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Partial Desingularizations of Good Moduli Spaces of Artin Toric Stacks

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1. Introduction

If G is a linear algebraic group acting properly on a smooth algebraic variety X then there is a geometric quotient $X \to X/G$, where X/G is an algebraic space with only finite quotient singularities (cf. [KeMo; Ko]). Unfortunately, if G does not act properly there is no general method for determining if a quotient exists, even as an algebraic space. However, if G is reductive and X is a projective variety then geometric invariant theory (GIT) can be used to construct quotients of the open sets $X^s \subset X^{ss}$ of the G-stable and semi-stable points. The quotient X^{ss}/G is projective, but in general highly singular, while the open subspace X^s/G has only finite quotient singularities.

In a landmark paper [K] Kirwan described, when $X^s \neq \emptyset$, a systematic sequence of blowups along nonsingular G-invariant subvarieties that yields a birational G-equivariant morphism $f: X' \to X$ such that every semi-stable point (with respect to a suitable linearization) of the G-variety X' is stable. The quotient $(X')^{ss}/G$ is a projective variety with only finite quotient singularities. Furthermore, there is an induced projective birational morphism $\bar{f}: (X')^{ss}/G \to X^{ss}/G$, which is an isomorphism over the open set X^s/G. Hence the quotient $(X')^{ss}/G$ may be viewed as a partial resolution of singularities of the highly singular quotient X^{ss}/G.

A natural problem is to try to understand to what extent Kirwan's procedure can be replicated for nonprojective quotients where the techniques of geometric invariant theory do not apply. Precisely, given a smooth G-variety X and a good quotient (see Section 2.2 for definitions) $X \xrightarrow{q} X/G$ we would like to find a systematic way of producing a birational map $X' \to X$ such that G acts properly on X' and the induced map of quotients $X'/G \to X/G$ is proper. More generally, suppose $\mathcal{X}$ is a smooth algebraic stack with a good moduli space (in the sense of [A]) $\mathcal{X} \to M$. Is there a systematic way of producing a smooth separated Deligne–Mumford stack $\mathcal{X}'$ and a morphism $\mathcal{X}' \to \mathcal{X}$, which is generically an isomorphism, such that the induced map $M' \to M$ is proper and birational, where M' is the coarse moduli space of the Deligne–Mumford stack $\mathcal{X}'$?

The main result of this paper (Theorem 5.2) is to solve this problem when $\mathcal{X}$ is an Artin toric stack (as defined by Borisov, Chen, and Smith [BChS]). From our

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perspective, toric stacks are a class of stacks with good moduli spaces that are not in general geometric invariant theory quotients.

To prove our result we introduce certain birational transformations of Artin stacks with good moduli spaces that we call Reichstein transforms (Section 2.4). If $\mathcal{C} \subset \mathcal{X}$ is a closed substack then the *Reichstein transform* $\mathcal{R}(\mathcal{X}, \mathcal{C})$ of $\mathcal{X}$ relative to $\mathcal{C}$ is defined to be the complement of the strict transform of the saturation of $\mathcal{C}$ (relative to the quotient map $q: \mathcal{X} \to M$) in the blowup of $\mathcal{X}$ along $\mathcal{C}$. The main technical result of the paper is Theorem 4.7. It states that the Reichstein transform of a toric stack along a toric substack produces a toric stack combinatorially related to the original stack by *stacky star subdivision*. Successively applying Theorem 4.7 yields the procedure for obtaining a toric Deligne–Mumford stack.

Finally, in Section 6 we discuss the relationship between divisorial Reichstein transformations and changes of linearizations in geometric invariant theory. We prove (Theorem 6.1) that every toric stack contains a separated Deligne–Mumford substack that can be obtained via a (noncanonical) sequence of Reichstein transformations relative to divisors. We also give an example (Example 6.4) of a toric Artin stack that has a projective good moduli space (and so is in fact a geometric invariant theory quotient) but that contains a complete open Deligne–Mumford toric stack whose moduli space is a complete nonprojective toric variety.

In a subsequent paper we will study the behavior of Reichstein transforms on arbitrary quotient stacks with good moduli spaces.

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2. Good Quotients and Reichstein Transformations

2.1. Standing Assumptions

Throughout this paper we work over the field $\mathbb{C}$ of complex numbers. All algebraic groups are assumed to be linear, that is, isomorphic to subgroups of $\mathrm{GL}_n(\mathbb{C})$ for some n. Because we work in characteristic 0, all reductive groups are linearly reductive—that is, every representation decomposes into a direct sum of irreducibles.

2.2. Good Quotients and Good Moduli Spaces

Let G be an algebraic group acting on an algebraic variety X (or more generally an algebraic space). Following [A], we make the following definition.

DEFINITION 2.1. A map $q: X \to M$ with M an algebraic space is called a *good quotient* if the following conditions are satisfied:

(i) the functor $G - \mathcal{Q}\operatorname{coh} X \to \mathcal{Q}\operatorname{coh} M$, $\mathcal{F} \mapsto (q_* \mathcal{F})^G$ is exact, where $G - \mathcal{Q}\operatorname{coh} X$ is the category of G-linearized quasi-coherent sheaves on X;

(ii) $(q_* \mathcal{O}_X)^G = \mathcal{O}_M$.

The map $q\colon X \to M$ is a *good geometric quotient* if, in addition, the orbits of closed points are closed.

REMARK 2.2. Condition (i) implies that the quotient map is affine, and if G is linearly reductive it is equivalent to the quotient map being affine. However, if G is not linearly reductive then the quotient map may be affine without condition (i) being satisfied.

In [A], Alper generalized the concept of good quotient to Artin stacks.

DEFINITION 2.3 [A]. Let $\mathcal{X}$ be an Artin stack. A map $q\colon \mathcal{X} \to M$ with M an algebraic space is a *good moduli space* for $\mathcal{X}$ if the following statements hold.

(i) The map $q\colon \mathcal{X} \to M$ is cohomologically affine; that is, the functor $q_*\colon \mathcal{Q}\,\mathrm{coh}\,\mathcal{X} \to \mathcal{Q}\,\mathrm{coh}\,M$ is exact.
(ii) $q_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_M$.

Good quotients and moduli spaces enjoy a number of natural properties. The following theorem of Alper summarizes these properties and shows that Alper's notion of good quotient is equivalent to other definitions in the literature.

THEOREM 2.4 [A, Thm. 4.16, Thm. 6.6]. *If $q\colon \mathcal{X} \to M$ is the good moduli space of an Artin stack, then*

(i) *q is surjective and universally closed.*
(ii) *If Z_1, Z_2 are closed substacks then $q(Z_1 \cap Z_2) = q(Z_1) \cap q(Z_2)$; in particular, the images of two disjoint closed substacks are disjoint.*
(iii) *q is a universal categorical quotient in the category of algebraic spaces.*

REMARK 2.5. If G acts on X and there exists a good quotient $X \to M$ then, by Alper's definition, the induced map $[X/G] \to M$ is the good moduli space of the quotient stack $[X/G]$. Theorem 2.4 implies that for a linearly reductive group G, Alper's notion of a good quotient is equivalent the definition given by Seshadri [Se, Def. 1.5].

2.3. Geometric Invariant Theory

Let G be a reductive group acting on scheme X and let L be a G-linearized line bundle on X.

DEFINITION 2.6 [MFoK, Def. 1.7]. (i) A point $x \in X$ is *semi-stable* (with respect to L) if there is a section $s \in H^0(X, L^n)^G$ such that $s(x) \neq 0$ and X_s is affine.

(ii) A point $x \in X$ is *stable* if it is semi-stable and has finite stabilizer and if the action of G on X_s is closed.

REMARK 2.7. If L is assumed to be ample then the condition that X_s be affine is automatic.

Our use of the term "stable" differs slightly from the terminology used in [MFoK] in that we require that the stabilizer of a stable point be finite. In [MFoK] such points are referred to as "properly stable".

Note that there can be strictly semi-stable points that have finite stabilizer. For example, consider the $\mathbb{C}^*$-action on $\mathbb{A}^4$ given by $t \cdot (x, y, z, w) = (tx, t^{-1}y, tz, t^{-1}w)$. Since any constant function is $\mathbb{C}^*$-invariant, all points are semi-stable with respect to the trivial linearization, and every point except the origin has trivial stabilizer. However, the stable locus is the complement of the linear subspaces $V(x, z)$ and $V(y, w)$. We will mention other aspects of this action in Example 4.11.

The main result of geometric invariant theory can be stated in the language of good quotients as follows.

THEOREM 2.8 [MFoK]. *Let G be a reductive algebraic group acting on a scheme X and let L be a line bundle on X linearized with respect to the action of G. If X^{ss} and X^s denote the open subsets of semi-stable and stable points, respectively, then a good quotient X^{ss}/G exists as a quasi-projective scheme and contains as an open set a good geometric quotient X^s/G. If X is complete and L is ample then the quotient X^{ss}/G is also projective.*

2.4. *Reichstein Transforms*

Let G be an algebraic group acting on a scheme (or algebraic space) X and let $q : X \to M$ be a good quotient.

DEFINITION 2.9. Let C be a closed G-invariant subscheme of X and let $\tilde{C} = q^{-1}(q(C))$ be the saturation of C relative to the quotient map. The *Reichstein transform* $\mathcal{R}(X, C)$ of X relative to C is defined as $(\mathrm{Bl}_C X) \setminus (\tilde{C})'$, where $\tilde{C}'$ is the strict transform of $\tilde{C}$ in the blowup $\mathrm{Bl}_C X$ of X along C.

More generally, if $\mathcal{X}$ is an Artin stack with good moduli space $q : \mathcal{X} \to M$ and if $\mathcal{C} \subset \mathcal{X}$ is a closed substack, then we define the Reichstein transform $\mathcal{R}(\mathcal{X}, \mathcal{C})$ to be $(\mathrm{Bl}_{\mathcal{C}} \mathcal{X}) \setminus \tilde{\mathcal{C}}'$, where $\tilde{\mathcal{C}} = q^{-1}(q(\mathcal{C}))$ and $'$ indicates strict transform.

If C (resp. $\mathcal{C}$) is a Cartier divisor then $\mathcal{R}(X, C)$ (resp. $\mathcal{R}(\mathcal{X}, \mathcal{C})$) is called a *divisorial Reichstein transform*.

REMARK 2.10. A closed subvariety $C \subset X$ is saturated if $\tilde{C} = C$. If C is saturated then the Reichstein transform is just the blowup $\mathrm{Bl}_C X$. When $q : X \to M$ is a good geometric quotient, every closed G-invariant subset is saturated. At the other extreme, if $\tilde{C} = X$ then $\mathcal{R}(X, C)$ is empty.

REMARK 2.11. If C (resp. $\mathcal{C}$) is a Cartier divisor then $\mathcal{R}(X, C)$ (resp. $\mathcal{R}(\mathcal{X}, \mathcal{C})$) is open in X (resp. $\mathcal{X}$). Indeed it is the complement of the closure of $\tilde{C} \setminus C$ (resp. the closure of $\tilde{\mathcal{C}} \setminus \mathcal{C}$).

Our terminology is based on a theorem of Reichstein in geometric invariant theory.

