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A Property of Quasi-diagonal Forms

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The aim of this paper is to prove the following result.

Theorem. *Let k be a positive integer and let $F_i \in \mathbb{Z}[\mathbf{x}_i]$ be a form of degree k, where $\mathbf{x}_i$ $(i = 1, 2, \dots)$ are disjoint vectors of variables. Assume that*

$(*)$ *either not all forms are semidefinite of the same sign, or all forms are nonsingular.*

Then there exists a positive integer s_0 such that, for all s, every integer represented by $\sum_{i=1}^{s} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$ is represented by $\sum_{i=1}^{s_0} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$.

If $k = 2$, then the condition $(*)$ can be omitted. J. Szejko has conjectured that the condition $(*)$ is superfluous.

Corollary. *Let k_i be a bounded infinite sequence of positive integers, and let $F_i[\mathbf{x}_i]$ be an infinite sequence of nonsingular forms of degree k_i with the $\mathbf{x}_i$ disjoint. Then there exists a positive integer s_0 such that, for all s, every integer represented by $\sum_{i=1}^{s} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$ is also represented by $\sum_{i=1}^{s_0} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$.*

Remark 1. The assertion is false when $k_i = 2^i$ and $F_i = x_i^{k_i}$ $(i = 1, 2, \dots)$. It may be enough to assume that $\sum_{i=1}^{\infty} \frac{1}{k_i} = \infty$.

Notation. For a given field K and a form $F \in K[x_1, \dots, x_r]$, we use $D(F)$ to denote the Netto discriminant of F—that is, the resultant of $\frac{\partial F}{\partial x_i}$ $(i = 1, 2, \dots, r)$; note that $D(F)$ differs from the true discriminant of F by a constant factor (see [6, p. 434]). Also, $h(F, K)$ is the least h such that $F = \sum_{i=1}^{h} G_i H_i$, where $G_i, H_i \in K[x_1, \dots, x_r]$ are forms of positive degree and $h(F) = h(F, \mathbb{Q})$.

For $a \in \mathbb{Z} \setminus \{0\}$ and p a prime, $\operatorname{ord}_p a$ is the highest exponent e such that $p^e | a$ (i.e., $p^e \| a$); $\operatorname{ord}_p 0 = \infty$. For $\mathbf{x} = [x_1, \dots, x_r] \in \mathbb{Z}^r$, we have $\operatorname{ord}_p \mathbf{x} = \min_{1 \le i \le r} \operatorname{ord}_p x_i$. Finally, $e(x) = \exp\{2\pi i x\}$.

Our proof of the theorem is based on the following series of seventeen lemmas.

Lemma 1. *Let p be a prime, $F \in \mathbb{Z}[\mathbf{x}]$ a form of degree $k = p^\tau k_0$, and $k_0 \in \mathbb{Z} \setminus p\mathbb{Z}$. Let $\gamma = \tau + 2$ if $p = 2$ and $\tau > 0$ and let $\gamma = \tau + 1$ otherwise. If*

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$a \in \mathbb{Z}_p \setminus \{0\}$, $p^\nu \| a$, and the congruence $F(\mathbf{x}) \equiv a \pmod{p^{\gamma+\nu}}$ is solvable, then the equation $F(\mathbf{x}) = a$ is solvable in $\mathbb{Z}_p$.

Proof. If $F(\mathbf{x}_0) \equiv a \pmod{p^{\gamma+\nu}}$, then $p^\nu \| F(\mathbf{x}_0)$ and $F(\mathbf{x}_0)p^{-\nu} \equiv ap^{-\nu} \pmod{p^\gamma}$. Then, by [4, Lemma 9] (or its proof for the case $p = 2$, $\tau = 0$), the congruence $h^k F(\mathbf{x}_0)p^{-\nu} \equiv ap^{-\nu} \pmod{p^n}$ is solvable for every n. Thus, by compactness of $\mathbb{Z}_p$, the equation $h^k F(\mathbf{x}_0)p^{-\nu} = ap^{-\nu}$ is solvable in $\mathbb{Z}_p$ and it suffices to take $\mathbf{x} = h\mathbf{x}_0$. □

