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Discrepancies of Non-$\mathbb{Q}$-Gorenstein Varieties

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1. Introduction

The aim of this paper is to investigate some surprising features of singularities of normal varieties in the non-$\mathbb{Q}$-Gorenstein case as defined by de Fernex and Hacon [dFH]. In that paper the authors focus on the difficulties of extending some invariants of singularities when the canonical divisor is not $\mathbb{Q}$-Cartier. Instead of the classical approach—-in which we modify the canonical divisor by adding a boundary, an effective $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier—they introduce a notion of *pullback* of (Weil) $\mathbb{Q}$-divisors that agrees with the usual one for $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. In this way, for any birational morphism of normal varieties $f : Y \to X$, they are able to define relative canonical divisors $K_{Y/X} = K_Y + f^*(-K_X)$ and $K^-_{Y/X} = K_Y - f^*(K_X)$. The two definitions coincide when K_X is $\mathbb{Q}$-Cartier; using $K_{Y/X}$ and $K^-_{Y/X}$, de Fernex and Hacon extended the definitions of canonical singularities, klt singularities, and multiplier ideal sheaves to this more general context.

In this setting, some of the properties characterizing the usual notions of singularity (see [KoMo, Sec. 2.3]) seem to fail owing to the asymptotic nature of the definitions of the canonical divisors.

We focus on three properties that for $\mathbb{Q}$-Gorenstein varieties are straightforward.

- The relative canonical divisor always has rational valuations (cf. [Ko2, Thm. 92]).
- A canonical variety is always kawamata log terminal (klt; cf. [KoMo, Def. 2.34]).
- The jumping numbers are a set of rational numbers that have no accumulation points (cf. [L2, Lemma 9.3.21]).

In this paper we investigate these properties for non-$\mathbb{Q}$-Gorenstein varieties. Section 2 is devoted to recalling the necessary definitions.

In Section 3, we show that if X is klt in the sense of [dFH] then the relative canonical divisor has rational valuations. We also give an example of a (non-klt) variety X with an irrational valuation and then use it to find an irrational jumping number (Theorem 3.6).

In Section 4 we give an example of a variety with canonical but not klt singularities (Theorem 4.1), and we prove that the finite generation of the canonical ring implies that the relative canonical model has canonical singularities (Proposition 4.4). Finally, in Section 5 we use one of the main results in [dFH]—namely, that every

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effective pair (X, Z) admits m-compatible boundaries for $m \geq 2$ (see Theorem 2.10)—we show that, for a normal variety whose singularities are either klt or isolated, it is never possible to have accumulation points for the jumping numbers (Theorem 5.2).

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2. Basic Definitions

The following notation and definitions are taken from [dFH].

NOTATION 2.1. Throughout this paper, X will be a normal variety over the complex numbers.

Let us denote by $v = \mathrm{val}_F$ a divisorial valuation on X with respect to the prime divisor F over X. Given a proper closed subscheme $Z \subset X$, we define $v(Z)$ as

$$v(Z) = v(\mathcal{I}_Z) := \min\{v(\phi) \mid \phi \in \mathcal{I}_Z(U),\ U \cap c_X(v) \neq \emptyset\};$$

here $\mathcal{I}_Z \subset \mathcal{O}_Z$ is the ideal sheaf of Z. The definition extends to $\mathbb{R}$-linear combinations of proper closed subschemes. The same definition works in a natural way for linear combinations of fractional ideal sheaves.

To any fractional ideal sheaf $\mathcal{I}$ on X we associate the divisor

$$\mathrm{div}(\mathcal{I}) := \sum_{E \subset X} \mathrm{val}_E(\mathcal{I}) \cdot E,$$

where the sum is over all prime divisors E on X and where val_E denotes the divisorial valuation with respect to E.

DEFINITION 2.2. Let X be as in Notation 2.1. The $\natural$-valuation (or natural valuation) along a valuation v of a divisor F on X is

$$v^\natural(F) := v(\mathcal{O}_X(-F)).$$

Let D be a $\mathbb{Q}$-divisor on X. Then the valuation along v of D is

$$v(D) := \lim_{k \to \infty} \frac{v^\natural(k!\,D)}{k!} = \inf_{k \geq 1} \frac{v^\natural(kD)}{k} \in \mathbb{R}.$$

NOTATION 2.3. Let X be as in Notation 2.1. Let us consider a projective birational morphism $f : Y \to X$ from a normal variety Y.

We have the following definitions.

DEFINITION 2.4. Using Notation 2.3, for any divisor D on X we define the $\natural$-pullback of D to Y to be

$$f^{\natural}D = \mathrm{div}(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y).$$

This is the natural choice for obtaining a reflexive sheaf,

$$\mathcal{O}_Y(-f^{\natural}D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}.$$

We also need a good definition of *pullback* of D to Y that coincides with the classical one when we restrict to nonsingular varieties. We have

$$f^*D := \sum \mathrm{val}_E(D) \cdot E,$$

where the sum is taken over all the prime divisors E on Y.

We now give the main definitions that characterize multiplier ideal sheaves.