THEOREM 2.12 (cf. [R, Thm. 2.4]). *Let X be a smooth projective variety and let $C \subset X$ be a G-invariant smooth subvariety. Let L be a G-linearized line bundle and let X^{ss} be the set of L-semi-stable points. Then, for a suitable linearization of the G-action on $\mathrm{Bl}_C X$, we have $(\mathrm{Bl}_C X)^{ss} = \mathcal{R}(X^{ss}, C \cap X^{ss})$. In particular, a good quotient $\mathcal{R}(X^{ss}, C \cap X^{ss})/G$ exists as a projective variety.*

3. Toric Stacks

The goal of this section is to define toric stacks. Toric stacks were originally defined in [BChS] using *stacky fans.* Iwanari [I] and subsequently Fantechi, Mann, and Nironi [FMaN] gave an intrinsic definition of Deligne–Mumford toric stacks and showed the relationship between simplicial stacky fans and toric Deligne–Mumford stacks. (Note that the toric variety does not uniquely determine a stacky fan). In this paper we are mainly interested in toric Artin stacks, which we define to be Artin stacks associated to possibly nonsimplicial stacky fans. We do not give an intrinsic definition in this case. Our definition of stacky fan is less general than the one given by Satriano in [Sa].

3.1. Toric Notation

We establish the notation that we will use for toric varieties. A basic reference is the book by Fulton [Fu]. Let N be a lattice of rank r and let $M = \mathrm{Hom}(N, \mathbb{Z})$ be the dual lattice. Given a fan $\Sigma \subset N_{\mathbb{R}} = \mathbb{R}^r$, we denote the associated toric variety by $X = X(\Sigma)$. Let $\Sigma(1)$ denote the rays in Σ. We assume throughout that $\Sigma(1)$ spans $N_{\mathbb{R}}$. Given a cone $\sigma \in \Sigma$, we denote by $X_\sigma \subseteq X$ the affine open set associated to σ. By definition, $X_\sigma = \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$ and the open sets X_σ form an affine cover of the toric variety $X(\Sigma)$. Likewise let γ_σ be the distinguished point of X_σ corresponding to the irrelevant ideal of the M-graded ring $\mathbb{C}[\sigma^\vee \cap M]$. The notation $V(\sigma)$ described next will come up often.

Definition 3.1. If σ is a cone, let $O(\sigma)$ denote the orbit of γ_σ under the torus T_N of X and let $V(\sigma) = \overline{O(\sigma)}$ be its closure in X.

The orbit–cone correspondence states that

$$V(\sigma) = \coprod_{\tau \text{ contains } \sigma \text{ as a face}} O(\tau). \tag{1}$$

We also recall some facts about toric morphisms. Let N' be a lattice and let Σ' be a fan in $N'_{\mathbb{R}}$. A map of lattices $\bar{\phi}\colon N' \to N$ is said to be compatible with Σ' and Σ if for every cone $\sigma' \in \Sigma'$ there is a cone $\sigma \in \Sigma$ such that $\bar{\phi}_{\mathbb{R}}(\sigma') \subseteq \sigma$ (where $\bar{\phi}_{\mathbb{R}}\colon (N')_{\mathbb{R}} \to N_{\mathbb{R}}$ is the map induced by $\bar{\phi}$). Such a compatible map $\bar{\phi}$ induces a toric morphism $\phi\colon X(\Sigma') \to X(\Sigma)$. For $\sigma' \in \Sigma'$, if σ is the minimal cone in Σ such that $\bar{\phi}_{\mathbb{R}}(\sigma') \subseteq \sigma$, then $\phi(\gamma_{\sigma'}) = \gamma_\sigma$ and $\phi(O(\sigma')) \subseteq O(\sigma)$ [CLSc, Lemma 3.3.21a].

Suppose $\bar{\phi}$ has finite cokernel or, equivalently, that $\phi_{\mathbb{R}}$ is surjective. Then $\bar{\phi} \otimes \mathbb{C}^*\colon N' \otimes \mathbb{C}^* \to N \otimes \mathbb{C}^*$ is surjective and hence $\phi|_{T_{N'}}\colon T_{N'} \to T_N$ is surjective. In this case, $\phi(O(\sigma')) = \phi(T_{N'} \cdot \gamma_{\sigma'}) = T_N \cdot \gamma_\sigma = O(\sigma)$.

3.2. Cox Construction of Toric Varieties as Quotients

As a first step in our discussion of toric stacks we recall Cox's ([C] or [CLSc, Thm. 5.1.11]) description of $X(\Sigma)$ as a quotient of an open subset of $\mathbb{A}^n$ by a diagonalizable group.

DEFINITION 3.2 (Cox construction of $X(\Sigma)$). Given a fan $\Sigma \subseteq N$, let $\tilde{N} = \mathbb{Z}^n$ for $n = |\Sigma(1)|$ and let $\{e_\rho \mid \rho \in \Sigma(1)\}$ be the standard basis of $\tilde{N}$. For each cone $\sigma \in \Sigma$, define $\tilde{\sigma} = \mathrm{Cone}(e_\rho \mid \rho \in \sigma(1)) \subseteq \mathbb{R}^n$. These cones form a fan $\tilde{\Sigma} = \{\tilde{\sigma} \mid \sigma \in \Sigma\}$. Define $\bar{q}\colon \tilde{N} \to N$ by $\bar{q}(e_\rho) = u_\rho$, where u_ρ is the primitive vector along the ray $\rho \in \Sigma(1)$. Then $\bar{q}$ is compatible with $\tilde{\Sigma}$ and Σ and so induces a toric morphism $q\colon X(\tilde{\Sigma}) \to X(\Sigma)$, called the Cox construction of $X(\Sigma)$. Cox [C, Thm. 2.1] showed that q is a good quotient for the group $G_\Sigma = \ker(T_{\tilde{N}} \to T_N)$, and if Σ is a simplicial fan, then q is a good geometric quotient.

A more geometric description of $X(\tilde{\Sigma})$ is as follows. The homogeneous coordinate ring of $X(\Sigma)$ is the ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$. For a cone $\sigma \in \Sigma$, let $x^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$ be the corresponding monomial. Let $B = B_\Sigma \subset S$ be the ideal of S generated by the $x^{\hat{\sigma}}$; that is, $B = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma\rangle$. Let $Z_\Sigma \subset \mathbb{A}^n$ be the subvariety $\mathbb{V}(B)$ associated to B. By [CLSc, Prop. 5.1.9a] we have that $X(\tilde{\Sigma}) = \mathbb{A}^n \setminus Z_\Sigma$.

Define a map $M \to \mathbb{Z}^n$ by $m \mapsto (\langle m, u_\rho\rangle)_{\rho\in\Sigma(1)}$, where u_ρ is a primitive generator for the ray ρ. This map induces a short exact sequence of diagonalizable groups

$$1 \to G_\Sigma \to (\mathbb{C}^*)^{\Sigma(1)} \to T \to 1, \tag{2}$$

where $T = \mathrm{Hom}(M, \mathbb{C}^*)$ is the torus acting on the toric variety $X(\Sigma)$.

Since $(\mathbb{C}^*)^{\Sigma(1)}$ acts naturally on $\mathbb{A}^n$, we have an induced action of G_Σ on $\mathbb{A}^n$.

THEOREM 3.3 [C, Thm. 2.1]. *The toric variety $X(\Sigma)$ is naturally isomorphic to the good quotient of $X(\tilde{\Sigma}) = \mathbb{A}^n \setminus Z_\Sigma$ by G_Σ. Also, this quotient is a geometric quotient if and only if Σ is simplicial.*

DEFINITION 3.4. We refer to $X(\tilde{\Sigma}) = \mathbb{A}^n \setminus Z_\Sigma$ as the *Cox space* of the toric variety $X(\Sigma)$. We denote the corresponding toric quotient stack $[X(\tilde{\Sigma})/G_\Sigma]$ as $\mathcal{X}(\Sigma)$ and refer to it as the *Cox stack* of the toric variety $X(\Sigma)$.

PROPOSITION 3.5. *The toric variety $X(\Sigma)$ is a good moduli space (in the sense of* [A]*) for the quotient stack $\mathcal{X}(\Sigma)$.*

Proof. This can be checked locally on $\mathcal{X}(\Sigma)$. The open set $X_\sigma \in X(\Sigma)$ is the quotient of an affine open subset of $\mathbb{A}^{\Sigma(1)}$ by the action of the linearly reductive group G_Σ, and the result then follows from [A]. □

REMARK 3.6. If the toric variety $X(\Sigma)$ is not projective or quasi-projective then $\mathcal{X}(\Sigma)$ is an example of an Artin stack with a good moduli space that is not a quotient stack arising from GIT; in other words, it is not of the form $[X^{ss}/G]$, where X^{ss} is the set of semi-stable points for some linearization on a projective variety X.

3.3. Toric Stacks

Given a fan Σ, the Cox construction can be generalized to produce other quotient stacks whose good moduli spaces are the toric variety $X(\Sigma)$. Such stacks were defined in [BChS] and were called *toric stacks.* (It should be noted that Lafforgue also used the term "toric stack". For Lafforgue, a toric stack is the quotient stack $[X(\Sigma)/T_N]$ and is not the same as those constructed in [BChS].) To give Borisov, Chen, and Smith's definition of toric stacks, we begin by defining the notion of a stacky fan.

DEFINITION 3.7. A *stacky fan* $\mathbf{\Sigma} = (N, \Sigma, (v_1, \dots, v_n))$ consists of:

1. a finitely generated abelian group N of rank d;
2. a (not necessarily simplicial) fan $\Sigma \subset N_{\mathbb{Q}}$, whose rays we will denote by $\rho_1, \dots, \rho_n$;
3. for every $1 \le i \le n$, an element $v_i \in N$ such that its image $\bar{v}_i \in N_{\mathbb{Q}}$ lies on ρ_i.

The data of a tuple $(v_1, \dots, v_n)$ is equivalent to a homomorphism $\mathbb{Z}^n \xrightarrow{\beta} N$ such that $\beta(e_i) = v_i$. Hence sometimes the stacky fan $\mathbf{\Sigma}$ is written as $\mathbf{\Sigma} = (N, \Sigma, \beta)$.

Let $\mathrm{DG}(\beta)$ be the Gale dual group of N (as defined in [BChS]) and let $G_{\mathbf{\Sigma}} = \mathrm{Hom}(\mathrm{DG}(\beta), \mathbb{C}^*)$. There is an induced map of abelian groups $\beta^\vee\colon (\mathbb{Z}^n)^* \to \mathrm{DG}(\beta)$. Applying the functor $\mathrm{Hom}(\cdot, \mathbb{C}^*)$ to the homomorphism $\beta^\vee$ yields a map $G_{\mathbf{\Sigma}} \to (\mathbb{C}^*)^n$, thereby defining an action of $G_{\mathbf{\Sigma}}$ on $\mathbb{A}^n$.

DEFINITION 3.8. The toric stack $\mathcal{X}(\mathbf{\Sigma})$ is defined to be the quotient stack $[X(\tilde{\Sigma})/G_{\mathbf{\Sigma}}]$, where (as before) $X(\tilde{\Sigma}) = \mathbb{A}^n \setminus Z_\Sigma$.

REMARK 3.9. Note that whereas the stack $\mathcal{X}(\mathbf{\Sigma})$ depends on the full stacky fan, the Cox space $X(\tilde{\Sigma})$ depends only on the underlying fan Σ.

To understand the relationship between the Cox construction and toric stacks, we give (following [BChS]) an explicit presentation for the dual group $\mathrm{DG}(\beta)$ and corresponding diagonalizable group $G_{\mathbf{\Sigma}}$. This will allow us to see that, if N is free and the b_i are minimal generators for the rays in the underlying fan Σ, then $G_{\mathbf{\Sigma}} = G_\Sigma$.

Since N is a finitely generated abelian group of rank d, it has a projective resolution as a $\mathbb{Z}$-module of the form $0 \longrightarrow \mathbb{Z}^r \xrightarrow{[Q]} \mathbb{Z}^{d+r} \longrightarrow N \longrightarrow 0$, where Q is a $(d+r) \times r$ integer-valued matrix. The map $\beta\colon \mathbb{Z}^n \to N$ lifts to a map $\mathbb{Z}^n \to \mathbb{Z}^{d+r}$ given by an integer matrix B. Since we have two maps with target $\mathbb{Z}^{d+r}$ we obtain a map $\mathbb{Z}^{n+r} \xrightarrow{[BQ]} \mathbb{Z}^{d+r}$. Borisov, Chen, and Smith show that the dual group $\mathrm{DG}(\beta)$ may be identified with the cokernel of the dual map $(\mathbb{Z}^{d+r})^* \xrightarrow{[BQ]^*} (\mathbb{Z}^{n+r})^*$.