LEMMA 2. *Let $F \in \mathbb{Z}[x_1, \dots, x_r]$ be a form of degree k, and let $c \in \mathbb{Z} \setminus \{0\}$. For p a prime, if a congruence $F(x) \cong c \pmod{p^{2 \operatorname{ord}_p kc+1}}$ is solvable then, for all n, the number $L(F, c, p^n)$ of solutions of the congruence*

$$F(x) \cong c \pmod{p^n} \tag{1}$$

satisfies

$$L(F, c, p^n) \geq p^{(n-2 \operatorname{ord}_p kc-1)(r-1)}. \tag{2}$$

Proof. By Euler's theorem (see [8, Satz 27]),

$$\sum_{i=1}^{r} \frac{\partial F}{\partial x_i} x_i = kF. \tag{3}$$

For a certain $\boldsymbol{\xi} \in \mathbb{Z}^r$ we have

$$F(\boldsymbol{\xi}) \cong c \pmod{p^{2 \operatorname{ord}_p kc+1}}, \tag{4}$$

so it follows from (3) that, for a certain $h \leq r$,

$$\delta = \operatorname{ord}_p \frac{\partial F}{x_h}(\boldsymbol{\xi}) \leq \operatorname{ord}_p kc;$$

now, by (4), we have

$$F(\boldsymbol{\xi}) \equiv c \pmod{p^{2\delta+1}}.$$

It follows from the proof of Theorem 3 in [1, Chap. I, Sec. 5] that, if

$$x_i \equiv \xi_i \pmod{p^{2\delta+1}} \quad \text{for } i \neq h,$$

then there exists an x_h such that

$$F(\mathbf{x}) \equiv c \pmod{p^n}.$$

Clearly, (2) holds. □

LEMMA 3. *Let $F \in \mathbb{Z}[x_1, \dots, x_r]$ be a form of degree k with $D(F) \neq 0$, and let p be a prime. If a congruence $F(\mathbf{x}) \equiv c \pmod{p^{2 \operatorname{ord}_p D(F)+1+k\nu}}$ is solvable with $\operatorname{ord}_p \mathbf{x} = \nu$ then, for all n, the number $L(F, c, p^n)$ of solutions of the congruence* (1) *satisfies*

$$L(F, c, p^n) \geq p^{(n-2 \operatorname{ord}_p D(F)-1-\nu)(r-1)}. \tag{5}$$

Proof. Consider first the case $\nu = 0$. Since $D(F)$ is the resultant of $\partial F/\partial x_i$ ($i = 1, 2, \dots, r$), we have (see [8, Satz 124]) that

$$\sum_{i=1}^{r} \frac{\partial F}{\partial x_i} \phi_{ij}(x_1, \ldots, x_r) = D(F) x_j^{k^r} \tag{6}$$

for all $j \le r$, where $\phi_{ij} \in \mathbb{Z}[x_1, \ldots, x_r]$. Since

$$F(\mathbf{x}) \equiv c \pmod{p^{2\,\mathrm{ord}_p\, D(F)+1}} \tag{7}$$

has a solution $\boldsymbol{\xi}$ with $\mathrm{ord}_p\, \boldsymbol{\xi} = 0$, we obtain from (6) that, for a certain $h \le r$,

$$\delta = \mathrm{ord}_p \frac{\partial F}{\partial x_h}(\boldsymbol{\xi}) \le \mathrm{ord}_p\, D(F);$$

then, by (7),

$$F(\boldsymbol{\xi}) \equiv c \pmod{p^{2\delta+1}}.$$

It follows, as in the proof of Lemma 2, that

$$L(F, c, p^n) \ge p^{(n-2\,\mathrm{ord}_p\, D(F)-1)(r-1)}.$$

Consider now the general case. Since

$$F(\boldsymbol{\xi}) \equiv c \pmod{p^{2\,\mathrm{ord}_p\, D(F)+1+\nu k}} \quad \text{and} \quad \mathrm{ord}_p(\boldsymbol{\xi}) = \nu,$$

we have

$$F(p^{-\nu}\boldsymbol{\xi}) \equiv cp^{-\nu k} \pmod{p^{2\,\mathrm{ord}_p\, D(F)+1}}.$$

By the already proved case of the lemma, we have

$$L(F, cp^{-\nu k}, p^{n-\nu}) \ge p^{(n-2\,\mathrm{ord}_p\, D(F)-1-\nu)(r-1)}.$$

Every solution of the congruence

$$F(\mathbf{y}) \equiv cp^{-\nu k} \pmod{p^{n-\nu}}$$

gives rise to a solution of the congruence (1) by the substitution $\mathbf{x} = p^{\nu}\mathbf{y}$, and solutions that are distinct (mod $p^{n-\nu}$) give rise to solutions that are distinct (mod p^n). Thus (5) holds. □

LEMMA 4. *Let $l = 2k^2(k,2)^2 - k(k,2)$, $s \ge l+1$, p be a prime, and d_i $(1 \le i \le s)$ be p-adic units. Then, for every integer c and all positive integers n, the congruence*

$$c \equiv \sum_{i=1}^{s} d_i x_i^k \pmod{p^n}$$

is solvable with at least one $x_i \not\equiv 0 \pmod{p}$, and the relevant equation is solvable in $\mathbb{Z}_p$.

