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Gluck Twist on a Certain Family of 2-Knots

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1. Introduction

The paper of Freedman, Gompf, Morrison, and Walker [5] about the potential application of Khovanov homology in solving the 4-dimensional smooth Poincaré conjecture (SPC4) revitalized this important subfield of topology. A sequence of papers appeared, some settling 30-year-old problems [1; 7], some introducing new potential exotic 4-spheres [14], and still others showing that the newly introduced examples are, in fact, standard [2; 15].

One underlying construction for producing examples of potential exotic 4-spheres is the Gluck twist along an embedded S^2 (a *2-knot*) in the standard 4-sphere $\mathbb{S}^4$. In this construction we remove the tubular neighborhood of the 2-knot and glue it back with a specific diffeomorphism. (For a more detailed discussion, see Section 3.) In turn, any 2-knot in $\mathbb{S}^4$ admits a *normal form* and hence can be described by an ordinary knot in S^3 together with two sets of ribbon bands (determining the "southern" and "northern" hemispheres of the 2-knot). Applying standard ideas of Kirby calculus (see e.g. [8]), we can explicitly draw the complement of a 2-knot and, from there, the result of the Gluck twist. From such a presentation we derive the following result.

Theorem 1.1. *Consider the knot $K(p,q)$ depicted by Figure 1, and use the bands b_1 and b_2 to construct the southern and northern hemispheres of a 2-knot $K^2_{pq} \subset \mathbb{S}^4$. Then a Gluck twist along the 2-knot K^2_{pq} provides the 4-sphere with its standard smooth structure.*

Remark 1.2. For the first appearance of the 1-knots $K(p,q)$ see [11; 12]. For certain choices of p and q, the 1-knot $K(p,q)$ can be identified more familiarly; for instance, $K(0,0)$ is isotopic to $F\#F = F\#m(F)$, where F is the figure-eight knot (isotopic to its mirror image $m(F)$), $K(1,-1)$ is the 8_9 knot, and $K(1,1)$ is 10_{155} in the standard knot tables. Notice that in [3] the knot 8_9 defines the 2-knot along which the Gluck twist is performed, although the bands used in [3] are potentially different from the b_1 and b_2 used here in Theorem 1.1 (cf. [3, Fig. 16]).

Before proving the theorem, in Sections 2 and 3 we briefly invoke basic facts about 2-knots, the Gluck twist, and the derivation of a Kirby diagram for the result of the Gluck twist along a 2-knot given by a ribbon 1-knot and two sets of

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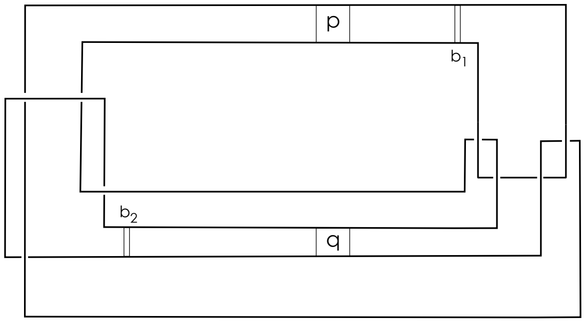

Figure 1 The knot $K(p,q)$ with the two ribbon bands b_1 and b_2, giving rise to the 2-knot $K^2_{pq} \subset \mathbb{S}^4$

ribbons. Then, in Section 4, a simple Kirby calculus argument provides the proof of Theorem 1.1. (A slightly different Kirby calculus argument for the same result is given in the Appendix.)

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2. Ribbon 2-Disks and Related 2-Knots

Every 2-knot is equivalent to one in normal form [4]. In other words, for a 2-knot $K \subset \mathbb{S}^4$ there is an ambiently isotopic $K' \subset \mathbb{R}^4$ (i.e. $\mathbb{S}^4 \setminus \infty$) with a projection $p\colon \mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ such that p restricted to K' gives a Morse function with the following properties:

(1) $K' \subset \mathbb{R}^3 \times [-c, c]$ for some $c > 0$,
(2) all index-0 critical points are in $K' \cap \mathbb{R}^3 \times \{-c\}$,
(3) all index-1 critical points with negative p-value give fusion bands within $K' \cap \mathbb{R}^3 \times (-c, 0)$,
(4) $K' \cap \mathbb{R}^3 \times \{0\}$ is a single 1-knot k,
(5) all index-1 critical points with positive p-value give fission bands within $K' \cap \mathbb{R}^3 \times (0, c)$, and
(6) all index-2 critical points are in $K' \cap \mathbb{R}^3 \times \{c\}$.