Definition 2.5. Let $f : Y \to X$ be as in Notation 2.3. Then, for every $m \geq 1$, the mth limiting relative canonical $\mathbb{Q}$-divisor $K_{m,Y/X}$ of Y over X is

$$K_{m,Y/X} := K_Y - \frac{1}{m} \cdot f^{\natural}(mK_X).$$

The relative canonical $\mathbb{R}$-divisor $K_{Y/X}$ of Y over X is

$$K_{Y/X} := K_Y + f^*(-K_X).$$

In particular, $K_{m,Y/X} \leq K_{mq,Y/X} \leq K_{Y/X}$. Also, taking the limsup of the coefficients of the components of the $\mathbb{Q}$-divisor $K_{m,Y/X}$, one obtains the $\mathbb{R}$-divisor $K^-_{Y/X} := K_Y - f^*K_X$, which satisfies $K^-_{Y/X} \leq K_{Y/X}$ (the two divisors coincide if X is $\mathbb{Q}$-Gorenstein—i.e., if K_X is $\mathbb{Q}$-Cartier).

Recall that an *effective* $\mathbb{Q}$-divisor Δ is a boundary on X if $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

Definition 2.6. Let $f : Y \to X$ as in Notation 2.3, let Δ be a boundary on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and let Δ_Y be the proper transform of Δ on Y. The log relative canonical $\mathbb{Q}$-divisor of (Y, Δ_Y) over (X, Δ) is then given by

$$K^{\Delta}_{Y/X} := K_Y + \Delta_Y - f^*(K_X + \Delta) = K_Y + \Delta_Y + f^*(-K_X - \Delta).$$

In particular, for every boundary Δ on X and every $m \geq 1$ such that $m(K_X + \Delta)$ is Cartier, we have

$$K_{m,Y/X} = K^{\Delta}_{Y/X} - \frac{1}{m} \cdot f^{\natural}(-m\Delta) - \Delta_Y \quad \text{and} \quad K_{Y/X} = K^{\Delta}_{Y/X} + f^*\Delta - \Delta_Y.$$

Observe that $K^{\Delta}_{Y/X} \leq K_{m,Y/X} \leq K^-_{Y/X}$.

Definition 2.7. Consider a pair (X, I), where X is a normal quasi-projective variety and $I = \sum a_k \mathcal{I}_k$ is a formal $\mathbb{R}$-linear combination of nonzero fractional ideal sheaves on X. Let us denote by $Z = \sum a_k Z_k$ the associated subscheme, where Z_k is the subscheme generated by $\mathcal{I}_k$.

We define a *log resolution* of this pair as a proper birational morphism $f : Y \to X$, where Y is a smooth variety, such that for every k the following statements hold.

- The sheaf $\mathcal{I}_k \cdot \mathcal{O}_Y$ is an invertible sheaf corresponding to a divisor E_k on Y.
- The exceptional locus $\mathrm{Ex}(f)$ is a divisor.
- The union of the supports of E_k and $\mathrm{Ex}(f)$ has simple normal crossing.

If Δ is a boundary on X, then a log resolution for $((X, \Delta); I)$ is given by a resolution of (X, I) such that $\mathrm{Ex}(f)$, E, and $\mathrm{Supp}\, f^*(K_X + \Delta)$ are divisors and their union $\mathrm{Ex}(f) \cup E \cup \mathrm{Supp}\, f^*(K_X + \Delta)$ has simple normal crossings.

DEFINITION 2.8. Let (X, Z) be as in Definition 2.7. Let $f : Y \to X$ be a log resolution with Y normal, and let F denote a prime divisor on Y. For any integer $m \geq 1$, we define the mth limiting log discrepancy of (X, Z) along F to be

$$a_{m,F}(X; Z) := \mathrm{ord}_F(K_{m,Y/X}) + 1 - \mathrm{val}_F(Z).$$

DEFINITION 2.9. With notation as before, the pair (X, Z) is said to be *log terminal* if there is an integer m_0 such that $a_{m_0,F}(X, Z) > 0$ for every prime divisor F over X.

We say that an effective pair is *klt* if and only if there exists a boundary Δ such that $((X, \Delta); Z)$ is kawamata log terminal in the usual sense.

In particular, the notions of log terminal and klt are equivalent because of the following theorem (cf. [dFH, Thm. 5.4]).

THEOREM 2.10. *Every effective pair* (X, Z) *admits* m*-compatible boundaries for* $m \geq 2$, *where a boundary* Δ *is said to be* m*-compatible if*:

(i) $m\Delta$ *is integral and* $\lfloor \Delta \rfloor = 0$;
(ii) *no component of* Δ *is contained in the support of* Z;
(iii) f *is a log resolution for the log pair* $((X, \Delta); Z + \mathcal{O}_X(-mK_X))$; *and*
(iv) $K^{\Delta}_{Y/X} = K_{m,Y/X}$.

NOTATION 2.11. Given this theorem and the notation generally used in the literature, from now on we will abuse that notation and say that a normal variety X is klt whenever it is log terminal according to Definition 2.9. A pair (X, Δ) will be klt in the usual sense.