PROPOSITION 3.10. *If N is free and $\beta_1 = \rho_1, \dots, \beta_n = \rho_n$ are generators for the rays of Σ, then $G_{\mathbf{\Sigma}} = G_\Sigma$.*

Proof. If N is free then we can take Q to be the 0 matrix, since $N = (\mathbb{Z})^d$ and $\beta = B$ as maps $\mathbb{Z}^n \to \mathbb{Z}^d$. The dual map is the map $M = (\mathbb{Z}^d)^* \to (\mathbb{Z}^n)^*$ given by $m \mapsto (\langle m, \rho_1\rangle, \langle m, \rho_2\rangle, \dots, \langle m, \rho_n\rangle)$. The cokernel of this map is the Weil divisor class group $A_{n-1}(X(\Sigma))$, and taking duals gives equation (2) defining G_Σ. $\square$

If Σ is simplicial, then $\mathcal{X}(\mathbf{\Sigma})$ is a Deligne–Mumford stack.

PROPOSITION 3.11. *If* $\mathbf{\Sigma}$ *is a stacky fan then the toric variety* $X(\Sigma)$ *is the good moduli space of the stack* $\mathcal{X}(\mathbf{\Sigma})$. *If* Σ *is simplicial then* $X(\Sigma)$ *is the coarse moduli space of* $\mathcal{X}(\mathbf{\Sigma})$.

3.4. The Nonsimplicial Index of a Fan

To analyze toric stacks (or toric varieties) when Σ is nonsimplicial, we introduce and prove some basic properties of what we call the *nonsimplicial index* of a strongly convex polyhedral cone or, more generally, a fan.

DEFINITION 3.12. Let $\sigma \subset \mathbb{R}^n$ be a strongly convex rational polyhedral cone. Define the *nonsimplicial index of* σ to be $\mathrm{ns}(\sigma) = |\sigma(1)| - d_\sigma$, where $|\sigma(1)|$ is the number of rays in σ and $d_\sigma = \dim \sigma$ is the dimension of the linear subspace spanned by σ.

REMARK 3.13. With this definition, a cone is simplicial if and only if $\mathrm{ns}(\sigma) = 0$.

LEMMA 3.14. *If* τ *is a face of a cone* σ *then* $\mathrm{ns}(\sigma) \geq \mathrm{ns}(\tau)$.

Proof. The statement is trivial if $\sigma = \tau$, so assume $\sigma \neq \tau$. By induction we may assume that τ is a facet (i.e., a codimension-1 face). Then σ is spanned by at least $|\tau(1)| + 1$ rays. Since $d_\sigma = d_\tau + 1$, it follows that $\mathrm{ns}(\sigma) \geq |\tau(1)| + 1 - d_\sigma = |\tau(1)| - d_\tau = \mathrm{ns}(\tau)$. □

The next lemma is slightly more subtle.

LEMMA 3.15. *If* $\mathrm{ns}(\sigma) > 0$ *then there is a unique minimal face* $\tau \subset \sigma$ *such that* $\mathrm{ns}(\tau) = \mathrm{ns}(\sigma)$.

Proof. Let $k = \mathrm{ns}(\sigma)$. We give a proof by contradiction. Suppose that τ_1 and τ_2 are two distinct minimal faces of σ with $\mathrm{ns}(\tau_1) = \mathrm{ns}(\tau_2) = k$. By assumption, $\mathrm{ns}(\tau_1 \cap \tau_2) < k$ and thus $\tau_1 \cap \tau_2$ is spanned by fewer than $k + \dim(\tau_1 \cap \tau_2)$ rays. Hence $|\tau_1(1) \cup \tau_2(1)| > |\tau_1(1)| + |\tau_2(1)| - k - \dim(\tau_1 \cap \tau_2)$. Let d_{12} be the dimension of the linear subspace of $\mathbb{R}^n$ spanned by τ_1 and τ_2. We have $d_{12} = d_{\tau_1} + d_{\tau_2} - \dim(\tau_1 \cap \tau_2)$. Then

$$\begin{aligned}
\sigma(1) &\geq |\tau_1(1) \cup \tau_2(2)| + d_\sigma - d_{12} \\
&> |\tau_1(1)| + |\tau_2(1)| - k - \dim(\tau_1 \cap \tau_2) + d_\sigma - d_{12} \\
&= |\tau_1(1)| + |\tau_2(1)| - k - \dim(\tau_1 \cap \tau_2) + d_\sigma - (d_{\tau_1} + d_{\tau_2} - \dim(\tau_1 \cap \tau_2)) \\
&= k + d_\sigma,
\end{aligned}$$

where we used $|\tau_i(1)| = \mathrm{ns}(\tau_i) + d_{\tau_i} = k + d_{\tau_i}$ to get the last line. But $\sigma(1) > k + d_\sigma$ contradicts $k = \mathrm{ns}(\sigma) = \sigma(1) - d_\sigma$. □

DEFINITION 3.16. If σ is a nonsimplical cone, then the minimal face $\tau \subset \sigma$ such that $\mathrm{ns}(\tau) = \mathrm{ns}(\sigma)$ is called the *minimal nonsimplicial face* of σ.

More generally, we can define the nonsimplicial index of a fan.

DEFINITION 3.17. Let $\Sigma \subset \mathbb{R}^n$ be a fan. The nonsimplicial index of Σ, denoted by $\mathrm{ns}(\Sigma)$, is defined to be $\max_{\sigma\in\Sigma}\{\mathrm{ns}(\sigma)\}$. A minimal cone $\sigma \in \Sigma$ with $\mathrm{ns}(\sigma) = \mathrm{ns}(\Sigma)$ is called a *minimal nonsimplicial cone* of Σ.

The following is an immediate consequence of Lemma 3.15.

LEMMA 3.18. *No two minimal nonsimplicial cones of Σ are contained in a cone of Σ.*

3.5. The Fibers of the Quotient Map $q\colon X(\tilde{\Sigma}) \to X(\Sigma)$

The following variety L_σ will be used throughout the paper.

DEFINITION 3.19 (L_σ). Let L_σ be the orbit closure $V(\tilde{\sigma}) = \mathbb{V}(\{x_\rho\}_{\rho\in\sigma(1)}) \cap X(\tilde{\Sigma})$. The subspace L_σ is clearly $(\mathbb{C}^*)^n$-invariant.

LEMMA 3.20. *The $G_{\mathbf{\Sigma}}$-stabilizer at a general point of L_σ has rank* $\mathrm{ns}(\sigma)$, *where* $\mathrm{ns}(\sigma)$ *is the nonsimplicial index of the cone σ (Definition 3.17).*

Proof. The $(\mathbb{C}^*)^{\Sigma(1)}$-stabilizer of a general point of L_σ is the torus $(\mathbb{C}^*)^l$, where $l = |\sigma(1)|$. The $G_{\mathbf{\Sigma}}$-stabilizer is the group $G_{\mathbf{\Sigma},\sigma} := \mathrm{Hom}(\mathrm{DG}(\beta)_\sigma, \mathbb{C}^*)$, where $\mathrm{DG}(\beta)_\sigma := (\mathbb{Z}^l)^* \otimes_{(\mathbb{Z}^n)^*} \bigotimes \mathrm{DG}(\beta)$ and the map $(\mathbb{Z}^n)^* \to (\mathbb{Z}^l)^*$ is dual to the inclusion map $\mathbb{Z}^l \to \mathbb{Z}^n$. Thus to compute the rank of the stabilizer $G_{\mathbf{\Sigma},\sigma}$ we need to compute the rank of the abelian group $\mathrm{DG}(\beta)_\sigma$. Let $N' = N/T$ for T the torsion subgroup of N and let β' be the composite of β with the projection $N \to N'$. As noted in the proof of [Sa, Prop. 5.4], there is an injection of abelian groups $\mathrm{DG}(\beta') \to \mathrm{DG}(\beta)$. Since both groups have the same rank, the map also has maximal rank.

Thus, to compute the rank of $\mathrm{DG}(\beta)_\sigma$ it suffices to compute the rank of the group $\mathrm{DG}(\beta')_\sigma := \mathrm{DG}(\beta') \otimes_{(\mathbb{Z}^n)^*} (\mathbb{Z}^l)^*$. The map β' maps standard basis vectors in $\mathbb{R}^n$ to rational generators of the cones of the fan Σ. Thus the rank of $\mathrm{DG}(\beta')_\sigma$ is $|\sigma(1)| - \dim\sigma = \mathrm{ns}(\sigma)$. □

Let $q\colon X(\tilde{\Sigma}) \to X(\Sigma)$ be the quotient map. For the next lemma, recall that $V(\sigma)$ was introduced in Definition 3.1 as the closure of the orbit of the point γ_σ.

LEMMA 3.21. *If σ is a cone of Σ then*

$$q(L_\sigma) = V(\sigma).$$

Proof. Again let N' be the quotient N/T, where T is the torsion subgroup, and let $\mathbf{\Sigma}' = (N', \beta', \Sigma)$ be the associated stacky fan. There is a finite morphism $\mathcal{X}(\mathbf{\Sigma}) = [X(\tilde{\Sigma})/G_{\mathbf{\Sigma}}] \to \mathcal{X}(\mathbf{\Sigma}') = [X(\tilde{\Sigma})/G_{\mathbf{\Sigma}'}]$ and both stacks have $X(\Sigma)$ as their good moduli spaces. Thus it suffices prove the lemma for the quotient by $G_{\mathbf{\Sigma}'}$. As a consequence, we may assume that the group N is free abelian.

The map q is constructed by patching quotient maps $q_\tau\colon U_\tau \to X_\tau$ for each cone τ of the fan. Here $X_\tau = \mathrm{Spec}\,\mathbb{C}[\tau^\vee \cap M]$ is the affine toric variety corresponding to the cone τ and U_τ is the affine open set $\mathbb{A}^n \setminus \mathbb{V}(x^{\hat{\tau}})$. The map q_τ

is induced by the map of rings $\mathbb{C}[\tau^\vee \cap M] \to S_\tau = \mathbb{C}[\{x_\rho\}_{\rho\in\Sigma(1)}]_{x^{\hat\tau}}$ given by $\chi^m \mapsto x^{D_m}$, where $x^{D_m} = \prod_\rho x_\rho^{\langle m,b_\rho\rangle}$. Given a cone σ, the linear space L_σ has nonempty intersection with the open set U_σ; since $q(L_\sigma)$ is closed (because q is a good categorical quotient), it suffices to prove that the generic point of L_σ maps to the generic point of $V(\sigma) \subset X_\sigma$. The generic point of L_σ is the ideal $I_\sigma = \langle \{x_\rho\}_{\rho\in\sigma(1)}\rangle$. Its inverse image under the ring homomorphism $\chi^m \mapsto x^{D_m}$ is the ideal generated by $\{\chi^m \mid \langle m,\rho\rangle > 0\}_{\rho\in\sigma(1)}$. But this is exactly the ideal of $V(\sigma)\cap X_\sigma = \operatorname{Spec}\mathbb{C}[\sigma^\perp \cap (\sigma^\vee \cap M)]$ in $X_\sigma = \operatorname{Spec}\mathbb{C}[\sigma^\vee\cap M]$. □

LEMMA 3.22. *Let $J \subseteq \Sigma(1)$ and let τ be the minimal cone (if such exists) of Σ that contains J. Then $q(\mathbb{V}(\{x_\rho\}_{\rho\in J})) \subseteq V(\tau)$.*

REMARK 3.23. If J does not lie in any cone of Σ, then $\mathbb{V}(\{x_\rho\}_{\rho\in J}) \cap X(\tilde\Sigma)$ is empty.