In particular this means that any 2-knot K is formed from the union of two *ribbon 2-disks,* glued together along their boundaries, which is the same (ribbon) 1-knot k for both. Because such a (ribbon disk) hemisphere D of a 2-knot has a handlebody with only 0- and 1-handles, we can construct a Kirby diagram for any ribbon disk complement in the 4-disk D^4 and, from there, for any 2-knot in $\mathbb{S}^4$ as follows (cf. [8, Chap. 6]).

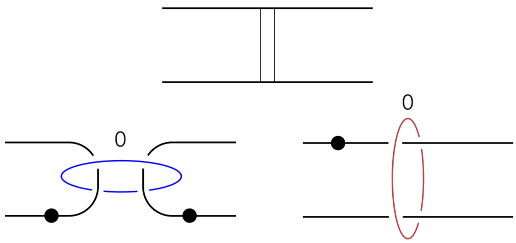

Figure 2 Handles from a ribbon move in the lower hemisphere (left) and upper hemisphere (right)

LEMMA 2.1. *Let K be a 2-knot given, as just described, by the union of two ribbon disks with equatorial 1-knot k, lower hemisphere ribbon presentation $\mathcal{B}_1 = \{b_1, b_2, \ldots, b_m\}$, and upper hemisphere ribbon presentation $\mathcal{B}_2 = \{b'_1, b'_2, \ldots, b'_n\}$. Then a handlebody for $\mathbb{S}^4 \setminus K$ can be constructed by the following algorithm*:

(1) *at each $\mathcal{B}_1$ ribbon, split from k a dotted circle component and add a 2-handle as in the left-hand side of Figure 2*;
(2) *at each $\mathcal{B}_2$ ribbon, add the 2-handle as in the right-hand side of Figure 2*; *and*
(3) *add a 3-handle for each ribbon of $\mathcal{B}_2$ and then add a single 4-handle.*

Proof. We shall start by describing the complement of one ribbon disk D; let $X = D^4 \setminus D$ denote the ribbon disk complement. Its handlebody starts with a 0-handle (4-ball) X_0. Then, for each 0-handle of D that is carved out, a (4-dimensional) 1-handle is added to X_0 to form the 1-handlebody X_1. Finally, the ribbons (or 1-handles) of D each yield 2-handles in the complement, and these are attached along curves formed from the union of push-offs of the core 1-disk of the ribbons (cf. [8, Sec. 6.2]). The attaching circles have 0-framings because push-offs of the core do not link. Hence the result follows easily from a diagram of the equatorial knot k by locally replacing the bands with the diagram presented on the left of Figure 2.

Next consider the special case when K is the 2-knot, which we obtain by doubling the disk D (i.e., $K = D \cup \bar{D}$). In this case, a Kirby diagram for the knot exterior $Y = \mathbb{S}^4 - (D \cup \bar{D})$ can be built up easily from the handlebody decomposition of D. This amounts to taking the disk complement X just defined and adding a second "upside down" copy of X (relative to the carved-out 2-disk D, so that the result is still a manifold with boundary). For each ribbon in the upper hemisphere, again we add a 2-handle to the complement. But with D turned upside down in the upper hemisphere, the ribbons have cores and co-cores that are opposite to their counterparts in the lower hemisphere. Consequently, the 2-handles added in the upper hemisphere's complement have attaching curves formed from the union of two co-cores of the original ribbons of D. This is shown in Figure 3, in which a pair of 0-handles of K and a 1-handle fusing them together gives the handlebody configuration in the complement on the right, with the vertical 2-handle depicting the upside-down copy coming from the upper hemisphere of K (and with "dotted"

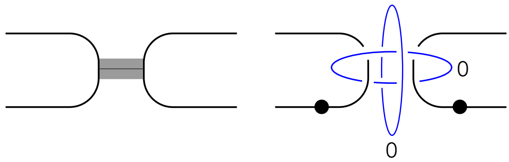

Figure 3 Handles in Y from a 1-handle in (each copy of) D

circles depicting the 4-dimensional 1-handles). Moreover, for each of the upper hemisphere 2-handles (corresponding to 0-handles of D) we obtain a 3-handle and then finally a 4-handle to complete the description of the complement of K. For a similar discussion, see [8, Exr. 6.2.11(b)].

