Michigan Math. J. 62 (2013), 259–292

Restricted Carathéodory Measure and Restricted Volume of the Canonical Bundle

Shin Kikuta

1. Introduction

This paper is concerned with relations between Carathéodory measure hyperbolicity and algebro-geometric positivity of the canonical bundle or the cotangent bundle over a compact complex manifold. On the one hand, positivity of vector bundles over a compact complex manifold is an important notion in algebraic geometry. On the other hand, Carathéodory measure hyperbolicity for a complex manifold is one of the principal properties in geometric function theory or the theory of hyperbolic complex spaces. It is therefore of fundamental interest to investigate how these notions are related.

In the author's previous paper [20] it is proved that the curvature function of the Carathéodory pseudo-volume form over a complex manifold is not larger than -1. As an easy application of the curvature property, we obtain the following explicit comparison formula between the volume $\mathrm{vol}_X(K_X)$ of the canonical bundle K_X over a compact complex manifold X and the total volume of X with respect to the Carathéodory measure $\mu^C_{\tilde{X}}$ (see Definition 2.1) of its universal cover $\tilde{X}$.

Theorem 1.1 [20, Cor. 1.2]. *Let X be an n-dimensional compact complex space with at most normal singularities, and let $\tilde{X}$ be its universal covering space. Then*

$$\mathrm{vol}_X(K_X) := \limsup_{m\to\infty} \frac{\dim \mathrm{H}^0(X, \mathcal{O}(mK_X))}{m^n/n!} \geq \frac{n!\,(n+1)^n}{(4\pi)^n}\mu^C_{\tilde{X}}(X),$$

where the Carathéodory measure $\mu^C_{\tilde{X}}$ of $\tilde{X}$ is considered as a measure on X.

Note that differently from the original theorem in [20], the complex space in this theorem is allowed to have at most normal singularities, but the point can be solved easily by taking a resolution of the singularities of X.

This result tells us not only that the Carathéodory measure hyperbolicity of the universal cover (i.e., $\mu^C_{\tilde{X}}(X) > 0$) implies the bigness of the canonical bundle (i.e., $\mathrm{vol}_X(K_X) > 0$) but also how the bigness increases as the Carathéodory measure hyperbolicity becomes stronger.

On the other hand, for a line bundle L over a compact complex manifold X, a restricted version of the volume of L recently appears as an algebro-geometric

Received August 11, 2011. Revision received January 16, 2013.

The author is supported by the Grant-in-Aid for Scientific Research (KAKENHI no. 22-3509) and the Grant-in-Aid for JSPS fellows.

quantity measuring positivity of L along a closed subvariety Z. It is called the restricted volume of L along Z and denoted by $\mathrm{vol}_{X|Z}(L)$. The notion is a useful tool for extending sections from the subvariety to the ambient space, and so various properties and various applications of the notion have been developed [13; 18; 29]. The restricted volume $\mathrm{vol}_{X|Z}(L)$ of L along Z is indeed defined by replacing global sections in the definition of the volume with extendable ones on Z to X.

Therefore, it seems a worthwhile goal to find a natural restricted version of the Carathéodory pseudo-volume form along a subvariety of a complex manifold. Furthermore, it is feasible to conjecture (after the comparison obtained in Theorem 1.1) that, over a compact complex manifold, the restricted volume of the canonical bundle can be estimated explicitly from below by the restricted version of the Carathéodory measure for its universal cover.

This paper presents a natural restricted version of the Carathéodory pseudo-volume form or the Carathéodory measure for a subvariety of a complex manifold. In fact, it is called the restricted Carathéodory pseudo-volume form or the restricted Carathéodory measure (respectively) and is defined as follows. For a general complex manifold X and its smooth subvariety Z, the restricted Carathéodory pseudo-volume form $v^C_{X|Z}$ on Z is defined by setting

$$v^C_{X|Z} := \sup\{(\tilde{f}|_Z)^* v_1^{(d)};\ \tilde{f}\colon X \to \mathbb{B}^d : \text{holomorphic}\}.$$

Here $\mathbb{B}^d$ and $v_1^{(d)}$ denote, respectively, the d-dimensional complex unit ball and the Poincaré volume form there. For singular Z we define an invariant measure, called the restricted Carathéodory measure, in a similar fashion and denote it by $\mu^C_{X|Z}$. As far as the author knows, this restricted version of Carathéodory measure has already appeared essentially in Eisenman's paper [14]. However, this definitional formula is insufficient for our purpose, which is to compare this notion with the restricted volume of the canonical bundle along irreducible closed subvarieties of a compact complex manifold. Namely, we need some properties of them. Section 2 is devoted to introducing these properties, and a property on curvature functions defined in [20] will be investigated in Section 4.1.

Following Theorem 1.1 (i.e., Corollary 1.2 in [20]), the author conjectures that its restricted version would also hold for the two restricted objects just described and attempts to verify this. But except for some very special cases, the author has been unable to establish whether this conjecture holds or not. The main theorem in this paper is that the conjecture is true if we substitute another, smaller restricted version for the usual one defined previously. In other words, a weak version of this conjecture is established. We denote the substitute by $\bar{v}^C_{X|Z}$, which is defined using Carathéodory extremal maps as follows. For a subvariety Z of a complex manifold X and a point x of the smooth locus Z_{reg} of Z,

$$(\bar{v}^C_{X|Z})_x := \sup\left\{(g_x|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar{\partial}\log v_1^{(n)}\right)^d\right)_x ;\ (g_x^* v_1)_x = (v^C_X)_x\right\},$$

where v^C_X denotes the Carathéodory pseudo-volume form of X. The weak version is stated as follows.

THEOREM 1.2 (Theorem 3.1). *Let X be an n-dimensional compact complex manifold, and let Z be its d-dimensional irreducible closed subvariety. Take any Galois covering space $\tilde{X} \xrightarrow{p} X$ and denote by $\tilde{Z}$ the pull-back of Z by p. Suppose that $\tilde{Z} \not\subset \mathrm{Zero}(v^C_{\tilde{X}})$. Then*

$$\mathrm{vol}_{X|Z}(K_X) \geq \frac{d!\,(n+1)^d}{(4\pi)^d} \int_{Z_{\mathrm{reg}}} \bar{v}^C_{\tilde{X}|\tilde{Z}}.$$

This weak version still generalizes the previous comparison in Theorem 1.1. Moreover, the corresponding theorem does not hold after the measure in the right-hand side is replaced with the original restricted Carathéodory measure $\mu^C_{\tilde{X}|\tilde{Z}}$. As regards the condition $\tilde{Z} \not\subset \mathrm{Zero}(v^C_{\tilde{X}})$ on subvarieties, it cannot be removed from either the conjecture or the theorem. This condition can be checked by constructing a counterexample with a blow-up.

Our procedure for proving Theorem 1.2 is similar to the one used in [20] to prove Theorem 1.1 (given as Corollary 1.2 there). In that proof, a central role is played by the Boucksom–Popovici formula [4; 30] on the volume of a line bundle. In this paper we apply its restricted version, which is due to Hisamoto [18] and Matsumura [29]. Before doing so, we must verify that the assumptions on Z and curvature currents hold in our case. The following lemma is needed for that.

LEMMA 1.1 (Lemma 3.4). *If X is a projective manifold and $\tilde{X}$ denotes its universal cover, then*

$$\mathbb{B}_+(K_X) \subset \mathrm{Zero}(v^C_{\tilde{X}});$$

here $\mathbb{B}_+(K_X)$ is the nonample locus of K_X (see Section 3.1 for the definition).

This lemma is of interest in its own right because it compares the Carathéodory measure hyperbolicity and positivity of the canonical bundle in terms of the inclusion relation. After proving the lemma we establish the following pluripotential theoretic inequality, which corresponds to the curvature inequality used in the proof of Theorem 1.1.

THEOREM 1.3 (Theorem 3.2). *Let X be an n-dimensional complex manifold, and let Z be a smooth d-dimensional subvariety of X not contained in $\mathrm{Zero}(v^C_X)$. If the left-hand side is well-defined then*

$$\big\langle \big(\big(\sqrt{-1}\partial\bar{\partial}\log v^C_X\big)\big|_Z\big)^d \big\rangle \geq \frac{d!\,(n+1)^d}{2^d} \bar{v}^C_{X|Z},$$

where $\langle\cdot\rangle$ denotes the non-pluripolar Monge–Ampère product (see Section 3.1 for the definition).

The results just stated are all proved in Section 3.1.

As an immediate application of Theorem 1.2, we have some numerical comparisons between the ampleness of the canonical bundle and the strong Carathéodory measure hyperbolicity over a compact complex manifold. A complex manifold is said to be strongly Carathéodory measure hyperbolic if its Carathéodory pseudo-volume form is positive everywhere. As shown in [20] and also stated by Wu [33], every compact complex manifold turns out to be a projective manifold with the ample canonical bundle if it has the strongly Carathéodory measure hyperbolic universal cover. However, their proofs do not yield any numerical relations.

Nonetheless, we can use Theorem 1.2 to estimate the intersection numbers $(K_X^{\dim Z} \cdot Z)$ of the canonical bundle with subvarieties Z from below by our substitutes for the restricted Carathéodory measures over a compact complex manifold. This result may be formally stated as follows.

COROLLARY 1.1 (Theorem 3.7). *Let X be an n-dimensional compact complex manifold. Take any Galois cover $\tilde{X} \xrightarrow{p} X$, and let $\tilde{Z}$ denote the pull-back of a subvariety Z of X by p. Suppose that $\tilde{X}$ is strongly Carathéodory measure hyperbolic. Then*

$$(K_X^d \cdot Z) \geq \frac{d!\,(n+1)^d}{(4\pi)^d} \int_{Z_{\mathrm{reg}}} \bar{v}^C_{\tilde{X}|\tilde{Z}_{\mathrm{reg}}}$$

for every d-dimensional irreducible closed subvariety Z of X with $d > 0$.

The Nakai–Moishezon–Kleiman criterion implies that we can regard all intersection numbers with all positive-dimensional irreducible closed subvarieties as quantities measuring the ampleness of the canonical bundle. Hence this result gives a numerical comparison between the ampleness of the canonical bundle and the strongly Carathéodory measure hyperbolicity. Further details about this corollary are given in Section 3.2.

In Section 4.1, we prove the following result on the curvature function of the restricted Carathéodory pseudo-volume form on a subvariety of a complex manifold.

THEOREM 1.4 (Theorem 4.2). *Let X be a complex manifold and Z its smooth complex submanifold. Then the curvature $K_{v^C_{X|Z}}$ of the restricted Carathéodory pseudo-volume form $v^C_{X|Z}$ is bounded above by -1.*

Here the curvature function $K_{v^C_{X|Z}}$ is defined in the same way as in [20]. Note that Theorem 1.1 in [20] coincides with this theorem when $Z = X$.

Theorem 1.4 can be proved by the same procedure used to prove Theorem 1.1 in [20]. Yet we can also prove a pluripotential version of this curvature inequality by a simpler method: the so-called viscosity approach to complex Monge–Ampère equations due to Eyssidieux, Guedj, and Zeriahi [15]. We shall describe an approach for calculating the curvature function of the restricted Carathéodory pseudo-volume form in the sense of pluripotential theory.

We can apply this curvature property to derive a numerical version of Kratz's result [24] on the nefness of the cotangent bundle over a compact complex manifold whose universal cover is strongly Carathéodory measure hyperbolic. For a compact complex manifold X, a positivity condition of the d-times wedge product $\Omega_X^d = \bigwedge^d T^*X$ of the cotangent bundle $\Omega_X^1 = T^*X$ is defined by the corresponding positivity condition of a certain line bundle denoted by $\mathcal{O}_{\mathbb{P}(\bigwedge^d TX)}(1)$.

Here the line bundle $\mathcal{O}_{\mathbb{P}(\bigwedge^d TX)}(1)$ is the dual bundle of the tautological line bundle $\mathcal{O}_{\mathbb{P}(\bigwedge^d TX)}(-1)$ over the projectivized bundle $\mathbb{P}\big(\bigwedge^d TX\big)$ of $\bigwedge^d TX$. Our result on the nef property of Ω_X^d is as follows.

COROLLARY 1.2 (Corollary 4.1). *Let X be an n-dimensional compact complex manifold. Take any Galois cover $\tilde{X} \xrightarrow{p} X$, and denote by $\tilde{Z}$ the pull-back of Z by p for a d-dimensional irreducible closed subvariety Z of X. Also assume that $Z \cap \mathrm{Zero}(v_{\tilde{X}}^C) = \emptyset$. Then*

$$(\mathcal{O}_{\mathbb{P}(\bigwedge^d TX)}(1)^d \cdot I_Z(Z)) \geq \frac{d!\,(d+1)^d}{(4\pi)^d} \mu^C_{\tilde{X}|\tilde{Z}}(Z).$$

Here $I_Z \colon Z \to \mathbb{P}\big(\bigwedge^d TX\big)$ is an embedding induced from the original embedding $Z \to X$ (the precise definition is given in Section 4.2).

ACKNOWLEDGMENTS. The author would like to express his deep gratitude to Professor Shigetoshi Bando and Professor Hajime Tsuji for their kind and valuable advice. He would also like to thank the referee for a careful reading of the manuscript and helpful comments that improved the paper's presentation.

2. Definition and Basic Properties of the Restricted Carathéodory Pseudo-volume Form

Let X be a connected paracompact complex manifold of complex dimension n, $d \in \{1, 2, \ldots, n-1\}$, and let Z be its d-dimensional subvariety throughout the paper unless otherwise noted.

First we define the usual Carathéodory pseudo-volume form of X and the usual Carathéodory measure of X.

DEFINITION 2.1. The *Carathéodory pseudo-volume form* v_X^C on X is defined by

$$v_X^C := \sup\{g^* v_1;\ g \in \mathrm{Hol}(X, \mathbb{B}^n)\},$$

where $v_1 := 2^n/(1-|t|^2)^{n+1} \times \bigwedge_{\alpha=1}^n \sqrt{-1} dt^\alpha \wedge d\bar{t}^\alpha$ is the Poincaré volume form on the n-dimensional complex unit ball $\mathbb{B}^n$ and $\mathrm{Hol}(X, \mathbb{B}^n)$ is the space of all holomorphic mappings from X to $\mathbb{B}^n$. The *Carathéodory measure* μ_X^C on X is defined similarly as follows: for each Borel set $B \subset X$, we set

$$\mu_X^C(B) := \sup\Bigg\{\sum_{i=1}^\infty \mu_1(g_i(B_i));\ g_i \in \mathrm{Hol}(X, \mathbb{B}^n)\ (i \in \mathbb{N}) \text{ and } (B_i)_{i\in\mathbb{N}} : \text{mutually disjoint Borel sets of } X \text{ s.t. } \bigcup_{i=1}^\infty B_i = B\Bigg\},$$

where μ_1 denotes the standard measure on $\mathbb{B}^n$ with v_1 as its density. It can be seen that μ_X^C coincides with the measure on X with v_X^C as its density in this case when X is smooth. Yet the Carathéodory measure μ_X^C clearly makes sense even if X is singular.

By a positivity condition for this Carathéodory measure of X, the definition of the Carathéodory measure hyperbolicity of X is given as follows.

DEFINITION 2.2. Let X be an n-dimensional complex analytic space. Then X is said to be *Carathéodory measure hyperbolic* if the Carathéodory measure μ_X^C has full support—that is, if $\mu_X^C(B) > 0$ for each nonempty open subset $B \subset X$. In addition, assume that X is smooth. If the Carathéodory pseudo-volume form v_X^C is positive everywhere, then X is a *strongly* Carathéodory measure hyperbolic manifold.

Note that the degenerate locus $\mathrm{Zero}(v_X^C)$ defined as $\{x \in X;\ (v_X^C)_x = o\}$ is a closed analytic subset of X (we use o to denote the origin). Hence a complex analytic space X is Carathéodory measure hyperbolic if and only if its Carathéodor measure μ_X^C is not identically zero. We also remark that if X is compact then the Carathéodory measure of X is identically zero. Thus hereafter we substitute for it the Carathéodory measure $v_{\tilde{X}}^C$ of the universal cover $\tilde{X}$ of X. The measure actually descends to X because of its invariance under the deck transformations, and for simplicity we also use $v_{\tilde{X}}^C$ to denote the obtained measure on X.