REMARK 2.12. Since $K^{\Delta}_{Y/X} \leq K^{-}_{Y/X}$, by Theorem 2.10 and Definition 2.2 we have

$$\mathrm{val}_F(K^{-}_{Y/X}) = \sup\{\mathrm{ord}_F(K^{\Delta}_{Y/X}) \mid (X, \Delta) \text{ is a log pair}\}.$$

Note that, because we are considering a limit, there may be irrational valuations.

DEFINITION 2.13. Let X be as in Notation 2.1. Let $X' \to X$ be a proper birational morphism with X' normal, and let F be a prime divisor on X'. The *log discrepancy* of a prime divisor F over X with respect to (X, Z) is

$$a_F(X, Z) := \mathrm{ord}_F(K_{X'/X}) + 1 - \mathrm{val}_F(Z).$$

Using the notation in Definition 2.7, the pair (X, Z) is said to be *canonical* (resp. *terminal*) if $a_F(X, Z) \geq 1$ (resp. > 1) for every exceptional prime divisor F over X.

Recall that, by [dFH, Prop. 8.2], a normal variety X is canonical if and only if, for sufficiently divisible $m \geq 1$ and for every sufficiently high log resolution $f\colon Y \to X$ of $(X, \mathcal{O}_X(mK_X))$, there is an inclusion

$$\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \hookrightarrow \mathcal{O}_Y(mK_Y).$$

We have the following useful lemma.

LEMMA 2.14. *Using Notation 2.1, let $f\colon Y \to X$ be a proper birational morphism such that Y is canonical. If $\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \hookrightarrow \mathcal{O}_Y(mK_Y)$ for sufficiently divisible $m \geq 1$, then X is canonical.*

Proof. Let $g\colon Y' \to X$ be a log resolution of $(X, \mathcal{O}_X(mK_X))$. Without loss of generality, we can assume that g factors through f so that we have $h\colon Y' \to Y$ with $g = f \circ h$. In particular,

$$\mathcal{O}_{Y'} \cdot \mathcal{O}_X(mK_X) \hookrightarrow \mathcal{O}_{Y'} \cdot \mathcal{O}_Y(mK_Y) \hookrightarrow \mathcal{O}_{Y'}(mK'_Y),$$

where the first inclusion is given by assumption and the second by Y being canonical. □

3. Irrational Valuations

Given a Weil $\mathbb{R}$-divisor D on a normal variety X, we define the corresponding divisorial ring as

$$\mathscr{R}_X(D) := \bigoplus_{m \geq 0} \mathcal{O}_X(mD).$$

REMARK 3.1. If X is klt as in Notation 2.11, then there exists a Δ such that (X, Δ) is klt in the usual sense (cf. Theorem 2.10). By [Ko2, Thm. 92], $\mathscr{R}_X(D)$ is finitely generated if and only if D is a $\mathbb{Q}$-divisor.

PROPOSITION 3.2. *If a normal variety X is klt then, for any prime divisor F over X, the valuation $\mathrm{val}_F(K^-_{Y/X})$ is rational; here $f\colon Y \to X$ is a projective birational morphism such that F is a divisor on Y.*

Proof. By Remark 4.4 and [G, Lemma 2.1.6] we know that $\mathscr{R}_X(m_0K_X)$ is generated by $\mathcal{O}_X(m_0K_X)$ over $\mathcal{O}_X$ for some $m_0 > 0$. It follows that $K^-_{Y/X} = K_{m_0,Y/X}$ and hence $\mathrm{val}_F(K^-_{Y/X}) = \mathrm{val}_F(K_{m_0,Y/X})$, which is a rational number (Remark 2.12). □

Next we will construct an example of a threefold whose relative canonical divisor $K^-_{Y/X}$ has an irrational valuation. The example is given by the resolution of a cone singularity over an abelian surface.

Let us consider the abelian surface $X = E \times E$, where E is an elliptic curve. For this surface we have that $\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X) \subset \mathrm{N}^1(X)$, where $\mathrm{N}^1(X)$ is generated by the classes

$$f_1 = [\{P\} \times E], \quad f_2 = [E \times \{P\}], \quad \delta = [\Delta].$$

The intersection numbers are given by

$$((f_1)^2) = ((f_2)^2) = (\delta^2) = 0 \quad \text{and} \quad (f_1.f_2) = (f_1.\delta) = (f_2.\delta) = 1.$$

Let the class $\alpha = xf_1 + yf_2 + z\delta$; then α is numerically effective (nef) if and only if

$$xy + xz + yz \geq 0 \quad \text{and} \quad x + y + z \geq 0,$$

so we obtain that $\mathrm{Nef}(X)$ is a circular cone (cf. [Ko1, Chap. II, Exm. 4.16]).

Next, we consider a double covering of this surface ramified over a general very ample divisor $H \in |2\mathscr{L}|$, where $\mathscr{L}$ is an ample line bundle. This cover is given by $W = \mathrm{Spec}_X(\mathcal{O}_X \oplus \mathscr{L}^\vee)$ with projection $p\colon W \to X$ induced by the inclusion $i\colon \mathcal{O}_X \hookrightarrow \mathcal{O}_X \oplus \mathscr{L}^\vee$. In particular,

$$\omega_W = p^*(\omega_X \otimes \mathscr{L}).$$

There is an induced involution $\sigma\colon W \to W$. For any Cartier divisor D on W, $D + \sigma^*(D)$ is the pullback of a Cartier divisor on X. Since $H \in |2\mathscr{L}|$ is general, the pullbacks of the generators p^*f_i and $p^*\delta$ are irreducible curves on W. Since the map is finite, the pullback of an ample divisor (resp. nef, effective) on X is ample (resp. nef, effective) on W.

