, at a.e. $x_1 \in \mathbb{C}$, f has nonzero complex derivative. In particular, $f : \mathbb{C} \to \mathbb{C}$ is a 1-quasi-conformal map. It follows that f is a similarity. Since f has complex derivative, the linear part of f cannot be a reflection, and so f must be a complex affine map. □

After composing F with a graded isomorphism and a left translation, we may assume that $f(x_1) = x_1$. So F has the following form:

$$F\left(x_1e_1 * \left(\sum_{i=2}^{n+1} x_ie_i\right)\right) = x_1e_1 * \sum_{i=2}^{n+1}\{x_i + h_i(x_1, x_{i+1}, \dots, x_{n+1})\}e_i.$$

Now the proof of Lemma 3.17 shows that h_i is a function of x_1 only. We shall show that h_i is a holomorphic function of x_1.

Let $H_3 = \mathbb{C}e_4 \oplus \cdots \oplus \mathbb{C}e_{n+1}$. Then H_3 is a closed normal subgroup of $F^n_{\mathbb{C}}$, and $F^n_{\mathbb{C}}/H_3$ is isomorphic to $H^1_{\mathbb{C}}$ (the first complex Heisenberg group). It is easy to see from the expression of F that F maps left cosets of H_3 to left cosets of H_3. Hence, F induces a map $\overline{F} : F^n_{\mathbb{C}}/H_3 = H^1_{\mathbb{C}} \to H^1_{\mathbb{C}} = F^n_{\mathbb{C}}/H_3$, and $\overline{F}$ admits the following expression:

$$\overline{F}(x_1e_1 * (x_2e_2 + x_3e_3)) = x_1e_1 * [(x_2 + h_2(x_1))e_2 + (x_3 + h_3(x_1))e_3].$$

It follows that $\overline{F}$ is the lifting of the map $f : \mathbb{C}^2 \to \mathbb{C}^2$, $f(x_1, x_2) = (x_1, x_2 + h_2(x_1))$. Here we identified the first layer V_1 of $H^1_{\mathbb{C}}$ with $\mathbb{C}^2$ via $x_1e_1 + x_2e_2 \to (x_1, x_2)$.

Let $\pi : F^n_{\mathbb{C}} \to F^n_{\mathbb{C}}/H_3 = H^1_{\mathbb{C}}$ be the quotient map. Since H_3 is normal in $F^n_{\mathbb{C}}$, and the quotient group $F^n_{\mathbb{C}}/H_3$ is also Carnot, it is not hard to check that for any $p, q \in F^n_{\mathbb{C}}/H_3 = H^1_{\mathbb{C}}$ and any $x \in \pi^{-1}(p)$, we have $d_c(\pi^{-1}(p), \pi^{-1}(q)) = d_c(x, \pi^{-1}(q)) = d_c(p, q)$. Since F is quasi-symmetric, it now follows from the following lemma of Tyson that $\overline{F} : F^n_{\mathbb{C}}/H_3 = H^1_{\mathbb{C}} \to H^1_{\mathbb{C}} = F^n_{\mathbb{C}}/H_3$ is also quasi-symmetric.

LEMMA 5.8 [T, Lemma 15.9]. *Let $g : X_1 \to X_2$ be an η-quasi-symmetry, and $A, B, C \subset X_1$. If $d(A, B) \le t d(A, C)$ for some $t \ge 0$, then there is $a \in A$ such that*

$$d(g(A), g(B)) \le \eta(t) d(g(a), g(C)).$$

Now we apply Proposition 4.3 to $\overline{F}$ to conclude that $h_2(x_1)$ is a complex affine function of x_1. So there are constants $a, b \in \mathbb{C}$ such that $h_2(x_1) = ax_1 + b$. After composing F with the map F_{-h_2} (see the Introduction), we may assume that $h_2(x_1) = 0$. It follows that the Pansu differential of F is a.e. the identity isomorphism. Lemma 2.5 implies that it is a left translation. Notice that F_{h_2} is a composition of a graded isomorphism and a left translation. Hence, every quasi-conformal map of $F^n_{\mathbb{C}}$ is a finite composition of left translations and graded isomorphisms. This finishes the proof of Theorem 1.2.