Finally consider the general case, when the 2-knot K is formed from two disks D_1 and D_2 (as opposed hemispheres); that is, $K = D_1 \cup \bar{D}_2$. The recipe described previously yields a diagram for $D^4 - D_1$ and for $\mathbb{S}^4 - (D_2 \cup \bar{D}_2)$. We need only replace $D^4 - D_2$ with $D^4 - D_1$ to obtain a diagram for $\mathbb{S}^4 - K$. This simply amounts to finding a diffeomorphism between the boundaries of $D^4 - D_1$ and $D^4 - D_2$ and then pulling back the attaching circles of the 2-handles of $D^4 - \bar{D}_2$ to the diagram of $D^4 - D_1$. By converting the dots to 0-framings, sliding the (once dotted, now 0-framed) circles on each other, and then canceling (in the 3-dimensional sense) the obvious handle pairs, we see that both $\partial(D^4 - D_1)$ and $\partial(D^4 - D_2)$ are diffeomorphic to the result of 0-surgery along the equatorial knot k. Using this diffeomorphism, the pull-back provides the attaching circle given in the statement, concluding the proof. □

3. The Gluck Twist and Kirby Diagrams

Suppose that $K \subset \mathbb{S}^4$ is a given 2-knot in the 4-sphere. Remove a normal neighborhood νK of $K \subset \mathbb{S}^4$ from the 4-sphere and reglue $S^2 \times D^2$ by the diffeomorphism of the boundary $\partial(S^2 \times D^2) \approx \partial(\mathbb{S}^4 \setminus \nu K) \approx S^2 \times S^1$,

$$\mu \colon S^2 \times S^1 \to S^2 \times S^1,$$

which is given by $(x, \theta) \overset{\mu}{\mapsto} (\mathrm{rot}_\theta(x), \theta)$ for rot_θ the rotation with angle θ of the 2-sphere about the axis through its poles.

DEFINITION 3.1. The preceding construction is called the *Gluck twist* along the 2-knot $K \subset \mathbb{S}^4$. The result of this construction will be denoted by $\Sigma(K)$.

Since the result $\Sigma(K)$ of a Gluck twist is simply connected, Freedman's celebrated theorem implies that $\Sigma(K)$ is homeomorphic to $\mathbb{S}^4$.

For a 2-sphere K embedded in the 4-sphere, a handlebody for νK consists of a 0-handle plus one 2-handle attached along a 0-framed unknot. This can also be built upside down from its boundary $S^2 \times S^1$ by attaching the (dualized) 2-handle h_K along any meridian $\{\mathrm{pt}\} \times S^1$ of the sphere and then attaching the dualized

0-handle as a 4-handle. Therefore, if a handlebody diagram for the knot exterior $Y = \mathbb{S}^4 \setminus \nu K$ is given, then one can reconstruct $\mathbb{S}^4$ by attaching the 2-handle h_K along a 0-framed meridian of any 1-handle h corresponding to a 0-handle of K. The homotopy sphere $\Sigma(K)$ resulting from the Gluck twist on K then can be formed from Y by attaching the 2-handle h_K with (± 1)-framing along the same meridional circle of the 1-handle h (see also [8, Exr. 6.2.4]). Note that all the further attaching circles of 2-handles linking the 1-handle h can be slid off h by the use of h_K, after which h and h_K can be canceled against each other. Therefore, in practice the presentation of the Gluck twist along K amounts to blowing down one of the dotted circles corresponding to a 0-handle of K as if the dotted circle were a (-1)-framed (or a ($+1$)-framed, up to our choice) unknot. Recall that in Section 2 we presented a diagram for Y that admits a 4-handle. Since in gluing $S^2 \times D^2$ back we add an additional 4-handle, one of them can be canceled against a 3-handle.