Proof. By Lemma 3.21 we know that $q(\mathbb{V}(x_\rho)) \subseteq V(\rho)$. Hence

$$\begin{aligned} q(\mathbb{V}(\{x_\rho\}_{\rho\in J})) &= q\left(\bigcap_{\rho\in J}\mathbb{V}(x_\rho)\right) \\ &\subseteq \bigcap_{\rho\in J} q(\mathbb{V}(x_\rho)) \\ &\subseteq \bigcap_{\rho\in J} V(\rho) = V(\tau), \end{aligned}$$

where the equality in the last line follows from the discussion in [Fu, pp. 99–100]. □

DEFINITION 3.24. Let σ be a cone in the fan Σ. Define $\tilde L_\sigma = q^{-1}(V(\sigma)) = q^{-1}(q(L_\sigma))$.

DEFINITION 3.25. Let μ be a cone containing σ. Let $\tilde I_{\mu,\sigma}$ be the ideal generated by the monomials $x_\mu^{\hat\tau} := \prod_{\rho\in\mu(1)\setminus\tau(1)} x_\rho$, where τ runs over all proper faces of μ such that τ does not contain σ (i.e., $\tau\cap\sigma$ is a proper face of σ). Let $\tilde I_\sigma = \bigcap_{\mu\supset\sigma} \tilde I_{\mu,\sigma}$.

REMARK 3.26. Observe that if $\mu' \subset \mu$ then $\tilde I_{\mu',\sigma} \supset \tilde I_{\mu,\sigma}$, so we may assume that the cones μ in Definition 3.25 are maximal.

PROPOSITION 3.27. $\tilde L_\sigma = \mathbb{V}(\tilde I_\sigma)$.

Proof. It suffices to prove the statement for each open set in the covering U_μ of $X(\tilde\Sigma)$, where μ is a (maximal) cone. Note that if μ is a cone then $U_\mu \cap \tilde L_\sigma = \emptyset$ unless $\sigma\subset\mu$ (because $p\in \tilde L_\sigma\cap U_\mu$ implies $q(p)\in V(\sigma)\cap X_\mu$, which is empty unless $\sigma\subset\mu$). We also claim that $U_\mu\cap\mathbb{V}(\tilde I_\sigma) = \emptyset$ if μ does not contain σ. To see this, note that if μ does not contain σ and if μ' is a cone containing σ, then $\tau' = \mu'\cap\mu$ is a face of μ' not containing σ and $x_{\mu'}^{\hat\tau'} \neq 0$ on U_μ.

Thus it suffices to prove the proposition for the quotient map $U_\mu \to X_\mu$ for any (maximal) cone μ containing σ. On U_μ, $x_\rho \neq 0$ if $\rho\notin\mu$ and so we may assume that $\tilde{I}_\sigma$ is generated by the products $x^{\hat{\tau}}$, where τ runs over all faces of μ that do not contain σ.

Step I: $\mathbb{V}(\tilde{I}_\sigma) \supset \tilde{L}_\sigma$. To prove Step I we will show that $\sqrt{\tilde{I}_\sigma} \subset I(\tilde{L}_\sigma)$. Let $x^{\hat{\tau}} \in \tilde{I}_\sigma$ be a generator, where τ is a face of μ that does not contain σ. By definition, $I(\tilde{L}_\sigma)$ is the radical of the ideal generated by x^{D_m} for all $m \in (\mu^\vee \cap M)$ such that $\langle m, \rho \rangle > 0$ for some ray $\rho \in \sigma(1)$. If $\tau \cap \sigma$ is a proper face of σ then there is an $m \in \tau^\perp \cap (\mu^\vee \cap M)$ such that $\langle m, \rho \rangle > 0$ for some ray $\rho \in \sigma(1) \cap \tau(1)^c$. Hence, for $N \gg 0$, $x^{D_m} | (x^{\hat{\tau}})^N$. Thus, $\sqrt{\tilde{I}_\sigma} \subset I(\tilde{L}_\sigma)$ as claimed.

Step II: $q(\mathbb{V}(\tilde{I}_\sigma)) \subset V(\sigma)$. Since $\tilde{I}_\sigma$ is a monomial ideal, the primes in its primary decomposition are all generated by the x_ρ with $\rho \in \mu(1)$. For a subset $J \subseteq \mu(1)$, let $\mathfrak{p}_J$ be the prime ideal generated by $\{x_\rho\}_{\rho \in J}$. Let ν be the minimal face of μ containing the rays in J. Suppose that $\nu \supset \sigma$. If τ is a face of μ not containing σ then there is a ray $\rho \in J$ such that $\rho \notin \tau(1)$. Hence $x_\rho | x_\mu^{\hat{\tau}}$. Thus $\mathfrak{p}_J \supset \tilde{I}_\sigma$. On the other hand, if J spans a face ν that does not contain σ then $x_\mu^{\hat{\nu}} \in \tilde{I}_\sigma$ but $x_\mu^{\hat{\nu}} \notin \mathfrak{p}_J$. Thus $\mathfrak{p}_J$ contains $\tilde{I}_\sigma$ if and only if J spans a face of μ that contains σ. By Lemma 3.22, $q_\sigma(\mathbb{V}(\mathfrak{p}_J)) \subset V(\sigma)$ if J spans a cone containing σ. This proves Step II and with it Proposition 3.27. □

LEMMA 3.28. *$\tilde{L}_\sigma = L_\sigma$ if and only if every cone μ containing σ is simplicial.*

Proof. If $\mu \supset \sigma$ is simplicial then the quotient map $q_\mu \colon U_\mu \to X_\mu$ is a geometric quotient. Since L_σ is closed and $G_{\mathbf{\Sigma}}$-invariant, it follows that $q_\sigma^{-1}(q_\sigma(L_\sigma)) = L_\sigma$. □

4. Blowups and the Stacky Star Subdivision of a Stacky Fan

Let $\mathbf{\Sigma} = (N, \Sigma, (v_1, \ldots, v_n))$ be a stacky fan, where the fan $\Sigma \in N_{\mathbb{R}}$ is not necessarily simplicial.

DEFINITION 4.1 (Stacky star subdivision of a stacky fan). Let σ be a cone in Σ. Let $v_0 = \sum_{\{k \in \sigma(1)\}} v_k$, and let ρ_0 be the ray in $N_{\mathbb{R}}$ generated by v_0. Set Σ_σ to be the fan obtained by replacing every cone of Σ containing σ with the joins of its faces with ρ_σ (cf. [Fu, p. 47]). The stacky fan $\mathbf{\Sigma}_\sigma = (N, \Sigma_\sigma, (v_0, v_1, \ldots, v_n))$ will be called the "star subdivision of σ"; for brevity, mention of $\mathbf{\Sigma}$ is omitted. Let $\beta_\sigma : \mathbb{Z}^{n+1} \to N$ be the homomorphism associated to the tuple $(v_0, v_1, \ldots, v_n)$; that is, we have $\beta_\sigma(e_i) = v_i$ for $0 \le i \le n$.

REMARK 4.2. Observe that if $\dim \sigma \ge 2$ then the fan Σ_σ has exactly $n + 1$ rays. The cones of Σ_σ can be described as follows. For each cone μ of Σ not containing σ there is a corresponding cone μ of Σ_σ. Each cone μ of Σ that contains σ is replaced by cones (which we will refer to as the "new" cones of Σ_σ) of the form $\tau' = \mathrm{Cone}(\tau, \rho_0)$, where τ is a face of μ such that $\rho_0 \notin \tau$. Since the cones are convex, $\rho_\sigma \notin \tau$ if and only if τ does not contain σ. Thus the new cones of Σ_σ are in bijective correspondence with the cones of Σ that were used in the definition of the ideal $\tilde{I}_\sigma$.

Also observe that σ is replaced by the cones $\mathrm{Cone}(\tau, \rho_\sigma)$, where τ runs over the facets of σ.

LEMMA 4.3. *If σ is a minimal nonsimplicial cone of Σ, then every new cone τ' of Σ_σ has* $\mathrm{ns}(\tau') < \mathrm{ns}(\sigma)$.

Proof. Because σ is a minimal nonsimplicial cone, every cone μ of Σ containing σ is generated by exactly $\mathrm{ns}(\sigma) + \dim \mu$ rays, and hence $\mathrm{ns}(\mu) = \mathrm{ns}(\sigma)$. If τ is a face of μ not containing ρ_0, then the uniqueness of minimal cones (Lemma 3.15) implies that $\mathrm{ns}(\tau) < \mathrm{ns}(\sigma)$. The new cone $\tau' = \mathrm{Cone}(\tau, \rho_0)$ has dimension $\dim(\tau) + 1$ and is spanned by $\mathrm{ns}(\tau) + \dim \tau + 1$ rays. Therefore $\mathrm{ns}(\tau') < \mathrm{ns}(\sigma)$. □

Next we compare the groups acting on $X(\tilde{\Sigma})$ and $X(\tilde{\Sigma}_\sigma)$. For notational convenience, we order the elements $v_1, \ldots, v_n \in N$ so that $\sigma(1) = \{\rho_1, \ldots, \rho_l\}$, where ρ_i is the ray through $\bar{v}_i \in N_{\mathbb{R}}$. Choose coordinates $(t_0, \ldots, t_n)$ on $(\mathbb{C}^*)^{n+1}$ and $(x_0, \ldots, x_n)$ on $\mathbb{A}^{n+1}$ so that x_0 corresponds to v_0 and $x_1, \ldots, x_l$ correspond to the rays in $\sigma(1)$. Consider the exact sequence

$$1 \to \mathbb{C}^* \xrightarrow{\lambda_0} (\mathbb{C}^*)^{n+1} \xrightarrow{\theta} (\mathbb{C}^*)^n \to 1, \tag{3}$$

where $\lambda_0(t) = (t^{-1}, t, \ldots, t, 1, \ldots, 1)$, with l copies of t and $n - l$ copies of 1, and where $\theta(t_0, \ldots, t_n) = (t_0 t_1, \ldots, t_0 t_l, t_{l+1}, \ldots, t_n)$.

