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Parshin Residues via Coboundary Operators

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1. Introduction

Let X be a compact complex curve and let ω be a meromorphic 1-form on X. In an open neighborhood of each point $x \in X$ we can write

$$\omega = f(t)\,dt, \quad f(t) = \sum_{i>N} \lambda_i t^i,$$

where t is a local normalizing parameter at x. The coefficient λ_{-1} in the series does not depend on the choice of parameter t; it is called the *residue* of ω at x. The residue is nonzero only at the finitely many points $\Sigma \subset X$ where ω has a pole. The well-known residue formula states that the sum of residues of ω over all points of Σ is zero:

$$\sum_{x \in \Sigma} \operatorname{res}_x \omega = 0.$$

Indeed, the residue at $x \in \Sigma$ is equal to the integral of ω over any sufficiently small cycle enclosing x, divided by $2\pi i$. In the complement $X \setminus \Sigma$ the form ω is closed, and the sum of cycles is homologous to zero. Thus, the residue formula follows from the Stokes theorem.

Although this proof is topological, the residue itself can be defined purely algebraically. In fact, one can give an algebraic proof of the residue formula that works in a much more general situation, not only in the case of complex curves (see e.g. [S; T]).

In the late 1970s, Parshin introduced his notion of multidimensional residue for a rational n-form ω on an n-dimensional algebraic variety V_n. (Although [P] deals mostly with the 2-dimensional case, Beilinson [Bei] and Lomadze [L] generalized Parshin's ideas to the multidimensional case.) The main difference between the Parshin residue and the classical 1-dimensional residue is that, in higher dimensions, one computes the residue not at a point but instead at a complete flag of subvarieties $F = \{V_n \supset \cdots \supset V_0\}$, $\dim V_k = k$.

Parshin, Beilinson, and Lomadze proved the "reciprocity law" for multidimensional residues, which generalizes the classical residue formula and reads as follows.

> *Fix a partial flag of irreducible subvarieties* $\{V_n \supset \cdots \supset \hat{V}_k \supset \cdots \supset V_0\}$, *where* V_k *is omitted* $(0 < k < n)$. *Then*

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$$\sum_{V_{k+1} \supset X \supset V_{k-1}} \mathrm{res}_{V_n \supset \cdots \supset X \supset \cdots \supset V_0}(\omega) = 0,$$

where the sum is taken over all irreducible k-dimensional subvarieties X such that $V_{k+1} \supset X \supset V_{k-1}$.

More precisely, this theorem states that (a) there are only finitely many summands that are not zero and (b) the sum of these nonzero summands is zero.

In addition, if V_1 is proper (compact in the complex case) then one has the same relation for $k = 0$:

$$\sum_{x \in V_1} \mathrm{res}_{V_n \supset \cdots \supset V_1 \supset \{x\}}(\omega) = 0.$$

Again, there are finitely many nonzero summands and their sum is zero.

All these papers are purely algebraic. The methods used by Parshin, Beilinson, and Lomadze are applicable in very general settings; they are not restricted to complex numbers. However, in the complex case one would expect a more geometric variant of the theory.

Brylinski and McLaughlin [BMc] offer a more topological treatment of the complex case. Given a flag $F = \{V_n \supset \cdots \supset V_0\}$, they introduce *flag-localized* homology groups $H_*^{V_i}(V_n; F)$ and a homology class $k_F \in H_n^{V_n}(V_n; F)$ such that

$$\mathrm{res}_F\, \omega = \frac{1}{(2\pi i)^n} \int_{k_F} \omega$$

for any meromorphic n-form ω. The class k_F is obtained from the fundamental class $c_{V_0} \in H_{2n}(V_n, V_n \backslash V_0)$ by applying the boundary homomorphisms in the appropriate flag-localized homology groups n times. Brylinski and McLaughlin mention that the class k_F could be constructed in a more geometric way so that it is naturally represented by a union of certain real n-tori. However, they describe the construction only for the case when all elements of the flag F are smooth.

In this paper we develop a different approach to the construction of the class k_F. Namely, we use the geometry of the Whitney stratified spaces to introduce the operators $\phi_{X,Y} \colon H_*(X) \to H_{*+k-n-1}(Y)$ for couples of consecutive strata $X < Y$ ($\dim X = n$, $\dim Y = k$; see Definition 2.4) of a stratified space. We call these operators the *Leray coboundary operators* by analogy with the Leray operator $\phi\colon H_*(N) \to H_{*+m-1}(M \backslash N)$ for a smooth manifold M and a submanifold $N \subset M$ of codimension m.

Given a flag $F = \{V_n \supset \cdots \supset V_0\}$ and a meromorphic top-form ω on V_n, one can choose a stratification of V_n such that the flag F consists of closures of strata and ω is regular on the top-dimensional stratum. Then one can construct the homology class $\Delta_F := \phi_{\check{V}_{n-1}, \check{V}_n} \circ \cdots \circ \phi_{\check{V}_0, \check{V}_1}([V_0]) \in H_n(\check{V}_n)$, where $\check{V}_k$ is the unique k-dimensional stratum in V_k. In Section 3.2 we prove the following formula (Theorem 3.2):

$$\mathrm{res}_F\, \omega = \frac{1}{(2\pi i)^n} \int_{\Delta_F} \omega.$$

The construction of the Leray coboundary operators is very geometric. In particular, the class Δ_F is naturally represented by a smooth submanifold $\tau_F \subset \check{V}_n$, which is a union of smooth n-dimensional tori $\tau_F = \bigcup \tau_{F,a_i}$.

In the original Parshin construction, the residue at the flag F is actually defined as a sum of certain more delicate residues (we briefly review Parshin's definitions in Section 3.1). We will show that the tori τ_{F,a_i} naturally correspond to the summands in Parshin's definition.

EXAMPLE 1.1. Let $S \subset \mathbb{C}^3$ be the algebraic surface given by the equation $\{xyz^2 + x^4 + y^4 = 0\}$. Consider the flag $F = \{V_2 \supset V_1 \supset V_0\}$, where V_2 is the surface S, V_1 is the z-axis (which is the singular locus of S), and V_0 is the origin. The intersection of S with the real plane is the cone over a figure eight; see Figure 1, which helps to visualize this example.

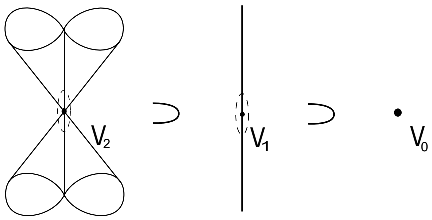

Figure 1 Intersection of the flag F with the real space (Example 1.1)

There is a natural stratification of S consisting of three strata: the origin, the z-axis without the origin, and the regular part of S. In line with our previous notation, we denote the strata $\check{V}_0$, $\check{V}_1$, and $\check{V}_2$, respectively.

The point V_0 is on the complex line V_1. One can consider a small circle τ^1 going counterclockwise around V_0 on V_1. This circle τ^1 naturally represents the class $\phi_{\check{V}_0,\check{V}_1}([V_0]) \in H_1(\check{V}_1)$.

In the next step, we have two branches of V_2 at each point of τ^1. Take a point $x \in \tau^1$. Consider a transversal section to V_1 through x; its intersection with V_2 is a curve with two local branches at x. Consider two small circles S_1 and S_2 around x, one on each branch.

One can choose transversal sections to V_1 at each point of τ^1 such that they depend nicely on the point $x \in \tau^1$. Furthermore, one can choose the circles in such a way that they form a fiber bundle over τ^1. Clearly, the local branches of V_2 do *not* interchange as one goes around the origin along τ^1. Hence in $\check{V}_2$ there are two tori, τ_{F,a_1} and τ_{F,a_2}.