It is easy to see that the map induced at the level of cones $p^*\colon \mathrm{NE}(X) \to \mathrm{NE}(W)$ is well-defined and injective and that

$$p^*\,\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(W) \cap p^*\,\mathrm{N}^1(X).$$

Let us now consider any ample divisor L on X such that p^*L defines an embedding $W \subset \mathbb{P}^n$. Let $C \subset \mathbb{P}^{n+1}$ be the projective cone over W. We want to investigate the properties of the relative canonical divisor.

THEOREM 3.3. *Given the construction just described, if* $H \sim 6(f_1 + f_2)$, $L \sim (3f_1 + 6f_2 + 6\delta)$, *and* $f\colon Y \to C$ *is the blowup of the cone at the vertex, then the relative canonical divisor* $K^-_{Y/X}$ *has an irrational valuation.*

Proof. We know that p^*L defines an embedding; in fact,

$$p_*p^*L \cong L \otimes (\mathcal{O}_X \oplus \mathscr{L}^\vee) = L \oplus (L \otimes \mathscr{L}^\vee) \sim (3f_1 + 6f_2 + 6\delta) \oplus (3f_2 + 6\delta)$$

is a sum of very ample divisors. Since $f\colon Y \to C$ is the blowup at the vertex, it follows that Y is isomorphic to the projective space bundle $\mathbb{P}(\mathcal{O}_W \oplus \mathcal{O}_W(p^*L))$ with the natural projection $\pi\colon Y \to W$. If we denote by W_0 the negative section, then $\mathcal{O}_{W_0}(W_0) \cong \mathcal{O}_W(-p^*L)$. Let us also denote by $W_\infty \sim W_0 + \pi^*p^*L$ the section at infinity. The canonical divisor K_Y is given by $K_Y \sim \pi^*K_W - 2W_0 + \pi^*(-p^*L)$.

REMARK 3.3.1. Recall that we have an isomorphism $\mathrm{Cl}\,W \cong \mathrm{Cl}\,C$ defined by the map that associates to a divisor $D \subset W$ the cone over D, $C_D \subset C$. A divisor C_D is $\mathbb{R}$-Cartier if and only if $D \sim_{\mathbb{R}} kp^*L$ for $k \in \mathbb{R}$.

We have that $K_C = f_*K_Y = C_{K_W} - C_{(p^*L)}$ and $C_{k(p^*L)}$ is an $\mathbb{R}$-Cartier divisor on C such that $f^*(C_{k(p^*L)}) = \pi^*(k(p^*L)) + kW_0$. Let Γ be a boundary on C. Then $\Gamma \equiv C_\Delta$ and, since $K_C + \Gamma$ is $\mathbb{Q}$-Cartier, we have $K_C + C_\Delta = C_{K_W} - C_{(p^*L)} + C_\Delta \equiv C_{k(p^*L)}$ for some $k \in \mathbb{Q}$. In particular, given $s = k + 1$, we have $s(p^*L) - K_W \equiv \Delta \geq 0$. Therefore,

$$\Delta \equiv s(p^*L) - K_W \equiv s(p^*L) - \frac{1}{2}p^*H$$
$$\equiv p^*\left(s(3f_1 + 6f_2 + 6\delta) - \frac{1}{2}(6f_1 + 6f_2)\right).$$

By Remark 2.12, we have

$$\mathrm{val}_{W_0}(K^-_{Y/C}) = \sup\{\mathrm{ord}_{W_0}(K^\Gamma_{Y/C}) \mid (C, \Gamma) \text{ is a log pair}\}$$
$$\geq \sup\{\mathrm{ord}_{W_0}(K^{C_\Delta}_{Y/C}) \mid (C, C_\Delta) \text{ is a log pair}\}.$$

Note that

$$K^\Gamma_{Y/C} \equiv K_Y + f_*^{-1}\Gamma - f^*(K_C + \Gamma)$$
$$\equiv K_Y + f_*^{-1}\Gamma - f^*(C_{K_W} - C_{(p^*L)} + \Gamma)$$
$$\equiv \pi^*K_W - 2W_0 + \pi^*(-p^*L) + f_*^{-1}\Gamma$$
$$- \pi^*(K_W - p^*L + \Delta) - (s-1)W_0$$
$$\equiv -(s+1)W_0 + f_*^{-1}\Gamma - \pi^*\Delta.$$

In particular, $K^{C_\Delta}_{Y/C} = -(s+1)W_0$. So if we let

$$t = \inf\{s \in \mathbb{R} \mid \exists\Delta \geq 0,\ K_W + \Delta \equiv s(p^*L)\},$$

then

$$\mathrm{val}_{W_0}(K^-_{Y/C}) \geq -(1+t).$$

Remark 3.3.2. Note that Δ is ample if $s > t$; in particular, it is always possible to choose $\Delta = A/m$ for A a smooth and very ample Cartier divisor.