LEMMA 4.4. *The map θ restricts to a map $G_{\mathbf{\Sigma}_\sigma} \to G_{\mathbf{\Sigma}}$ and induces a short exact sequence*

$$1 \to \mathbb{C}^* \to G_{\mathbf{\Sigma}_\sigma} \to G_{\mathbf{\Sigma}} \to 1 \tag{4}$$

such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C}^* & \longrightarrow & G_{\mathbf{\Sigma}_\sigma} & \longrightarrow & G_{\mathbf{\Sigma}} & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle (\beta_\sigma)^\vee} & & \Big\downarrow{\scriptstyle \beta^\vee} & & \\ 1 & \longrightarrow & \mathbb{C}^* & \xrightarrow{\lambda_0} & (\mathbb{C}^*)^{n+1} & \xrightarrow{\theta} & (\mathbb{C}^*)^n & \longrightarrow & 1. \end{array} \tag{5}$$

Proof. Consider the short exact sequence of free abelian groups

$$0 \to \mathbb{Z} \to \mathbb{Z}^{n+1} \to \mathbb{Z}^n \to 0,$$

where the first map is given by $k \mapsto (-k, k, \ldots, k, 0, \ldots, 0)$ and the second is given by $(a_0, a_1, \ldots, a_n) \mapsto (a_0 + a_1, \ldots, a_0 + a_l, a_{l+1}, \ldots, a_n)$. There is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & (\mathbb{Z})^{n+1} & \longrightarrow & \mathbb{Z}^n & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow{\scriptstyle \beta_\sigma} & & \Big\downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & N & \xrightarrow{=} & N & \longrightarrow & 0. \end{array} \tag{6}$$

The Gale dual of the map $\mathbb{Z} \to 0$ is the identity map $\mathbb{Z}^* \to \mathbb{Z}^*$. Moreover, all of the vertical arrows in (6) have finite cokernels. Thus, by [BChS, Lemma 2.3] we obtain a commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & (\mathbb{Z}^*)^n \arrow[r] \arrow[d, "\beta^\vee"] & (\mathbb{Z}^*)^{n+1} \arrow[r] \arrow[d, "\beta_\sigma^\vee"] & \mathbb{Z}^* \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \mathrm{DG}(\beta) \arrow[r] & \mathrm{DG}(\beta_\sigma) \arrow[r] & \mathbb{Z}^* \arrow[r] & 0. \end{tikzcd} \tag{7}$$

Applying the functor $\mathrm{Hom}(\cdot, \mathbb{C}^*)$ to (7) yields (5). □

DEFINITION 4.5 (Toric Reichstein transform). Given a stacky fan $\mathbf{\Sigma}$ and a cone σ in the underlying fan Σ, let Y_σ be the blowup of $X(\tilde{\Sigma}) = \mathbb{A}^n \setminus Z_\Sigma$ along the linear subspace L_σ and let Y'_σ be the Reichstein transform (Definition 2.9) of $X(\tilde{\Sigma})$ relative to the linear subspace L_σ. The quotient stack $[Y'_\sigma/G_{\mathbf{\Sigma}}]$ is called the *toric Reichstein transform* of $\mathcal{X}(\mathbf{\Sigma})$ relative to the cone σ, or relative to the substack $[L_\sigma/G_{\mathbf{\Sigma}}] \subset \mathcal{X}(\mathbf{\Sigma})$.

REMARK 4.6. $Y'_\sigma = Y_\sigma$ if and only if $\tilde{L}_\sigma = L_\sigma$, which is equivalent (by Lemma 3.28) to the condition that every cone containing σ is simplicial.

We now come to our main technical result. It shows that the toric Reichstein transform of a toric stack is again a toric stack. As a result, the toric Reichstein transform of a toric stack also has a good moduli space. The proof will be given in Section 4.3.

THEOREM 4.7. *Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan, and let σ be a cone in Σ. Then the quotient stack $[Y'_\sigma/G_{\mathbf{\Sigma}}]$ obtained from the Reichstein transform of $\mathcal{X}(\mathbf{\Sigma})$ relative to the substack $[L_\sigma/G_{\mathbf{\Sigma}}]$ is isomorphic to the toric stack $\mathcal{X}(\mathbf{\Sigma}_\sigma)$ formed by stacky star subdivision of σ* (*Definition 4.1*); *in other words, there is an isomorphism of stacks $\mathcal{X}(\mathbf{\Sigma}_\sigma) \simeq [Y'_\sigma/G_{\mathbf{\Sigma}}]$.*

REMARK 4.8. If one is interested primarily in toric varieties and not in toric stacks, Theorem 4.7 says that the toric variety $X(\Sigma_\sigma)$ formed by star subdivision of the cone σ is isomorphic to the quotient by G_Σ of Y'_σ, the Reichstein transform of $L_\sigma \subset X(\tilde{\Sigma})$. However, as we show in Example 4.9, the stack language is convenient and natural for describing some proper birational morphisms between toric varieties.

4.1. Examples

EXAMPLE 4.9. Let $\Sigma \subset \mathbb{R}^2$ be the fan with a single maximal cone generated by $v_1 = e_1$ and $v_2 = e_1 + 2e_2$. Let Σ' be the star subdivision of Σ. Then $X(\Sigma')$ is a smooth toric variety with maximal cones $\sigma_1 = \mathrm{Cone}(e_1, e_1 + e_2)$ and $\sigma_2 = \mathrm{Cone}(e_1+e_2, e_1+2e_2)$. There is a map of toric varieties $X(\Sigma') \to X(\Sigma)$, but there is no map of Cox stacks $\mathcal{X}(\Sigma') \to \mathcal{X}(\Sigma)$ because the vector e_1+e_2 is not an integral linear combination of v_1 and v_2. However, if we let $\mathbf{\Sigma}' = (\mathbb{Z}^2, \Sigma', 2(e_1+e_2), e_1, e_2)$ then the map of toric varieties $X(\Sigma') \to X(\Sigma)$ is induced by the map of toric stacks $\mathcal{X}(\mathbf{\Sigma}') \to \mathcal{X}(\Sigma)$ corresponding to the Reichstein transform of the stack $\mathcal{X}(\Sigma)$ relative to the maximal cone σ of Σ.

If $\mathbf{\Sigma} = (N, \Sigma, v_1, \dots, v_n)$ is a toric fan, then the good moduli space of $\mathcal{X}(\mathbf{\Sigma})$ is the toric variety $X(\Sigma)$ and does not depend on the choice of elements $v_1, \dots, v_n \in N$.

However, as the next example shows, the toric variety associated to a stacky subdivision relative to a cone σ depends on the choice of v_i corresponding to the rays of σ.

EXAMPLE 4.10. Let $\Sigma \in \mathbb{R}^3$ be the fan with a single maximal cone σ generated by $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$, and $v_4 = e_1 - e_2 + e_3$, where the e_i are the standard basis vectors. Let $\mathcal{X}(\Sigma)$ be the Cox stack of $X(\Sigma)$. The toric variety $X(\Sigma)$ is the quadric cone. Let Σ' be the star subdivision of σ. Then Σ' is a smooth (hence simplicial) toric variety. Since $v_0 = v_1 + v_2 + v_3 + v_4 = 2e_1 + 2e_3$ is not primitive, the map of toric varieties $X(\Sigma') \to X(\Sigma)$ is induced by a map of stacks $\mathcal{X}(\mathbf{\Sigma}') \to \mathcal{X}(\Sigma)$, where $\mathbf{\Sigma}' = (\mathbb{Z}^3, \Sigma', 2e_1 + 2e_3, e_1, e_2, e_3, e_1 - e_2 + e_3)$ is the stacky star subdivision of the stacky fan $\mathbf{\Sigma} = (\mathbb{Z}^3, \Sigma, e_1, e_2, e_3, e_1 - e_2 + e_3)$.

Now consider the stacky fan $\mathbf{\Sigma} = (\mathbb{Z}^3, \Sigma, \{2e_1, 2e_2, e_3, v_4\})$. The underlying toric variety of $\mathcal{X}(\mathbf{\Sigma})$ is also $X(\Sigma)$, but the stacky star subdivision of $\mathbf{\Sigma}$ relative to σ gives the stacky fan $\mathbf{\Sigma}_\sigma = (\mathbb{Z}^3, \Sigma_\sigma, \{3e_1 + e_2 + 2e_3, 2e_1, 2e_2, e_3, v_4\})$, where Σ_σ is the fan formed by star subdividing Σ along the ray through $3e_1 + e_2 + e_3$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{X}(\mathbf{\Sigma}_\sigma) \arrow[r, "\tilde{f}"] \arrow[d, "q(\Sigma_\sigma)"'] & \mathcal{X}(\mathbf{\Sigma}) \arrow[d, "q(\Sigma)"] \\
X(\Sigma_\sigma) \arrow[r, "f"'] & X(\Sigma),
\end{tikzcd}$$

where $\tilde{f}$ is birational, $\mathcal{X}(\mathbf{\Sigma}_\sigma)$ is isomorphic to the Reichstein transform of $[L_\sigma/G_\Sigma] \subset \mathcal{X}(\mathbf{\Sigma})$, and f is a proper birational map between toric varieties.

Suppose that $\mathbf{\Sigma} = (\Sigma, N, v_1, \dots, v_n)$ is a stacky fan. If $\sigma = \rho$ is a ray in Σ then L_σ is a Cartier divisor in $X(\tilde{\Sigma})$, so the blowup of $X(\tilde{\Sigma})$ along L_σ is identified with $X(\tilde{\Sigma})$. However, the hyperplane L_σ need not be saturated with respect to the quotient map $X(\tilde{\Sigma}) \to X(\Sigma)$, so the Reichstein transformation of L_σ may be a proper open set in $X(\tilde{\Sigma})$. As the next example shows, this phenomenon is related to wall crossing in geometric invariant theory and will be discussed further in Section 6.

EXAMPLE 4.11. Let Σ be the fan in $\mathbb{R}^3$ considered in Example 4.10. Let $\mathcal{X}$ be the Cox stack of the toric variety $X(\Sigma)$. Then $\mathcal{X} = [\mathbb{A}^4/\mathbb{C}^*]$, where $\mathbb{C}^*$ acts with weights $(1, -1, 1, -1)$. If L is any of the coordinate hyperplanes in $\mathbb{A}^4 = \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3, x_4]$, then L is not saturated and so the Reichstein transform with respect to L is an open set in $\mathbb{A}^4$. For example, if $L = L_1 = \mathbb{V}(x_1)$ then $\tilde{L} = \mathbb{V}(x_1) \cup \mathbb{V}(x_2, x_4)$. Thus the Reichstein transform of $\mathcal{X}(\Sigma)$ with respect to L produces the quotient stack $[(\mathbb{A}^4 \setminus \mathbb{V}(x_2, x_4))/\mathbb{C}^*]$. This is the Cox stack of the toric variety $X(\Sigma')$, where Σ' is the fan obtained by subdividing σ into two cones via the subdivision relative to the lattice point $v = e_1$.

This example can be interpreted in terms of change of linearizations in geometric invariant theory (cf. [T, Exm. 1.16]). The quotient $\mathbb{A}^4/\mathbb{C}^*$ is the GIT quotient of $\mathbb{A}^4$ linearized with respect to the trivial character, and the quotient $(\mathbb{A}^4 \setminus \mathbb{V}(x_2, x_4))/\mathbb{C}^*$ is the GIT quotient of $\mathbb{A}^4$ with respect to the character of weight 1. Likewise, the

GIT quotient $(\mathbb{A}^4 \setminus \mathbb{V}(x_1, x_3))/\mathbb{C}^*$ corresponds to the GIT quotient where the linearization has weight -1. The second quotient corresponds to the Reichstein transformation with respect to the divisor $L_2 = \mathbb{V}(x_2)$.

4.2. *Reichstein Transforms and the Factorization of Birational Toric Morphisms*

The phenomena of Examples 4.9–4.11 can be generalized.

PROPOSITION 4.12. *Let Σ be a fan in a lattice N, and let Σ' be the subdivision of Σ relative to a ray $\rho_0 \in |\Sigma|$ in the support of Σ. Then there exist stacky fans $\mathbf{\Sigma} = (N, \Sigma, v_1, \dots, v_n)$ and $\mathbf{\Sigma}' = (N, \Sigma', v_0, v_1, \dots, v_n)$ with $v_0 = v_1 + \cdots + v_l$, where v_0 is a lattice point on ρ_0 and $v_1, \dots, v_l$ are lattice points on the rays of a cone $\sigma \in \Sigma$. Hence for any toric blowup $X(\Sigma') \to X(\Sigma)$ there is a toric Reichstein transform of toric stacks $\mathcal{X}' \to \mathcal{X}$ and a commutative diagram of stacks and good moduli spaces*:

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ X(\Sigma') \arrow[r] & X(\Sigma). \end{tikzcd}$$

Proof. Let σ be the unique cone of Σ containing ρ_0 in its interior. If $\rho_1, \dots, \rho_l$ are the rays of σ and if w_i is the primitive lattice point on ρ_i, then we can write $w_0 = \sum_{i=1}^{l} a_i w_i$ for $a_i \in \mathbb{Q}_{>0}$. Clearing denominators, we see that $b_0 w_0 = \sum b_i w_i$ with $b_i \in \mathbb{Z}_{>0}$. If we let $v_i = b_i w_i$ for $i = 0, \dots, l$ and let $v_{l+1}, \dots, v_n$ be any lattice vectors on the remaining rays $\rho_{l+1}, \dots, \rho_n$ of Σ, then $v_0 = v_1 + \cdots + v_l$. $\square$

Combining Proposition 4.12 with weak factorization for birational toric morphisms [W, Thm. A] yields the following result relating birational toric morphisms and toric Reichstein transformations.