Since $\mathbb{B}^n$ is homogeneous, the Ascoli–Arzelà theorem tells us that, for any $x \in X$, there exists a $g_x \in \mathrm{Hol}(X, \mathbb{B}^n)$ with $g_x(x) = o$ such that

$$(v_X^C)_x = \sup\{(g^* v_1)_x : g \in \mathrm{Hol}(X, \mathbb{B}^n),\ g(x) = o\} = (g_x^* v_1)_x.$$

This g_x is called a Carathéodory extremal map of X at x. From the existence of such maps one can see easily that the pseudo-volume form is continuous on X.

We give several simple examples of Carathéodory pseudo-volume forms.

EXAMPLE 2.1. (i) On the n-dimensional complex unit ball $\mathbb{B}^n$, we have $v_{\mathbb{B}^n}^C = v_1$. This fact ensures that Carathéodory pseudo-volume forms generalize the Poincaré volume form v_1 on $\mathbb{B}^n$.

(ii) On the n-dimensional unit polydisk Δ^n, we have $v_{\Delta^n}^C = (v_\Delta^C)^n/n^n$ [7]. Moreover, the corresponding Carathéodory extremal map g_o of Δ^n at o is given by $\Delta^n \ni z \mapsto z/\sqrt{n} \in \mathbb{B}^n$.

Next we define "restricted Carathéodory pseudo-volume form" and "restricted Carathéodory measure", which are restricted versions of the respective form and measure along subvarieties. These restricted versions are among the main objects in this paper. As far as the author knows, these notions first appeared in Eisenman's paper [14].

DEFINITION 2.3. The definition of the *restricted Carathéodory pseudo-volume* form $v_{X|Z}^C$ on the regular locus Z_{reg} of Z is given by setting, for any $z \in Z_{\mathrm{reg}}$,

$$(v_{X|Z}^C)_z = \sup\{((\tilde{f}|_Z)^* v_1^{(d)})_z;\ \tilde{f} \in \mathrm{Hol}(X, \mathbb{B}^d)\};$$

here $v_1^{(d)}$ is the Poincaré volume form on $\mathbb{B}^d$. A measure on Z, called the *restricted Carathéodory measure* $\mu_{X|Z}^C$, is defined as follows: for each Borel set $B \subset Z$, we set

$$\mu^C_{X|Z}(B) := \sup\left\{\sum_{i=1}^{\infty}\mu_1^{(d)}(\tilde{f}_i(B_i));\ \tilde{f}_i \in \mathrm{Hol}(X,\mathbb{B}^d)\ (i\in\mathbb{N}) \text{ and } (B_i)_{i\in\mathbb{N}} : \text{mutually disjoint Borel sets of } Z \text{ s.t. } \bigcup_{i=1}^{\infty} B_i = B\right\},$$

where $\mu_1^{(d)}$ denotes the measure on $\mathbb{B}^d$ with $v_1^{(d)}$ as its density.

Observe that, since the restricted Carathéodory pseudo-volume form is computed using holomorphic maps defined on the whole ambient space X, it follows immediately that $v^C_{X|Z} = v^C_{X|Z_{\mathrm{reg}}}$ on Z_{reg}. Combining this with the equality $\mu^C_{X|Z}(Z \setminus Z_{\mathrm{reg}}) = 0$, we can derive the next result straightforwardly from the same proposition in the smooth case.

Proposition 2.1 [14, Prop. 2.23]. *We have*

$$\int_B \mathbf{1}_{Z_{\mathrm{reg}}} v^C_{X|Z} = \mu^C_{X|Z}(B) \quad \textit{for any Borel subset } B \textit{ of } Z,$$

where $\mathbf{1}_{Z_{\mathrm{reg}}}$ *is the characteristic function of* Z_{reg}.

Remark. There is a crucial point on the definition of the restricted Carathéodory pseudo-volume form. It is easy to calculate its curvature function by direct use of the formula in Definition 2.3, and that calculation will be applied (in Section 4.2) to a study of some nef properties of exterior powers of the cotangent bundle over a compact complex manifold. However, this formulation seems unsuitable for our purpose of making a connection between the restricted volume of the canonical bundle and the Carathéodory measure hyperbolicity along subvarieties. Therefore, in order to show Theorem 1.2 (a.k.a. Theorem 3.1), we rewrite the formula in Definition 2.3 as described next.

Proposition 2.2. *On* Z_{reg},

$$v^C_{X|Z} = \sup\left\{(f|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar{\partial}\log v_1\right)^d\right);\ f \in \mathrm{Hol}(X,\mathbb{B}^n)\right\}.$$

Proof. Let p be the standard projection from $\mathbb{B}^n$ to $\mathbb{B}^d$. We identify $\mathbb{B}^d$ with the subset $\{(t^1,\dots,t^d,0,\dots,0);\ (t^1,\dots,t^d)\in\mathbb{B}^d\}$ of $\mathbb{B}^n$ and consider $v_1^{(d)}$ as a positive (d,d)-form on $\mathbb{B}^n$ in the standard way.

For every $\tilde{f} \in \mathrm{Hol}(X,\mathbb{B}^d)$, we have

$$\begin{aligned}(\tilde{f}|_Z)^*&\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar{\partial}\log v_1\right)^d\right)\\ &= (\tilde{f}|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\frac{n+1}{d+1}\sqrt{-1}\partial\bar{\partial}\log v_1^{(d)}\right)^d\right)\\ &= (\tilde{f}|_Z)^*\left(\frac{2^d}{d!\,(d+1)^d}\left(\sqrt{-1}\partial\bar{\partial}\log v_1^{(d)}\right)^d\right) = (\tilde{f}|_Z)^* v_1^{(d)};\end{aligned}$$

thus it follows that

$$v^C_{X|Z} \leq \sup\left\{(f|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar\partial\log v_1\right)^d\right);\ f\in \mathrm{Hol}(X,\mathbb{B}^n)\right\}.$$

Next we will prove the converse inequality at a fixed point $x\in Z_{\mathrm{reg}}$. At first, it should be remarked that the right-hand side at x is equal to

$$\sup\left\{(f|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar\partial\log v_1\right)^d\right)_x\right\},$$

where f runs through all $f\in \mathrm{Hol}(X,\mathbb{B}^n)$ satisfying $f(x)=o$ and $d_xf(T_xZ)\subset T_o\mathbb{B}^d$. This fact can be easily seen by making a composition of f with an automorphism of $\mathbb{B}^n$ that transforms $f(x)$ to the origin o and transforms $d_xf(T_xZ)$ to a subspace of $T_o\mathbb{B}^d$. Then, for such a map f, we can verify by direct calculations that

$$(f|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar\partial\log v_1\right)^d\right)_x = ((p\circ f|_Z)^*v_1^{(d)})_x.$$

Therefore, we have the converse inequality at all $x\in Z_{\mathrm{reg}}$:

$$(v^C_{X|Z})_x \geq \sup\left\{(f|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar\partial\log v_1\right)^d\right)_x;\ f\in \mathrm{Hol}(X,\mathbb{B}^n)\right\}. \qquad \square$$

We present a simple example of a restricted Carathéodory pseudo-volume form.

EXAMPLE 2.2. If $X=\Delta^2$ and $Z=\Delta\times o$, then Schwarz's lemma yields $v^C_{X|Z} = \sup\{(\tilde f|_\Delta)^*v_1^{(1)};\ \tilde f\in \mathrm{Hol}(\Delta^2,\Delta)\} = v_1^{(1)}$.

Next we introduce two basic properties of restricted Carathéodory pseudo-volume forms; both correspond to known properties of the usual Carathéodory pseudo-volume forms. The first property is that, for every $x\in Z_{\mathrm{reg}}$, we can take a map $f_x\in \mathrm{Hol}(X,\mathbb{B}^n)$ with $f_x(x)=o$ that attains the supremum at x in the right-hand side of the formula in Proposition 2.2:

$$\begin{aligned}(v^C_{X|Z})_x &= \sup_{f(x)=o}\left\{(f|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar\partial\log v_1\right)^d\right)_x\right\}\\ &= (f_x|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar\partial\log v_1\right)^d\right)_x.\end{aligned}$$

As a consequence, we can immediately observe that the pseudo-volume form $v^C_{X|Z}$ is also continuous on Z_{reg}.

The second basic property, formalized in our next proposition, is a distinguishing characteristic of restricted Carathéodory measures corresponding to the volume decreasing property of usual Carathéodory measures.

PROPOSITION 2.3. *Let Y be an n-dimensional complex manifold and W its d-dimensional complex subvariety. Then, for all $\varphi\in \mathrm{Hol}(X,Y)$ such that $\varphi(Z)\subset W$,*

$$(\varphi|_Z)^*\mu^C_{Y|W} \leq \mu^C_{X|Z}.$$

In particular, $\mu^C_{X|Z}$ is invariant under holomorphic automorphisms of X preserving Z.

For compact X, the restricted Carathéodory measure $\mu^C_{X|Z}$ and the restricted Carathéodory pseudo-volume form $v^C_{X|Z}$ vanish everywhere. In contrast, for a Galois cover $p\colon \tilde{X} \to X$, the restricted Carathéodory measure $\mu^C_{\tilde{X}|\tilde{Z}}$ on $\tilde{Z} := p^{-1}(Z)$ and the restricted Carathéodory pseudo-volume form $v^C_{\tilde{X}|\tilde{Z}}$ on $\tilde{Z}_{\mathrm{reg}}$ can descend to Z and Z_{reg} (respectively) because $\mu^C_{\tilde{X}|\tilde{Z}}$ and $v^C_{\tilde{X}|\tilde{Z}}$ are both invariant under the deck transformations from Proposition 2.3. Namely, there exist a measure on Z and a pseudo-volume form on Z_{reg} whose pull-backs by p are, respectively, the restricted Carathéodory measure $\mu^C_{\tilde{X}|\tilde{Z}}$ on $\tilde{Z}$ and the restricted Carathéodory pseudo-volume form $v^C_{\tilde{X}|\tilde{Z}}$ on $\tilde{Z}_{\mathrm{reg}}$. For simplicity, the obtained measure on Z (resp. the obtained pseudo-volume form on Z_{reg}) is also denoted by $\mu^C_{\tilde{X}|\tilde{Z}}$ (resp. $v^C_{\tilde{X}|\tilde{Z}}$) and will be regarded as a substitute for $\mu^C_{X|Z}$ (resp. $v^C_{X|Z}$) from now on.

3. Positivity of the Canonical Bundle along Subvarieties

3.1. Restricted Carathéodory Measure and Restricted Volume of the Canonical Bundle

In this section we prove Theorem 1.2, which is one restricted version of Theorem 1.1. As stated in the Introduction, our main purpose is to establish a restricted version of the inequality in Theorem 1.1 for two restricted objects: restricted Carathéodory measure and the so-called restricted volume of the canonical bundle. Next we define precisely the restricted volume of a line bundle.

DEFINITION 3.1. Let L be a line bundle over a compact complex manifold X, and let Z be a d-dimensional irreducible closed complex subvariety of X. Denote by $i_Z\colon Z \hookrightarrow X$ the inclusion map. We consider the spaces

$$\mathrm{H}^0(X|Z, \mathcal{O}(mL)) := \mathrm{Im}[i_Z^*\colon \mathrm{H}^0(X, \mathcal{O}(mL)) \to \mathrm{H}^0(Z, \mathcal{O}(mL|_Z))]$$

for all $m \in \mathbb{N}$. Such spaces consist of all global sections of $L^{\otimes m}$ over Z that can be extended to X. Then the restricted volume $\mathrm{vol}_{X|Z}(L)$ of L along Z is defined as

$$\mathrm{vol}_{X|Z}(L) := \limsup_{m\to\infty} \frac{\dim \mathrm{H}^0(X|Z, \mathcal{O}(mL))}{m^d/d!},$$

which measures the asymptotic growth of $\dim \mathrm{H}^0(X|Z, \mathcal{O}(mL))$ as $m \to \infty$.

Our conjecture is that, over a compact complex manifold X, the restricted volume $\mathrm{vol}_{X|Z}(K_X)$ of the canonical bundle K_X can be estimated explicitly from below by the restricted Carathéodory measure for any generic subvariety Z.

CONJECTURE. *Let X be an n-dimensional compact complex manifold and Z a d-dimensional irreducible closed subvariety of X. Take a Galois cover $\tilde{X} \xrightarrow{p} X$, and denote by $\tilde{Z}$ the pull-back of Z by p. Suppose that $\tilde{Z}$ is not contained in* $\mathrm{Zero}(v^C_{\tilde{X}})$. *Then there exists a positive constant $C_{n,d}$, depending only on the dimensions d and n, such that*

$$\mathrm{vol}_{X|Z}(K_X) \geq C_{n,d}\,\mu^C_{\tilde{X}|\tilde{Z}}(Z).$$

Moreover, the constant $C_{n,d}$ can be chosen explicitly so that $C_{n,n}$ is exactly the constant $n!\,(n+1)^n/(4\pi)^n$ in Theorem 1.1.

Thanks to the condition on the constants, this conjecture will give a generalization of Theorem 1.1 if it is true. Furthermore, we can construct a counterexample to the inequality when the condition $\tilde{Z} \not\subset \mathrm{Zero}(v^C_{\tilde{X}})$ is removed. That will be explained at the end of this section.

Unfortunately, so far the author can show this conjecture only for some special cases. Yet we can prove a weak version of this conjecture by substituting another restricted version of Carathéodory measure for the usual one. Thus, for general noncompact X and general (possibly singular) Z, let $\bar{v}^C_{X|Z}$ denote the substitute pseudo-volume form on Z_{reg}. It is defined with the use of Carathéodory extremal maps g_x (in the same notation as in Section 2) as follows: for $x \in Z_{\mathrm{reg}}$, set

$$(\bar{v}^C_{X|Z})_x := \sup\left\{(g_x|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar{\partial}\log v_1\right)^d\right)_x\right\}$$

(this supremum is actually a maximum by the Ascoli–Arzelà theorem). Here we have taken into account another description of restricted Carathéodory pseudo-volume form in Proposition 2.2. Then it obviously holds that $\bar{v}^C_{X|Z} \leq v^C_{X|Z}$.

From the invariance of Carathéodory pseudo-volume forms it easily follows that $\bar{v}^C_{X|Z}$ is invariant under holomorphic automorphisms of X preserving Z. Therefore, just as in the case of (restricted) Carathéodory pseudo-volume forms, for any Galois cover $p\colon \tilde{X} \to X$ and the pull-back $\tilde{Z} := p^{-1}(Z)$, the invariant pseudo-volume form $\bar{v}^C_{\tilde{X}|\tilde{Z}}$ on $\tilde{Z}_{\mathrm{reg}}$ can be regarded as a pseudo-volume form on Z_{reg} (and we denote the resultant pseudo-volume form on Z_{reg} by the same symbol, $\bar{v}^C_{\tilde{X}|\tilde{Z}}$). However, the author has not yet determined whether $\bar{v}^C_{X|Z}$ is continuous on $Z_{\mathrm{reg}} \setminus \mathrm{Zero}(v^C_X)$—although $\bar{v}^C_{X|Z}$ is upper semicontinuous by the continuity of Carathéodory pseudo-volume forms. For example, if we know that the Carathéodory extremal map is unique at every $x \in Z_{\mathrm{reg}} \setminus \mathrm{Zero}(v^C_X)$ up to unitary linear transformations of $\mathbb{B}^n$, then $\bar{v}^C_{X|Z}$ will turn out to be continuous on $Z_{\mathrm{reg}} \setminus \mathrm{Zero}(v^C_X)$. But as far as the author knows, little is known concerning uniqueness in such a general setting. For example, Kubota [25] shows that there is uniqueness on X when X is a bounded symmetric domain. Furthermore, the Riemann mapping theorem tells us that the uniqueness on X holds also when X is a simply connected planar domain.