Claim 3.3.3. $\mathrm{val}_{W_0}(K^-_{Y/C}) = -(1+t)$.

Proof. Let us consider any effective boundary $\Gamma \geq 0$. It suffices to show that, in the previous construction, it is always possible to choose a boundary $\Delta \equiv s(p^*L) - K_W \subseteq W$ such that $\mathrm{ord}_{W_0} K^\Gamma_{Y/C} = \mathrm{ord}_{W_0} K^{C_\Delta}_{Y/C}$. If $f^*(K_C + \Gamma) = K_Y + f_*^{-1}\Gamma + kW_0$, let $\Delta = f_*^{-1}\Gamma|_{W_0} \geq 0$. Note that

$$\Delta = f_*^{-1}\Gamma|_{W_0} \equiv -(K_Y + kW_0)|_{W_0} \equiv -K_W + (k-1)p^*L.$$

By what we have already seen (with $s = k-1$), $K^{C_\Delta}_{Y/C} = -kW_0$. Hence

$$\mathrm{ord}_{W_0} K^\Gamma_{Y/C} = \mathrm{ord}_{W_0} K^{C_\Delta}_{Y/C}. \qquad \square$$

We now return to the proof of Theorem 3.3.

Since $p^*\,\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(W) \cap p^*\,\mathrm{N}^1(X)$ and $\Delta \geq 0$, it follows that the sum of the coefficients of $p^*(f_1)$, $p^*(f_2)$, and $p^*\delta$ must be positive and so $s \geq \frac{2}{5}$. Again, because of the isomorphism of cones just described, we have that Δ is effective if and only if it is nef:

$$\frac{\Delta^2}{4} = 9(8s^2 - 7s + 1) \geq 0 \iff s \geq \frac{7+\sqrt{17}}{16}\left(> \frac{2}{5}\right).$$

Thus we obtain the following irrational valuation of the relative canonical divisor:

$$\mathrm{val}_{W_0}(K^-_{Y/C}) = -\frac{23+\sqrt{17}}{16}. \qquad \square$$

Using the result of Theorem 3.3, we now give an example of an irrational jumping number. The definitions of multiplier ideal sheaf and jumping numbers given here follow those in [dFH].

DEFINITION 3.4. As in Definition 2.7, let (X, Z) be an effective pair. The multiplier ideal sheaf of (X, Z), denoted by $\mathcal{I}(X, Z)$, is the unique maximal element of $\{\mathcal{I}_m(X, Z)\}_{m\geq 1}$, where

$$\mathcal{I}_m(X, Z) := f_{m*}\mathcal{O}_{Y_m}(\lceil K_{m,Y_m/X} - f_m^{-1}(Z)\rceil)$$

for $f_m\colon Y_m \to X$ a log resolution of the pair $(X, Z + \mathcal{O}_X(-mK_X))$.

DEFINITION 3.5. A number $\mu \in \mathbb{R}_{>0}$ is a *jumping number* of an effective pair (X, Z) if $\mathcal{I}(X, \lambda\cdot Z) \neq \mathcal{I}(X, \mu\cdot Z)$ for all $0 \leq \lambda < \mu$.

A relevant feature of the jumping numbers in the $\mathbb{Q}$-Gorenstein case is that they are always rational.

THEOREM 3.6. *With the same construction as in Theorem 3.3, there exist irrational jumping numbers for the pair* (C, P), *where* P *is the vertex of the projective cone.*

Proof. We are considering $Z = P \subset C$ the vertex of the projective cone. Let us denote by $\mathrm{Bl}_P\, C := f : Y \to C$ the blowup of the vertex; then we have that $f^{-1}(k\cdot Z) = k\cdot W_0$. By Theorem 2.10, for every $m \geq 1$ there exists an m-compatible boundary Γ_m such that $K_{m,Y/X} = K^{\Gamma_m}_{Y/X}$ and, in particular, $\mathcal{I}_m(X, Z) = \mathcal{I}((X, \Gamma_m); Z)$; hence

$$\mathcal{I}(X, k\cdot Z) = \bigcup_m \mathcal{I}_m(X, k\cdot Z) = \bigcup_{\Gamma_m} \mathcal{I}((X, \Gamma_m); k\cdot Z).$$

Also, by Remark 3.3.2, the blowup is a log resolution of $((X, \Gamma_m); Z)$ for every $m \geq 1$ and so

$$\mathcal{I}((X, \Gamma_m); k\cdot Z) = f_*\mathcal{O}_Y(\lceil K^{\Gamma_m}_{Y/X} - k\cdot W_0\rceil).$$

We can therefore compute the jumping numbers simply by considering the log resolution given by the blowup $Y \to C$, and thus we have

$$\mathcal{I}(X, k\cdot Z) = \bigcup_{\Gamma_m} f_*\mathcal{O}_Y(\lceil K^{\Gamma_m}_{Y/X} - k\cdot W_0\rceil).$$

Since $\mathrm{val}_{W_0}(K^-_{Y/X}) = -\frac{23+\sqrt{17}}{16}$, the jumping numbers are of the form $k = t - \frac{23+\sqrt{17}}{16}$ for t any integer ≥ 1. $\square$

4. Canonical Singularities

We begin by giving an example of a canonical singularity that is not klt.