COROLLARY 4.13. *Let Σ' and Σ'' be two fans in $\mathbb{R}^d$ such that $|\Sigma'| = |\Sigma''|$. Then there exists a sequence of stacky fans $\mathbf{\Sigma}_i = (\mathbb{Z}, \Sigma_i, \beta_i)$, $i = 0, \dots, n$, with $\Sigma_0 = \Sigma'$ and $\Sigma_n = \Sigma''$ such that the toric stack $\mathcal{X}(\mathbf{\Sigma}_i)$ is related to $\mathcal{X}(\mathbf{\Sigma}_{i-1})$ by a composition of toric Reichstein transformations for $i = 1, \dots, n$.*

Proof. By toric resolution of singularities, the fans Σ' and Σ'' can be subdivided into regular fans. By Proposition 4.12 these subdivisions correspond to Reichstein transformations of toric stacks with underlying fans Σ' and Σ''. Hence we are reduced to the case that Σ' and Σ'' are regular. By [W, Thm. A] there is a sequence of fans Σ_i, $i = 0, \dots, n$, such that $\Sigma_0 = \Sigma'$, $\Sigma_n = \Sigma''$, and Σ_i is related to Σ_{i-1} via a sequence of subdivisions. Once again using Proposition 4.12, we may interpret these subdivisions in terms of toric Reichstein transforms. $\square$

4.3. *Proof of Theorem 4.7*

Label the rays of Σ as $\rho_1, \dots, \rho_l, \rho_{l+1}, \dots, \rho_N$ with $\rho_1, \dots, \rho_l$ the rays of σ. Let $x_1, \dots, x_n$ be the corresponding coordinates on $\mathbb{A}^n$. The linear subspace L_σ is cut out by the regular sequence $(x_1, \dots, x_l)$, so the blowup Y_σ is embedded as the subvariety $X(\tilde{\Sigma}) \times \mathbb{P}^{l-1} \subset \mathbb{A}^n \times \mathbb{P}^{l-1}$ defined by the equations $x_i y_j = x_j y_i$ for $1 \le i < j \le l$.

Likewise label the rays of Σ_σ as $\rho_0, \rho_1, \dots, \rho_l, \rho_{l+1}, \dots, \rho_N$ with $\rho_0 = \rho_\sigma$ (introduced in Definition 4.1). Let $z_0, z_1, \dots, z_N$ be the corresponding coordinates on $\mathbb{A}^{n+1}$.

Observe that, for each ray ρ of σ, there is a cone $\tau' = \operatorname{Cone}(\tau, \rho_s)$ of Σ_σ such that ρ does not lie in τ' and such that τ is a face of σ. Hence not all of $z_1, \dots, z_l$ vanish on the open set $X(\tilde{\Sigma}_\sigma) \subset \mathbb{A}^{n+1}$; that is, $X(\tilde{\Sigma}_\sigma) \subset \mathbb{A}^{n+1} \setminus V(z_1, \dots, z_l)$.

Define a map $p\colon \mathbb{A}^{n+1} \setminus V(z_1, \dots, z_l) \to \mathbb{A}^n \times \mathbb{P}^{l-1}$ by the formula

$$(z_0, \dots, z_n) \mapsto (z_0 z_1, z_0 z_2, \dots, z_0 z_l, z_{l+1}, \dots, z_n) \times [z_1 : \cdots : z_l].$$

By definition, the image of p satisfies the equations $x_i y_j = x_j y_i$ for $1 \le i < j \le l$ and so is contained in the blowup $\operatorname{Bl}_{L_\sigma} \mathbb{A}^n$. Since not all of $z_1, \dots, z_l$ vanish on $\mathbb{A}^{n+1} \setminus V(z_1, \dots, z_l)$, the action of $\lambda_0(\mathbb{C}^*)$ is free and the map p is $\lambda_0(\mathbb{C}^*)$-invariant.

LEMMA 4.14. *The map* $p\colon \mathbb{A}^{n+1} \setminus V(z_1, \dots, z_l) \to \operatorname{Bl}_{L_\sigma} \mathbb{A}^n$ *is a* $\lambda_0(\mathbb{C}^*)$*-torsor.*

Proof. Observe that $\operatorname{Bl}_{L_\sigma} \mathbb{A}^n = \operatorname{Bl}_0(\mathbb{A}^l \times \mathbb{A}^{n-l})$ and that the map p is obtained by base change from the map $q\colon \mathbb{A}^{l+1} \setminus V(z_1, \dots, z_l) \to \operatorname{Bl}_0 \mathbb{A}^l$ given by

$$(z_0, z_1, \dots, z_l) \mapsto (z_0 z_1, \dots, z_0 z_l) \times [z_1 : \cdots : z_l].$$

To prove the lemma it suffices to show that q is a $\mathbb{C}^*$-torsor. Again the action is free, so it suffices to show that $\operatorname{Bl}_0 \mathbb{A}^l$ is the quotient. To see this, cover $\mathbb{C}^{l+1} \setminus V(z_1, \dots, z_l)$ by the invariant open affines $\operatorname{Spec} \mathbb{C}[z_0, z_1, \dots, z_l, 1/z_i]$, where $1 \le i \le l$. The subring of invariants is $\operatorname{Spec} \mathbb{C}[z_0 z_1, \dots, z_0 z_l, z_1/z_i, \dots, z_l/z_i]$, which is the natural affine covering of $\operatorname{Bl}_0 \mathbb{A}^l$. □

Recall (Definition 4.5) that we defined Y'_σ to be the Reichstein transform of $X(\tilde{\Sigma})$ relative to the linear subspace L_σ. We show in Lemma 4.15 that $p^{-1}(Y'_\sigma) = X(\tilde{\Sigma}_\sigma)$. Theorem 4.7 then follows: by Lemma 4.15 and the exact sequence (4), the quotient stack $\mathcal{X}(\boldsymbol{\Sigma}_\sigma) := [X(\tilde{\Sigma}_\sigma)/G_{\boldsymbol{\Sigma}_\sigma}]$ is equivalent to the quotient stack $[Y'_\sigma/G_{\boldsymbol{\Sigma}}]$.

LEMMA 4.15. $p^{-1}(Y'_\sigma) = X(\tilde{\Sigma}_\sigma)$. *Hence the map* $p|_{X(\tilde{\Sigma}_\sigma)}\colon X(\tilde{\Sigma}_\sigma) \to Y'_\sigma$ *is a* $\lambda_0(\mathbb{C}^*)$*-torsor.*

Proof. First we prove $p^{-1}(Y'_\sigma) \subset X(\tilde{\Sigma}_\sigma)$. Let $\mathbf{y} = (x_1, \dots, x_l, x_{l+1}, \dots, x_N) \times [y_1 : \cdots : y_l]$ be a point of Y'_σ. We distinguish two cases.

Case I: $\mathbf{y}$ *is not in the exceptional divisor.* In this case, at least one of the x_i is nonzero and $p^{-1}(\mathbf{y}) = \{(t, t^{-1}x_1, \dots, t^{-1}x_l, x_{l+1}, \dots, x_N) \mid t \ne 0\}$. Let $\mathbf{x} := (x_1, \dots, x_l, x_{l+1}, \dots, x_N)$. Since $\mathbf{x} \in \mathbb{A}^n \setminus (Z_\Sigma \cup \tilde{L}_\sigma)$, there is a cone $\mu \subset \Sigma$ such that the function $x^{\hat{\mu}}$ does not vanish at $\mathbf{x}$. If μ does not contain σ, then the argument used at the beginning of the proof of Proposition 3.27 implies that $\mathbf{x} \notin \tilde{L}_\sigma$ as well. Since μ does not contain σ, it is also a cone of the fan Σ_σ. Moreover, $t \ne 0$ and so the function $z^{\hat{\mu}}$ is nonzero on $p^{-1}(\mathbf{y})$. Hence $p^{-1}(\mathbf{y}) \subset X(\tilde{\Sigma}_\sigma)$.

On the other hand, if μ contains σ then the assumption that $\mathbf{x} \notin \tilde{L}_\sigma$ implies that there is a face $\tau \subset \mu$ such that $x_\mu^{\hat{\tau}}$ does not vanish at $\mathbf{x}$. Thus there is a cone $\mu \supset \sigma$ and a face $\tau \subset \mu$ not containing μ such that the function $x_\sigma^{\hat{\tau}}$ is nonzero at $\mathbf{x}$. Since $x^{\hat{\tau}} = x_\sigma^{\hat{\tau}} x^{\hat{\sigma}}$, it follows that the function $x^{\hat{\tau}}$ also does not vanish at x. Let $\tau' = \mathrm{Cone}(\tau, \rho_0)$ be the corresponding cone of the fan Σ_σ. Then the function $z^{\hat{\tau}'}$ is nonzero on $p^{-1}(\mathbf{y})$.

Case II: $\mathbf{y}$ *is on the exceptional divisor* (*i.e.,* $x_1 = \cdots = x_l = 0$). In this case, $p^{-1}(\mathbf{y}) = \{(0, y_1^*, \ldots, y_l^*, x_{l+1}, \ldots, x_N) \mid [y_1^* : \cdots : y_l^*] = [y_1 : \cdots : y_l] \in \mathbb{P}^{l-1}\}$. Again $\mathbf{x} = (0, 0, \ldots, 0, x_{l+1}, \ldots, x_n)$ is in $X(\tilde{\Sigma})$, so there is a cone μ necessarily containing σ and such that $x^{\hat{\mu}}$ is nonzero at $\mathbf{x}$. Also, since $\mathbf{y}$ is not in the proper transform of $\tilde{L}_\sigma$, we know that $\mathbf{x}^* = (y_1^*, \ldots, y_l^*, x_{l+1}, \ldots, x_n)$ is not in $\tilde{L}_\sigma$. Thus there is a face τ of μ such that the function $x_\mu^{\hat{\tau}}$ does not vanish at $\mathbf{x}^*$. Since $x^{\hat{\mu}}$ also does not vanish at $\mathbf{x}^*$, again we conclude that $x^{\hat{\tau}}$ is nonzero at $\mathbf{x}^*$. If we let $\tau' = \mathrm{Cone}(\tau, \rho_0)$ then again the function $z^{\hat{\tau}'}$ is nonzero on $p^{-1}(\mathbf{y})$.

Therefore, $p^{-1}(Y'_\sigma) \subset X(\tilde{\Sigma}_\sigma)$.

The proof that $p(X(\tilde{\Sigma}_\sigma)) \subset Y'_\sigma$ is similar. Let $\mathbf{z} = (z_0, z_1, \ldots, z_l, z_{l+1}, \ldots, z_n)$ be a point of $X(\tilde{\Sigma}_\sigma)$. Again we have two cases.

Case I: $z_0 \neq 0$. Thus $p(\mathbf{z})$ is not contained in the exceptional divisor. In this case it suffices to show that $\mathbf{x} = (z_0 z_1, \ldots, z_0 z_l, z_{l+1}, \ldots, z_n)$ is in the complement of $Z_\Sigma \cup \tilde{L}_\sigma$. By hypothesis, there is a cone τ' of Σ_σ such that $z^{\hat{\tau}'} \neq 0$ at $\mathbf{z}$. If τ' corresponds to a cone τ of Σ not containing σ then $\mathbf{x} \notin \tilde{L}_\sigma$ and the function $x^{\hat{\tau}}$ does not vanish at $\mathbf{x}$, so $p(\mathbf{z}) \in Y'_\sigma$.