Our result—namely, a weak version of the conjecture—is stated as follows.

THEOREM 3.1. *Let X be an n-dimensional compact complex manifold, and let Z be its d-dimensional irreducible closed subvariety. Take any Galois covering space $\tilde{X} \xrightarrow{p} X$ and denote by $\tilde{Z}$ the pull-back of Z by p. Suppose that $\tilde{Z} \not\subset \mathrm{Zero}(v^C_{\tilde{X}})$. Then*

$$\mathrm{vol}_{X|Z}(K_X) \geq \frac{d!\,(n+1)^d}{(4\pi)^d}\int_{Z_{\mathrm{reg}}} \bar{v}^C_{\tilde{X}|\tilde{Z}}.$$

This theorem is also a generalization of Theorem 1.1, although it is naturally weaker than the conjecture.

We remark that the constant $C_{n,d}$ of the conjecture cannot be the same as the one $d!\,(n+1)^d/(4\pi)^d$ in Theorem 3.1. In fact, we need only consider the case when $X = C \times C$ and Z is its first component $C \times [o]$, where $C = \Delta/\Gamma$ for some discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ and $[o]$ is the point in C represented by the origin $o \in \Delta$. Then it is obvious that $\mathrm{vol}_{X|Z}(K_X) = \deg_C(K_C) = -\chi(C)$. At the same time, Example 2.2 yields that

$$\frac{d!\,(n+1)^d}{(4\pi)^d}\mu^C_{\tilde{X}|\tilde{Z}}(Z) = \frac{3}{4\pi}\int_{C\times[o]} v^C_{\Delta^2|\Delta\times\Gamma o} = \frac{3}{2}\int_C \frac{v_1^{(1)}}{2\pi} = -\frac{3}{2}\chi(C),$$

which is indeed larger than $\mathrm{vol}_{X|Z}(K_X) = -\chi(C)$. Hence this example ensures that the conjecture with the constant $C_{n,d} = d!\,(n+1)^d/(4\pi)^d$ does not hold. As regards Theorem 3.1 in this case, since at $o \in \Delta$ we have

$$\begin{aligned}\frac{2^1}{1!\,(2+1)^1}(g_{o\times o}|_{\Delta\times o})^*\sqrt{-1}\partial\bar{\partial}\log v_1^{(2)} &= \frac{2}{3}\sqrt{-1}\partial\bar{\partial}\log\left(1-\left|\frac{z_1}{\sqrt{2}}\right|^2\right)^{-3}\\ &= \frac{2}{3}\times 3\cdot\frac{1}{2}dz^1\wedge d\bar{z}^1 = \frac{1}{2}v_1^{(1)}\end{aligned}$$

(by Example 2.1), it follows that

$$\frac{d!\,(n+1)^d}{(4\pi)^d}\int_Z \bar{v}^C_{\tilde{X}|\tilde{Z}} = \frac{3}{4\pi}\int_C \frac{1}{2}v_1^{(1)} = -\frac{3}{4}\chi(C).$$

Therefore Theorem 3.1 is surely true in this case.

Before proceeding with the proof of Theorem 3.1, we first explain that we can reduce to the case when X is projective.

LEMMA 3.1. *It is sufficient to prove Theorem 3.1 when X is a projective manifold.*

Proof. First note that, since $\tilde{Z} \not\subset \mathrm{Zero}(v^C_{\tilde{X}})$ is assumed, we obviously have $\mathrm{Zero}(v^C_{\tilde{X}}) \neq \tilde{X}$; that is, $\tilde{X}$ is Carathéodory measure hyperbolic. Therefore, Theorem 1.1 leads to the bigness of K_X and so $\Phi_{|mK_X|}\colon X \dashrightarrow \Phi_{|mK_X|}(X)$, the canonical meromorphic map associated with the complete linear system $|mK_X|$, is bimeromorphic for sufficiently large $m \in \mathbb{N}$.

Let U be any open subset of X satisfying $\bar{U} \cap \mathrm{Zero}(v^C_{\tilde{X}}) = \emptyset$ and $U \cap Z \neq \emptyset$. Then, thanks to standard L^2 estimates for the $\bar{\partial}$ operator [11], $\Phi_{|mK_X|}$ satisfies the following property for sufficiently large m.

CLAIM. *We can take m sufficiently large that $\Phi_{|mK_X|}$ is biholomorphic over U.*

Proof. Consider the $(m-1)$-times product $(v^C_{\tilde{X}})^{-(m-1)}$ of the inverse of the Carathéodory pseudo-volume form of $\tilde{X}$ as a singular Hermitian metric on $K_X^{\otimes(m-1)}$. Note that the metric $(v^C_{\tilde{X}})^{-(m-1)}$ is finite and continuous over $X \setminus \mathrm{Zero}(v^C_{\tilde{X}})$. The metric's most useful property in this proof is that its curvature current,

$$\Theta_{(v^C_{\tilde{X}})^{-(m-1)}} = -\sqrt{-1}\partial\bar{\partial}(v^C_{\tilde{X}})^{-(m-1)} = (m-1)\sqrt{-1}\partial\bar{\partial}\log v^C_{\tilde{X}},$$

is semi-positive on X and strictly positive outside of $\mathrm{Zero}(v^C_{\tilde{X}})$ (see [20, Cor. 1.2] or Theorem 3.5 to follow). It should be also mentioned that $X \setminus (\mathrm{Zero}(v^C_{\tilde{X}}) \cup S)$ has a complete Kähler metric for some analytic subset S of X with $\mathrm{codim}_X S \geq 2$ (see [9] or [20, (4.3) and Lemma 4.3]). We can therefore apply the standard argument by the L^2 estimates for the $\bar{\partial}$ operator [11] over $X \setminus (\mathrm{Zero}(v^C_{\tilde{X}}) \cup S)$ with $K_X^{\otimes(m-1)}$-valued $(n,0)$-forms and the continuous Hermitian metric $(v^C_{\tilde{X}})^{-(m-1)}$ on $K_X^{\otimes(m-1)}$. It then follows that, for sufficiently large m, global sections of $K_X^{\otimes m} = K_X^{\otimes(m-1)} \otimes K_X$ separate points in U and give local coordinates at any points in U. Hence $\Phi_{|mK_X|}$ is biholomorphic over U. □

Because K_X is big, we can use Moishezon's theorem to obtain a proper modification $\sigma : \hat{X} \to X$ obtained by a finite number of blow-ups with smooth centers such that $\hat{X}$ is a projective manifold. Moreover, according to the proof of Moishezon's theorem due to Hironaka ([17]; see also [1; 28]), we can take such a proper modification $\sigma : \hat{X} \to X$ so that the centers of the blow-ups are disjoint from U. Then the bimeromorphic map σ^{-1} can be restricted to Z, so we let $i_{\hat{Z}} : \hat{Z} \hookrightarrow \hat{X}$ denote the irreducible subvariety of $\hat{X}$ obtained as an image of Z by σ^{-1}.

CLAIM. $\mathrm{vol}_{X|Z}(K_X) = \mathrm{vol}_{\hat{X}|\hat{Z}}(K_{\hat{X}})$.

Proof. Since $\sigma(\hat{Z}) = Z$, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0(X, \mathcal{O}_X(mK_X)) \arrow[r, "\sigma^*"] \arrow[d, "i_Z^*"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{H}^0(\hat{X}, \mathcal{O}_{\hat{X}}(mK_{\hat{X}})) \arrow[d, "i_{\hat{Z}}^*"] \\
\mathrm{H}^0(Z, \mathcal{O}_Z(mK_X|_Z)) \arrow[r, "\sigma^*"] & \mathrm{H}^0(\hat{Z}, \mathcal{O}_{\hat{Z}}(mK_{\hat{X}}|_{\hat{Z}})).
\end{tikzcd}$$

Here the homomorphisms in the rows are the pull-backs as forms, and the homomorphisms in the columns are the restriction maps. The homomorphism in the upper row is an isomorphism because the target manifold and the domain manifold are smooth, and the homomorphism in the lower row is obviously injective. We can therefore conclude that the equality $\mathrm{vol}_{X|Z}(K_X) = \mathrm{vol}_{\hat{X}|\hat{Z}}(K_{\hat{X}})$ holds. □

On the other hand, consider the right-hand side of the inequality in Theorem 3.1. An argument in [20, Sec. 4] (especially near the commutative diagram (4) there) gives a Galois cover $\hat{p} : \tilde{\hat{X}} \to \hat{X}$ of $\hat{X}$ and a lift $\tilde{\sigma} : \tilde{\hat{X}} \to \tilde{X}$ of σ such that $\tilde{\sigma}^* v^C_{\tilde{X}} = v^C_{\tilde{\hat{X}}}$. Therefore, since the Carathéodory extremal maps satisfy the relation $g_{\hat{x}} = g_{\tilde{\sigma}(\hat{x})} \circ \tilde{\sigma}$ for all $\hat{x} \in \tilde{\hat{X}}$, the following claim holds.

CLAIM.

$$\int_{Z_{\mathrm{reg}}} \bar{v}^C_{\tilde{X}|\tilde{Z}} = \int_{\hat{Z}_{\mathrm{reg}}} \bar{v}^C_{\tilde{\hat{X}}|\tilde{\hat{Z}}},$$

where $\tilde{\hat{Z}} = \hat{p}^{-1}(\hat{Z})$.

Thus we have proved Lemma 3.1, so it is sufficient to show Theorem 3.1 for $\hat{X}$ and $\hat{Z}$ (and also $\tilde{\hat{X}}$ and $\tilde{\hat{Z}}$), where $\hat{X}$ is projective. $\square$

We prove Theorem 3.1 in the projective case by the same procedure as in the proof of Theorem 1.1. To carry out the procedure, the most important point is that $\Theta_{(v^C_{\tilde{X}})^{-1}} = \sqrt{-1}\partial\bar{\partial}\log v^C_{\tilde{X}}$ is semi-positive on X (as mentioned in the proof of Lemma 3.1) and so its coefficients are Radon measures.

The procedure requires that we first prove a curvature inequality corresponding to the curvature inequality (see Theorem 4.1 below) used in the proof of Theorem 1.1. After that, the obtained curvature inequality is applied to the result (Theorem 3.4 to follow), due to Hisamoto [18] and Matsumura [29], which is a restricted version of the Boucksom–Popovici formula [4; 30] on the volume of a line bundle. Their result will be stated later, but in order to use it we must express the curvature inequality in the terminology of pluripotential theory (and in absolutely continuous parts) as follows.

Theorem 3.2. *Let X be an n-dimensional complex manifold, and let Z be a smooth d-dimensional subvariety of X not contained in* $\mathrm{Zero}(v^C_X)$. *Then, on Z,*

$$(\Theta_{(v^C_X)^{-1}}|_Z)^d_{\mathrm{ac}} = \big(\big(\sqrt{-1}\partial\bar{\partial}\log v^C_X\big)|_Z\big)^d_{\mathrm{ac}} \geq \frac{d!\,(n+1)^d}{2^d}\bar{v}^C_{X|Z}; \tag{1}$$

here $\big(\big(\sqrt{-1}\partial\bar{\partial}\log v^C_X\big)|_Z\big)_{\mathrm{ac}}$ represents the absolutely continuous part of the closed semi-positive current $\big(\sqrt{-1}\partial\bar{\partial}\log v^C_X\big)|_Z$ on Z with respect to the Lebesgue measure. Moreover,

$$\langle(\Theta_{(v^C_X)^{-1}}|_Z)^d\rangle = \big\langle\big(\big(\sqrt{-1}\partial\bar{\partial}\log v^C_X\big)|_Z\big)^d\big\rangle \geq \frac{d!\,(n+1)^d}{2^d}\bar{v}^C_{X|Z} \tag{2}$$

follows if the left-hand side is well-defined, where $\langle\cdot\rangle$ denotes the non-pluripolar Monge–Ampère product.

Before commencing with a proof of this theorem, we briefly recall some necessary facts about the non-pluripolar Monge–Ampère product for closed semi-positive $(1,1)$-currents on X. This notion first appeared in [5] and was systematically developed there by Boucksom, Eyssidieux, Guedj, and Zeriahi. We shall next summarize several results introduced in that paper.

For a pseudo-volume form v on a general n-dimensional complex manifold X, we regard v^{-1} as a singular Hermitian metric on the canonical bundle K_X of X as before. We always suppose that v^{-1} has the semi-positive curvature current $\sqrt{-1}\partial\bar{\partial}\log v$.

The non-pluripolar Monge–Ampère product $\big\langle\big(\sqrt{-1}\partial\bar{\partial}\log v\big)^n\big\rangle$ is uniquely characterized by two properties: being local in the plurifine topology and putting no mass on pluripolar subsets. On general X, however, the product is known not to be well-defined for all such pseudo-volume forms v, where by *well-defined* we mean that the product for v has locally finite mass. Surprisingly, we find that the products for all such pseudo-volume forms v are well-defined over every compact Kähler manifold X. Furthermore, if such a pseudo-volume form v on general X is continuous over X and is positive outside some proper analytic subset S of X, then its non-pluripolar product can be rather concretely described. Hence $\langle(\sqrt{-1}\partial\bar\partial\log v)^n\rangle$, if it is well-defined, is nothing but the zero extension $\mathbf{1}_{X\setminus S}(\sqrt{-1}\partial\bar\partial\log v)^n$ of the product (due to Bedford and Taylor [2]) on $X \setminus S$. Here $\mathbf{1}_{X\setminus S}$ is the characteristic function of $X \setminus S$. For instance, this description is valid also for the Carathéodory pseudo-volume form v_X^C.

We should also mention that, even if absolutely continuous products are replaced with non-pluripolar products in the Boucksom–Popovici formula on the volume of a line bundle, the formula is also true over a compact Kähler manifold. This claim may be stated formally as follows.

THEOREM 3.3 [5]. *Let X be an n-dimensional compact Kähler manifold, and let L be a holomorphic line bundle over X. Then*

$$\mathrm{vol}_X(L) = \sup\left\{\int_X \langle T^n\rangle;\ T \in c_1(L) : \textit{semi-positive } (1,1)\textit{-current}\right\}.$$

The following proof of Theorem 3.2 was, in part, described to me by S. Bando.

Proof of Theorem 3.2. In order to show these inequalities, we need minor changes to the proof of the first part of [20, Thm. 1] (see also Theorem 4.1 to follow). We so often use Lemma 3.2 in [20] that we forgo citing it on each occasion.

We begin a proof of (1). First note that it is sufficient to prove that

$$\big(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)\big)^d_{\mathrm{ac}} \ge \frac{(n+1)^d d!}{2^d}\bar v^C_{X|Z}$$

on a small compact subset $K \subset Z \setminus \mathrm{Zero}(v_X^C)$. Hence for all $x \in K$ we may consider $(g_x|_Z)^*\big(2^d/d!\,(n+1)^d \times (\sqrt{-1}\partial\bar\partial\log v_1)^d\big)$ and $g_x^* v_1 \circ i_Z$ as nonnegative functions on K. Also, the Carathéodory extremal map g_x is assumed to attain the supremum of $\bar v^C_{X|Z}$ at $x \in Z$:

$$(\bar v^C_{X|Z})_x = (g_x|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\big(\sqrt{-1}\partial\bar\partial\log v_1\big)^d\right)_x.$$

Then we can see that there exists some $\delta_0 > 0$ such that, for any $x \in K$, this $g_x = (g_x^1,\dots,g_x^n)$ maps an open neighborhood U_x of x biholomorphically onto the set $\{t \in \mathbb{B}^n;\ |t| < \delta_0\}$. In particular, $g_x = (g_x^1,\dots,g_x^n)$ itself gives holomorphic local coordinates on U_x with center x. Hereafter let K be even smaller, so that $K \subset U_x$ for all $x \in K$.