Proof. For $n = \gamma$ the assertion is proved in [4, pp. 53–54]. Assume without loss of generality that

$$c \equiv \sum_{i=1}^{s} d_i \xi_i^k \pmod{p^{\gamma}} \quad \text{and} \quad \xi_s \not\equiv 0 \pmod{p}.$$

Applying Lemma 1 with $F(x) = d_s x^k$ and $a = c - \sum_{i=1}^{s-1} d_i \xi_i^k$ allows us to infer the existence of an $\eta \in \mathbb{Z}$ such that

$$c \equiv \sum_{i=1}^{s-1} d_i \xi_i^k + d_s \eta^k \pmod{p^n};$$

clearly, $\eta^k \equiv \xi_s^k \pmod{p^n}$ and so $\eta \not\equiv 0 \pmod p$. Solvability of the relevant equation in $\mathbb{Z}_p$ follows from compactness of $\mathbb{Z}_p$. □

LEMMA 5. *Let $F_i(\mathbf{x}_i)$ be a nonsingular form of degree k in r_i variables $(1 \le i \le s)$, let p be a prime, and let $p^{\delta_{pi}}$ be the highest power of p dividing $F_i(\boldsymbol{\eta}_i)$ for all $\boldsymbol{\eta}_i \in \mathbb{Z}^{r_i}$. If $s \ge kl + 1$ and if the equation*

$$F(\mathbf{x}) := \sum_{i=1}^{s} F_i(\mathbf{x}_i) = N \tag{8}$$

is solvable in $\mathbb{Z}_p$, then for all n we have

$$L(F, N, p^n) \ge p^{(n-\gamma_p-\delta_p)(R-1)}, \tag{9}$$

where

$$\gamma_p = 2 \operatorname{ord}_p D(F) + 1,$$
$$\delta_p = \max_{1 \le i \le s} \delta_{pi}, \quad \textit{and}$$
$$R = \sum_{i=1}^{s} r_i.$$

Proof. We note first that, by assumption, $D(F_i) \ne 0$ $(1 \le i \le s)$; hence $D(F) \ne 0$ by the Laplace formula (see [7, 5.10]). Let

$$N = p^\delta d \quad \text{for } d \text{ a } p\text{-adic unit,}$$

and assume first that $\delta < \delta_p$. Then equation (8) gives

$$\operatorname{ord}_p \mathbf{x} \le \left\lfloor \frac{\delta}{k} \right\rfloor < \delta_p$$

and so, by Lemma 3, (9) holds.

Assume now that $\delta \ge \delta_p$ (or $N = d = 0$) and that $F_i(\boldsymbol{\eta}_i) = p^{\delta_{pi}} d_i$ for $d_i =$ a p-adic unit. Because $s \ge kl + 1$, there is a residue $r \pmod k$ such that $S = \{i \le s : \delta_{pi} \equiv r \pmod k\}$ satisfies $|S| \ge l + 1$. Let $\delta_{pm} = \max_{i \in S} \delta_{pi}$. Then by Lemma 4 we have

$$p^{\delta - \delta_{pm}} d \equiv \sum_{i \in S} d_i \xi_i^k \pmod{p^{\gamma_p + \delta_p}},$$

where not all the ξ_i are divisible by p. Suppose $\xi_j \not\equiv 0 \pmod p$. Now put $\mathbf{x}_i \equiv 0 \pmod{p^{\gamma_p+\delta_p}}$ for $i \notin S$ and

$$\mathbf{x}_i \equiv p^{(\delta_{pm} - \delta_{pi})/k} \boldsymbol{\eta}_i \xi_i \pmod{p^{\gamma_p+\delta_p}} \quad \text{for } i \in S \setminus \{j\}.$$