Gompf [6] presented an alternative effect of the Gluck twist. The rotation rot_θ involved in the gluing map fixes both poles N and S of the 2-sphere, resulting in two fixed circles $\{N\} \times S^1$ and $\{S\} \times S^1$ of μ. Presenting $S^2 \times D^2$ as the union of a 0-handle, a 1-handle, and two 2-handles (or, in the upside-down picture, as the union of two 2-handles h_K and h'_K, a 3-handle, and a 4-handle), one can construct $\Sigma(K)$ from Y by attaching the two 2-handles h_K and h'_K (one along $\{N\} \times S^1$ and one along $\{S\} \times S^1$ with framings ($+1$) and (-1), respectively) as well as the 3- and 4-handles, where the two attaching circles are meridional circles of two dotted circles (corresponding to two 0-handles of K). Once again, because the 2-handles can be slid over h_K and h'_K—and then these 2-handles can cancel the corresponding dotted circles—in practice the Gluck twist along K amounts to simply blowing down two dotted circles as if one were a ($+1$)- and the other a (-1)-framed unknot (and then adding a 3- and a 4-handle). As before, one 3-handle cancels one of the two 4-handles appearing in the decomposition.

In conclusion, if a 2-knot K in normal form is given in $\mathbb{S}^4$ by a ribbon knot k with two sets of bands $\mathcal{B}_1$ and $\mathcal{B}_2$, then the foregoing description provides a simple algorithmic way to produce a handle decomposition of the result $\Sigma(K)$ of the Gluck twist along K. Observe, moreover, that if $|\mathcal{B}_1| = 1$ (or $|\mathcal{B}_2| = 1$) then the resulting decomposition can be chosen not to contain any 1-handles.

Remark 3.2. For some special classes of 2-knots K, the diffeomorphism type of $\Sigma(K)$ is well understood. In [9], Gordon proved that $\Sigma(K)$ is diffeomorphic to $\mathbb{S}^4$ for any twist-spun 2-knot K. In [13], Melvin showed that every ribbon 2-knot K has $\Sigma(K)$ standard as well. Additionally, since any ribbon 2-knot is the double of a ribbon 2-disk, it follows that [8, Exr. 6.2.11(b)] gives an alternate proof of this second result.

4. A Family of 2-Knots

For p, q (possibly nondistinct) integers, let $K(p, q)$ be the knot of Figure 1. This is a ribbon knot of 1-fusion; that is, there is a ribbon presentation of $K(p, q)$ such that performing the indicated single ribbon move transforms the knot into a two-component unlink. In fact, there are two apparent choices for the single ribbon move (or two apparently distinct ribbon presentations). These are indicated by the fine-lined bands b_i $(i = 1, 2)$ of Figure 1. Either of these ribbon presentations corresponds to a ribbon 2-disk, which we will denote by $D(p,q)_i$ $(i = 1, 2)$.

DEFINITION 4.1. Define the 2-knot K^2_{pq} as the union

$$(D^4, D(p,q)_1) \cup \overline{(D^4, D(p,q)_2)}.$$

Following the recipe of Section 3, we give a handlebody description of the result $\Sigma(K^2_{pq})$ of the Gluck twist along K^2_{pq}. In so doing, we first present a diagram for $Y_{pq} = \mathbb{S}^4 - \nu K^2_{pq}$; see Figure 4.

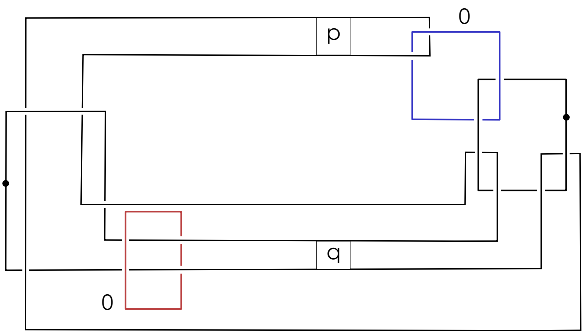

Figure 4 Kirby diagram for Y_{pq} minus two 3-handles and one 4-handle

Figures 5 through 8 demonstrate an isotopy of the Figure 4 diagram of Y_{pq} into a form where the 1-handles are visibly separated. In Figure 5 we have the result of