The Parshin residue of a meromorphic 2-form ω on S can be computed via integration over $\tau_F := \tau_{F,a_1} \cup \tau_{F,a_2}$ as follows:

$$\operatorname{res}_F \omega = \frac{1}{(2\pi i)^2} \int_{\tau_F} \omega.$$

EXAMPLE 1.2. Consider the Whitney umbrella, which is the surface $S \subset \mathbb{C}^3$ given by the equation $\{y^2 - zx^2 = 0\}$. Consider the flag $F = \{V_2 \supset V_1 \supset V_0\}$;

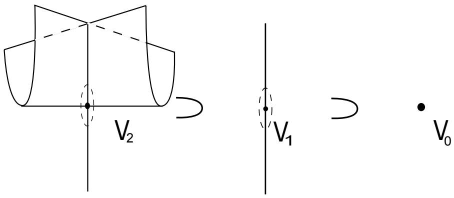

Figure 2 Intersection of the flag F with the real space (Example 1.2)

here, as before, V_2 is the surface, V_1 is the z-axis, and V_0 is the origin (see Figure 2). Observe that V_1 again coincides with the singular locus of S.

In the same way as in the previous example, one can consider a small loop around the origin on the V_1. And once again, V_2 has two branches at each point of the loop. In this case, however, the branches *do* interchange as one goes around the origin on V_1. So here the class F_Δ is represented by just one torus and there is only one summand in Parshin's definition of the residue.

Coboundary operators satisfy an interesting relation. Let $X < Y$ be two strata such that there exist k intermediate strata $Z_1, \ldots, Z_k$, and these intermediate strata are incomparable (equivalently, for any m from 1 to k, $X < Z_m < Y$ are consecutive strata). Then

$$\phi_{Z_1,Y} \circ \phi_{X,Z_1} + \phi_{Z_2,Y} \circ \phi_{X,Z_2} + \cdots + \phi_{Z_k,Y} \circ \phi_{X,Z_k} = 0$$

(see Theorem 2.2.)

This relation is illustrated by the following example.

EXAMPLE 1.3. In Figure 3, X is the origin, Z_1 is a half-line, Z_2 is a surface with an isolated singularity at the origin, and $Y = \mathbb{R}^3 \setminus (X \cup Z_1 \cup Z_2)$. We take a small

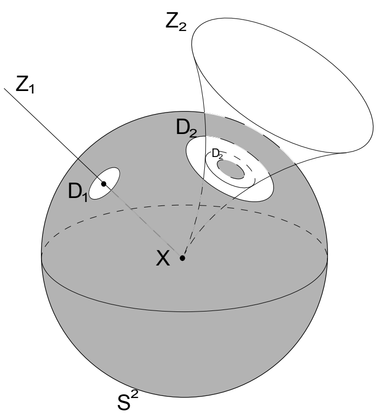

Figure 3 Example 1.3

sphere S^2 with center at the origin. Then $\phi_{X,Z_i}([X]) \in H_{\dim Z_i - 1}(Z_i)$ is represented by the intersection $N_i = S^2 \cap Z_i$. Take a small neighborhood of N_i in S^2. Its boundary D_i represents the class $\phi_{Z_i,Y} \circ \phi_{X,Z_i}([X]) \in H_1(Y)$. Then the sphere S^2 with the neighborhoods of the N_i deleted gives a 2-dimensional chain in Y whose boundary is the union $D_1 \cup D_2$.

Our approach also allows one to prove an interesting result about Parshin residues. Let ω be a meromorphic top-form on V_n, and consider any Whitney stratification of V_n such that ω is regular on the top-dimensional stratum. Then the residue $\operatorname{res}_F \omega$ can be nontrivial only if all elements of the flag F are closures of strata of the stratification (Theorem 3.3). In particular, there are only finitely many nontrivial residues for a given form.

The reciprocity law for Parshin residues in the complex case follows from the preceding results. Indeed, given a partial flag of subvarieties and a meromorphic form, one can choose a Whitney stratification such that (a) the elements of the flag are closures of strata and (b) the form is regular on the open stratum. Then, by Theorem 3.3, the only nonzero summands in the reciprocity law correspond to the flags consisting of closures of strata, and the reciprocity law then follows from the relation on coboundary operators (Theorem 2.2) and the formula for Parshin residues via coboundary operators (Theorem 3.2).

The rest of the paper proceeds as follows. In Section 2 we introduce the Leray coboundary operators for stratified spaces and prove the relation (Theorem 2.2). In Section 3 we use the results from Section 2 to express the Parshin residue as an integral over a real smooth cycle and to prove the reciprocity law. Section 2.1 offers a short introduction to the theory of stratified spaces, and in Section 3.1 we review Parshin's original definitions and formulation of the reciprocity law.

This paper constitutes the first part of the author's Ph.D. thesis under the supervision of Professor Askold Khovanskii. A short announcement of the main results of the thesis is available in [M2]. In this paper we include complete proofs as well as some examples. The second part of the thesis, which concerns applying "resolution of singularities" techniques to the theory of Parshin residues, is available in [M1].

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2. Leray Coboundary Operators for Stratified Spaces

2.1. Whitney Stratifications and Mather's Abstract Stratified Spaces

Definition 2.1. Let M be a smooth manifold, and let V be a locally closed subset of M. By a *Whitney stratification* $\mathbf{S}$ of V we mean a subdivision of V into smooth strata such that the following statements hold.

1. The subdivision **S** is *locally finite*—in other words, each point of V has an open neighborhood that intersects only finitely many strata.
2. *Condition of the frontier:* For each stratum $X \in \mathbf{S}$, its boundary $(\bar{X} \setminus X) \cap V$ is a union of strata.
3. Each pair (X, Y) of strata satisfies the *Whitney conditions* (a) and (b).
   (a) For any $x \in X$ and any sequence $\{y_n\} \in Y$ such that $y_n \to x$, if the sequence of tangent planes $T_{y_n}Y$ converges to some plane $\tau \subset T_x M$ (in the appropriate Grassmanian bundle over M) then $T_x X \subset \tau$.
   (b) For any $x \in X$, any sequence $\{y_n\} \in Y$, and any sequence $\{x_n\} \in X$ such that $y_n \to x$ and $x_n \to x$, if the sequence of tangent planes $T_{y_n}Y$ converges to some plane $\tau \subset T_x M$ and if the sequence of secants $\overline{x_n y_n}$ converges to some line l (in some smooth coordinate system in M) then $l \subset \tau$.

REMARK. One can show that condition (a) is implied by condition (b); therefore, it is enough to require the latter. One can prove that if a pair of strata (X, Y) satisfies condition (b) and if $\bar{Y} \cap X \neq \emptyset$, then $\dim X < \dim Y$.

NOTATION. We say that $X < Y$ if $\bar{Y} \cap X \neq \emptyset$. One can see that this defines a partial order on the set of strata **S**.

EXAMPLE 2.1. Consider the surface in $\mathbb{C}^3$ given by the equation $y^2 + x^3 - z^2 x^2 = 0$ (see Figure 4). The singular locus of the surface coincides with the z-axis. Thus, the z-axis and its complement give a subdivision of the surface into two smooth pieces. It is easy to show that this pair satisfies condition (a) but does not satisfy condition (b) at the origin. Note that the small neighborhood of the origin looks much different from the neighborhood of any other point of the z-axis.

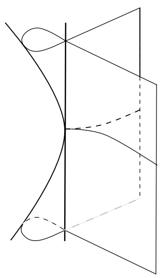

Figure 4 Intersection of the surface with the real space (Example 2.1)

It is easy to improve the subdivision in such a way that it satisfies condition (b); one need only consider the origin as a separate stratum.

Whitney showed that if conditions (a) and (b) are satisfied for the pair (X, Y), then Y “behaves regularly” along X.

THEOREM 2.1 (see e.g. [GMac]). *Let V be a closed subvariety in a smooth algebraic variety M, and let Σ be a locally finite family of subvarieties in V. Then* *there exists a Whitney stratification of the set V such that each element of Σ is a union of strata and all strata are algebraic.*

A detailed review of the theory of Whitney stratifications can be found in [GMac].

The notion of an abstract stratified space, introduced by Mather [Ma], provides a convenient setup for working with "nice" stratifications: subdivisions into smooth pieces with regular behavior along strata. Mather proved that any Whitney stratification can be endowed with a structure of an abstract stratified space, a notion that will be described next.