We approximate v_X^C by using the volume form $(1-|t|^2)^N v_1$ on $\mathbb{B}^n$ and its pullback $(1-|g_x|^2)^N g_x^* v_1$ on X for two appropriate positive constants N. We have $(1-|t|^2)^N v_1 = v_1$ at $o \in \mathbb{B}^n$ and $(1-|t|^2)^N v_1 < v_1$ on $\mathbb{B}^n \setminus \{o\}$. Then the approximation is denoted by v_ε^C and is defined as follows. For an arbitrarily small positive constant ε, we take such finite points $(x_i)_{i=1,2,\dots,k(\varepsilon)} \subset K$ with $k = k(\varepsilon) \to \infty$ (as $\varepsilon \searrow 0$) so that $\{x_i\}_{i=1,2,\dots}$ is dense in K, and we define a pseudo-volume form by

$$v_\varepsilon^C := \sup_{i=1,\dots,k(\varepsilon)} \{(1-|g_{x_i}|^2)^N g_{x_i}^* v_1|_{U_{x_i}}\}.$$

Then we have

$$v_\varepsilon^C \circ i_Z \nearrow v_X^C \circ i_Z \ \text{ (as } \varepsilon \searrow 0) \ \text{ on } K,$$

which follows from three properties: $v_\varepsilon^C \circ i_Z = v_X^C \circ i_Z$ at x_i $(i = 1, 2, \dots, k(\varepsilon))$; $v_\varepsilon^C \circ i_Z \le v_X^C \circ i_Z$ on K; and $\{x_i\}_{i=1,2,\dots}$ is dense in K. Moreover, if N and finite points $(x_i)_{i=1,2,\dots,k(\varepsilon)}$ in K are suitably chosen for ε, then we obtain something more about the curvature current of $v_\varepsilon^C \circ i_Z$ as follows.

LEMMA 3.2. *For an arbitrarily small positive constant ε, there exist a constant $N = N(\varepsilon) > 0$ and finite points $(x_i)_{i=1,2,\dots,k(\varepsilon)}$ in K such that the pseudo-volume form v_ε^C defined previously satisfies*

$$\left(\sqrt{-1}\partial\bar\partial \log(v_\varepsilon^C \circ i_Z)\right)_{\text{ac}}^d \ge \frac{d!\,(n+1)^d}{2^d}(1-\varepsilon)^2 \bar v_{X|Z}^C \ \text{ on } K. \tag{3}$$

In the derivation of inequality (1) from this lemma, the following lemma of Boucksom [4] plays a key role.

LEMMA 3.3 [4]. *Let $(T_k)_{k\in\mathbb{N}}$ and T be semi-positive $(1,1)$-currents on X. If $T_k \xrightarrow{k\to 0} T$, then $(T_{\text{ac}})^n \ge \liminf_{k\to\infty}(T_k)^n_{\text{ac}}$.*

This lemma is applied with

$$T_k = \sqrt{-1}\partial\bar\partial \log(v_{1/k}^C \circ i_Z), \qquad T = \sqrt{-1}\partial\bar\partial \log(v_X^C \circ i_Z)$$

to obtain, via inequality (3) in Lemma 3.2,

$$\left(\sqrt{-1}\partial\bar\partial \log(v_X^C \circ i_Z)\right)_{\text{ac}}^d \ge \frac{d!\,(n+1)^d}{2^d}\bar v_{X|Z}^C \ \text{ on } K.$$

Therefore, we now have only to show Lemma 3.2.

Proof of Lemma 3.2. Let $(|g_x| < r)$ denote the set $\{z \in X;\ |g_x(z)| < r\}$.

CLAIM. *There exist a positive number $\delta_1 = \delta_1(\varepsilon) \in (0, \delta_0)$ and an $N = N(\delta)$ such that*

$$(g_x|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar\partial \log(1-|t|^2)^N v_1\right)^d\right) > (1-\varepsilon)^2 \bar v_{X|Z}^C \tag{4}$$

on $(|g_x| < \delta_1)$ for all $x \in K$.

Proof. By upper semicontinuity of $\bar v_{X|Z}^C$, we can take a positive constant $\delta_1 = \delta_1(\varepsilon) \in (0, \delta_0)$ such that

$$(g_x|_Z)^*\left(\frac{2^d}{d!\,(n+1)^d}\left(\sqrt{-1}\partial\bar\partial \log v_1\right)^d\right) > (1-\varepsilon)\bar v_{X|Z}^C$$

on $(|g_x| < \delta_1)$ for all $x \in K$. So for all $x \in K$, if $N < \delta_1 < (n+1)\varepsilon$ then

$$
\begin{aligned}
(g_x|_Z)^*&\biggl(\frac{2^d}{d!\,(n+1)^d}f\bigl(\sqrt{-1}\partial\bar\partial\log(1-|t|^2)^N v_1\bigr)^d\biggr)\\
&=\biggl(\frac{n+1-N}{n+1}\biggr)^d (g_x|_Z)^*\biggl(\frac{2^d}{d!\,(n+1)^d}\bigl(\sqrt{-1}\partial\bar\partial\log v_1\bigr)^d\biggr)\\
&>\biggl(1-\frac{\delta_1}{n+1}\biggr)^d(1-\varepsilon)\bar v^C_{X|Z}>(1-\varepsilon)^2\bar v^C_{X|Z}
\end{aligned}
$$

on $|g_x|<\delta_1$. □

Since v^C_X is Lipschitz continuous [23], there exists (after we make δ_1 smaller, if necessary) a positive constant L such that

$$
|g_x^*v_1-v^C_X|<L|g_x-g_x(x)|v^C_X=L|g_x|v^C_X \tag{5}
$$

on $(|g_x|<\delta_1)$ for all $x\in K$.

CLAIM. *There exist positive constants $\delta=\delta(\varepsilon)\in(0,\delta_1)$ and $N=N(\varepsilon)>0$ such that*

$$
(1-|g_x|^2)^N g_x^*v_1\big|_{U_x}\circ i_Z<(1-|g_y|^2)^N g_y^*v_1\big|_{U_y}\circ i_Z
$$

over $(|g_x|\geq\delta_1)\cap(|g_y|<\delta)$ for all $x\in K$.

Proof. We know that if $N=N(\varepsilon)$ satisfies $N\cdot\delta_1\geq 4L$ then

$$
\begin{aligned}
(1-|g_x|^2)^N g_x^*v_1&\leq(1-(\delta_1)^2)^N g_x^*v_1\leq(1-N(\delta_1)^2)g_x^*v_1\\
&\leq(1-4L\delta_1)v^C_X\leq(1-4L\delta)v^C_X
\end{aligned}
$$

on $(|g_x|\geq\delta_1)$ for all $x\in K$. Meanwhile, we can check from (5) that, for sufficiently small $\delta=\delta(\varepsilon)>0$,

$$
\begin{aligned}
(1-|g_x|^2)^N g_x^*v_1&>(1-|g_x|^2)^N(1-L\delta)v^C_X\\
&>(1-\delta^2)^N(1-L\delta)v^C_X>(1-2L\delta)v^C_X
\end{aligned}
$$

on $(|g_x|<\delta)$ for all $x\in K$. Hence the assertion follows. □

Fix ε for the time being. If we choose finite points $(x_i)_{i=1,2,\dots,k(\varepsilon)}$ in K such that $K\subset\bigcup_{i=1}^{k(\varepsilon)}(|g_{x_i}|<\delta)$ then this claim implies that, for each $i\in\{1,2,\dots,k(\varepsilon)\}$, the set in K defined by

$$
\log\bigl(g_{x_i}^*v_1(\delta)\big|_{U_{x_i}}\circ i_Z\bigr)>\max_{j=1,2,\dots,\hat{i},\dots,k(\varepsilon)}\log\bigl(g_{x_j}^*v_1(\delta)\big|_{U_{x_j}}\circ i_Z\bigr)
$$

is contained in the open set $(|g_{x_i}|<\delta_1)\cap K$. Hence for any sequences $(\varepsilon_j(l))_{l\in\mathbb{N}}$ $(j=1,2,\dots,k(\varepsilon))$ of positive numbers such that each $(\varepsilon_j(l))_{l\in\mathbb{N}}$ converges monotonically to 0 as $l\to\infty$, the open set defined by the inequality

$$
\log\bigl(g_{x_i}^*v_1(\delta)\big|_{U_{x_i}}\circ i_Z\bigr)+\varepsilon_i(l)>\max_{j=1,2,\dots,\hat{i},\dots,k(\varepsilon)}\bigl(\log g_{x_j}^*v_1(\delta)\big|_{U_{x_j}}\circ i_Z\bigr)+\varepsilon_j(l)
$$

is also contained in the open set $(|g_{x_i}|<\delta_1)\cap K$ for sufficiently large l. Therefore, on the open set in K defined by

$$\log\bigl(g_{x_i}^* v_1(\delta)\big|_{U_{x_i}} \circ i_Z\bigr) + \varepsilon_i(l) > \max_{j=1,2,\ldots,\hat{i},\ldots,k(\varepsilon)} \bigl(\log\bigl(g_{x_j}^* v_1(\delta)\big|_{U_{x_j}} \circ i_Z\bigr) + \varepsilon_j(l)\bigr),$$

we obtain from inequality (4) that

$$\begin{aligned}
&\Bigl(\sqrt{-1}\partial\bar{\partial} \max_{j=1,2,\ldots,k(\varepsilon)} \bigl(\log\bigl(g_{x_j}^* v_1(\delta)\big|_{U_{x_j}} \circ i_Z\bigr) + \varepsilon_j(l)\bigr)\Bigr)^d \\
&\quad = \bigl(\sqrt{-1}\partial\bar{\partial}\bigl(\log\bigl(g_{x_i}^* v_1(\delta)\big|_{U_{x_i}} \circ i_Z\bigr) + \varepsilon_i(l)\bigr)\bigr)^d \\
&\quad = \bigl(\sqrt{-1}\partial\bar{\partial}\log\bigl(g_{x_i}^* v_1(\delta)\big|_{U_{x_i}} \circ i_Z\bigr)\bigr)^d \\
&\quad \geq \bigl(\bigl(g_{x_i}|_{Z\cap U_{x_i}}\bigr)^* \sqrt{-1}\partial\bar{\partial}\log v_1(\delta)\bigr)^d \geq \frac{d!\,(n+1)^d}{2^d} \times (1-\varepsilon)^2 \bar{v}_{X|Z}^C.
\end{aligned}$$

If we additionally require these sequences $(\varepsilon_j(l))_{l\in\mathbb{N}}$ $(j = 1, 2, \ldots, k(\varepsilon))$ to satisfy the property that, for any $l \in \mathbb{N}$, the set defined by the equation

$$\log\bigl(g_{x_i}^* v_1(\delta)\big|_{U_{x_i}} \circ i_Z\bigr) + \varepsilon_i(l) = \log\bigl(g_{x_j}^* v_1(\delta)\big|_{U_{x_j}} \circ i_Z\bigr) + \varepsilon_j(l)$$

is of Lebesgue measure 0 for each $i, j \in \{1, 2, \ldots, k(\varepsilon)\}$ with $i < j$, then

$$\Bigl(\sqrt{-1}\partial\bar{\partial} \max_{j=1,2,\ldots,k(\varepsilon)} \bigl(\log\bigl(g_{x_j}^* v_1(\delta)\big|_{U_{x_j}} \circ i_Z\bigr) + \varepsilon_j(l)\bigr)\Bigr)^d_{\mathrm{ac}} \geq \frac{d!\,(n+1)^d}{2^d} (1-\varepsilon)^2 \bar{v}_{X|Z}^C$$

almost everywhere on K. Because the locally uniform convergence

$$\max_{j=1,2,\ldots,k(\varepsilon)} \bigl(\log\bigl(g_{x_j}^* v_1(\delta)\big|_{U_{x_j}} \circ i_Z\bigr) + \varepsilon_j(l)\bigr) \xrightarrow{l\to\infty} \log(v_\varepsilon^C \circ i_Z)$$

holds on K (by definition of the approximation v_ε^C), we can apply Lemma 3.3 again with

$$T_l = \sqrt{-1}\partial\bar{\partial} \max_{j=1,2,\ldots,k(\varepsilon)} \bigl(\log\bigl(g_{x_j}^* v_1(\delta)\big|_{U_{x_j}} \circ i_Z\bigr) + \varepsilon_j(l)\bigr)$$

and

$$T = \sqrt{-1}\partial\bar{\partial} \max_{j=1,2,\ldots,k(\varepsilon)} \log\bigl(g_{x_j}^* v_1(\delta)\big|_{U_{x_j}} \circ i_Z\bigr) = \sqrt{-1}\partial\bar{\partial}\log(v_\varepsilon^C \circ i_Z)$$

to obtain the desired inequality:

$$\bigl(\sqrt{-1}\partial\bar{\partial}\log(v_\varepsilon^C \circ i_Z)\bigr)^d_{\mathrm{ac}} \geq \frac{d!\,(n+1)^d}{2^d} (1-\varepsilon)^2 \bar{v}_{X|Z}^C \ \text{ on } K.$$

This concludes the proof of Lemma 3.2. □

Next we prove the pluripotential theoretic inequality (2) in Theorem 3.2. First we note that v_X^C is a continuous pseudo-volume form with the analytic subset $\mathrm{Zero}(v_X^C)$ as a degenerate locus and that, by our hypothesis, $\mathrm{Zero}(v_X^C)$ does not contain Z. So as we explained before the proof of Theorem 3.2, the non-pluripolar Monge–Ampère product of the current $\Theta_{(v_X^C)^{-1}}|_Z = \sqrt{-1}\partial\bar{\partial}\log(v_X^C \circ i_Z)$ on Z, if it is well-defined, coincides with the zero extension $\mathbf{1}_{Z\setminus Z\cap\mathrm{Zero}(v_X^C)}\bigl(\sqrt{-1}\partial\bar{\partial}\log(v_X^C \circ i_Z)\bigr)^d$ of the product on $Z \setminus Z \cap \mathrm{Zero}(v_X^C)$ in the sense of Bedford and Taylor:

$$\bigl\langle\bigl(\sqrt{-1}\partial\bar{\partial}\log(v_X^C \circ i_Z)\bigr)^d\bigr\rangle = \mathbf{1}_{Z\setminus Z\cap\mathrm{Zero}(v_X^C)}\bigl(\sqrt{-1}\partial\bar{\partial}\log(v_X^C \circ i_Z)\bigr)^d.$$

Hence the next claim follows easily.

CLAIM.
$$\left\langle\left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)\right)^d\right\rangle \geq \left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)\right)^d_{\mathrm{ac}}$$
over Z.

NOTE. Given (1), this inequality immediately implies the desired inequality (2).

Proof of Claim. Since we must only prove this inequality locally on $Z \setminus Z \cap \mathrm{Zero}(v_X^C)$, it follows from the preceding note that $\left\langle\left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)\right)^d\right\rangle = \left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)\right)^d$ there. For standard regularization kernels $(\rho_\varepsilon)_{\varepsilon>0}$, locally we have the weak convergence

$$\left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)*\rho_\varepsilon\right)^d \xrightharpoonup{\varepsilon\to0} \left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)\right)^d,$$

and the almost everywhere convergence

$$\left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)*\rho_\varepsilon\right)^d \xrightarrow{\varepsilon\to0} \left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)\right)^d_{\mathrm{ac}}$$

also holds (by Lebesgue's theorem). Therefore, by Fatou's lemma we have

$$\left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)\right)^d \geq \left(\sqrt{-1}\partial\bar\partial\log(v_X^C\circ i_Z)\right)^d_{\mathrm{ac}}. \qquad \square$$

The proof of Theorem 3.2 is now complete. □

This proof could be simplified considerably if we used the method of viscosity solutions for complex Monge–Ampère equations [15]. However, the author cannot yet prove this theorem in that way. Instead, a proof of the pluripotential theoretic inequality (7) of Theorem 4.2 by means of viscosity subsolutions will be given in the last part of Section 4.1. We can also prove (7) using an argument similar to that in the proof of Theorem 3.2.