Let us consider a construction similar to the one in the previous section. Let $S = \mathbb{P}^1 \times \mathcal{E}$, where $\mathcal{E}$ is an elliptic curve. The canonical sheaf is then

$$\omega_S \sim \mathcal{O}_{\mathbb{P}^1}(-2) \boxtimes \mathcal{O}_{\mathcal{E}}.$$

Let $\mathscr{A}$ be an ample line bundle on $\mathcal{E}$, and consider the embedding $S \subseteq \mathbb{P}^n$ given by the very ample divisor $L = \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathscr{A}^{\otimes 2}$. Let $C \subseteq \mathbb{P}^{n+1}$ be the projective cone over S.

THEOREM 4.1. *With the construction just described, the singularity of C at its vertex is canonical but not klt.*

Proof. Using the same computation as in Theorem 3.3, let $f : Y \to C$ be the blowup of the cone at the origin P, let $\pi : Y \to S$ be the natural projection, and denote by S_0 the negative section. The canonical divisor K_Y is given by $K_Y \sim \pi^*(K_S) - 2S_0 + \pi^*(-L)$. Let us compute s in this case. We have $\Delta \equiv sL - K_S \sim \mathcal{O}_{\mathbb{P}^1}(2s + 2) \boxtimes \mathscr{A}^{\otimes 2s}$. In particular, Δ is effective if and only if $s > 0$. Hence we have

$$\mathrm{val}_{S_0}(K^-_{Y/C}) = -1.$$

In particular, C is not klt.

We will use a similar computation to show that C has canonical singularities. The relative canonical divisor used to characterize such singularities is $K_{Y/X} = K_Y + f^*(-K_X)$; by the notion of pullback given in Definition 2.4, this divisor is given by an approximation of the form

$$K^+_{m,Y/X} = K_Y + \frac{1}{m} f^\natural(-mK_X).$$

In this new definition we have $K^+_{m,Y/X} \geq K^+_{mq,Y/X} \geq K_{Y/X}$. In particular, the proof of the existence of an m-compatible boundary given in [dFH] works also in this case with small modifications.

We now introduce the following corollary of Lemma 2.14.

PROPOSITION 4.2. *Let $f : Y \to X$ be a proper birational morphism such that Y is canonical. If $\mathrm{val}_F(K_{Y/X}) \geq 0$ for all divisors F on Y, then X is canonical.*

Proof. For all sufficiently divisible $m \geq 1$ we have $\mathrm{val}_F(K^+_{m,Y/X}) \geq 0$ (i.e., $mK_Y \geq -f^\natural(-mK_X)$) and so

$$\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \hookrightarrow (\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X))^{\vee\vee} = \mathcal{O}_Y(-f^\natural(-mK_X)) \hookrightarrow \mathcal{O}_Y(mK_Y).$$

Lemma 2.14 now implies the claim. □

Since $K_Y + f^*(-K_C + \Gamma') \geq K_{Y/C}$, it follows (as in Remark 2.12) that if we denote by S_0 the negative section then

$$\begin{aligned}&\operatorname{val}_{S_0}(K_{Y/C})\\&\quad=\inf\{\operatorname{ord}_{S_0}(K_Y+f^*(-K_C+\Gamma'))\mid(-K_C+\Gamma')\text{ is }\mathbb{R}\text{-Cartier},\ \Gamma'\geq 0\}\end{aligned}$$

for $\Gamma'\equiv C_{\Delta'}$, where $\Delta'\equiv rL+K_S$. Therefore, if

$$t=\inf\{r\in\mathbb{R}\mid\exists\Delta'\geq 0,\ -K_S+\Delta'\equiv rL\}$$

then

$$\operatorname{val}_{S_0}(K_{Y/C})=t-1.$$

As before, we want to control the values of r for which Δ' is numerically equivalent to an effective class. In this case, $\Delta'\equiv rL+K_S\sim\mathcal{O}_{\mathbb{P}^1}(2r-2)\boxtimes\mathscr{A}^{\otimes 2r}$; hence

$$\Delta'\geq 0\iff r\geq 1.$$

In particular, $\operatorname{val}_{S_0}(K_{Y/C})=0$ and so C is canonical. □

Next we will show that, if X is canonical and $\mathscr{R}_X(K_X)$ is finitely generated, then X has a canonical model with canonical singularities. To begin, we introduce the following useful lemma.

LEMMA 4.3. *Let Y be a normal algebraic variety and B a Weil divisor on Y. Then the following statements are equivalent.*

(1) *$\mathscr{R}_Y(B)$ is a finitely generated sheaf of $\mathcal{O}_Y$-algebras.*
(2) *There exists a projective birational morphism $\pi: Y^+\to Y$ such that Y^+ is normal,* $\operatorname{Ex}(\pi)$ *has codimension ≥ 2, and $B'=\pi_*^{-1}B$ is $\mathbb{Q}$-Cartier and π-ample over Y for $Y^+:=\operatorname{Proj}_Y\sum_{m\geq 0}\mathcal{O}_Y(mB)$.*

The mapping $\pi: Y^+\to Y$ is the unique morphism with the properties listed in (2).