Let V be a Hausdorff locally compact topological space that satisfies the second countability axiom (i.e., there is a countable basis in the topology of V). Let $\mathbf{S}$ be a locally finite subdivision of V into topological manifolds endowed with smoothness structures; the elements of $\mathbf{S}$ are called *strata*. Let $\mathbf{S}$ satisfy the condition of the frontier: the boundary of any stratum is a union of strata. Similarly as for Whitney stratifications, the set of strata $\mathbf{S}$ inherits the natural partial order ($X < Y$ if $X \subset \partial Y$).

For every $X \in \mathbf{S}$, let U_X be a neighborhood of X in V, let $\rho_X\colon U_X \to \mathbb{R}_{\geq 0}$ be a continuous function, and let $\pi_X\colon U_X \to X$ be a retraction. One should think of ρ_X as of the distance to X. Therefore, we require that $X = \{\rho_X = 0\}$. It is also convenient to say that $\rho_X(y) = \infty$ if $y \notin U_X$. We call ρ_X the *tubular function* and call U_X the *tubular neighborhood*.

Let $X, Y \in \mathbf{S}$, $X \neq Y$. We use the following notation:

$$\begin{aligned} U_{X,Y} &:= U_X \cap Y; \\ \rho_{X,Y} &:= \rho_X|_{U_{X,Y}}\colon U_{X,Y} \to \mathbb{R}_+; \\ \pi_{X,Y} &:= \pi_X|_{U_{X,Y}}\colon U_{X,Y} \to X. \end{aligned}$$

We assume that $U_{X,Y}$ is empty unless $X < Y$. We also assume that if X and Y are incomparable then $U_X \cap U_Y$ is empty.

We have the compatibility conditions

$$\begin{aligned} \pi_{X,Y}(\pi_{Y,Z}(v)) &= \pi_{X,Z}(v), \\ \rho_{X,Y}(\pi_{Y,Z}(v)) &= \rho_{X,Z}(v) \end{aligned}$$

whenever both sides of these equations are defined.

The following two conditions ensure that the space V behaves regularly along strata: (i) for any $X \in \mathbf{S}$, the map

$$(\pi_X, \rho_X)|_{U_X \setminus X}\colon U_X \setminus X \to X \times \mathbb{R}_+$$

is a locally trivial fibration with a compact fiber; (ii) for any $Y > X$, the restriction

$$(\pi_{X,Y}, \rho_{X,Y})\colon U_{X,Y} \to X \times \mathbb{R}_+$$

is a smooth fibration.

Finally, we want the fiber of π_X over a point $x \in X$ to be a cone with the vertex at x. Because that does not follow from the conditions stated so far, we must add one more. Let $U_X^{\leq 1} = \{y \in U_X \mid \rho_X(y) \leq 1\}$ and $N_X = \partial U_X^{\leq 1} = \{y \in U_X \mid \rho_X(y) = 1\}$. Then $\pi_X|_{U_X^{\leq 1}}\colon U_X^{\leq 1} \to X$ is the mapping cone over $\pi_X|_{N_X}\colon N_X \to X$.

DEFINITION 2.2. The triple $\mathbf{J} = \{\{U_X\}, \{\pi_X\}, \{\rho_X\}\}$ is called *control data*.

DEFINITION 2.3. The triple $\{V, \mathbf{S}, \mathbf{J}\}$ under the conditions just specified is called an *abstract stratified space*.

It follows that N_X has a natural structure of an abstract stratified space, as obtained by intersecting the strata of V with N_X and restricting tubular functions and retractions.

The original definition (in [Ma]) of abstract stratified spaces is slightly different and less restrictive. However, it is clear that shrinking the tubular neighborhoods and rescaling the tubular functions allows one to change the control data so that they satisfy the conditions stipulated here.

2.2. *Leray Coboundary Operators and Relations*

Let $f: M \to N$ be a smooth fibration with compact oriented k-dimensional fiber F. Then one can define the Gysin homomorphism on homology $f^*: H_*(N) \to H_{*+k}(M)$. In essence one can simply set $f^*(a) = [f^{-1}(A)]$, where A is a representative of the homology class $a \in H_*(N)$.

REMARK. We use the following convention about the orientations. Let $y \in M$ and $x = f(y) \in N$. Let $A \subset N$ be a smooth representative of a homology class $a \in H_*(N)$ with $x \in A$. Let the differential form ω_A on N be such that its restriction to A defines the orientation of A at x, and let the differential form ω_F on M be such that its restriction to the fiber F_x defines the orientation of F_x at y. Then the orientation of the preimage $f^{-1}(A) \subset M$ at the point y is given by the restriction of the form $f^*(\omega_A) \wedge \omega_F$.

Let now M be an oriented manifold with boundary, and let $f: M \to N$ be a proper map to an oriented manifold N such that its restriction, both to the boundary $\partial M \subset M$ and the interior $\breve{M} \subset M$, are submersions. Then the Ehresmann lemma for manifolds with boundary implies that f is a locally trivial fibration and that its restrictions to ∂M and $\breve{M}$ are smooth fibrations.

Let $\phi := (f|_{\partial M})^*: H_*(N) \to H_{*+\dim M - \dim N - 1}(\partial M)$ be the Gysin homomorphism.

LEMMA 2.1. $i_* \circ \phi = 0$, *where* $i: \partial M \hookrightarrow M$ *is the embedding.*

Proof. One can generalize the Gysin homomorphism to the case just described when the fiber of f is a manifold with boundary. The only difference is that now the homomorphism lands in the relative homology group: $f^*: H_*(N) \to H_{*+k}(M, \partial M)$, where $k = \dim F = \dim M - \dim N$. Then one immediately sees that $\phi = \partial \circ f^*$, where $\partial: H_*(M, \partial M) \to H_{*-1}(\partial M)$ is the boundary homomorphism from the long exact sequence of the pair $(M, \partial M)$. However, by the long exact sequence, $i_* \circ \partial = 0$. □

We next apply the foregoing constructions to the stratified spaces.

Let all the strata of a stratified space V be oriented. Let $X \in \mathbf{S}$ be a stratum. Restriction of the retraction $\pi_X: U_X \to X$ to $N_X = \{y \in U_X \mid \rho_X(y) = 1\}$ is a locally trivial fibration. Moreover, for any stratum Y such that $X < Y$, the restriction to $N_{X,Y} = N_X \cap Y = \{y \in U_{X,Y} \mid \rho_X(y) = 1\}$ is a smooth fibration.

DEFINITION 2.4. Let $X < Y$ be two strata. We say that X and Y are *consecutive strata* if there is no Z such that $X < Z < Y$.

LEMMA 2.2. *Let $X < Y$ be consecutive strata. Then the fiber of $\pi_X|_{N_{X,Y}} : N_{X,Y} \to X$ is compact.*

Proof. Since $X < Y$ are consecutive strata, it follows that $N_{X,Y} = Y \cap N_X$ is a closed stratum of N_X (indeed, otherwise the closure of $N_{X,Y}$ in N_X would contain a smaller stratum). The fiber of the restriction of π_X to $N_{X,Y}$ is the intersection of the fiber of the restriction of π_X to N_X and $N_{X,Y}$. Therefore, it is compact as a closed subset of a compact set. □

Note that $N_{X,Y}$ is orientable; indeed, it is the level set of a smooth function $\rho_{X,Y}$ in $U_{X,Y} \subset Y$. Let us fix the orientation of $N_{X,Y}$ as follows. We say that the restriction of a differential $(\dim Y - 1)$-form $\omega_{N_{X,Y}}$ on Y defines the positive orientation of $N_{X,Y}$ if the form $d\rho_{X,Y} \wedge \omega_{N_{X,Y}}$ defines the positive orientation of Y.

Let $\dim X = n$ and $\dim Y = k$.

DEFINITION 2.5. The *Leray coboundary operator* $\phi_{X,Y} : H_*(X) \to H_{*+k-n-1}(Y)$ is given by the composition $\phi_{X,Y} = i_* \circ \phi$, where $i \colon N_{X,Y} \hookrightarrow Y$ is the embedding and $\phi \colon H_*(X) \to H_{*+k-n-1}(N_{X,Y})$ is the Gysin homomorphism.