We proceed with the proof of Theorem 3.1 in the case when X is projective. A crucial tool for our proof is another expression formula of restricted volumes for a line bundle L, which is obtained by Hisamoto [18] and Matsumura [29]. Their formula is described in an analytic form with non-pluripolar products. In order to state their result, we prepare several concepts. The algebraic subset of X defined as

$$\mathbb{B}_+(L) = \left\{ x\in X;\ \begin{array}{r} \text{there is no singular Hermitian metric } h \text{ on } L \\ \text{such that} \quad h : \text{smooth around } x \\ \Theta_h : \text{strictly positive on } X \end{array}\right\}$$

and is called the augmented base locus or the nonample locus of L. Furthermore, a semi-positive $(1,1)$-current $T\in c_1(L)$ is said to have a small unbounded locus if the unbounded locus of a potential ϕ of T is contained in a certain complete pluripolar closed proper subset of X. Here the unbounded locus of a quasi-plurisubharmonic function ϕ is the set consisting of all points $x\in X$ such that ϕ is unbounded in every neighborhood of x.

The Hisamoto–Matsumura formula is stated by using these concepts as follows.

THEOREM 3.4 [18; 29]. *Let X be an n-dimensional projective manifold, L a big line bundle over X, and Z a d-dimensional irreducible closed complex subvariety of X. Furthermore, suppose that $Z \not\subset \mathbb{B}_+(L)$. Then*

$$\mathrm{vol}_{X|Z}(L) = \sup_T \int_{Z_{\mathrm{reg}}} \langle (T|_{Z_{\mathrm{reg}}})^d \rangle,$$

where T runs through all semi-positive $(1,1)$-currents $T \in c_1(L)$ that have small unbounded loci and whose unbounded loci do not contain Z.

To apply this formula with $L = K_X, T = 1/2\pi \times \Theta_{(v^C_{\tilde{X}})^{-1}} = 1/2\pi\sqrt{-1}\partial\bar{\partial}\log v^C_{\tilde{X}}$, and our Z, we must verify that the assumptions of Theorem 3.4 for the currents and the subvariety hold in our situation. In the situation of Theorem 3.1, the unbounded locus of the curvature current $\sqrt{-1}\partial\bar{\partial}\log v^C_{\tilde{X}}$ is $\mathrm{Zero}(v^C_{\tilde{X}})$, which is certainly an analytic subset—especially a complete pluripolar closed proper subset—of X. In particular, the unbounded locus $\mathrm{Zero}(v^C_{\tilde{X}})$ does not contain Z by the hypothesis of Theorem 3.1. Moreover, by our hypothesis and the next lemma, we can check that Z is not contained in the augmented base locus $\mathbb{B}_+(K_X)$ of the canonical bundle K_X, since Z is not contained in $\mathrm{Zero}(v^C_{\tilde{X}})$.

LEMMA 3.4.

$$\mathbb{B}_+(K_X) \subset \mathrm{Zero}(v^C_{\tilde{X}}).$$

This lemma also allows us to compare the Carathéodory measure hyperbolicity with positivity of the canonical bundle in terms of relation of inclusion.

Proof of Lemma 3.4. Take any point $x \in X \setminus \mathrm{Zero}(v^C_{\tilde{X}})$ and let $z = (z^1, z^2, \dots, z^n)$ be local holomorphic coordinates around x with center x. Since X is projective, there exist a Kähler form ω and a sufficiently ample smooth divisor H on X not containing x. In addition, let χ_x (resp. ρ) be a cutoff function that is identically equal to 1 in a small neighborhood of x (resp. a small neighborhood of $\mathrm{Zero}(v^C_{\tilde{X}})$) and such that the support of χ_x does not touch H and $\mathrm{Zero}(v^C_{\tilde{X}})$.

We consider the singular Hermitian metric

$$h = (v^C_{\tilde{X}})^{-m} \times \exp(-(1-\rho)\log h_H(e,e) - 2n\chi_x \log|z-x|)$$

on $K_X^{\otimes m}$ over a Stein manifold $X \setminus H$, where e is the canonical section of the line bundle $[H]$ associated with H and h_H is any positively curved smooth Hermitian metric on $[H]$ over X. If m is sufficiently large, then the curvature current $\Theta_h = -\sqrt{-1}\partial\bar{\partial}\log h$ satisfies the following two positivity properties:

$$\Theta_h = m\sqrt{-1}\partial\bar{\partial}\log v^C_{\tilde{X}} + \sqrt{-1}\partial\bar{\partial}((1-\rho)\log h_H(e,e) + n\chi_x \log|z-x|^2)$$

is semi-positive everywhere on $X \setminus H$ and is strictly positive on a neighborhood of x. These facts stem essentially from the strict positivity of the current $\sqrt{-1}\partial\bar{\partial}\log v^C_{\tilde{X}}$

on $X \setminus \mathrm{Zero}(v^C_{\tilde{X}})$, which is obtained in [20, Thm. 1] (see also Theorem 3.5 to follow). We can therefore apply the L^2 estimates for the $\bar{\partial}$ operator in the present situation. As a result, we obtain a smooth section α of $K_X^{\otimes(m+1)}$ (i.e., a $K_X^{\otimes m}$-valued $(n,0)$-form) over $X \setminus H$ that is a solution of

$$\bar{\partial}\alpha = \bar{\partial}(\chi_x(dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n)^{\otimes(m+1)})$$

with the estimate

$$\int_{X\setminus H} |\alpha|^2_{h,\omega}\frac{\omega^n}{n!} \le C\int_{X\setminus H} |\bar{\partial}(\chi_x(dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n)^{\otimes(m+1)})|^2_{h,\omega}\frac{\omega^n}{n!} < \infty$$

for some constant $C > 0$ depending only on a lower bound of Θ_h around x. Here $|\cdot|_{h,\omega}$ denotes the standard norm with respect to the metrics h and ω. We thus find a holomorphic section

$$\sigma := \chi_x(dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n)^{\otimes(m+1)} - \alpha$$

of $(m+1)K_X$ on $X \setminus H$. By the L^2 condition for α, this σ can be extended holomorphically to the whole X. Hence α can eventually be extended to the whole X. Since the L^2 integral of α with $\exp(-(1-\rho)\log h_H(e,e) - 2n\chi_x \log|z-x|)$ as a weight is finite, α must vanish at x and along $\rho^{-1}(0) \cap H$. Therefore, we find a holomorphic section

$$\sigma := \chi_x(dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n)^{\otimes(m+1)} - \alpha \in \mathrm{H}^0(X, \mathcal{O}((m+1)K_X))$$

that is nonvanishing at x and vanishing along $\rho^{-1}(0) \cap H$. Furthermore, since $\rho^{-1}(0) \cap H$ contains a nonempty open set in H, it follows that σ vanishes on the whole of H and so brings a section $\sigma \in \mathrm{H}^0(X, \mathcal{O}((m+1)K_X - H))$ nonvanishing at x. Hence we let $\alpha_0, \alpha_1, \ldots, \alpha_N$ be a basis of $\mathrm{H}^0(X, \mathcal{O}(H))$. Then

$$\mathcal{H} = \left(\sum_{i=0}^{N} |\sigma \otimes \alpha_i|^2\right)^{-1/(m+1)}$$

defines a singular Hermitian metric on K_X with the globally strictly positive curvature current, and H is smooth at x. Thus we have the required condition $x \notin \mathbb{B}_+(K_X)$. □

Thanks to Lemma 3.4, $1/2\pi \times \Theta_{(v^C_{\tilde{X}})^{-1}}$ and Z satisfy (respectively) the hypotheses on T and Z in Theorem 3.4. Finally, by (2) in Theorem 3.2 we conclude that

$$\mathrm{vol}_{X|Z}(K_X) \ge \int_{Z_{\mathrm{reg}}} \left\langle \left(\frac{1}{2\pi}\Theta_{(v^C_{\tilde{X}})^{-1}}|_{Z_{\mathrm{reg}}}\right)^d \right\rangle \ge \frac{d!\,(n+1)^d}{(4\pi)^d}\int_{Z_{\mathrm{reg}}} \bar{v}^C_{\tilde{X}|\tilde{Z}_{\mathrm{reg}}}.$$

The proof of Theorem 3.1 is now complete.

We finish this section by giving a counterexample to the inequality in the Conjecture (and in Theorem 3.1) under the condition $\tilde{Z} \subset \mathrm{Zero}(v^C_{\tilde{X}})$.

Let C be a smooth projective curve whose genus is greater than 1 and let $\varphi\colon X \to C^3$ be the blow-up of the smooth 3-fold $C^3 = C \times C \times C$ along $C = C \times [o] \times [o]$. Because the unit disk Δ is here the universal cover of C, we can write $C = \Delta/\Gamma$

for some discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ and denote by $[o]$ the point in C represented by the origin $o \in \Delta$. We have already seen that the blow-up $\psi: \tilde{X} \to \Delta^3$ of Δ^3 along $\Delta \times \Gamma o \times \Gamma o$ induces the covering space $\pi: \tilde{X} \to X$ over X satisfying the commutative diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, "\psi"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \Delta^3 \arrow[d] \\ X \arrow[r, "\varphi"] & C^3. \end{tikzcd}$$

The exceptional divisor of φ is given by $E = \varphi^{-1}(C) = C \times \mathbb{P}^1 = \mathrm{Zero}(v_{\tilde{X}}^C)$. We define a 1-dimensional irreducible closed subvariety $Z \subset X$ in E as $Z = C \times (1:0) \subset C \times \mathbb{P}^1$ and set $\tilde{Z} = \pi^{-1}(Z) = \Delta \times \Gamma o \times \Gamma o \times (1:0) \subset \tilde{X}$.

We can show that the volume comparison in Theorem 3.1 and/or the Conjecture surely fails for this X and this Z. Indeed, we first identify $\Delta \times y \times z \times (1:0)$ with Δ for any $(y,z) \in \Gamma o \times \Gamma o$. By Riemann's extension theorem,

$$\begin{aligned} v_{\tilde{X}|\tilde{Z}}^C &= \sup\{(f \circ \psi|_{\tilde{Z}})^* v_1;\ f \in \mathrm{Hol}(\Delta^3, \Delta)\} \\ &= \sup\{\tilde{\psi}^*(f|_{\Delta\times\Gamma o\times\Gamma o})^* v_1;\ f \in \mathrm{Hol}(\Delta^3, \Delta)\} \end{aligned}$$

for $\tilde{\psi} := \psi|_{\tilde{Z}}: \tilde{Z} = \Delta \times \Gamma o \times \Gamma o \times (1:0) \to \Delta \times \Gamma o \times \Gamma o$. Since $\tilde{\psi}$ is an isomorphism by definition, it follows that for any $(y,z) \in \Gamma o \times \Gamma o$ we have

$$\begin{aligned} v_{\tilde{X}|\tilde{Z}}^C|_{\Delta\times y\times z\times(1:0)} &= \sup\{(f|_{\Delta\times y\times z})^* v_1;\ f \in \mathrm{Hol}(\Delta^3, \Delta)\} \\ &= \sup\{f^* v_1;\ f \in \mathrm{Hol}(\Delta \times y \times z, \Delta)\} \\ &= v_1. \end{aligned}$$

Note that $v_{\tilde{X}|\tilde{Z}}^C = \bar{v}_{\tilde{X}|\tilde{Z}}^C$ since $\tilde{Z} \subset \mathrm{Zero}(v_{\tilde{X}}^C) = E$. Therefore,

$$\int_Z v_{\tilde{X}|\tilde{Z}}^C = \int_Z \bar{v}_{\tilde{X}|\tilde{Z}}^C = \int_C v_1 > 0.$$

On the other hand, we consider the restricted volume $\mathrm{vol}_{X|Z}(K_X)$ of K_X along Z. Since the stable base locus $\mathbb{B}(K_X) = \bigcap_{m\in\mathbb{N}} \bigcap_{\sigma\in \mathrm{H}^0(\mathcal{O}(mK_X))} \sigma^{-1}(o)$ of K_X is just equal to E, it follows that $Z \subset \mathbb{B}(K_X)$ and so clearly $\mathrm{vol}_{X|Z}(K_X) = 0$.

3.2. Ampleness of the Canonical Bundle via Strong Carathéodory Measure Hyperbolicity

Next we deal mainly with the strong Carathéodory measure hyperbolicity. Our purpose in this section is to give a numerical comparison between the Carathéodory measure hyperbolicity and the ampleness of the canonical bundle over a compact complex manifold when its universal cover is strongly Carathéodory measure hyperbolic. In fact, this is a direct application of Theorem 3.1.

Yet if one merely wants to derive the ampleness of the canonical bundle from the strong Carathéodory measure hyperbolicity, that can be done without much difficulty. In particular, it follows from the strict positivity of the curvature current on the canonical bundle associated with the Carathéodory pseudo-volume form as follows.

THEOREM 3.5 [20, Thm. 1.1]. *The curvature current* $\Theta_{(v_X^C)^{-1}} = \sqrt{-1}\partial\bar{\partial}\log v_X^C$ *of the singular Hermitian metric* $(v_X^C)^{-1}$ *on the canonical bundle is strictly positive on the open set where* v_X^C *is nondegenerate.*

This was also stated (without proof) by Wu [33].

Theorem 3.5 can actually lead to the ampleness of the canonical bundle.

COROLLARY 3.1 [20, Cor. 1.2]. *Let* X *be a compact complex manifold and* $\tilde{X}$ *its universal covering space. If* $\tilde{X}$ *is strongly Carathéodory measure hyperbolic* (*i.e., if* $v_{\tilde{X}}^C$ *is positive everywhere on* $\tilde{X}$), *then* X *is projective algebraic with the ample canonical bundle.*

In the proof of this corollary in [20], Richberg's regularization technique [10] is applied to the continuous strictly plurisubharmonic function $\log v_{\tilde{X}}^C$. The technique enables us to regularize the singular Hermitian metric $(v_{\tilde{X}}^C)^{-1}$ on the canonical bundle K_X of X while keeping the strict positivity of its curvature current. Hence we obtain a smooth Hermitian metric on K_X with the strictly positive curvature form on X. By Kodaira's embedding theorem, X turns out to be a projective algebraic manifold with ample K_X.

However, neither this corollary nor the proof can tell us how much more ample the canonical bundle becomes as the Carathéodory pseudo-volume form becomes larger. Hence our aim is to numericalize this corollary. To do that, we make use of the Nakai–Moishezon–Kleiman numerical criterion for a line bundle being ample.

THEOREM 3.6. *Let* L *be a line bundle over a projective manifold* X. *Then* L *is ample if and only if the intersection number*

$$(L^{\dim Z} \cdot Z) > 0$$

for every positive-dimensional irreducible closed subvariety Z *of* X.

In other words, the intersection numbers with all such subvarieties can be thought of as quantities measuring the ampleness of the line bundle. Hence our next theorem surely means that the ampleness of the canonical bundle is explicitly bounded from below by the Carathéodory measure hyperbolicity.

THEOREM 3.7. *Let* X *be an* n*-dimensional compact complex manifold. Take any Galois cover* $\tilde{X} \xrightarrow{p} X$ *and, for every subvariety* Z *of* X, *denote by* $\tilde{Z}$ *the pull-back of* Z *by* p. *Suppose that* $\tilde{X}$ *is strongly Carathéodory measure hyperbolic. Then we have*

$$(K_X^{\dim Z} \cdot Z) \geq \frac{(\dim Z)!\,(n+1)^{\dim Z}}{(4\pi)^{\dim Z}} \int_{Z_{\mathrm{reg}}} \bar{v}^C_{\tilde{X}|\tilde{Z}_{\mathrm{reg}}}$$

for every positive-dimensional irreducible closed subvariety Z *of* X.

This theorem is obtained as an easy application of Theorem 3.1 and Theorem 1.1 together. Hence the left-hand sides of the inequalities in Theorem 3.1 and Theorem 1.1 become $(K_X^{\dim Z} \cdot Z)$ and K_X^n, respectively; the reasons are that $\mathrm{vol}_{X|Z}(L) = (L^{\dim Z} \cdot Z)$ follows for any nef and big line bundle L [13] and that K_X is indeed ample (by Corollary 3.1).