Notice that left translations in $F^n_{\mathbb{C}}$ are polynomial maps with polynomial inverse (this follows from the BCH formula). By Lemma 5.4 each graded isomorphism is complex linear after possibly composing with τ (taking complex conjugation). Hence, Theorem 1.2 implies that for $n \ge 3$, every quasi-conformal map of $F^n_{\mathbb{C}}$ is a polynomial map with polynomial inverse after possibly composing with τ.

ACKNOWLEDGMENTS. This work was completed while the author was attending the workshop "Interactions between analysis and geometry" at IPAM, University of California at Los Angeles from March to June 2013. I would like to thank IPAM for financial support, excellent working conditions, and conducive atmosphere. I also would like to thank David Freeman and Tullia Dymarz for discussions about Carnot groups. The author is partially supported by NSF grant DMS-1265735.

References

[B] Z. Balogh, *Hausdorff dimension distribution of quasiconformal mappings on the Heisenberg group,* J. Anal. Math. 83 (2001), 289–312.

[BKR] Z. Balogh, P. Koskela, and S. Rogovin, *Absolute continuity of quasiconformal mappings on curves,* Geom. Funct. Anal. 17 (2007), no. 3, 645–664.

[BR] A. Bellaiche and J. J. Risler, *Sub-Riemannian geometry,* Progr. Math., 144, Birkhäuser Verlag, Basel, 1996.

[CC] L. Capogna and M. Cowling, *Conformality and Q-harmonicity in Carnot groups,* Duke Math. J. 135 (2006), no. 3, 455–479.

[CG] L. Corwin and F. Greenleaf, *Representations of nilpotent Lie groups and their applications, Part I. Basic theory and examples,* Cambridge Stud. Adv. Math., 18, Cambridge University Press, Cambridge, 1990.

[H] E. Heintze, *On homogeneous manifolds of negative curvature,* Math. Ann. 211 (1974), 23–34.

[HK] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry,* Acta Math. 181 (1998), no. 1, 1–61.

[O] A. Ottazzi, *A sufficient condition for nonrigidity of Carnot groups,* Math. Z. 259 (2008), no. 3, 617–629.

[OW] A. Ottazzi and B. Warhurst, *Contact and 1-quasiconformal maps on Carnot groups,* J. Lie Theory 21 (2011), no. 4, 787–811.

[P1] P. Pansu, *Croissance des boules et des géodésiques fermées dans les nilvariétés,* Ergodic Theory Dynam. Systems 3 (1983), no. 3, 415–445.

[P2] ———, *Metriques de Carnot-Caratheodory et quasiisometries des espaces symetriques de rang un,* Ann. of Math. (2) 129 (1989), no. 1, 1–60.

[RR] H. M. Reimann and F. Ricci, *The complexified Heisenberg group,* Proceedings on analysis and geometry (Russian) (Novosibirsk Akademgorodok, 1999), pp. 465–480, Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.

[SX] N. Shanmugalingam and X. Xie, *A rigidity property of some negatively curved solvable Lie groups,* Comment. Math. Helv. 87 (2012), no. 4, 805–823.

[T] J. Tyson, *Metric and geometric quasiconformality in Ahlfors regular Loewner spaces,* Conform. Geom. Dyn. 5 (2001), 21–73.

[V] J. Väisälä, *The free quasiworld. Freely quasiconformal and related maps in Banach spaces,* Quasiconformal geometry and dynamics (Lublin, 1996), Banach Center Publ., 48, pp. 55–118, Polish Acad. Sci., Warsaw, 1996.

[W] B. Warhurst, *Contact and quasiconformal mappings on real model filiform groups,* Bull. Aust. Math. Soc. 68 (2003), no. 2, 329–343.

[X1] X. Xie, *Some examples of quasiisometries of nilpotent Lie groups,* J. Reine Angew. Math. (to appear).

[X2] ———, *Rigidity of quasiconformal maps on Carnot groups,* ⟨http://front.math.ucdavis.edu/1308.3028⟩.

[X3] ———, *Quasisymmetric maps on reducible Carnot groups,* Pacific J. Math. 265 (2013), no. 1, 113–122.

[X4] ———, *Quasiconformal maps on non-rigid Carnot groups,* ⟨http://front.math.ucdavis.edu/1308.3031⟩.

Dept. of Mathematics and Statistics
Bowling Green State University
Bowling Green, OH 43403
USA

xiex@bgsu.edu