On the other hand, suppose that $\tau' = \mathrm{Cone}(\tau, \sigma)$. Then the function $x^{\hat{\tau}} \neq 0$ on $p(\mathbf{z})$. For every cone μ that contains τ and σ, we may factor $x^{\hat{\tau}} = x^{\hat{\mu}} x_\mu^{\hat{\tau}}$. Thus, for every $\mu \supset \sigma$, $x_\mu^{\hat{\tau}}$ does not vanish at $\mathbf{x}$ and so $p(\mathbf{z}) \notin \tilde{L}_\sigma$ as well.

Case II: $z_0 = 0$. Here $p(\mathbf{z}) = (0, 0, \ldots, 0, z_{l+1}, \ldots, z_n) \times [z_1 : \cdots : z_l]$ is contained in the exceptional divisor. By hypothesis, there is a cone τ' of Σ_σ such that $z^{\hat{\tau}'} \neq 0$ at $\mathbf{z}$. Since $z_0 = 0$, it follows that ρ_0 must be a ray of τ'. Hence $\tau' = \mathrm{Cone}(\tau, \rho_0)$. Thus, for every cone μ containing τ, the functions $x^{\hat{\mu}}$ and $x_\mu^{\hat{\tau}}$ do not vanish at $\mathbf{x} = (z_1, \ldots, z_l, z_{l+1}, \ldots, z_n) \in \mathbb{A}^n$. Hence $p(\mathbf{z}) = (0, 0, \ldots, 0, z_{l+1}, \ldots, z_n)$ is not in the proper transform of $\tilde{L}_\sigma$. Also, for any cone μ containing τ and σ, the function $x^{\hat{\mu}}$ does not vanish at $(0, 0, \ldots, 0, z_{l+1}, \ldots, z_n)$. Therefore, $p(\mathbf{z}) \in Y'_\sigma$.

This completes the proof of Lemma 4.15 and thus of Theorem 4.7. □

5. Relation with Kirwan–Reichstein Partial Desingularization

Kirwan [K] gave a method for obtaining partial desingularizations of geometric invariant theory quotients of smooth projective varieties. Let X be a smooth projective variety with the linearized action of a reductive group G. Let L be a

G-linearized ample line bundle, and let $X^{ss} = X^{ss}(L)$ and $X^s = X^s(L)$ be (respectively) the sets of semi-stable and stable points.

Kirwan's procedure for partial desingularization can be described as a canonical sequence of Reichstein transformations. Framed in this language, Kirwan's main result is the following.

THEOREM 5.1 [K; MFoK, Chap. 8, Sec. 4]. *With notation as before, let r be the maximum dimension of a reductive subgroup of G that fixes a point of X^{ss}.*

(i) *If $r = 0$ then every semi-stable point is stable.*

Let $Z_r \subset X^{ss}$ be the locus of points fixed by a reductive subgroup of dimension r.

(ii) *$Z_r \subset X^{ss}$ is nonsingular.*

Let $X_{r-1} = \mathcal{R}(X^{ss}, Z_r)$. Then the following statements hold.

(iii) *Any reductive subgroup of G that fixes a point of X_{r-1} has dimension strictly less than r.*

(iv) *The morphism $\mathcal{R}(X^{ss}, Z_r) \to X^{ss}$ is an isomorphism over the open set $X^s \subset X^{ss}$.*

(v) *There exists a G-equivariant projective birational morphism $X' \to X$ with X' smooth and such that $X_{r-1} = (X')^{ss}(L')$ for a suitable line bundle L' on X'. In particular, there exist a projective good quotient X_1/G and a projective birational morphism $X_1/G \to X^{ss}/G$ that is an isomorphism over the open set $X^s/G \subset X^{ss}/G$.*

Hence, after this process is iterated a finite number of times, every semi-stable point becomes stable and we get a G-equivariant birational morphism $X_0 \to X$ and a smooth Deligne–Mumford stack $\mathcal{X}' = [X_0/G]$. The projective variety X_0/G has finite quotient singularities. The map $X_0/G \to X^{ss}/G$ is projective and birational and is an isomorphism over the quotient of the stable locus $X^s/G \subset X^{ss}/G$.

We now come to the main result of this paper—an analogue of Kirwan's result for toric stacks. Since toric varieties need not be projective or quasi-projective, toric stacks are not necessarily of the form $[X^{ss}/G]$; in such cases, Kirwan's result does not apply. In a subsequent paper we will consider analogues of Kirwan's result for arbitrary quotient stacks that admit good moduli spaces.

THEOREM 5.2 (Partial desingularization of Artin toric stacks). *Let $\mathbf{\Sigma} = (N, \Sigma, \beta)$ be a stacky fan. Then there exist a simplicial stacky fan $\mathbf{\Sigma}' = (N, \Sigma', \beta')$ and a birational map of toric stacks $\mathcal{X}(\mathbf{\Sigma}') \to \mathcal{X}(\mathbf{\Sigma})$ that induces a proper birational morphism of toric varieties $X(\Sigma') \to X(\Sigma)$.*

The map $\mathcal{X}(\mathbf{\Sigma}') \to \mathcal{X}(\mathbf{\Sigma})$ is obtained by a finite sequence of stacky star subdivisions along minimal nonsimplicial cones. In addition, $\mathcal{X}(\mathbf{\Sigma}')$ is isomorphic to the stack obtained by a finite sequence of Reichstein transformations of the locus of points with maximal dimensional stabilizer.

If $X(\Sigma)$ is a projective toric variety, then this sequence of Reichstein transformations is the same as the one in Kirwan's theorem applied to a weighted projective space.

Proof. We proceed by induction on the nonsimplicial index $\mathrm{ns}(\Sigma)$ (Definition 3.17). If $\mathrm{ns}(\Sigma) = 0$, then Σ is simplicial and there is nothing to prove.

Assume $\mathrm{ns}(\Sigma) > 0$. Let $S = S_\Sigma$ be the set of minimal nonsimplicial cones of Σ (Definition 3.17). For $\sigma \in \Sigma$, recall (Definitions 3.19 and 3.24) that $L_\sigma = \mathbb{V}(x_\rho \mid \rho \in \sigma(1)) \cap X(\tilde{\Sigma})$ and $\tilde{L}_\sigma = q^{-1}(q(L_\sigma)) = q^{-1}(V(\sigma))$, where $q\colon X(\tilde{\Sigma}) \to X(\Sigma)$ is the quotient map.

Note that for any two distinct elements $\sigma, \sigma' \in S$ we have $V(\sigma) \cap V(\sigma') = \emptyset$: if $p \in V(\sigma) \cap V(\sigma')$ then, by the orbit–cone correspondence (equation (1)), p is in the orbit $O(\tau)$ for a cone τ containing σ and σ'; but by Lemma 3.18, no such cone τ exists.

It follows that for any two distinct elements $\sigma, \sigma' \in S$, $\tilde{L}_\sigma$ and $\tilde{L}_{\sigma'}$ are disjoint (since if p were a common point, then $q(p) \in V(\sigma) \cap V(\sigma')$). Let $L_S = \bigcup_{\sigma \in S} L_\sigma$ and $\tilde{L}_S = \bigcup_{\sigma \in S} \tilde{L}_\sigma$; both of these unions are in fact disjoint unions. The following calculation shows that $\tilde{L}_S = q^{-1}(q(L_S))$:

$$q(L_S) = q\left(\coprod_{\sigma \in S} L_\sigma\right) = \coprod_{\sigma \in S} q(L_\sigma) = \coprod_{\sigma \in S} V(\sigma);$$

$$q^{-1}(q(L_S)) = q^{-1}\left(\coprod_{\sigma \in S} V(\sigma)\right) = \coprod_{\sigma \in S} q^{-1}(V(\sigma)) = \coprod_{\sigma \in S} \tilde{L}_\sigma = \tilde{L}_S.$$

Let Y_S be the blowup of $\mathcal{X}(\Sigma)$ along L_S, and let Y_S' be the complement in Y_S of the strict transform of $\tilde{L}_S$. In other words, Y_S' is the Reichstein transform of $L_S \subseteq \mathcal{X}(\Sigma)$. By Lemma 3.20, the general point of L_S has $G_{\mathbf{\Sigma}}$-stabilizer of rank $\mathrm{ns}(\Sigma)$, and by the same lemma, L_S is the closure of the locus with maximal dimensional stabilizer.

Since the $\tilde{L}_\sigma$ are disjoint as $\sigma \in S$ varies, Y_S' is isomorphic to successively blowing up $\mathcal{X}(\Sigma)$ along each L_σ and removing the strict transform of $\tilde{L}_\sigma$ as $\sigma \in S$ varies. Let $\mathbf{\Sigma}_S$ be the stacky fan formed by starting with $\mathbf{\Sigma}$ and then stacky star subdividing (Definition 4.1) each cone $\sigma \in S$. By repeatedly applying Theorem 4.7, we conclude that $[Y_S'/G_{\mathbf{\Sigma}}]$ is isomorphic to the toric stack $\mathcal{X}(\mathbf{\Sigma}_S)$. By Lemma 4.3 we have $\mathrm{ns}(\Sigma_S) < \mathrm{ns}(\Sigma)$, and the inductive step is complete.

If $X(\Sigma)$ is projective, then by [CLSc, Prop. 14.1.9] there is a $G = G_{\mathbf{\Sigma}}$-equivariant line bundle $L_{\mathbf{a}}$ on $\mathbb{A}^n$ (where $n = |\Sigma(1)|$) such that $X(\tilde{\Sigma}) = \mathbb{A}^n(L_{\mathbf{a}})^{ss}$. Hence the argument of [D, Prop. 12.2] expresses $X(\Sigma)$ as a GIT quotient of a weighted projective space, so we can apply Kirwan's result. Kirwan's algorithm proceeds by successively taking the Reichstein transforms of the locus of points with maximal dimensional stabilizer, and hence our algorithm agrees with Kirwan's in the projective toric case. □

6. Divisorial Reichstein Transformations and Change of Linearizations

Recall [DH, T], that if X is a projective variety with the action of a reductive group G, then the cone of G-linearized ample divisors is divided into a finite number of chambers such that the GIT quotient is constant on the interior of each chamber. In many (but not all) examples, if L is a line bundle corresponding to a point on a wall between two chambers then $X^{ss}(L) \neq X^{s}(L)$, but if L is deformed to a line bundle L' in the interior of a chamber then $X^{s}(L') = X^{ss}(L') \subset X^{ss}(L)$. From the stack point of view, this means that the nonseparated quotient stack $[X^{ss}(L)/G]$ with complete good moduli space $X^{ss}(L)/G$ contains the complete Deligne–Mumford (DM) open substack $[X^{s}(L')/G]$. The induced map of quotients $X^{s}(L')/G \to X^{ss}(L)/G$ is proper and birational. Our final result shows we can use Reichstein transformations relative to divisors to find complete open DM substacks of Artin toric stacks with complete good moduli space. In a subsequent paper we will further investigate the relationship between divisorial Reichstein transformations and changes of linearizations.