4. Curvature of Restricted Carathéodory Pseudo-volume Form and Its Application to Positivity of a Cotangent Bundle

4.1. Curvature of Restricted Carathéodory Pseudo-volume Form

In this section we use several methods to investigate the curvature function of a restricted Carathéodory pseudo-volume.

First we recall the curvature function for a pseudo-volume form v whose curvature current $\Theta_{v^{-1}} = \sqrt{-1}\partial\bar{\partial}\log v$ is semi-positive; this function is defined in [20]. Here v^{-1} is regarded as a singular Hermitian metric on the canonical bundle K_X of X.

Definition 4.1 [20]. For a pseudo-volume form v on an n-dimensional complex manifold X such that the curvature current $\Theta_{v^{-1}}$ is semi-positive, we define the curvature function K_v of v by setting

$$K_v := -\frac{2^n}{(n+1)^n n!}\frac{\left(\sqrt{-1}\partial\bar{\partial}\log v\right)^n_{\mathrm{ac}}}{v}.$$

Here the coefficients of $\sqrt{-1}\partial\bar{\partial}\log v$ are Radon measures owing to the semi-positivity, so we denote by $\left(\sqrt{-1}\partial\bar{\partial}\log v\right)_{\mathrm{ac}}$ the absolutely continuous part of $\sqrt{-1}\partial\bar{\partial}\log v$ with respect to the Lebesgue measure.

Under this definition, we calculate the curvature function of the Carathéodory pseudo-volume form v_X^C in [20].

Theorem 4.1 [20, Thm. 1.1]. *If a complex manifold X is Carathéodory measure hyperbolic, then*

$$K_{v_X^C} \le -1.$$

Our purpose here is to establish a restricted version of this theorem in various ways. First we should point out that the curvature current $\Theta_{(v_{X|Z}^C)^{-1}}$ of the restricted Carathéodory pseudo-volume form $v_{X|Z}^C$ is semi-positive by the definition of restricted Carathéodory pseudo-volume form. Therefore, the curvature function makes sense for restricted Carathéodory pseudo-volume forms under the preceding definition.

Theorem 4.2. *Let X be a complex manifold of dimension n and let Z be its smooth complex subvariety of dimension d. Then the curvature function $K_{v_{X|Z}^C}$ of the restricted Carathéodory pseudo-volume form $v_{X|Z}^C$ is bounded above by -1; that is,*

$$\big(\sqrt{-1}\partial\bar{\partial}\log v^C_{X|Z}\big)^d_{\mathrm{ac}} \geq \frac{(d+1)^d d!}{2^d} v^C_{X|Z}. \tag{6}$$

Furthermore, the following pluripotential version of (6) *holds if the non-pluripolar Monge–Ampère product in the left-hand side is well-defined*:

$$\big\langle\big(\sqrt{-1}\partial\bar{\partial}\log v^C_{X|Z}\big)^d\big\rangle \geq \frac{(d+1)^d d!}{2^d} v^C_{X|Z}. \tag{7}$$

The inequality (7) for $d=1$ corresponds to the holomorphic sectional curvature of the Carathéodory metric being bounded above by -1 [8; 32].

We shall begin a proof of Theorem 4.2. In the same way as in the corresponding part of the proof of Theorem 3.2, we can prove that (6) implies (7). Moreover, one of the proofs of (6) is almost the same as the proof of Theorem 4.1 given in [20]. One difference with proving Theorem 4.1 is that here we apply the inequality

$$\sqrt{-1}\partial\bar{\partial}\log((\tilde{f}|_Z)^* v_1^{(d)}) \geq (\tilde{f}|_Z)^*\sqrt{-1}\partial\bar{\partial}\log v_1^{(d)} \tag{8}$$

for every $\tilde{f} \in \mathrm{Hol}(X, \mathbb{B}^d)$ that is nondegenerate on Z (we omit the details). However, we can also prove (6) by using another expression given in Proposition 2.2 of restricted Carathéodory pseudo-volume forms. For that, the following inequality is used in place of (8).

LEMMA 4.1. *For any $f \in \mathrm{Hol}(X, \mathbb{B}^n)$ that is nondegenerate on Z*,

$$\sqrt{-1}\partial\bar{\partial}\log\big((f|_Z)^*\sqrt{-1}\partial\bar{\partial}\log v_1\big)^d \geq \frac{d+1}{n+1}(f|_Z)^*\sqrt{-1}\partial\bar{\partial}\log v_1.$$

Note that this lemma follows more easily if $f|_Z$ is an immersion into $\mathbb{B}^n$. Actually, it amounts to the elementary fact that the holomorphic bisectional curvature of a complex submanifold is not larger than the one of its ambient space. If this were not the case then we could not prove the inequality in Lemma 4.1 directly from the general theory because the left-hand side is not regular.

Proof of Lemma 4.1. We give a proof by direct albeit rather long calculations. For simplicity, we use $f|_Z$ to denote $f = (f^\alpha)_{\alpha=1,2,\ldots,n} = (f^1, f^2, \ldots, f^n)\colon Z \to \mathbb{B}^n$. The result is local, so it is sufficient to prove the inequality on any local holomorphic chart $(U; z^1, \ldots, z^d)$ of Z. We use the Einstein convention and omit $\bigwedge_{\alpha=1}^d \sqrt{-1}dz^\alpha \wedge d\bar{z}^\alpha$ in what follows.

First we calculate the term $\big(f^*\sqrt{-1}\partial\bar{\partial}\log v_1\big)^d$ in the inequality of Lemma 4.1 using the quantity

$$\Delta^\alpha = \det\bigg(\frac{\partial f^{\alpha_i}}{\partial z^j}\bigg)_{i,j=1,2,\ldots,d}$$

for a pair $\alpha = (\alpha_1, \ldots, \alpha_d)$ of d integers with $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_d \leq n$.

CLAIM.

$$\big(f^*\sqrt{-1}\partial\bar{\partial}\log v_1\big)^d = \frac{d!\,(n+1)^d}{(1-|f|^2)^{d+1}}\sum_{\alpha,\beta}\bigg(\delta_{\alpha\bar{\beta}} - \sum_{i,j=1}^{d+1}\sum_{\gamma}(-1)^{i+j} f^{\gamma_i}\overline{f^{\gamma_j}}\bigg)\Delta^\alpha\overline{\Delta^\beta};$$

here $\alpha = (\alpha_1, \dots, \alpha_d)$ *and* $\beta = (\beta_1, \dots, \beta_d)$ *run over all pairs of* d *integers such that* $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_d \le n$ *and* $1 \le \beta_1 < \beta_2 < \cdots < \beta_d \le n$, *and* $\gamma = (\gamma_1, \dots, \gamma_d, \gamma_{d+1})$ *runs over all pairs of* $d+1$ *integers with* $1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_d < \gamma_{d+1} \le n$ *satisfying* $(\gamma_1, \dots, \widehat{\gamma_i}, \dots, \gamma_{d+1}) = \alpha$ *and* $(\gamma_1, \dots, \widehat{\gamma_j}, \dots, \gamma_{d+1}) = \beta$ (*where the wide hat marks terms that are to be omitted*).

Proof. At first, we have

$$\begin{aligned}
&f^*\sqrt{-1}\partial\bar\partial \log v_1 \\
&\quad = \sum_{\alpha,\beta=1,2,\dots,n} \frac{\sqrt{-1}(n+1)}{1-|f|^2}\left(\delta_{\alpha\bar\beta} + \frac{\overline{f^\alpha}\cdot f^\beta}{1-|f|^2}\right)\frac{\partial f^\alpha}{\partial z^i}\cdot\overline{\frac{\partial f^\beta}{\partial z^j}}\, dz^i \wedge d\bar z^j \\
&\quad = \frac{\sqrt{-1}(n+1)}{1-|f|^2}\left(\left\langle \frac{\partial f}{\partial z^i}, \frac{\partial f}{\partial z^j}\right\rangle + \frac{1}{1-|f|^2}\left\langle \frac{\partial f}{\partial z^i}, f\right\rangle\left\langle f, \frac{\partial f}{\partial z^j}\right\rangle\right) dz^i \wedge d\bar z^j.
\end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ denotes the usual Hermitian inner product for complex n-vectors; thus, for example,

$$\left\langle \frac{\partial f}{\partial z^i}, \frac{\partial f}{\partial z^j}\right\rangle = \sum_{\alpha=1}^n \frac{\partial f^\alpha}{\partial z^i}\cdot\overline{\frac{\partial f^\alpha}{\partial z^j}}.$$

If we set $A = (A_{i\bar j}) = \left(\left\langle \frac{\partial f}{\partial z^i}, \frac{\partial f}{\partial z^j}\right\rangle\right)_{i,j}$ and $A^{-1} = (A^{\bar i j})_{i,j}$, then $\left(f^*\sqrt{-1}\partial\bar\partial \log v_1\right)^d$ can be calculated as follows:

$$\begin{aligned}
&\left(f^*\sqrt{-1}\partial\bar\partial \log v_1\right)^d \\
&\quad = \frac{d!\,(n+1)^d}{(1-|f|^2)^d}\det\left(\left\langle \frac{\partial f}{\partial z^i}, \frac{\partial f}{\partial z^j}\right\rangle + \frac{1}{1-|f|^2}\left\langle \frac{\partial f}{\partial z^i}, f\right\rangle\left\langle f, \frac{\partial f}{\partial z^j}\right\rangle\right)_{i,j=1,\dots,d} \\
&\quad = \frac{d!\,(n+1)^d}{(1-|f|^2)^d}\det A\left(1 + \frac{1}{1-|f|^2}A^{\bar j i}\left\langle \frac{\partial f}{\partial z^i}, f\right\rangle\left\langle f, \frac{\partial f}{\partial z^j}\right\rangle\right) \\
&\quad = \frac{d!\,(n+1)^d}{(1-|f|^2)^{d+1}}\left(\det A\cdot(1-|f|^2) + \det A\cdot A^{\bar j i}\left\langle \frac{\partial f}{\partial z^i}, f\right\rangle\left\langle f, \frac{\partial f}{\partial z^j}\right\rangle\right).
\end{aligned}$$

Here the second term in the last line becomes

$$\begin{aligned}
&\det A\cdot A^{\bar j i}\left\langle \frac{\partial f}{\partial z^i}, f\right\rangle\left\langle f, \frac{\partial f}{\partial z^j}\right\rangle \\
&\quad = (-1)^{i+j}\det\left(\left\langle \frac{\partial f}{\partial z^k}, \frac{\partial f}{\partial z^l}\right\rangle\right)_{k\ne i, l\ne j}\left\langle \frac{\partial f}{\partial z^i}, f\right\rangle\left\langle f, \frac{\partial f}{\partial z^j}\right\rangle \\
&\quad = \det\left(\left\langle \frac{\partial f}{\partial z^i}, \frac{\partial f}{\partial z^j}\right\rangle\right)|f|^2 - \det\begin{pmatrix} \left\langle \frac{\partial f}{\partial z^i}, \frac{\partial f}{\partial z^j}\right\rangle & \left\langle \frac{\partial f}{\partial z^i}, f\right\rangle \\ \left\langle f, \frac{\partial f}{\partial z^j}\right\rangle & \langle f, f\rangle \end{pmatrix}.
\end{aligned}$$

Let $\alpha = (\alpha_1, \dots, \alpha_d)$ denote a pair of d integers with $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_d \le n$ and let $\gamma = (\gamma_1, \dots, \gamma_d, \gamma_{d+1})$ denote a pair of $d+1$ integers satisfying the same property $1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_d < \gamma_{d+1} \le n$. Then

$$\begin{aligned}
&\frac{(1-|f|^2)^{d+1}}{d!\,(n+1)^d}\left(f^*\sqrt{-1}\partial\bar{\partial}\log v_1\right)^d\\
&\quad= \det A - \det\begin{pmatrix} \left\langle \dfrac{\partial f}{\partial z^i}, \dfrac{\partial f}{\partial z^j}\right\rangle & \left\langle \dfrac{\partial f}{\partial z^i}, f\right\rangle \\ \left\langle f, \dfrac{\partial f}{\partial z^j}\right\rangle & \langle f, f\rangle \end{pmatrix}\\
&\quad= \sum_{\alpha}|\Delta^{\alpha}|^2 - \sum_{\gamma}\left|\det\begin{pmatrix} \dfrac{\partial f^{\gamma_1}}{\partial z^1} & \cdots & \dfrac{\partial f^{\gamma_1}}{\partial z^d} & f^{\gamma_1} \\ \dfrac{\partial f^{\gamma_2}}{\partial z^1} & \cdots & \dfrac{\partial f^{\gamma_2}}{\partial z^d} & f^{\gamma_2} \\ \vdots & \ddots & \vdots & \vdots \\ \dfrac{\partial f^{\gamma_{d+1}}}{\partial z^1} & \cdots & \dfrac{\partial f^{\gamma_{d+1}}}{\partial z^d} & f^{\gamma_{d+1}} \end{pmatrix}\right|^2,
\end{aligned}$$

where α runs over all such pairs $\alpha = (\alpha_1, \dots, \alpha_d)$ of d integers as before and likewise γ runs over all such pairs $\gamma = (\gamma_1, \dots, \gamma_d, \gamma_{d+1})$ of $d+1$ integers as before. Calculating the second term further, we obtain

$$\begin{aligned}
&\frac{(1-|f|^2)^{d+1}}{d!\,(n+1)^d}\left(f^*\sqrt{-1}\partial\bar{\partial}\log v_1\right)^d\\
&\quad= \sum_{\alpha}|\Delta^{\alpha}|^2 - \sum_{\gamma}\left|\sum_{i=1}^{d+1}(-1)^i f^{\gamma_i}\Delta^{\gamma_1\cdots\widehat{\gamma_i}\cdots\gamma_{d+1}}\right|^2\\
&\quad= \sum_{\alpha}|\Delta^{\alpha}|^2 - \sum_{\gamma}\sum_{i,j=1}^{d+1}(-1)^{i+j} f^{\gamma_i}\overline{f^{\gamma_j}}\Delta^{\gamma_1\cdots\widehat{\gamma_i}\cdots\gamma_{d+1}}\overline{\Delta^{\gamma_1\cdots\widehat{\gamma_j}\cdots\gamma_{d+1}}}\\
&\quad= \sum_{\alpha}|\Delta^{\alpha}|^2 - \sum_{\alpha,\beta}\sum_{i,j=1}^{d+1}\sum_{\gamma}(-1)^{i+j} f^{\gamma_i}\overline{f^{\gamma_j}}\Delta^{\alpha}\overline{\Delta^{\beta}},
\end{aligned}$$

where α, β, and γ run over all such pairs as in the Claim. □

Define a Hermitian metric h on the trivial bundle $U \times \mathbb{C}^{n(n-1)/2}$ over U of rank $\frac{n(n-1)}{2}$ by

$$h(\xi, \zeta) = \sum_{\alpha,\beta} h_{\alpha\bar{\beta}}\xi\overline{\zeta^{\beta}} = \sum_{\alpha,\beta}\left(\delta_{\alpha\bar{\beta}} - \sum_{i,j=1}^{d+1}\sum_{\gamma}(-1)^{i+j} f^{\gamma_i}\overline{f^{\gamma_j}}\right)_{\alpha,\beta}\xi^{\alpha}\overline{\zeta^{\beta}}$$

for two sections $(\xi^{\alpha})_{\alpha}$ and $(\zeta^{\beta})_{\beta}$ of the bundle $U\times\mathbb{C}^{n(n-1)/2}$. Here $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ run over all pairs of d integers such that $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_d \le n$ and $1 \le \beta_1 < \beta_2 < \cdots < \beta_d \le n$, and $\gamma = (\gamma_1, \dots, \gamma_d, \gamma_{d+1})$ runs over all pairs of $d+1$ integers with $1 \le \gamma_1 < \gamma_2 < \cdots < \gamma_d < \gamma_{d+1} \le n$ satisfying $(\gamma_1, \dots, \widehat{\gamma_i}, \dots, \gamma_{d+1}) = \alpha$ and $(\gamma_1, \dots, \widehat{\gamma_j}, \dots, \gamma_{d+1}) = \beta$. Then the preceding Claim means that

$$\left(f^*\sqrt{-1}\partial\bar\partial \log v_1\right)^d = \frac{d!\,(n+1)^d}{(1-|f|^2)^{d+1}} h(\Delta, \Delta)$$

holds if $\Delta := (\Delta^\alpha)_\alpha$ is regarded as a holomorphic section of $U \times \mathbb{C}^{n(n-1)/2}$.