PROPOSITION 4.4. *Let X be a normal quasi-projective variety with canonical singularities whose canonical ring $\mathscr{R}_X(K_X)$ is a finitely generated $\mathcal{O}_X$-algebra. Then the relative canonical model $X_{\mathrm{can}}:=\operatorname{Proj}_X(\mathscr{R}_X(K_X))$ exists and has canonical singularities.*

Proof. Since X is canonical, it follows from [dFH, Prop. 8.2] that, for any sufficiently high log resolution $f: Y\to X$, we have $K_Y-\frac{1}{m}f^{\natural}(-mK_X)\geq 0$.

By Lemma 4.3, there exists a small birational morphism $\pi: X^+\to X$ such that K_{X^+} is a relatively ample $\mathbb{Q}$-Cartier divisor. Also, for this morphism we have that $\pi^{\natural}(-mK_X)=-mK_{X^+}$. Let us now consider $f: Y\to X$ and $g: Y\to X^+$, which are the common log resolutions of X and X^+, respectively.

We consider the map $\mathcal{O}_{X^+}\cdot\mathcal{O}_X(mK_X)\to\mathcal{O}_{X^+}(mK_{X^+})$. Because $\pi_*^{-1}(K_X)=K_{X^+}$ is π-ample, $\mathcal{O}_{X^+}(mK_{X^+})$ is globally generated over X for m sufficiently divisible; hence we have an isomorphism of sheaves. Thus

$$K_Y-g^*(K_{X^+})=K_Y+\frac{1}{m}g^*(-mK_{X^+})=K_Y+\frac{1}{m}f^{\natural}(-mK_X)\geq 0,$$

where the last equality holds by [dFH, Lemma 2.7]. Therefore, the canonical model X^+ has canonical singularities. □

5. Accumulation Points for Jumping Numbers

In this final section we shall use definitions and results from [dFH].

Given an effective pair (X, Z), we want to consider a family of ideal sheaves in the form

$$\mathcal{I}_k = \{\mathcal{I}(X, t_k \cdot Z)\}_k$$

for $k \in \mathbb{N}$, $t_k > 0$.

If t_k is a decreasing sequence then $\mathcal{I}_k \subset \mathcal{I}_{k+1}$ and, by the Noetherian property, the sequence stabilizes.

If we consider an increasing sequence t_k, then $\mathcal{I}_k \supset \mathcal{I}_{k+1}$ and the ascending chain condition does not apply. We will show (under appropriate hypotheses) that the set of ideals stabilizes even in this case. Thus there are no accumulation points for the jumping numbers of the pair (X, Z). We will use the following lemma.

LEMMA 5.1. *Let X be a projective variety and let $I = \{\mathcal{I}_k\}_k$ be the family of ideals defined previously. If there exists a line bundle $\mathscr{L}$ on X such that $\mathscr{L} \otimes \mathcal{I}_k$ is globally generated for all k, then it is not possible to have an infinite sequence of ideal sheaves $\mathcal{I}_r \subseteq I$ such that*

$$\mathcal{O}_X \supseteq \cdots \supseteq \mathcal{I}_r \supsetneq \mathcal{I}_{r+1} \supsetneq \mathcal{I}_{r+2} \supsetneq \cdots .$$

Proof. Tensoring by $\mathscr{L}$ and considering cohomology yields

$$0 \leq \cdots \lneq h^0(\mathscr{L} \otimes \mathcal{I}_{r+1}) \lneq h^0(\mathscr{L} \otimes \mathcal{I}_r) \leq h^0(\mathscr{L}) = n,$$

which is impossible. □

The following theorem is the main result of this section.

THEOREM 5.2. *Let (X, Z) be an effective pair, where X is a projective normal variety such that X has either log-terminal or isolated singularities. Then the set of jumping numbers has no accumulation points*; *that is, given any sequence $\{t_i\}_{i \in \mathbb{N}}$ such that $t_i > 0$ and $\lim_{i \to \infty} t_i = t$, we have*

$$\bigcap_i \mathcal{I}(X, t_i \cdot Z) = \mathcal{I}(X, t_{i_0} \cdot Z)$$

for some $i_0 > 0$.

We will need the following three results.