THEOREM 2.2. *Let $X < Y$ be two strata and let $Z_1, \dots, Z_m$ be all strata such that $X < Z_i < Y$. Suppose that $Z_1, \dots, Z_m$ are incomparable. Then*

$$\phi_{Z_1,Y} \circ \phi_{X,Z_1} + \phi_{Z_2,Y} \circ \phi_{X,Z_2} + \cdots + \phi_{Z_m,Y} \circ \phi_{X,Z_m} = 0.$$

Proof. We want to apply Lemma 2.1. Consider $D_i := N_{X,Y} \cap N_{Z_i,Y} = \{y \in Y \mid \rho_{Z_i}(y) = \rho_X(y) = 1\}$, and note that $D_i = (\pi_{Z_i}|_{N_{Z_i,Y}})^{-1}(N_{X,Z_i})$. Therefore, $\pi_{Z_i}|_{D_i}$ is a smooth fibration over N_{X,Z_i}. Let $p_i := \pi_X|_{N_{X,Z_i}} \circ \pi_{Z_i}|_{D_i} : D_i \to X$. By construction of the Leray coboundary operators we have

$$\phi_{Z_i,Y} \circ \phi_{X,Z_i} = i_* \circ \phi_i,$$

where $i \colon D_i \hookrightarrow Y$ is the embedding and $\phi_i \colon H_*(X) \to H_{*+\dim Y - \dim X - 2}$ is the Gysin homomorphism of $p_i \colon D_i \to X$. Here we fix the orientation of D_i as follows. We say that the restriction of a differential $(\dim Y - 2)$-form ω_{D_i} on Y defines the positive orientation of D_i if the form $d\rho_{Z_i,Y} \wedge d\rho_{X,Y} \wedge \omega_{D_i}$ defines the positive orientation of Y.

Consider now $N_{X,Y} = \{y \in Y \mid \rho_X(y) = 1\}$. The restriction $\pi_X|_{N_{X,Y}}$ is a smooth fibration, but the fibers of this fibration are not compact. On the other hand, if we consider the restriction of π_X to the union $N_{X,Y \cup Z_1 \cup \cdots \cup Z_m} := N_{X,Y} \cup N_{X,Z_1} \cup \cdots \cup N_{X,Z_m} = N_X \cap (Y \cup Z_1 \cup \cdots \cup Z_m)$, then the fibers are compact.

Now $D_i \subset N_{X,Y}$ can be viewed as the boundary of the neighborhood $U_i = \{y \in N_{X,Y} \cap U_{Z_i} \mid \rho_{Z_i}(y) < 1\}$ of N_{X,Z_i} in $N_{X,Y \cup Z_1 \cup \cdots \cup Z_m}$. Denote $M = N_{X,Y} \backslash (U_1 \cup \cdots \cup U_m)$. By the Ehresmann lemma for manifolds with boundary, the restriction $\pi_X|_M \colon M \to X$ is a locally trivial fibration. Indeed, $\pi_X|_M$ is proper because M is a closed subset of $N_{X,Y\cup Z_1\cup\cdots\cup Z_m}$ and $\pi_X|_{N_{X,Y\cup Z_1\cup\cdots\cup Z_m}}$ is a fibration with compact fibers; the restrictions of π_X to the interior of M and the boundary $\partial M = D_1\cup\cdots\cup D_m$ are submersions.

To conclude the proof by Lemma 2.1, one needs to check that the orientation of D_i as a piece of the boundary of M always coincides with (or always is opposite to) the orientation of D_i used in the first part of the proof. Recall that we fixed the orientation of D_i in such a way that, if $\omega_{D_i}|_{D_i}$ gives the orientation of D_i, then $d\rho_{Z_i,Y}\wedge d\rho_{X,Y}\wedge\omega_{D_i}$ gives the orientation of Y. Let $\omega_{N_{X,Y}} := -d\rho_{Z_i,Y}\wedge\omega_{D_i}$. According to our convention about the orientation of $N_{X,Y}$, $\omega_{N_{X,Y}}$ gives the positive orientation of $N_{X,Y}$. Therefore, the orientation of D_i as a piece of the boundary of M is given by $-\omega_{D_i}$. □

2.3. Dual Homomorphism

In this section the coefficient ring is always $\mathbb{R}$. For simplicity, we omit this in the notation.

The following question naturally arises: Which operator is Poincaré dual to the coboundary operator $\phi_{X,Y}$?

The manifolds X and Y are not compact, so one must use the Borel–Moore homology to achieve the Poincaré duality. A nice review of the theory of Borel–Moore homology (in much more detail than needed here) is given in [Gin].

Consider $\phi_{X,Y}\colon H_m(X)\to H_{m+k-n-1}(Y)$ (here $\dim X = n$ and $\dim Y = k$). The dual operator is $(\phi_{X,Y})^*\colon H^{\mathrm{BM}}_{n-m+1}(Y)\to H^{\mathrm{BM}}_{n-m}(X)$. There is a natural candidate for the dual; indeed, one can show that

$$H^{\mathrm{BM}}_{n-m+1}(Y) = H^{\mathrm{BM}}_{n-m+1}(Y\cup X, X).$$

Hence there exits the boundary operator

$$\partial_{Y,X}\colon H^{\mathrm{BM}}_{n-m+1}(Y)\to H^{\mathrm{BM}}_{n-m}(X).$$

REMARK. It is crucial that $X<Y$ are consecutive strata. Otherwise, the union $X\cup Y$ would not be locally compact and so the boundary operator would not be defined.

THEOREM 2.3. *The Leray coboundary operator* $\phi_{X,Y}\colon H_m(X)\to H_{m+k-n-1}(Y)$ ($\dim X = n$, $\dim Y = k$) *is Poincaré dual to the boundary homomorphism* $\partial_{Y,X}\colon H^{\mathrm{BM}}_{n-m+1}(Y)\to H^{\mathrm{BM}}_{n-m}(X)$.

Proof. By Poincaré duality, the intersection form $H_*(M)\times H^{\mathrm{BM}}_{d-*}(M)\to\mathbb{R}$ is well-defined and nondegenerate (here M is a smooth oriented manifold and $\dim M = d$). Thus we need only check that, for any classes $a\in H_n(X)$ and $b\in H^{\mathrm{BM}}_{m-n+1}(Y)$,

$$\langle\partial_{Y,X}b, a\rangle = \langle b, \phi_{X,Y}(a)\rangle.$$

Let $i\colon U_{X,Y}\hookrightarrow Y$ be the embedding. According to the definition of the Leray coboundary operator, $\phi_{X,Y}$ can be factored: $\phi_{X,Y} = i_*\circ\phi_{X,U_{X,Y}}$, where $\phi_{X,U_{X,Y}}\colon H_m(X)\to H_{m+k-n-1}(U_{X,Y})$ is the Leray coboundary operator for the stratified space with two strata, X and $U_{X,Y}$. On the other hand, the boundary homomorphism $\partial_{Y,X}$ also can be factored: $\partial_{Y,X} = \partial_{U_{X,Y},X} \circ i^*$ (here $i^*\colon H^{\mathrm{BM}}(Y) \to H^{\mathrm{BM}}(U_{X,Y})$ is the restriction homomorphism induced by the inclusion i). Therefore, it is enough to assume that $Y = U_{X,Y}$.

We know that $U_{X,Y}$ is diffeomorphic to $N_{X,Y} \times \mathbb{R}_+$. Hence there is an isomorphism $\theta\colon H_*^{\mathrm{BM}}(U_{X,Y}) \xrightarrow{\sim} H_{*-1}^{\mathrm{BM}}(N_{X,Y})$ given by taking a representative that is transversal to $N_{X,Y}$ and then intersecting it with $N_{X,Y}$. The inverse isomorphism θ^{-1} is derived from multiplying a representative by $\mathbb{R}_+$.