Claim. *The metric h is positive definite.*

Proof. If $\alpha \neq \beta$ and $\alpha \setminus \{\alpha_k\} = (\alpha_1, \ldots, \widehat{\alpha_k}, \ldots, \alpha_d) = (\beta_1, \ldots, \widehat{\beta_l}, \ldots, \beta_d) = \beta \setminus \{\beta_l\}$ for some k and l, then

$$\sum_{i,j=1}^{d+1} \sum_\gamma (-1)^{i+j} f^{\gamma_i} \overline{f^{\gamma_j}} = -(-1)^{k+l} f^{\beta_l} \overline{f^{\alpha_k}}$$

in the component $h_{\alpha\bar\beta}$ of h; the reason is that $\gamma = (\gamma_1, \ldots, \gamma_d, \gamma_{d+1})$ runs over all pairs of $d+1$ integers with $1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_d < \gamma_{d+1} \leq n$ satisfying both $(\gamma_1, \ldots, \widehat{\gamma_i}, \ldots, \gamma_{d+1}) = \alpha$ and $(\gamma_1, \ldots, \widehat{\gamma_j}, \ldots, \gamma_{d+1}) = \beta$. If $\alpha = \beta$, then it follows for the same reason that

$$\sum_{i,j=1}^{d+1} \sum_\gamma (-1)^{i+j} f^{\gamma_i} \overline{f^{\gamma_j}} = |f|^2 - \sum_{i=1}^{d} f^{\alpha_i} \overline{f^{\beta_i}}$$

in the component $h_{\alpha\bar\beta}$ of h. Moreover, except in the two cases just described, we obviously have

$$\sum_{i,j=1}^{d+1} \sum_\gamma (-1)^{i+j} f^{\gamma_i} \overline{f^{\gamma_j}} = 0$$

in the component $h_{\alpha\bar\beta}$ of h for the same reason. If two pairs α and β satisfy $\alpha \setminus \beta = \{\alpha_k\}$ and $\beta \setminus \alpha = \{\beta_l\}$ for some k and l, then we set $\varepsilon_{\alpha\bar\beta} = (-1)^{k+l}$, $f^{\beta\setminus\alpha} = f^{\beta_l}$, and $f^{\alpha\setminus\beta} = f^{\alpha_k}$; otherwise, we set $\varepsilon_{\alpha\bar\beta} = 0$. Then we obtain the expression

$$h_{\alpha\bar\beta} = \left(\left(1 - |f|^2 + \sum_{i=1}^{d} f^{\alpha_i} \overline{f^{\beta_i}}\right) \delta_{\alpha\bar\beta} + \varepsilon_{\alpha\bar\beta} f^{\beta\setminus\alpha} \overline{f^{\alpha\setminus\beta}}\right).$$

This equality yields, for all complex vectors $(\xi^\alpha)_\alpha$,

$$\begin{aligned}
h_{\alpha\bar\beta}\xi^\alpha\overline{\xi^\beta} &= (1-|f|^2)|\xi|^2 + \sum_\alpha \sum_{i=1}^{d} |f^{\alpha_i}|^2 |\xi^\alpha|^2 \\
&\quad + \sum_\tau \sum_{a,b\notin\tau,\, a\neq b} \operatorname{sgn}(b\tau) f^b\, \overline{\operatorname{sgn}(a\tau) f^a} \xi^{\tau\cup\{a\}} \overline{\xi^{\tau\cup\{b\}}} \\
&= (1-|f|^2)|\xi|^2 + \sum_\tau \sum_{a=b\notin\tau} |f^a|^2 |\xi^{\tau\cup\{a\}}|^2 \\
&\quad + \sum_\tau \sum_{a,b\notin\tau,\, a\neq b} \operatorname{sgn}(b\tau) f^b\, \overline{\operatorname{sgn}(a\tau) f^a} \xi^{\tau\cup\{a\}} \overline{\xi^{\tau\cup\{b\}}} \\
&= (1-|f|^2)|\xi|^2 + \sum_\tau \left| \sum_{a\notin\tau} \operatorname{sgn}(a\tau) f^a \overline{\xi^{\tau\cup\{a\}}} \right|^2,
\end{aligned}$$

where τ in the summations runs over all pairs $\tau = (\tau_1, \dots, \tau_{d-1})$ of $d-1$ integers with the property $1 \leq \tau_1 < \cdots < \tau_{d-1} \leq n$. Furthermore, for $a \notin \tau$ and $b \notin \tau$ we use $\operatorname{sgn}(a\tau)$ and $\operatorname{sgn}(b\tau))$ to denote $\operatorname{sgn}(a, \tau_1, \dots, \tau_{d-1})$ and $\operatorname{sgn}(b, \tau_1, \dots, \tau_{d-1})$, respectively. By $\tau \cup \{a\}$ and $\tau \cup \{b\}$ we denote the respective pairs of d integers arranged in ascending order. Therefore, we can see from the preceding formula that the Claim is true. $\square$

Let D and R be, respectively, the Chern connection and the curvature form associated with the Hermitian structure h on the vector bundle $U \times \mathbb{C}^{n(n-1)/2}$ over U. In order to yield the inequality in Lemma 4.1, we take $\sqrt{-1}\partial\bar{\partial}\log$ on both sides of (9) to obtain

$$\begin{aligned}
&\sqrt{-1}\partial\bar{\partial}\log\bigl(f^*\sqrt{-1}\partial\bar{\partial}\log v_1\bigr)^d \\
&\quad= \frac{d+1}{n+1} f^*\sqrt{-1}\partial\bar{\partial}\log v_1 + \sqrt{-1}\partial\bar{\partial}\log h(\Delta, \Delta) \\
&\quad= \frac{d+1}{n+1} f^*\sqrt{-1}\partial\bar{\partial}\log v_1 \\
&\qquad + \sqrt{-1}\,\frac{h(\Delta,\Delta)h(D\Delta \wedge D\Delta) - h(D\Delta,\Delta) \wedge h(\Delta, D\Delta)}{h(\Delta,\Delta)^2} \\
&\qquad - \frac{h\bigl(\sqrt{-1}R\Delta, \Delta\bigr)}{h(\Delta,\Delta)} \\
&\quad\geq \frac{d+1}{n+1} f^*\sqrt{-1}\partial\bar{\partial}\log v_1 - \frac{h\bigl(\sqrt{-1}R\Delta, \Delta\bigr)}{h(\Delta,\Delta)}.
\end{aligned}$$

In the last line we use the Cauchy–Schwarz inequality. In addition, we can see the following about the last term of the right-hand side of the inequality.

CLAIM. *The term $R\Delta$ vanishes.*

Proof. In fact, for any pair $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$ of d integers with $1 \leq \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_d \leq n$, we have

$$\begin{aligned}
\sum_\alpha \frac{\partial h_{\alpha\bar{\varepsilon}}}{\partial z^k}\Delta^\alpha &= \sum_\alpha \frac{\partial}{\partial z^k}\Biggl(\delta_{\alpha\bar{\varepsilon}} - \sum_{i,j=1}^{d+1}\sum_\gamma (-1)^{i+j} f^{\gamma_i}\overline{f^{\gamma_j}}\Biggr)\Delta^\alpha \\
&= -\sum_{i,j=1}^{d+1}\sum_\alpha\sum_\gamma (-1)^{i+j}\frac{\partial f^{\gamma_i}}{\partial z^k}\overline{f^{\gamma_j}}\Delta^\alpha \\
&= -\sum_{j=1}^{d+1}\sum_\gamma (-1)^j\overline{f^{\gamma_j}}\sum_{i=1}^{d+1}(-1)^i\frac{\partial f^{\gamma_i}}{\partial z^k}\Delta^{\gamma_1\gamma_2\cdots\widehat{\gamma_i}\cdots\gamma_{d+1}};
\end{aligned}$$

in the last line, γ runs over all such pairs $\gamma = (\gamma_1, \dots, \gamma_{d+1})$ of $d+1$ integers as before satisfying $(\gamma_1, \dots, \widehat{\gamma_j}, \dots, \gamma_{d+1}) = \varepsilon$. Moreover, if we substitute the equality

$$\sum_{i=1}^{d+1}(-1)^{i+d+1}\frac{\partial f^{\gamma_i}}{\partial z^k}\Delta^{\gamma_1\gamma_2\cdots\widehat{\gamma_i}\cdots\gamma_{d+1}}$$
$$= \det\begin{pmatrix} \dfrac{\partial f^{\gamma_1}}{\partial z^1} & \dfrac{\partial f^{\gamma_1}}{\partial z^2} & \cdots & \dfrac{\partial f^{\gamma_1}}{\partial z^d} & \dfrac{\partial f^{\gamma_1}}{\partial z^k} \\ \dfrac{\partial f^{\gamma_2}}{\partial z^1} & \dfrac{\partial f^{\gamma_2}}{\partial z^2} & \cdots & \dfrac{\partial f^{\gamma_2}}{\partial z^d} & \dfrac{\partial f^{\gamma_2}}{\partial z^k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \dfrac{\partial f^{\gamma_{d+1}}}{\partial z^1} & \dfrac{\partial f^{\gamma_{d+1}}}{\partial z^2} & \cdots & \dfrac{\partial f^{\gamma_{d+1}}}{\partial z^d} & \dfrac{\partial f^{\gamma_{d+1}}}{\partial z^k} \end{pmatrix} = 0,$$

which holds by the definition of Δ, then

$$\sum_{\alpha}\frac{\partial h_{\alpha\bar{\varepsilon}}}{\partial z^k}\Delta^{\alpha} = 0$$

and we have in particular the β-component of $R\Delta$:

$$(R\Delta)^{\beta} = \sum_{k,j=1,2,\ldots,d}\frac{\partial}{\partial\bar{z}^j}\left\{\sum_{\alpha,\varepsilon}h^{\bar{\varepsilon}\beta}\frac{\partial h_{\alpha\bar{\varepsilon}}}{\partial z^i}\Delta^{\alpha}\right\}dz^i\wedge d\bar{z}^j = 0$$

for any $\beta = (\beta_1,\beta_2,\ldots,\beta_d)$ with $1 \le \beta_1 < \beta_2 < \cdots < \beta_d \le n$. Here α and ε in the summation run over all pairs of d integers such that $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_d \le n$ and $1 \le \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_d \le n$, respectively. □

This completes the proof of Lemma 4.1. □

Next we provide another proof of (7) in Theorem 4.2 via viscosity subsolutions of a complex Monge–Ampère equation; the concept is introduced in [15]. As mentioned before, it may (or may not) be possible to prove (2) in Theorem 3.2 in this way.

Recall that we must prove the inequality

$$\left\langle\left(\sqrt{-1}\partial\bar{\partial}\log v^C_{X|Z}\right)^d\right\rangle \ge \frac{(d+1)^d d!}{2^d}v^C_{X|Z}.$$

If the left-hand side is well-defined over Z, then we already know that it coincides with $\mathbf{1}_{Z\setminus\mathrm{Zero}(v^C_{X|Z})}\left(\sqrt{-1}\partial\bar{\partial}\log v^C_{X|Z}\right)^d$, where the wedge product $\left(\sqrt{-1}\partial\bar{\partial}\log v^C_{X|Z}\right)^d$ on $Z\setminus\mathrm{Zero}(v^C_{X|Z})$ is due to Bedford and Taylor. Hence we need only show that

$$\left(\sqrt{-1}\partial\bar{\partial}\log v^C_{X|Z}\right)^d \ge \frac{(d+1)^d d!}{2^d}v^C_{X|Z} \tag{10}$$

over $Z\setminus\mathrm{Zero}(v^C_{X|Z})$. Since $\log v^C_{X|Z}$ is continuous on $Z\setminus\mathrm{Zero}(v^C_{X|Z})$, we can apply the viscosity approach of [15] to the complex Monge–Ampère equation in the form

$$\left(\sqrt{-1}\partial\bar{\partial}\varphi\right)^d = \frac{(d+1)^d d!}{2^d}v^C_{X|Z}. \tag{11}$$

A viscosity subsolution for a general complex Monge–Ampère equation is defined as follows.

DEFINITION 4.2. Let M be a connected complex manifold of dimension m, and let v be a continuous pseudo-volume form on M. Then an upper semicontinuous function $\varphi\colon M \to \mathbb{R} \cup \{-\infty\}$ is called a *viscosity subsolution* of the complex Monge–Ampère equation $\left(\sqrt{-1}\partial\bar{\partial}\varphi\right)^m = v$ if it satisfies the following conditions:

(i) $\varphi \not\equiv -\infty$;

(ii) for every $x_0 \in M$ and any C^2 function q defined on a neighborhood of x_0, if $\varphi - q$ has a local maximum 0 at x_0 then $\left(\sqrt{-1}\partial\bar{\partial}q\right)^m_{x_0} \geq v_{x_0}$.

We need the following theorem (which is [15, Prop. 1.5]) on a relation between viscosity subsolutions and pluripotential subsolutions of a complex Monge–Ampère equation.

THEOREM 4.3. *Let M be a connected complex manifold of dimension m and let v be a continuous pseudo-volume form on M. Then, for any locally bounded and upper semicontinuous function φ in M, φ is a viscosity subsolution of the complex Monge–Ampère equation $\left(\sqrt{-1}\partial\bar{\partial}\varphi\right)^m = v$ if and only if it is plurisubharmonic and $\left(\sqrt{-1}\partial\bar{\partial}\varphi\right)^m \geq v$ holds in the sense of the Bedford–Taylor product.*

We apply this theorem with $M = Z \setminus \mathrm{Zero}(v^C_{X|Z})$, $v = (d+1)^d d!/2^d \times v^C_{X|Z}$, and $\varphi = \log v^C_{X|Z}$. We have already shown that φ is plurisubharmonic; hence, in order to prove (10), it suffices to check that $\varphi = \log v^C_{X|Z}$ is actually a viscosity subsolution of the complex Monge–Ampère equation (11). Take every $x_0 \in Z \setminus \mathrm{Zero}(v^C_{X|Z})$ and any C^2 function q defined on a neighborhood of x_0 such that $\log v^C_{X|Z} - q$ has a local maximum 0 at x_0. In that case, if $h_{x_0} \in \mathrm{Hol}(X, \mathbb{B}^d)$ attains the supremum of $v^C_{X|Z}$ at x_0 then $\log(h_{x_0}|_Z)^* v^{(d)}_1 - q$ also has a local maximum 0 at x_0, and what is most important is that the function is smooth around x_0. Therefore,

$$\left(\sqrt{-1}\partial\bar{\partial}q\right)_{x_0} \geq \left(\sqrt{-1}\partial\bar{\partial}\log(h_{x_0}|_Z)^* v^{(d)}_1\right)_{x_0} \geq (h_{x_0}|_Z)^*\left(\sqrt{-1}\partial\bar{\partial}\log v^{(d)}_1\right)_{x_0},$$

which implies that

$$\begin{aligned}\left(\sqrt{-1}\partial\bar{\partial}q\right)^d_{x_0} &\geq (h_{x_0}|_Z)^*\left(\sqrt{-1}\partial\bar{\partial}\log v^{(d)}_1\right)^d_{x_0} \\ &= \frac{(d+1)^d d!}{2^d}((h_{x_0}|_Z)^* v^{(d)}_1)_{x_0} = \frac{(d+1)^d d!}{2^d}(v^C_{X|Z})_{x_0}.\end{aligned}$$

As a result, q has the required properties in Definition 4.2. We can thus conclude that $\varphi = \log v^C_{X|Z}$ is a viscosity subsolution of the complex Monge–Ampère equation (11).