THEOREM 5.3 [dFH, Cor. 5.8]. *Let (X, Z) be an effective pair, where X is a projective normal variety and $Z = \sum a_k \cdot Z_k$. Let $m \geq 2$ be an integer such that $\mathcal{I}(X, Z) = \mathcal{I}_m(X, Z)$, and let Δ be an m-compatible boundary for (X, Z). For each k, let B_k be a Cartier divisor such that $\mathcal{O}_X(B_k) \otimes \mathcal{I}_{Z_k}$ is globally generated, where $\mathcal{I}_{Z_k}$ is the ideal sheaf of Z_k, and suppose that L is a Cartier divisor such that $L - (K_X + \Delta + \sum a_k B_k)$ is nef and big. Then*

$$H^i(\mathcal{O}_X(L) \otimes \mathcal{I}(X, Z)) = 0 \quad \textit{for } i > 0.$$

COROLLARY 5.4 [dFH, Cor. 5.9]. *With the same notation and assumptions as in Theorem 5.3, let A be a very ample Cartier divisor on X. Then the sheaf $\mathcal{O}_X(L+kA)\otimes\mathcal{I}(X,Z)$ is globally generated for every integer $k\geq\dim X+1$.*

PROPOSITION 5.5. *Let X be a projective normal variety that has either log-terminal or isolated singularities. Then, for any divisor $D\in\mathrm{WDiv}_{\mathbb{Q}}(X)$, there exists a very ample divisor A such that $\mathcal{O}_X(mD)\otimes\mathcal{O}_X(A)^{\otimes m}$ is globally generated for every $m\geq 1$.*

Proof. If X has log-terminal singularities then, by Remark 4.4, $\mathscr{R}_X(D)$ is a finitely generated $\mathcal{O}_X$-algebra. It is then easy to see that the proposition holds.

So let us assume that X has isolated singularities. We may assume $D\in\mathrm{WDiv}(X)$. Let us fix a log resolution $f:Y\to X$ of (X,D), where $\mathcal{O}_Y\cdot\mathcal{O}_X(D)=\mathcal{O}_Y(\tilde{D}+F)$ for $\tilde{D}=f_*^{-1}D$ and F an exceptional divisor. Let B be a general very ample divisor on X such that $\mathcal{O}_X(D+B)$ and $\mathcal{O}_X(-K_X+B)$ are globally generated, where $\mathcal{O}_Y\cdot\mathcal{O}_X(-K_X+B)=\mathcal{O}_Y(G)$. Then $\tilde{B}=f^*B$ and $\mathcal{O}_Y(\tilde{B}+m\tilde{D}+mF)$ is globally generated, hence nef and big, for every $m>0$. By Kawamata–Viehweg vanishing, if $\mathcal{G}=\mathcal{O}_Y(K_Y+m\tilde{B}+m\tilde{D}+mF+G)$ then $R^if_*(\mathcal{G})=0$ for all $i>0$; hence $H^i(Y,\mathcal{G})\cong H^i(X,f_*\mathcal{G})=0$ for all $i>0$. Then, by Mumford regularity, we may assume that $\mathcal{F}:=f_*\mathcal{O}_Y(K_Y+m((n\tilde{B}+\tilde{D}+F)+G))$ is globally generated for all $m>0$. Since $(f_*\mathcal{F})^{\vee\vee}\cong\mathcal{O}_X(K_X+mD+mnB+B-K_X)\cong\mathcal{O}_X(mD+(mn+1)B)$, we have the induced short exact sequence

$$0\to f_*\mathcal{F}\to\mathcal{O}_X(mD+(mn+1)B)\to Q\to 0,$$

where the quotient Q is supported on points and hence globally generated, so $mD+(mn+1)B$ is globally generated for all m. In particular, $mD+m(n+1)B$ is globally generated for every m. □

REMARK 5.6. It is apparently unknown whether Proposition 5.5 holds for any divisor $D\in\mathrm{WDiv}_{\mathbb{Q}}(X)$ on any projective normal variety (regardless of the singularity). We conjecture that this is the case. Note that, by Proposition 5.5, this conjecture holds for surfaces.

We can now prove Theorem 5.2.

Proof of Theorem 5.2. We follow the proof of [dFH, Thm. 5.4]. Let us consider an effective divisor D such that K_X-D is Cartier. By Proposition 5.5 we know that there exists an ample line bundle $\mathscr{A}$ such that

$$\mathscr{A}^{\otimes m}\otimes\mathcal{O}_X(-mD)$$

is globally generated for all $m\geq 0$.

For a general element G in the linear system $|\mathscr{A}^{\otimes m}-mD|$, let $G=M+mD$; we can then choose $\Delta_m:=\frac{1}{m}M$ as our boundary. Let B_k be Cartier divisors such that $\mathcal{O}_X(B_k)\otimes\mathcal{I}_{Z_k}$ is globally generated. As in Corollary 5.3, let H be an ample Cartier divisor such that $H-\left(K_X-D+\sum a_k\cdot B_k\right)$ is nef and big. Then the Cartier divisor $(\mathscr{A}+H)$ is such that

$$(\mathscr{A}+H)-\Big(K_X+\Delta_m+\sum a_kB_k\Big)$$

is nef and big for all m.

Let B be a very ample Cartier divisor on X. Then, for $\mathscr{L} := \mathcal{O}_X(\mathscr{A} + H + sB)$ with $s > \dim X$, we have that $\mathscr{L} \otimes \mathcal{I}_k(X, Z)$ is globally generated for all k. Now, by Lemma 5.1,

$$\bigcap_i \mathcal{I}(X, t_i \cdot Z) = \mathcal{I}(X, t_{i_0} \cdot Z)$$

for some $i_0 > 0$ and the theorem is proved. □

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