We remark that one should be careful with the orientations. We want the following condition to be satisfied. Let $B \subset U_{X,Y}$ be a cycle transversal to $N_{X,Y}$ such that $[B] = b \in H_*^{\mathrm{BM}}(U_{X,Y})$, and let $C = B \cap N_{X,Y}$ such that $[C] = \theta(b) \in H_*^{\mathrm{BM}}(N_{X,Y})$. Then a form ω_C gives the positive orientation of C at a point in C if and only if the form $d\rho_{X,Y} \wedge \omega_C$ gives the positive orientation of B at this point.

With the orientation conventions just described, we have

$$\langle b, \phi_{X,Y}(a)\rangle = \langle \theta(b), \phi(a)\rangle;$$

here $\phi\colon H_*(X) \to H_{*+k-n-1}(N_{X,Y})$ is the Gysin homomorphism and the intersection on the right is taken inside $N_{X,Y}$. Moreover,

$$\partial_{U_{X,Y},X} = (\pi_X|_{N_{X,Y}})_* \circ \theta.$$

Therefore, we need only check that the Gysin homomorphism is dual to $(\pi_X|_{N_{X,Y}})_*\colon H_{n-m}(N_{X,Y}) \to H_{n-m}(X)$, which is obvious. □

COROLLARY. *The Leray coboundary operator $\phi_{X,Y}$ does not depend on the choice of the control data, at least modulo torsion.*

One can also investigate the relation that is dual to the relation, proved in Theorem 2.2, on the coboundary operators.

Consider the strata $X, Z_1, \dots, Z_p, Y$ satisfying the conditions of Theorem 2.2, and let $Z = \bigcup Z_i$. Then the boundary operator

$$\partial_{Y,Z}\colon H_*^{\mathrm{BM}}(Y) \to H_{*-1}^{\mathrm{BM}}(Z) = \bigoplus H_{*-1}^{\mathrm{BM}}(Z_i)$$

is dual to the direct sum of the coboundary operators $\bigoplus \phi_{Z_i,Y}$. In turn, the boundary operator

$$\partial_{Z,X}\colon H_{*-1}^{\mathrm{BM}}(Z) \to H_{*-2}^{\mathrm{BM}}(X)$$

is dual to the direct sum $\bigoplus \phi_{X,Z_i}$. Therefore, the dual relation is

$$\partial_{Z,X} \circ \partial_{Y,Z} = 0.$$

However, it is not hard to prove this relation independently. Indeed, one can use that $H_*^{\mathrm{BM}}(Y) = H_*^{\mathrm{BM}}(Y \cup Z \cup X, Z \cup X)$ and $H_*^{\mathrm{BM}}(Z) = H_*^{\mathrm{BM}}(Z \cup X, X)$. Then, in essence, the equality states that the boundary of the boundary of a chain is zero, which is trivial. Thus we have given another proof of Theorem 2.2 modulo torsion.

REMARK. It is crucial for this argument that $Y \cup Z \cup X$ be locally compact. That is, the dual relation would not hold if one were to forget one of the intermediate strata Z_i.

3. Application to Parshin Residues

3.1. Parshin Residues and the Reciprocity Law

In this section we review the definition of the Parshin residue and the reciprocity law. As discussed in the Introduction, the Parshin residue at a flag F is defined as a sum of certain more delicate residues. In fact, every flag "contains" finitely many *Parshin points,* and the more delicate residues are computed at these points.

We start from the definition of a Parshin point. Let V_n be an algebraic variety of dimension n, and let $F =: \{V_n \supset \cdots \supset V_0\}$ be a flag of subvarieties of dimensions $\dim V_k = k$. Now consider the diagram

$$\begin{tikzcd}
V_n \arrow[r, phantom, "\supset"] & V_{n-1} \arrow[r, phantom, "\supset"] & \cdots \arrow[r, phantom, "\supset"] & V_1 \arrow[r, phantom, "\supset"] & V_0 \\
\tilde{V}_n \arrow[u, "p_n"] \arrow[r, phantom, "\supset"] & W_{n-1} \arrow[u, "p_n"] & & & \\
 & \widetilde{W}_{n-1} \arrow[u, "p_{n-1}"] \arrow[r, phantom, "\supset"] & \cdots & & \\
 & & \cdots \arrow[r, phantom, "\supset"] & W_1 & \\
 & & & \widetilde{W}_1 \arrow[u, "p_1"] \arrow[r, phantom, "\supset"] & W_0,
\end{tikzcd} \tag{$*$}$$

where the following statements hold.

1. $p_n\colon \tilde{V}_n \to V_n$ is the normalization.
2. $W_{n-1} \subset \tilde{V}_n$ is the union of $(n-1)$-dimensional irreducible components of the preimage of V_{n-1}.
3. For every $k = 1, 2, \ldots, n-1$:
   (a) $p_k\colon \widetilde{W}_k \to W_k$ is the normalization;
   (b) $W_{k-1} \subset \widetilde{W}_k$ is the union of $(k-1)$-dimensional irreducible components of the preimage of V_{k-1}.

DEFINITION 3.1. We call diagram $(*)$ the *normalization diagram* of the flag $V_n \supset \cdots \supset V_0$.

DEFINITION 3.2. The flag $F = \{V_n \supset \cdots \supset V_0\}$ of irreducible subvarieties together with the choice of a point $a_\alpha \in W_0$ is called a *Parshin point.*

Choosing a point $a_\alpha \in W_0$ is equivalent to choosing irreducible components in every W_i, $i = n-1, \ldots, 0$. Indeed, $\widetilde{W}_i$ is normal and therefore locally irreducible at every point. In particular, it is locally irreducible at the image of a_α. Let $\widetilde{W}_i^\alpha$ be the irreducible component of $\widetilde{W}_i$ that contains the image of a_α. Let $W_i^\alpha = p_i(\widetilde{W}_i^\alpha)$. Note that W_i^α is an irreducible component of W_i.

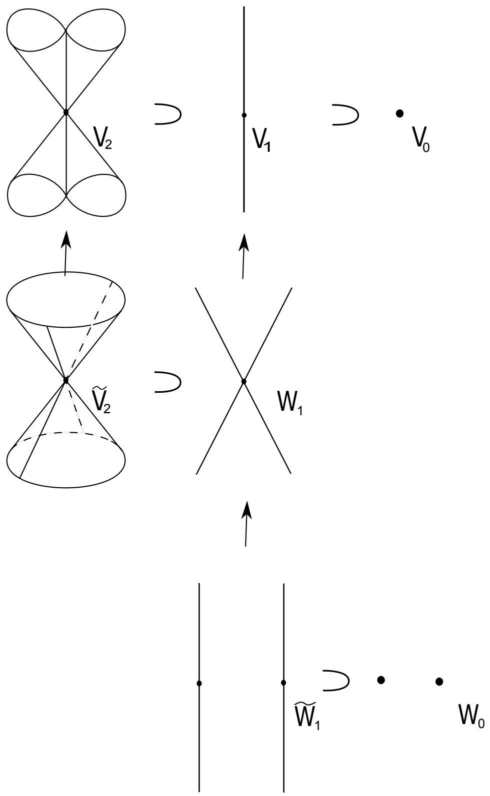

Figure 5 Intersection of the flag F with the real space (Example 3.1): normalization splits the local irreducible components at every point; hence normalization of the cone over the figure eight is the usual cone, and the preimage of the z-axis is two lines intersecting at the origin

EXAMPLE 3.1. Consider the flag from the Example 1.1 (see Figure 5). It follows that there are two Parshin points corresponding to the flag. Note that these points naturally correspond to the tori from Example 1.1.

EXAMPLE 3.2. The normalization of the Whitney umbrella (Example 1.2) is isomorphic to $\mathbb{C}^2$. The preimage of the z-axis is a line that covers the z-axis twice with a branching at the origin. Therefore, W_0 is just one point. This corresponds to the existence of only one torus in Example 1.2.

Defining the Parshin residue requires that one first define the *local parameters* at a Parshin point, which play the same role that the normalizing parameter does in the 1-dimensional case. After that, one uses these parameters to define a sequence of residual meromorphic forms $\omega_{n-1}, \ldots, \omega_0$ on $W_{n-1}^{\alpha}, \ldots, W_0^{\alpha}$.