4.2. Nef Properties of Exterior Products of the Cotangent Bundle

In this section, Theorem 4.2 is applied to derive some explicit positivity properties of the cotangent bundle—or, more generally, its exterior products over a compact complex manifold.

Referring to Chandler and Wong [8], Wong [32], Graham [16], Eisenman [14], and Kobayashi [23], we first recall briefly some standard terminology from complex Finsler geometry and the theory of hyperbolic complex spaces. This terminology will be used to state our result.

DEFINITION 4.3. The dth Carathéodory metric $\mathcal{C}_X^d$ of a complex manifold X is a complex Finsler metric on $\bigwedge^d TX$ defined as

$$(\mathcal{C}_X^d)^2 = \sup\{g^* v_1^{(d)};\, g \in \mathrm{Hol}(X, \mathbb{B}^d)\}.$$

Then we immediately prove the following relation between the dth Carathéodory metric and the restricted Carathéodory pseudo-volume form of Z over X.

PROPOSITION 4.1. *For local holomorphic coordinates $(z^1, \dots, z^d)$ on a smooth subvariety Z,*

$$(\mathcal{C}_X^d)^2\left(\frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^d}\right) \bigwedge_{\alpha=1}^{d} (\sqrt{-1} dz^\alpha \wedge d\bar{z}^\alpha) = v_{X|Z}^C \ \text{on } Z. \tag{12}$$

Let $\pi : E \to X$ be a general holomorphic vector bundle over a compact complex manifold X—say, the d-times exterior product $\bigwedge^d TX$ of the tangent bundle TX of X. We consider the projectivized bundle $\pi_q : \mathbb{P}(E) := E^\times/\mathbb{C}^\times \to X$ of E and the quotient map $q : E^\times \to \mathbb{P}(E)$, where $E^\times := E \setminus$(zero section). We set $[v] := q(v)$ for $v \in E^\times$. The tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ over $\mathbb{P}(E)$ is defined as

$$\mathcal{O}_{\mathbb{P}(E)}(-1) = \{([v], \lambda v) \in \pi_q^* E;\, [v] \in \mathbb{P}(E),\, \lambda \in \mathbb{C}\};$$

its dual $\mathcal{O}_{\mathbb{P}(E)}(1) := \mathcal{O}_{\mathbb{P}(E)}(-1)^*$ is called the hyperplane bundle, and it satisfies the condition that $\mathcal{O}_{\mathbb{P}(E)}(1)|_{\mathbb{P}(E_z)}$ be equal to the hyperplane bundle $\mathcal{O}_{\mathbb{P}(E_z)}(1)$ over the complex projective space $\mathbb{P}(E_z)$ for all $z \in X$, where $E_z := \pi^{-1}(z)$. A positivity condition of E^* is defined by the corresponding positivity condition of $\mathcal{O}_{\mathbb{P}(E)}(1)$. For instance, E^* is said to be nef if and only if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef.

As a matter of fact, $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is obtained after blowing up E along its zero section. We therefore denote the blow-up by $\beta : \mathcal{O}_{\mathbb{P}(E)}(-1) \to E$.

LEMMA 4.2. *Through the blow-up $\beta : \mathcal{O}_{\mathbb{P}(E)}(-1) \to E$ we have a one-to-one correspondence between the space of all complex Finsler metrics on E whose logarithms are locally integrable and the space of all singular Hermitian metrics on $\mathcal{O}_{\mathbb{P}(E)}(1)$. Hence the singular Hermitian metric h_F corresponding to a given locally logarithmic integrable complex Finsler metric F is of the form $h_F := (F^2 \circ \beta)^{-1}$.*

Then we obtain that

$$c_1(\mathcal{O}_{\mathbb{P}(E)}(1), h_F) = \frac{1}{2\pi} \sqrt{-1} \partial\bar{\partial} \log F^2 \tag{13}$$

on $\mathbb{P}(E)$. Here it should be pointed out that the right-hand side can be interpreted as a current on $\mathbb{P}(E)$ by the homogeneity of F.

We now begin to give a detailed explanation about an application of Theorem 4.2. For this purpose, we introduce a construction of a d-dimensional complex subvariety in $\mathbb{P}\big(\bigwedge^d TX\big)$ from any d-dimensional subvariety $i_Z\colon Z \hookrightarrow X$ of an n-dimensional complex manifold X, where $\pi_d\colon \mathbb{P}\big(\bigwedge^d TX\big) \to X$ is the projectivization of $\bigwedge^d TX$. If Z is smooth, then the subvariety is indeed defined as the image $I_Z(Z)$ of the following embedding $I_Z\colon Z \to \mathbb{P}\big(\bigwedge^d TX\big)$:

$$I_Z(z) = \left[\left(\frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^d}\right)_z\right] \in \mathbb{P}\Big(\bigwedge^d TX\Big)$$

for $z \in Z$ and holomorphic local coordinates $(z^1, \dots, z^d)$ of Z around z, where $[\cdot]$ denotes the equivalence class. Observe that, because of the projectivization, this definition does not depend on the choice of the holomorphic local coordinates. For singular Z, we define a subvariety $I_Z(Z)$ as the Zariski closure of $I_{Z_{\mathrm{reg}}}(Z_{\mathrm{reg}})$ in $\mathbb{P}\big(\bigwedge^d TX\big)$.

Unless stated otherwise, hereafter an n-dimensional complex manifold X is assumed to be compact. Take a Galois cover $\tilde{X} \xrightarrow{p} X$, and denote by $\tilde{Z}$ the pull-back by p of a d-dimensional irreducible closed subvariety Z of X.

Kratz [24] has shown that the cotangent bundle T^*X of X is nef if $\tilde{X}$ is strongly Carathéodory measure hyperbolic. His proof goes as follows. By the definition of the Carathéodory metric $\mathcal{C}^1_{\tilde{X}}$ on $\tilde{X}$, its logarithm is plurisubharmonic on TX. Moreover, its logarithm is finite and continuous on $(TX)^\times$ because the Carathéodory metric is nondegenerate everywhere (by the strong Carathéodory measure hyperbolicity of $\tilde{X}$). Therefore, since the curvature current of the singular Hermitian line bundle $(\mathcal{O}_{\mathbb{P}(TX)}(1), h_{\mathcal{C}^1_{\tilde{X}}})$ is $\sqrt{-1}\partial\bar{\partial}\log(\mathcal{C}^1_{\tilde{X}})^2$ by (13), it follows from the Richberg regularization theorem that the curvature current can be regularized with an arbitrarily small negative part. From this we have that $\mathcal{O}_{\mathbb{P}(TX)}(1)$ is nef. By definition, the nefness of the cotangent bundle T^*X holds.

However, the proof does not tell us how nef T^*X becomes as the Carathéodory measure hyperbolicity becomes stronger. So our aim is to numericalize Kratz's result, and we actually obtain the following result from Theorem 4.2.

COROLLARY 4.1. *Let X be an n-dimensional compact complex manifold and Z its d-dimensional irreducible closed subvariety. Take any Galois cover $\tilde{X} \xrightarrow{p} X$, and denote by $\tilde{Z}$ the pull-back of Z by p. Suppose that $Z \cap \mathrm{Zero}(v^C_{\tilde{X}}) = \emptyset$. Then*

$$(\mathcal{O}_{\mathbb{P}(\bigwedge^d TX)}(1)^d \cdot I_Z(Z)) \geq \frac{d!\,(d+1)^d}{(4\pi)^d}\,\mu^C_{\tilde{X}|\tilde{Z}}(Z). \tag{14}$$

Proof. By (13) we know that $c_1(\mathcal{O}_{\mathbb{P}(\bigwedge^d TX)}(1), h_{\mathcal{C}^d_{\tilde{X}}})$ is just $1/2\pi \times \sqrt{-1}\partial\bar{\partial}\log(\mathcal{C}^d_{\tilde{X}})^2$. Owing to the requisite hypothesis $Z \cap \mathrm{Zero}(v^C_{\tilde{X}}) = \emptyset$ for the theorem, we can apply Richberg's regularization technique to the semi-positive current $\big(1/2\pi \times \sqrt{-1}\partial\bar{\partial}\log(\mathcal{C}^d_{\tilde{X}})^2\big)\big|_{I_{Z_{\mathrm{reg}}}(Z_{\mathrm{reg}})}$ and obtain

$$\begin{aligned}(\mathcal{O}_{\mathbb{P}(\bigwedge^d TX)}(1)^d \cdot I_Z(Z)) &= \int_{I_{Z_{\mathrm{reg}}}(Z_{\mathrm{reg}})} \left\langle\left(\left(\frac{1}{2\pi}\sqrt{-1}\partial\bar{\partial}\log(\mathcal{C}^d_{\tilde{X}})^2\right)\bigg|_{I_{Z_{\mathrm{reg}}}(Z_{\mathrm{reg}})}\right)^d\right\rangle \\ &= \int_{I_{Z_{\mathrm{reg}}}(Z_{\mathrm{reg}})} \left\langle\left(\left(\frac{1}{2\pi}\sqrt{-1}\partial\bar{\partial}\log(\mathcal{C}^d_{\tilde{X}})^2\right)\bigg|_{I_{\tilde{Z}_{\mathrm{reg}}}(\tilde{Z}_{\mathrm{reg}})}\right)^d\right\rangle.\end{aligned}$$

Here the wedge product in the integral is taken in the sense of Bedford and Taylor, and in the first equality we used the continuity property of Bedford–Taylor products along locally uniformly convergent sequences of potentials. By Proposition 4.1,

$$(\mathcal{O}_{\mathbb{P}(\bigwedge^d TX)}(1)^d \cdot I_Z(Z)) = \int_{Z_{\mathrm{reg}}} \left\langle \left(\frac{1}{2\pi} \sqrt{-1}\partial\bar{\partial} \log v^C_{\tilde{X}|\tilde{Z}_{\mathrm{reg}}} \right)^d \right\rangle.$$

Therefore, the proof can be reduced to a curvature property of restricted Carathéodory pseudo-volume forms. Hence, by Theorem 4.2 and the Riemann extension theorem, we have the desired inequality:

$$(\mathcal{O}_{\mathbb{P}(\bigwedge^d TX)}(1)^d \cdot I_Z(Z)) \geq \frac{d!\,(d+1)^d}{(4\pi)^d} \int_{Z_{\mathrm{reg}}} v^C_{\tilde{X}|\tilde{Z}_{\mathrm{reg}}} = \frac{d!\,(d+1)^d}{(4\pi)^d} \mu^C_{\tilde{X}|\tilde{Z}}(Z). \qquad \square$$

References

[1] D. Abramovich, K. Karu, K. Matsuki, and J. Włodarczyk, *Torification and factorization of birational maps,* J. Amer. Math. Soc. 15 (2002), 531–572.

[2] E. Bedford and B. A. Taylor, *The Dirichlet problem for a complex Monge–Ampère equation,* Invent. Math. 37 (1976), 1–44.

[3] ———, *A new capacity for plurisubharmonic functions,* Acta Math. 149 (1982), 1–40.

[4] S. Boucksom, *On the volume of a line bundle,* Internat. J. Math. 13 (2002), 1043–1063.

[5] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, *Monge–Ampère equations in big cohomology classes,* Acta Math. 205 (2010), 199–262.

[6] J.-G. Cao and P.-M. Wong, *Finsler geometry of projectivized vector bundles,* J. Math. Kyoto Univ. 43 (2003), 369–410.

[7] C. Carathéodory, *Über die Abbildungen, die durch Systeme von analytischen Funktionen von mehreren Veränderlichen erzeugt werden,* Math. Z. 34 (1932), 758–792.

[8] K. Chandler and P.-M. Wong, *On the holomorphic sectional and bisectional curvatures in complex Finsler geometry,* Period. Math. Hungar. 48 (2004), 93–123.

[9] J.-P. Demailly, *Estimations L^2 pour l'opérateur $\bar{\partial}$ d'un fibré vectoriel holomorphe semi-positif au-dessus d'une variété kählérienne complète,* Ann. Sci. École Norm. Sup. (4) 15 (1982), 457–511.

[10] ———, *Regularization of closed positive currents and intersection theory,* J. Algebraic Geom. 1 (1992), 361–409.

[11] ———, *L^2 vanishing theorems for positive line bundles and adjunction theory,* Transcendental methods in algebraic geometry (Cetraro, 1994), Lecture Notes in Math., 1646, pp. 1–97, Springer-Verlag, Berlin, 1996.

[12] ———, *Complex analytic and differential geometry,* ⟨www.agbook.ps.gz⟩.

[13] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa, *Restricted volumes and base loci of linear series,* Amer. J. Math. 131 (2009), 607–651.

[14] D. A. Eisenman, *Intrinsic measures on complex manifolds and holomorphic mappings,* Mem. Amer. Math. Soc., 96, Amer. Math. Soc., Providence, RI, 1970.

[15] P. Eyssidieux, V. Guedj, and A. Zeriahi, *Viscosity solutions to degenerate complex Monge–Ampère equations,* Comm. Pure Appl. Math. 64 (2011), 1059–1094.

[16] I. Graham, *Intrinsic measures and holomorphic retracts,* Pacific J. Math. 130 (1987), 299–311.
[17] H. Hironaka, *Flattening theorem in complex-analytic geometry,* Amer. J. Math. 97 (1975), 503–547.
[18] T. Hisamoto, *Restricted Bergman kernel asymptotics,* Trans. Amer. Math. Soc. 364 (2012), 3585–3607.
[19] L. Hörmander, *An introduction to complex analysis in several variables,* 3rd ed., North-Holland Math. Library, 7, North-Holland, Amsterdam, 1990.
[20] S. Kikuta, *Carathéodory measure hyperbolicity and positivity of canonical bundles,* Proc. Amer. Math. Soc. 139 (2011), 1411–1420.
[21] S. Kobayashi, *Negative vector bundles and complex Finsler structures,* Nagoya Math. J. 57 (1975), 153–166.
[22] ———, *Complex Finsler vector bundles,* Finsler geometry (Seattle, 1995), Contemp. Math., 196, pp. 145–153, Amer. Math. Soc., Providence, RI, 1996.
[23] ———, *Hyperbolic complex spaces,* Grundlehren Math. Wiss., 318, Springer-Verlag, Berlin, 1998.
[24] H. Kratz, *Compact complex manifolds with numerically effective cotangent bundles,* Doc. Math. 2 (1997), 183–193.
[25] Y. Kubota, *An extremal problem on bounded symmetric domains,* Bull. London Math. Soc. 15 (1983), 126–130.
[26] R. Lazarsfeld, *Positivity in algebraic geometry. I,* Ergeb. Math. Grenzgeb. (3), 48, Springer-Verlag, Berlin, 2004.
[27] ———, *Positivity in algebraic geometry. II,* Ergeb. Math. Grenzgeb. (3), 49, Springer-Verlag, Berlin, 2004.
[28] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels,* Progr. Math., 254, Birkhäuser, Basel, 2007.
[29] S. Matsumura, *Restricted volumes and divisorial Zariski decompositions,* Amer. J. Math. (to appear).
[30] D. Popovici, *Regularization of currents with mass control and singular Morse inequalities,* J. Differential Geom. 80 (2008), 281–326.
[31] J. Sanderson, *Curvature properties of intrinsic metrics,* Ph.D. thesis, University of Notre Dame, 1995.
[32] P.-M. Wong, *A survey of complex Finsler geometry,* Finsler geometry (Sapporo, 2005), Adv. Stud. Pure Math., 48, pp. 375–433, Math. Soc. Japan, Tokyo, 2007.
[33] H. Wu, *Old and new invariant metrics on complex manifolds,* Several complex variables (Stockholm, 1987/1988), Math. Notes, 38, pp. 640–682, Princeton Univ. Press, Princeton, NJ, 1993.

Mathematical Institute
Tôhoku University
Sendai 980-8578
Japan

skikuta@sophia.ac.jp