Since $\gamma_p \ge 2 \operatorname{ord}_p D(F_j) + 1$ by the Laplace formula, it follows that

$$F_j(\mathbf{x}_j) \equiv N - \sum_{\substack{i=1\\ i\neq j}}^{s} F_i(\mathbf{x}_i) \pmod{p^{2\,\mathrm{ord}_p D(F_j)+1+\delta_p}}$$

has a solution $p^{(\delta_{pm}-\delta_{pj})/k}\boldsymbol{\eta}_j = \boldsymbol{\eta}'_j$ with $\mathrm{ord}_p\, \boldsymbol{\eta}'_j = \frac{\delta_{pm}-\delta_{pj}}{k}$. Hence, by Lemma 3,

$$L(F, N, p^n) \geq p^{\Sigma} \geq p^{(n-\gamma_p-\delta_p)(R-1)},$$

where

$$\Sigma = \sum_{\substack{i=1\\ i\neq j}}^{s}(n-\gamma_p-\delta_p)r_i + \left(n-\gamma_p-\left\lfloor\frac{\delta_p}{k}\right\rfloor\right)(r_j-1).$$

Therefore, (9) holds again. □

Lemma 6. *Let $\phi \in \mathbb{Z}[x_1, \ldots, x_r]$ be a polynomial of degree $k > 1$, F the leading form of ϕ, $\alpha \in \mathbb{R}$, and B a certain product of fixed intervals of length ≤ 1. Let:*

$$S(\alpha) = \sum_{\mathbf{x}\in PB\cap\mathbb{Z}^r} e(\alpha\phi(\mathbf{x}));$$

$$\sigma(2) = 1, \qquad \sigma(k+1) = \sum_{u=2}^{k}\binom{k-1}{u-2}\sigma(u) \quad (k \geq 2).$$

Then, for every positive $\Delta \leq k-1$ and $\varepsilon > 0$ and for all sufficiently large P, either

$$|S(\alpha)| \leq P^{r-\Delta\frac{h(F)}{(k-1)2^{k-1}\sigma(k)}+\varepsilon}$$

or there exists a positive integer q satisfying

$$q \leq cP^{\Delta} \quad \textit{and} \quad \|\alpha q\| < P^{-k+\Delta}, \tag{10}$$

where $c \geq 1$ depends only on $\phi - \phi(\mathbf{0})$ and B.

Proof. The lemma follows from statements 4A and 7A of [9, Chap. III] and roughly as in [9, p. 89], where we put $d = k$, $s = r$, $t = \Delta\frac{h(F)}{(k-1)2^{k-1}\sigma(k)} - \varepsilon$, and $\eta = \frac{\Delta}{k-1}$. The $\sigma(k)$ that we have defined recursively coincides with the $\sigma(k)$ defined in [9, p. 117]. Indeed, it is easily proved by induction that $\sigma(k)$ as defined in this paper satisfies $\sigma(k) \geq 2^{k-2} - 1$. Moreover, for every $k > 1$, we have $(k-2)!\,(\log 2)^{2-k} \geq \sigma(k) > \frac{1}{2}(k-2)!\,(\log 2)^{2k}$. Note also that $|S(\alpha)|$ depends only on α and $\phi - \phi(\mathbf{0})$. □

Lemma 7. *For integers a and q with $q > 0$ and $(a, q) = 1$, let*

$$S(a, q) = \sum_{\mathbf{z} \bmod q} e\left(\frac{a}{q}\phi(\mathbf{z})\right).$$

Then, for $k \geq 2$ and $\varepsilon > 0$,

$$S(a, q) \ll q^{r-(1-\varepsilon)\frac{h(F)}{(k-1)2^{k-1}\sigma(k)}+\varepsilon}.$$

Proof. In Lemma 6 we take $\alpha = a/q$, $\Delta = 1-\varepsilon$, $P = q$, and $B = [0,1]^r$. We obtain that either

$$|S(a,q)| \le q^{r-(1-\varepsilon)\frac{h(F)}{(k-1)2^{k-1}\sigma(k)}+\varepsilon}$$

or there exists a positive integer $q' \le cq^{1-\varepsilon}$ with $\|\alpha q'\| < q^{-k+1-\varepsilon}$. However, for $q^\varepsilon > c$ we have $\|\alpha q'\| \ge 1/q$ and so $q^{-k+2-\varepsilon} > 1$, which is impossible for $k \ge 2$.