The local parameters are defined as follows: $W_{i-1}^{\alpha} \subset \widetilde{W}_i$ is a hypersurface in a normal variety. It follows that there exists a (meromorphic) function u_i on $\widetilde{W}_i$ that has zero of order 1 at a generic point of W^{α}_{i-1}. Since meromorphic functions are the same on W_i and $\widetilde{W}_i$, one can consider u_i as a function on W_i. Then one can extend (in an arbitrary way) u_i to $\widetilde{W}_{i+1}$ and so on. For simplicity, we denote all these functions by u_i. Now the u_i are defined on V_n and can be consecutively restricted to W_j for $j \geq i$.

DEFINITION 3.3. Functions $(u_1, \dots, u_n)$ are called *local parameters* at the Parshin point $P = \{V_n \supset \cdots \supset V_0,\, a_\alpha\}$.

REMARK. One can choose local parameters in such a way that u_i has zero of order 1 at a generic point not only of W^{α}_{i-1}, but of the whole W_{i-1}. Then these local parameters work for all Parshin points with the flag $F = \{V_n \supset \cdots \supset V_0\}$. We only use local parameters with this property.

Let ω be a meromorphic n-form on V_n. One can show that the differentials $du_1, \dots, du_n$ are linearly independent at a generic point of V_n. Therefore, one can write

$$\omega = f\, du_1 \wedge \cdots \wedge du_n,$$

where f is a meromorphic function on V_n.

Now we define the residual forms ω_i. Take a generic point $p \in W^{\alpha}_{n-1}$. Both $\tilde{V}_n$ and W_{n-1} are smooth at p. Moreover, the parameters $u_1, \dots, u_n$ provide an isomorphism of a neighborhood of p to an open subset in $\mathbb{C}^n$, and W^{α}_{n-1} is given by the equation $u_n = 0$ in this neighborhood. Restrict the function f to the transversal section to W_{n-1} at p that is given by fixing the parameters $u_1, \dots, u_{n-1}$. The restriction can be expanded into a Laurent series in u_n. It is easy to see that the coefficients of this expansion depend analytically on p. Moreover, one can see that the coefficients are meromorphic functions on W^{α}_{n-1}. Let f_{-1} be the coefficient at u_n^{-1} in this expansion. Then $\omega_{n-1} = f_{-1}\, du_1 \wedge \cdots \wedge du_{n-1}$ is a meromorphic $(n-1)$-form on W^{α}_{n-1}.

Repeating this procedure one more time yields a meromorphic $(n-2)$-form on W^{α}_{n-2}. Finally, after n steps, one obtains a function ω_0 on the one-point set $W^{\alpha}_0 = \{a_\alpha\}$.

DEFINITION 3.4. The *residue of ω at the Parshin point* $P = \{V_n \supset \cdots \supset V_0,\, a_\alpha \in W_0\}$ is $\operatorname{res}_P(\omega) = \omega_0(a_\alpha)$.

Parshin proves that the residue does not depend on the choice of local parameters.

DEFINITION 3.5. The sum of residues over all $a \in W_0$ is called the *residue at the flag* $F = \{V_n \supset \cdots \supset V_0\}$ and is denoted $\operatorname{res}_F(\omega) = \sum_{a \in W_0} \operatorname{res}_{\{F,a\}}(\omega)$.

THEOREM 3.1 [Bei; L; P]. *Let ω be a meromorphic n-form on V_n. Fix a partial flag of irreducible subvarieties $\{V_n \supset \cdots \supset \hat{V}_k \supset \cdots \supset V_0\}$, where V_k is omitted $(0 < k < n)$. Then*

$$\sum_{V_{k+1} \supset X \supset V_{k-1}} \operatorname{res}_{V_n \supset \cdots \supset X \supset \cdots \supset V_0}(\omega) = 0;$$

here the sum is taken over all irreducible k-dimensional subvarieties X such that $V_{k-1} \supset X \supset V_{k+1}$, *and only finitely many summands are nonzero.*

In addition, if V_1 *is compact then one has the same relation for* $k = 0$.

3.2. Residues via Leray Coboundary Operators and the Reciprocity Law

We want to apply the stratification theory to study the Parshin points and residues. Hence we must stratify all the spaces in the normalization diagram in such a way that the stratifications respect the normalization maps $p_1, \dots, p_n$. The following lemma easily follows from well-known results on the existence of Whitney stratifications (see e.g. [GMac, Sec. 1.7]).

NOTATION. Let X be an irreducible (complex analytic) variety considered with a fixed Whitney stratification; then by $\breve{X}$ we denote the stratum of maximal dimension. If X is reducible, then by $\breve{X}$ we denote the union of strata of maximal dimension.

LEMMA 3.1. *Fix a Parshin point* $P = \{V_n \supset \cdots \supset V_0, a_\alpha \in W_0\}$ *and local parameters* $u_1, \dots, u_n$. *Then there exist Whitney stratifications* $\mathbf{S}, \mathbf{S}_{\tilde{V}}, \mathbf{S}_{\widetilde{W}_{n-1}}, \dots, \mathbf{S}_{\widetilde{W}_1}$ *of* (*respectively*) $V_n, \tilde{V}_n, \widetilde{W}_{n-1}, \dots, \widetilde{W}_1$ *such that*:

1. $V_{n-1}, \dots, V_0$ *are unions of strata of* $\mathbf{S}$;
2. $W_{n-1}, W_{n-2}, \dots, W_0$ *are unions of strata of* $\mathbf{S}_{\tilde{V}}, \mathbf{S}_{\widetilde{W}_{n-1}}, \dots, \mathbf{S}_{\widetilde{W}_1}$, *respectively*;
3. *for all* $i = 1, \dots, n$, *the local parameter* u_i *is regular and nonvanishing on* $\breve{V}_n, \breve{\tilde{V}}_n, \breve{\widetilde{W}}_{n-1}, \dots, \breve{\widetilde{W}}_i$; *and*
4. *for all* $i = 1, \dots, n$, *the restriction of the normalization map* p_i *to any stratum in the source is a covering over a stratum in the image*.

An important corollary about stratifications $\mathbf{S}_{\widetilde{W}_i}$ is expressed as the following lemma.

LEMMA 3.2. *The stratum* (*or the union of strata if* W_{i-1} *is reducible*) $\breve{W}_{i-1} \in \mathbf{S}_{\widetilde{W}_i}$ *consists of regular points of* $\widetilde{W}_i$.

Proof. We prove the lemma by contradiction. Assume there is a point $x \in \breve{W}_{i-1}$ such that $\widetilde{W}_i$ is singular at x. Observe that, by dimension reasons and the condition of the frontier, the only strata intersecting a small neighborhood of x are $\breve{W}_{i-1}$ and $\breve{\widetilde{W}}_i$. Note that u_i is regular in $\breve{\widetilde{W}}_i$ and at a generic point of $\breve{W}_{i-1}$. Therefore, by the extension theorem for normal varieties, u_i is regular at x.

Note also that u_i is nonvanishing in $\breve{\widetilde{W}}_i$ and has zero of order 1 at a generic point of W_{i-1}. Therefore, the $\{u_i = 0\}$ coincide with W_{i-1} near x. Moreover, it is easy to see that the germ of u_i at x generates the ideal of the germ of W_{i-1} at x. Indeed, if g is a function that is regular at x and vanishing on W_{i-1}, then g/u_i is regular at x by the extension theorem for normal varieties.

Now let $f_1, \dots, f_{i-1}$ be any coordinate system on W_{i-1} at x. Then it is clear that the functions $u_i, f_1, \dots, f_{i-1}$ generate the maximal ideal in the local ring of $\{x\} \subset \widetilde{W}_i$. Hence x is a smooth point of $\widetilde{W}_i$, which contradicts our assumption. □

Our goal is to show that

$$\operatorname{res}_F(\omega) = \frac{1}{(2\pi i)^n} \int_{\Delta_F} \omega,$$

where $F := \{V_n \supset \cdots \supset V_0\}$ and $\Delta_F = \phi_{\check{V}_{n-1},\check{V}_n} \circ \cdots \circ \phi_{\check{V}_0,\check{V}_1}([V_0]) \in H_n(\check{V}_n)$.