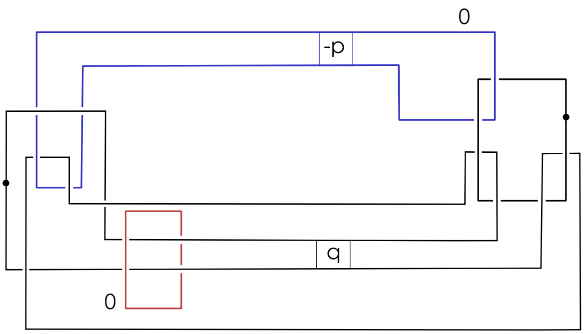

Figure 5 Transferring the p-twist from the dotted circle to the 0-framed unknot

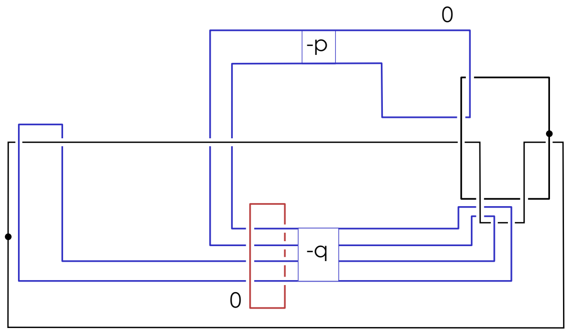

Figure 6 A further isotopy of the diagram of Figure 5

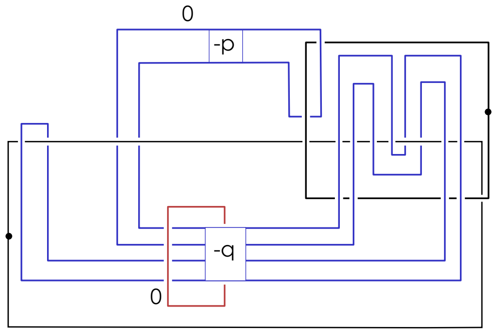

Figure 7 Isotopy to separate the dotted circles

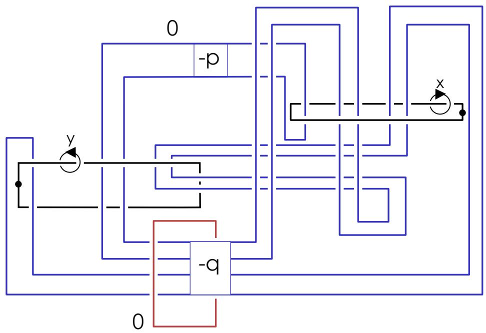

Figure 8 The knot complement Y_{pq} (minus two 3-handles and one 4-handle) with 1-handles separated (and generators of π_1 labeled)

undoing the p-twist in the first 1-handle and then starting to isotope the 2-handle through the second 1-handle. In Figure 6, the q-twist of the second 1-handle is undone by twisting the indicated four strands of the 2-handle. The additional isotopies in Figure 7 then produce Figure 8, where the 1-handles are conveniently separated.

This handlebody depiction of Y_{pq} allows us to analyze the 2-knot K^2_{pq}, compute its knot group directly, and prove the following statement.

PROPOSITION 4.2. *The two 2-knots K^2_{pq} and K^2_{rs} are distinct provided that the parities of the unordered pairs $\{p, q\}$ and $\{r, s\}$ are distinct.*

Proof. Choosing orientations on generators of π_1 as in Figure 8, we obtain $\pi_1(Y_{pq}) \cong \langle x, y \mid r_{pq} \rangle$, where the relation r_{pq} is expressed in one of four ways:

$$r_{pq} \text{ takes the form } \begin{cases} xyxy^{-1}x^{-1}yxyx^{-1}y^{-1} & \text{if } p \text{ is even and } q \text{ is even,} \\ xyxyx^{-1}y^{-1}xy^{-1}x^{-1}y^{-1} & \text{if } p \text{ is odd and } q \text{ is odd;} \\ xyxy^{-1}x^{-1}y^{-1}xyx^{-1}y^{-1} & \text{if } p \text{ is odd and } q \text{ is even,} \\ xyxyx^{-1}yxy^{-1}x^{-1}y^{-1} & \text{if } p \text{ is even and } q \text{ is odd.} \end{cases}$$

Using Fox calculus, it is easy to see that each 2-knot does have a principal first elementary ideal and hence an Alexander polynomial:

$$\Delta(t) = \begin{cases} -t^2 + 3t - 1 & \text{if } p \text{ is even and } q \text{ is even,} \\ 1 - t + 2t^2 - t^3 & \text{if } p \text{ is odd and } q \text{ is odd;} \\ 2 - 2t + t^2 & \text{if } p \text{ is odd and } q \text{ is even,} \\ 2t^2 - 2t + 1 & \text{if } p \text{ is even and } q \text{ is odd.} \end{cases}$$

This computation gives three distinguished cases for a pair $\{p, q\}$ (since the last two polynomials are equivalent as Alexander polynomials). In particular, K^2_{pq} and K^2_{rs} have distinct Alexander polynomials if the pairs of parities are distinct. □

REMARK 4.3. Except when p and q are both even, Δ_K is asymmetric and hence not the Alexander polynomial of a 1-knot. Consequently, K^2_{pq} cannot be a spun knot if p and q are not both even. Furthermore, since m-twist-spun 2-knots for $|m| > 1$ have deficiency ≤ 0 [10], none of the K^2_{pq} knots are m-twist-spun (for $|m| > 1$), either.