THEOREM 6.1. *Let $\mathbf{\Sigma} = (N, \Sigma, (v_1, \dots, v_n))$ be a stacky fan and let $\mathcal{X}(\mathbf{\Sigma})$ be the associated toric stack. Then there exists a sequence of divisorial toric Reichstein transforms relative to rays in Σ such that the resulting toric stack $\mathcal{X}(\mathbf{\Sigma}')$ is Deligne–Mumford. In particular, if the toric variety $X(\Sigma)$ is complete, then the toric stack $\mathcal{X}(\mathbf{\Sigma})$ contains a complete open Deligne–Mumford substack $\mathcal{X}(\mathbf{\Sigma}')$ such that the induced map on moduli spaces $X(\Sigma') \to X(\Sigma)$ is proper and birational.*

Proof. Our result follows from [CLSc, Prop. 11.1.7], which says that any fan can be made simplicial by a sequence of star subdivisions relative to the rays of the fan. Let Σ' be a fan obtained from Σ by subdividing relative to a sequence of rays $\rho_1, \dots, \rho_l$. If $\mathbf{\Sigma}' = (N, \Sigma', (v_1, \dots, v_n))$ then the toric stack $\mathcal{X}(\mathbf{\Sigma}')$ is obtained from $\mathcal{X}(\mathbf{\Sigma})$ by a sequence of divisorial Reichstein transformations. $\square$

REMARK 6.2. If $X(\Sigma)$ is projective, then $X(\tilde{\Sigma}) = (\mathbb{A}^n)^{ss}(\chi)$ for some character $\chi \in \hat{G}_{\mathbf{\Sigma}}$. If $\mathcal{X}(\mathbf{\Sigma}') \subset \mathcal{X}(\mathbf{\Sigma})$ is an open toric substack obtained by a divisorial Reichstein transform, then $X(\tilde{\Sigma}') = (\mathbb{A}^n)^{ss}(\chi')$ for some other character $\chi' \in \hat{G}_{\mathbf{\Sigma}}$. Thus the toric variety $X(\Sigma')$ is obtained by a change of linearization for the $G_{\mathbf{\Sigma}} = G_{\mathbf{\Sigma}'}$ action on $\mathbb{A}^n$.

If $G \subset (\mathbb{C}^*)^n$ and $\chi \in \hat{G}$ is a character, then the quotient $(\mathbb{A}^n)^{ss}(\chi)/G$ is again a toric variety—although the stack $[(\mathbb{A}^n)^{ss}/G]$ need not be a toric stack in our sense of the term. As χ varies through $\hat{G}$, the different quotients are related by birational transformations. As the next example shows, these transformations can also be interpreted in terms of divisorial Reichstein transforms.

EXAMPLE 6.3. The group $G = \{(t, t^{-1}u, t, u) \mid t, u \in \mathbb{C}^*\} \subseteq (\mathbb{C}^*)^4$ acts on $\mathbb{A}^4 = \operatorname{Spec} \mathbb{C}[x_1, x_2, x_3, x_4]$ in the natural way. Since $G \simeq (\mathbb{C}^*)^2$, the character group $\hat{G}$ is the lattice $\mathbb{Z}^2$. As noted in [CLSc, Exm. 14.3.7], the GIT quotient $\mathbb{A}^4//_\chi G$ is nonempty if and only if χ lies in the cone spanned by $(-1, 1)$ and $(1, 0)$. This cone is divided into two chambers separated by the wall spanned by the ray through $(0, 1)$. In the interior of the left chamber, as well as on the wall, the quotients are all isomorphic to $\mathbb{P}^2$, and in the interior of the right chamber the quotients are all isomorphic to the Hirzebruch surface H_1.

The point $\beta = (0, 1)$ lies on the wall, and the GIT quotient $\mathbb{A}^4//_{(0,1)}G$, is equal to $\operatorname{Proj}\mathbb{C}[x_1x_2, x_2x_3, x_4] \simeq \mathbb{P}^2$. This is the good quotient of

$$X = \mathbb{A}^4 \setminus (\mathbb{V}(x_1, x_3, x_4) \cup \mathbb{V}(x_2, x_4))$$

by G. Note, however, that the quotient stack $[X/G]$ is not a toric stack in the sense of this paper, although it is a toric stack in the sense of [Sa]. The reason is that $[X/G]$ is not a DM stack (since $(0, 0, 0, 1) \in X$ has a 1-dimensional stabilizer) but the moduli space of $[X/G]$ is a simplicial toric variety.

Consider the point $(-1, 2)$ in the interior of the left chamber. The semi-stable locus relative to this character is $\mathbb{A}^4 \setminus (\mathbb{V}(x_1, x_3, x_4) \cup \mathbb{V}(x_2))$ and the GIT quotient $\mathbb{A}^4//_{(-1,2)}G$ is $\mathbb{P}^2$. The Reichstein transform of X relative to the divisor $x_1 = 0$ is $\mathbb{A}^4 \setminus (\mathbb{V}(x_1, x_3, x_4) \cup \mathbb{V}(x_2))$, since the saturation (relative to the quotient map $q\colon X \to X/G$) of the divisor $x_1 = 0$ is $\mathbb{V}(x_1) \cup \mathbb{V}(x_2)$. Thus the Reichstein transform of the nontoric stack $[X/G]$ relative to the divisor $[\mathbb{V}(x_0)/G]$ is the representable toric stack $\big[(\mathbb{A}^4 \setminus (\mathbb{V}(x_1, x_3, x_4) \cup \mathbb{V}(x_2)))/G\big]$, which is represented by the smooth toric variety $\mathbb{P}^2$.

Now take the point $(1, 1)$ in the right chamber. The semi-stable locus relative to this character is $\mathbb{A}^4 \setminus (\mathbb{V}(x_1, x_3) \cup \mathbb{V}(x_2, x_4))$. The GIT quotient $\mathbb{A}^4//_{(1,1)}G$ is H_1. The Reichstein transform of X relative to the divisor $x_2 = 0$ is

$$\mathbb{A}^4 \setminus (\mathbb{V}(x_1, x_3) \cup \mathbb{V}(x_2, x_4)),$$

since the saturation of the divisor $x_2 = 0$ is $\mathbb{V}(x_1, x_3) \cup \mathbb{V}(x_2)$. Thus we see again that a divisorial Reichstein transform of the nontoric stack $[X/G]$ produces the new GIT quotient. The quotient stack $[\mathbb{A}^4((1, 1))^{ss}/G]$ is again a representable toric stack that is represented by the toric variety H_1.

This example will be generalized in a subsequent paper.

Our final example gives a complete open DM substack $\mathcal{X}(\Sigma'')$ of a Cox stack $\mathcal{X}(\Sigma)$ with projective moduli space $X(\Sigma)$ that cannot be formed by a sequence of Reichstein transforms starting with $\mathcal{X}(\Sigma)$. However, $\mathcal{X}(\Sigma'')$ is the blowdown (of a Reichstein transform) of a stack that is the result of a sequence of Reichstein transforms starting with $\mathcal{X}(\Sigma)$.

This example shows that there are projective or quasi-projective varieties $X^{ss}(L)/G$ such that $X^{ss}(L)$ contains open sets that have good geometric quotients but that are not GIT quotients. These quotients do not fit into the chamber decomposition description of [DH; T].

Example 6.4. We begin by copying verbatim the setup of [CLSc, Exm. 6.1.17]. The fan for $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ has the eight orthants of $\mathbb{R}^3$ as its maximal cones, and the ray generators are $\pm e_1, \pm e_2, \pm e_3$. Take the positive orthant $\mathbb{R}^3_{\geq 0}$ and subdivide it further by adding new ray generators

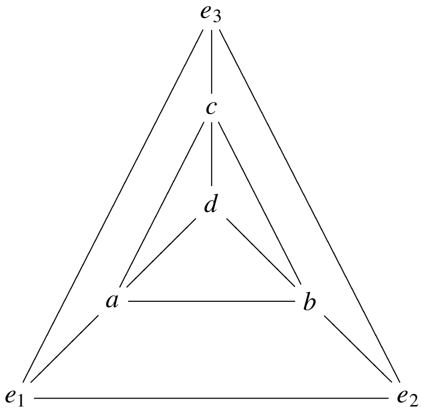
$$a = (2, 1, 1), \quad b = (1, 2, 1), \quad c = (1, 1, 2), \quad d = (1, 1, 1).$$

We obtain a complete fan Σ by filling the first orthant with the maximal cones

$$\sigma_1 = \langle e_1, e_2, a, b\rangle, \qquad \sigma_2 = \langle e_2, e_3, b, c\rangle,$$

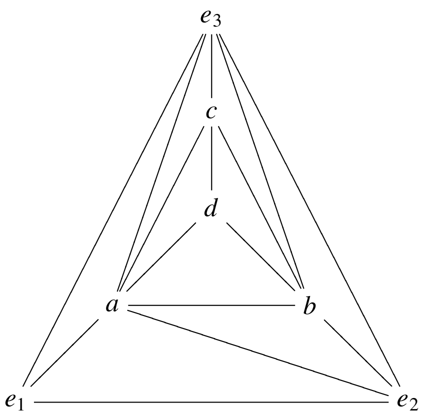
$$\sigma_3 = \langle e_1, e_3, a, c\rangle,\ \langle a, b, d\rangle,\ \langle b, c, d\rangle,\ \langle a, c, d\rangle.$$

The complete toric variety $X(\Sigma)$ is a projective variety (a calculation shows that the divisor $D = 16(D_1 + D_2 + D_3) + 56(D_a + D_b + D_c) + 41D_d$ on $X(\Sigma)$ is ample). In the positive orthant, the intersection of the plane $x + y + z = 1$ with Σ looks like this:



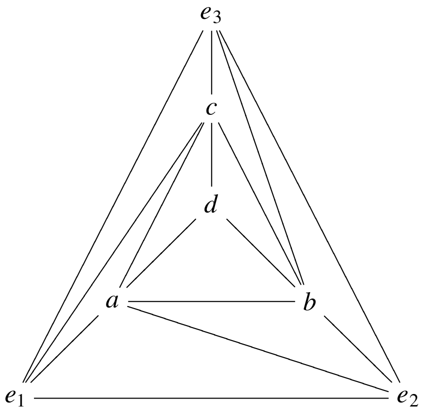
If we perform a star subdivision of Σ with respect to the ray through a followed by a star subdivision by the ray through b, we get a fan Σ' formed from Σ by dividing each of the maximal cones $\sigma_1, \sigma_2, \sigma_3 \in \Sigma$ into two maximal cones separated by the facets $\langle a, e_2\rangle, \langle b, e_3\rangle, \langle a, e_3\rangle$ respectively. The associated Cox stack $\mathcal{X}(\Sigma')$ is a complete open DM substack of the Cox stack $\mathcal{X}(\Sigma')$, and $\mathcal{X}(\Sigma')$ is isomorphic to a sequence of divisorial Reichstein transforms starting with $\mathcal{X}(\Sigma)$. Furthermore, we have a projective birational morphism $X(\Sigma') \to X(\Sigma)$ because a star subdivision of a fan gives rise to a projective morphism on the level of toric varieties [CLSc, Thm. 11.1.6].

In the positive orthant, the intersection of the plane $x + y + z = 1$ with Σ' looks like this:



However, not all complete open DM substacks of $\mathcal{X}(\Sigma)$ can be reached by Reichstein transforms of $\mathcal{X}(\Sigma)$. Form the fan Σ'' from Σ' by flipping the facet subdividing σ_3 from $\langle a, e_3\rangle$ to $\langle c, e_1\rangle$.

In the positive orthant, the intersection of the plane $x + y + z = 1$ with Σ'' looks like this:



Since the toric variety $X(\Sigma'')$ is not projective [CLSc, Exm. 6.1.17], $X(\Sigma'')$ cannot be obtained from $X(\Sigma)$ by star subdivisions. Hence the Cox stack $\mathcal{X}(\Sigma'')$ is a complete open Deligne–Mumford substack of $\mathcal{X}(\Sigma)$ that cannot be reached by Reichstein transforms of $\mathcal{X}(\Sigma)$.

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