Moreover, we will show that Δ_F naturally splits into the sum

$$\Delta_F = \sum_{a_i \in W_0} \Delta_{\{F,a_i\}}$$

such that

$$\operatorname{res}_{\{F,a_i\}}(\omega) = \frac{1}{(2\pi i)^n} \int_{\Delta_{\{F,a_i\}}} \omega.$$

According to the construction of the Leray coboundary operator, Δ_F is represented by a smooth compact real n-dimensional submanifold $\tau_F \subset \check{V}_n$. This submanifold is obtained from a point by the following procedure: there are n steps, and at each step we take the total space of an oriented fibration with 1-dimensional compact fiber over the result of the previous step. Thus, τ_F is a union of n-dimensional tori. We will show that (i) the connected components of τ_F are in natural one-to-one correspondence with the points of W_0 and (ii) the connected component τ_{F,a_i} corresponding to $a_i \in W_0$ represents Δ_{F,a_i}.

Fix control data on the stratification $\mathbf{S}$ of V_n. Let us use these control data to construct the representative $\tau_F \subset \check{V}_n$ of Δ_F. We also denote by $\tau_k \subset \check{V}_k$ the representative of $\Delta_k = \phi_{\check{V}_{k-1},\check{V}_k} \circ \cdots \circ \phi_{\check{V}_0,\check{V}_1}([V_0]) \in H_k(\check{V}_k)$ constructed in the same way.

Let us introduce the following notation:

- $\hat{U}_0 := \check{V}_0$;
- $\hat{U}_k := \pi^{-1}_{\check{V}_{k-1},\check{V}_k}(\hat{U}_{k-1})$ for $k = 1, \ldots, n$.

Note that, for $k > 0$, $\hat{U}_k$ is the preimage of $\hat{U}_{k-1} \times \mathbb{R}_+$ under the mapping $(\pi_{\check{V}_{k-1},\check{V}_k}, \rho_{\check{V}_{k-1},\check{V}_k})\colon U_{\check{V}_{k-1},\check{V}_k} \to \check{V}_{k-1} \times \mathbb{R}_+$. Since $\check{V}_{k-1}$ and $\check{V}_k$ are consecutive strata, it follows that the restriction $(\pi_{\check{V}_{k-1},\check{V}_k}, \rho_{\check{V}_{k-1},\check{V}_k})|_{\hat{U}_k}$ is a proper submersion to $\hat{U}_{k-1} \times \mathbb{R}_+$.

Composing these maps n times yields the following lemma.

LEMMA 3.3. $(\rho_{\check{V}_0}, \ldots, \rho_{\check{V}_{k-1}})\colon \hat{U}_k \to (\mathbb{R}_+)^k$ *is a proper submersion. Therefore,* $\hat{U}_k$ *is diffeomorphic to* $\tau_k \times (\mathbb{R}_+)^k$.

Consider the preimages $U_k = (p_n \circ \cdots \circ p_k)^{-1}(\hat{U}_k) \subset \widetilde{W}_k$. By Lemma 3.1, $U_k \subset \check{\widetilde{W}}_k$ and $(p_n \circ \cdots \circ p_k)|_{U_k}\colon U_k \to \hat{U}_k$ is a covering.

Let $\bar{U}_k := U_k \cup p_{k-1}(U_{k-1})$ for $k = n, n-1, \ldots, 1$.

LEMMA 3.4. $\bar{U}_k \subset \widetilde{W}_k$ *is an open subset consisting of regular points of* $\widetilde{W}_k$.

Proof. $\hat{U}_k \cup \hat{U}_{k-1}$ is an open subset in $\check{V}_k \cup \check{V}_{k-1}$. Indeed, it is the preimage of the $\hat{U}_{k-1}$ under the restriction of the projection $\pi_{\check{V}_{k-1}}$, restricted to $U_{\check{V}_{k-1}} \cap (\check{V}_{k-1} \cup \check{V}_k)$. In turn, $\bar{U}_k$ is the preimage of $\hat{U}_k \cup \hat{U}_{k-1}$ under $(p_n \circ \cdots \circ p_k)|_{\check{\tilde{W}}_k \cup \check{W}_{k-1}}$.

Finally, by Lemma 3.2, $\check{W}_{k-1}$ consists of regular points of $\widetilde{W}_k$. $\square$

We need the following lemma about lifting the control data.

LEMMA 3.5. *Let V and V' be two stratified spaces consisting of two strata each: $V = X \sqcup Y$ for $X < Y$, and $V' = X' \sqcup Y'$ for $X' < Y'$. Let $p\colon V' \to V$ be a map such that $p|_{X'}$ is a covering over X and $p|_{Y'}$ is a covering over Y. Let $U_X \subset V$, $\pi_X\colon U_X \to X$, and $\rho_X\colon U_X \to \mathbb{R}_{\geq 0}$ be the control data on V. Then there exist control data $U_{X'}, \pi_{X'}, \rho_{X'}$ on V' such that*

(1) $\rho_X \circ p = \rho_{X'}$ *and*
(2) $\pi_X \circ p = p \circ \pi_{X'}$.

Proof. We set the tubular neighborhood $U_{X'} := p^{-1}(U_X)$. The tubular function $\rho_{X'}$ is defined by property (1). The retraction $\rho_{X'}$ is defined uniquely by property (2) and continuity. □

We next apply Lemma 3.5 to the $V = \hat{U}_k \sqcup \hat{U}_{k-1}$ and $V' = \bar{U}_k = p_{k-1}(U_{k-1}) \sqcup U_k$. Let $\pi_{p_{k-1}(U_{k-1})}\colon \bar{U}_k \to p_{k-1}(U_{k-1})$ and $\rho_{p_{k-1}(U_{k-1})}\colon \bar{U}_k \to \mathbb{R}_{\geq 0}$ be the corresponding retraction and tubular function. We have the following corollary.

COROLLARY. *For any $k = n, n-1, \ldots, 1$, the connected components of U_k are in natural one-to-one correspondence with the connected components of U_{k-1}.*

Proof. Indeed, the map from the connected components of U_k to the connected components of $p_{k-1}(U_{k-1})$ is given by the retraction $\pi_{p_{k-1}(U_{k-1})}$. Existence of the inverse to this map follows because $p_{k-1}(U_{k-1}) \subset \bar{U}_k$ is a complex hypersurface in the manifold $\bar{U}_k$. Indeed, if $H \subset M$ is a hypersurface in a complex manifold M, then there is only one connected component of M in a neighborhood of a connected component of H.

Finally, $p_{k-1}|_{U_{k-1}}$ is an isomorphism to the image. □

Pick a point $a_\alpha \in W_0$. Let $U_1^\alpha, \ldots, U_n^\alpha$ be the corresponding connected components of $U_1, \ldots, U_n$, respectively, and let $\bar{U}_k^\alpha := U_k^\alpha \cup p_{k-1}(U_{k-1}^\alpha)$ be the corresponding connected components of $\bar{U}_k$.

Let $\tilde{\tau}_k := (p_n \circ \cdots \circ p_k)^{-1}(\tau_k)$. Note that $\tilde{\tau}_k \subset U_k$ is a union of connected components, one in each U_k^α. Let $\tilde{\tau}_k^\alpha \subset U_k^\alpha$ be the corresponding connected component.

LEMMA 3.6. $\phi_{p_{k-1}(U_{k-1}), U_k} \circ (p_{k-1}|_{U_{k-1}})_*([\tilde{\tau}_{k-1}^\alpha]) = [\tilde{\tau}_k^\alpha]$.

Proof. Since we have chosen the control data on $\bar{U}_k$ to be coherent with the control data on $\hat{U}_k \cup \hat{U}_{k-1}$ (given by restricting the control data from the ambient space), we have the equality on the level of representatives. □

Now we use the local parameters $(u_1, \ldots, u_n)$ to construct cycles $\gamma_k^\alpha \subset U_k^\alpha$ such that, on the one hand, it is obvious that

$$\operatorname{res}_{\{F, a_\alpha\}}(\omega) = \frac{1}{(2\pi i)^n} \int_{\gamma_n^\alpha} \omega,$$

and on the other hand γ_k^α is homologically equivalent to $\tilde{\tau}_k^\alpha$ in U_k^α.