Now we are ready to prove the paper's main result.

Proof of Theorem 1.1. After isotoping the small 2-handle until it becomes parallel to the rightmost dotted circle and then blowing down the two dotted circles (as required when implementing the Gluck twist), we arrive at Figure 9. Finally, Figure 10 (obtained by performing the indicated handle slide in Figure 9) unravels to give a pair of disjoint 0-framed 2-handles that cancel against the 3-handles to give $\mathbb{S}^4$. □

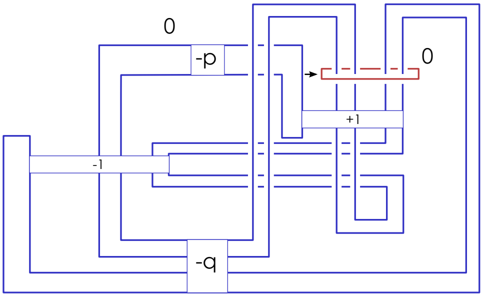

Figure 9 Diagram of $\Sigma(K^2_{pq})$ (minus two uniquely attached 3-handles and one 4-handle)

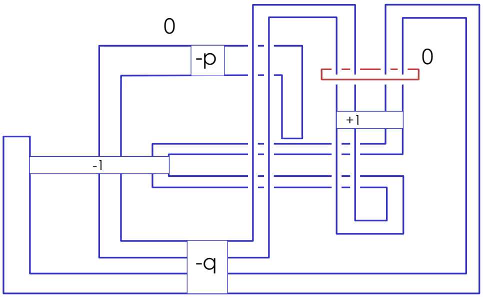

Figure 10 After sliding one 2-handle over the other one (as instructed by the arrow in Figure 9) we are left with two unlinked 2-handles, which can be seen to cancel against the 3-handles

Appendix: Alternate Proof of Theorem 1.1

For the particular case of the 2-knot K^2_{pq} there is, in fact, a way to see that the Gluck twist leaves $\mathbb{S}^4$ standard without separating the 1-handles first.

Second Proof of Theorem 1.1. Starting in Y_{pq} (cf. Figure 4), realize the Gluck twist on K^2_{pq} by adding a (-1)-framed 2-handle to a 1-handle (instead of just immediately blowing down the dotted 1-handle). Slide the lower 0-framed 2-handle over the upper 1-handle and off the q-twisted 1-handle to get Figure 11. Now, in that figure, slide the (-1)-framed 2-handle over the leftmost 0-framed 2-handle and off its 1-handle and the rightmost 2-handle. Next, in Figure 12 slide the 0-framed 2-handle on the left over the (-1)-framed 2-handle (which changes its own framing to -1) and then use the remaining 0-framed 2-handle to unhook the other two 2-handles from each other. The result is a collection of Hopf links, and standard handle cancellations now show that $\Sigma(K^2_{pq})$ is, indeed, diffeomorphic to the standard 4-sphere $\mathbb{S}^4$. □

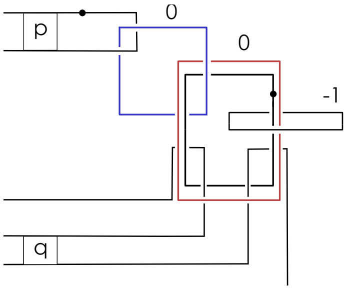

Figure 11 Relevant portion of the diagram of $\Sigma(K^2_{pq})$ (again, minus 3- and 4-handles) after one handle slide

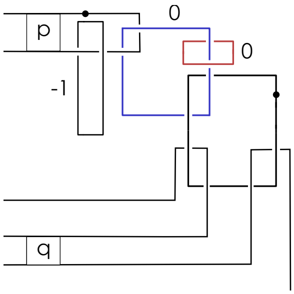

Figure 12 Relevant portion of the diagram after sliding the (-1)-framed 2-handle and isotoping the rightmost 0-framed component

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