The function u_k is regular and nonvanishing in $U_k^\alpha \subset \breve{\tilde{W}}_k$ and has zero of order 1 at a generic point of $p_{k-1}(U_{k-1}^\alpha) \subset \bar{U}_k^\alpha$. It follows immediately that u_k is regular on $\bar{U}_k^\alpha$ and that the equation $u_k = 0$ defines $p_{k-1}(U_{k-1}^\alpha)$ in $\bar{U}_k^\alpha$.

Our next lemma easily follows from the preceding observation.

LEMMA 3.7. *There exist smooth positive real functions* $\varepsilon_1, \dots, \varepsilon_n$ $(\varepsilon_k \colon \mathbb{C}^{k-1} \to \mathbb{R}_+)$ *and open subsets* $B_k \subset U_k^\alpha$ $(k = 1, \dots, n)$ *such that*

$$
\begin{aligned}
&(u_1, \dots, u_k) \colon B_k \to A_k \\
&\qquad := \{(z_1, \dots, z_k) : 0 < |z_i| < \varepsilon_i(z_1, \dots, z_{i-1}),\ i = 1, \dots, k\} \subset \mathbb{C}^k
\end{aligned}
$$

are biholomorphisms. (*Note that* ε_1 *is a constant.*)

Let $\delta_1, \dots, \delta_n \in \mathbb{R}_+$ be small enough that

$$\{(z_1, \dots, z_n) : |z_i| = \delta_i,\ i = 1, \dots, n\} \subset A_n.$$

DEFINITION 3.6. Let $\gamma_k^\alpha = \{x \in B_k : |u_i(x)| = \delta_i,\ i = 1, \dots, k\}$ and $\gamma_0^\alpha = a_\alpha$.

It follows immediately from the definition of the Parshin residue that

$$\operatorname{res}_{\{F, a_\alpha\}}(\omega) = \frac{1}{(2\pi i)^n} \int_{\gamma_n^\alpha} \omega.$$

LEMMA 3.8. γ_k^α *and* $\tilde{\tau}_k^\alpha$ *define the same homology class in* $H_k(U_k)$.

Proof. We prove this lemma by induction. For $k = 0$ one has $\gamma_0^\alpha = \tilde{\tau}_0^\alpha = a_\alpha$. For the induction step, use Lemma 3.6 and the similar observation for cycles γ_k^α. $\square$

We have thus proved the following theorem.

THEOREM 3.2.

$$\operatorname{res}_F(\omega) = \frac{1}{(2\pi i)^n} \int_{\Delta_F} \omega,$$

where $F := \{V_n \supset \cdots \supset V_0\}$ *and* $\Delta_F = \phi_{\check{V}_{n-1}, \check{V}_n} \circ \cdots \circ \phi_{\check{V}_0, \check{V}_1}([V_0]) \in H_n(\check{V}_n)$.

In order to derive the Parshin reciprocity law from Theorem 2.2 and Theorem 3.2, one needs a fixed stratification of V such that all nonzero residues of a given form ω are in the flags consisting of closures of strata of the stratification. As it happens, any Whitney stratification such that ω is regular on the top-dimensional stratum is good enough. More precisely, we have the following theorem.

THEOREM 3.3. *Let* V *be an* n*-dimensional variety and let* ω *be a meromorphic* n*-form on* V. *Let* $\mathbf{S}_\omega$ *be a Whitney stratification of* V *such that* ω *is regular on* $\check{V}$.

Let $F = \{V_n \supset \cdots \supset V_0\}$ *be a flag of irreducible subvarieties of* V, *where* $\dim V_i = i$. *Suppose that at least one of* V_i *is not the closure of a stratum of* $\mathbf{S}_\omega$. *Then* $\operatorname{res}_{\{F, a_\alpha\}} \omega = 0$ *for all* $a_\alpha \in W_0$.

Proof. Consider the normalization diagram for the flag F. Let $a_\alpha \in W_0$ and let $(u_1, \dots, u_n)$ be local parameters. Let $\mathbf{S}$ be a stratification of V satisfying the conditions of Lemma 3.1 and such that all strata of the stratification $\mathbf{S}_\omega$ are unions of strata of $\mathbf{S}$. As usual, we denote by $\check{V}_k$ the stratum of $\mathbf{S}$ that is open and dense in V_k.

The proof is based on the following two observations.

1. Let $X' < Y'$ be consecutive strata of $\mathbf{S}_\omega$, and let $X < Y$ be the consecutive strata of $\mathbf{S}$ such that X is an open dense subset in X' and Y is an open dense subset in Y'. Let $i_X : X \hookrightarrow X'$ and $i_Y : Y \hookrightarrow Y'$ be the embeddings. Then it easily follows—by the construction of coboundary operators $\phi_{X,Y}$ and $\phi_{X',Y'}$ and by the independence of these operators from the choice of the control data—that $\phi_{X',Y'} \circ i_{X*} = i_{Y*} \circ \phi_{X,Y}$.
2. Let Y' be a stratum of $\mathbf{S}_\omega$, and let $X < Y$ be consecutive strata of $\mathbf{S}$ such that $(X \cup Y) \subset Y'$ and Y is open and dense in Y'. Then $i_* \circ \phi_{X,Y} = 0$, where $i : Y \hookrightarrow Y'$ is the embedding. Moreover, if A is a representative of a homology class in $H_*(X)$ and if B is the representative of $\phi_{X,Y}([A]) \in H_{*+\dim Y - \dim X - 1}(Y)$ constructed in the standard way, then every connected component of B is homologically equivalent to 0 in Y. Indeed, one can use the control data on $\mathbf{S}$ to embed the mapping cone of $\pi_X|_B : B \to A$ into Y'.

Let k be the largest number such that $\check{V}_k$ is a subset of a stratum of $\mathbf{S}_\omega$ of dimension greater than k. For $m = k+1, \dots, n$, let $\check{V}'_m$ be the stratum of $\mathbf{S}_\omega$ such that $\check{V}_m \subset \check{V}'_m$. Observe that $\dim \check{V}'_m = m$ and that $\check{V}_m$ is open and dense in $\check{V}'_m$. Moreover, by dimension reasons and the condition of the frontier, $\check{V}_k \subset \check{V}'_{k+1}$.

Let $i_m : \check{V}_m \hookrightarrow \check{V}'_m$ be the embedding. Then, by observation 1,

$$(i_n)_* \circ \phi_{\check{V}_{n-1},\check{V}_n} \circ \cdots \circ \phi_{\check{V}_1,\check{V}_0} = \phi_{\check{V}'_{n-1},\check{V}'_n} \circ \cdots \circ \phi_{\check{V}'_{k+1},\check{V}'_{k+2}} \circ (i_{k+1})_* \circ \phi_{\check{V}_k,\check{V}_{k+1}} \circ \cdots \circ \phi_{\check{V}_1\check{V}_0}.$$

Yet by observation 2, $(i_{k+1})_* \circ \phi_{\check{V}_k,\check{V}_{k+1}} = 0$. Therefore,

$$(i_n)_* \circ \phi_{\check{V}_{n-1},\check{V}_n} \circ \cdots \circ \phi_{\check{V}_1\check{V}_0} = 0;$$

since ω is regular in $\check{V}'_n$, it follows that $\operatorname{res}_F \omega = 0$. Moreover, it is easy to see that each connected component of the standard representative of the $\phi_{\check{V}_{n-1},\check{V}_n} \circ \cdots \circ \phi_{\check{V}_1\check{V}_0}([V_0])$ is homologically equivalent to 0. Hence $\operatorname{res}_{F,a} \omega = 0$ for any $a \in W_0$. $\square$

COROLLARY. *There are only finitely many nonzero Parshin residues for a given meromorphic form.*

Note that the Parshin reciprocity law follows from Theorems 3.2, 3.3, and 2.2.

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