For $(a,q) = 1$, let $\mathfrak{M}_{a,q}$ be the set of $\alpha \in (0,1)$ satisfying

$$q \le cP^\Delta \quad \text{and} \quad \left|\alpha - \frac{a}{q}\right| \le P^{-k+\Delta},$$

and let $\mathfrak{m}$ be the complement of the union of all $\mathfrak{M}_{a,q}$ where $q \le cP^\Delta$ and $(a,q) = 1$. □

LEMMA 8. *If* $h(F) \ge (k-1)2^k\sigma(k)+1$, *then*

$$\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha \ll P^{r-k-\Delta/(k-1)2^{k-1}\sigma(k)}.$$

Proof (following [5, Sec. 4]). Let $\mathcal{E}(\Delta)$ be the set of those $\alpha \in [0,1)$ for which there exists a positive integer q satisfying (10). Plainly $\mathcal{E}(\Delta)$ increases with Δ. Since every α has a rational approximation satisfying $1 \le q \le P^{k/2}$ and $\|q\alpha\| < P^{-k/2}$ and since these inequalities imply (10) with $\Delta = k/2$, the whole interval $[0,1)$ is contained in $\mathcal{E}(k/2)$. On the other hand, for $P > P_0(\varepsilon)$, the set $\mathfrak{m}$ is contained in the complement of $\mathcal{E}(\Delta-\varepsilon)$. We choose numbers $\Delta_0, \Delta_1, \dots, \Delta_g$ such that

$$\Delta - \varepsilon = \Delta_0 < \Delta_1 < \cdots < \Delta_g = k/2.$$

Then $\mathfrak{m}$ is contained in the union of the sets

$$\mathcal{E}(\Delta_f) - \mathcal{E}(\Delta_{f-1}), \quad f = 1, \dots, g. \tag{11}$$

By Lemma 6 with $\Delta = \Delta_{f-1}$, we have

$$|S(\alpha)| \le P^{r-\frac{h(F)}{(k-1)2^{k-1}\sigma(k)}\Delta_{f-1}+\varepsilon}$$

for all α in the set (11). Furthermore, the set (11) is a part of $\mathcal{E}(\Delta_f)$ and so, by (10), the measure of $\mathcal{E}(\Delta_f)$ is

$$\ll \sum_{q \le cP^{\Delta_f}} \sum_{a=1}^{q} q^{-1}P^{-k+\Delta_f} \ll P^{-k+2\Delta_f}.$$

Therefore,

$$\begin{aligned}\int_{\mathfrak{m}} |S(\alpha)|\, d\alpha &\ll P^{r-\frac{h(F)}{(k-1)2^{k-1}\sigma(k)}\Delta_{f-1}+\varepsilon-k+2\Delta_f}\\ &\ll P^{r-k-\frac{\Delta_{f-1}}{(k-1)2^{k-1}\sigma(k)}+2(\Delta_f-\Delta_{f-1})+\varepsilon}.\end{aligned}$$

Provided the numbers $\Delta_0, \dots, \Delta_g$ are chosen sufficiently close together (but independent of P), the last exponent is less than

$$r-k-\frac{\Delta}{(k-1)2^{k-1}\sigma(k)}+2\varepsilon < r-k-\frac{\Delta}{(k-1)2^{k-1}\sigma(k)+1}. \qquad \square$$

LEMMA 9. *For α in $\mathfrak{M}_{a,q}$ we have*

$$S(\alpha) = q^{-r} S(a,q) I(\beta) + O(P^{r-1+2\Delta}), \tag{12}$$

where $\beta = \alpha - a/q$ and

$$I(\beta) = \int_{PB} e(\beta\phi(\boldsymbol{\xi}))\, d\boldsymbol{\xi}. \tag{13}$$

Proof (following [5, Sec. 4]). In the sum

$$S(\alpha) = \sum_{\mathbf{x} \in PB \cap \mathbb{Z}^r} e(\alpha\phi(x_1, \ldots, x_r)), \tag{14}$$

put $x_i = qy_i + z_i$ for $0 \le z_i < q$. Then

$$S(\alpha) = \sum_{\mathbf{z}} \sum_{\mathbf{y}} e(\alpha\phi(q\mathbf{y} + \mathbf{z})) = \sum_{\mathbf{z}} e\left(\frac{a}{q}\phi(\mathbf{z})\right) \sum_{\mathbf{y}} e(\beta(q\mathbf{y} + \mathbf{z})).$$

The inner sum is over all $\mathbf{y}$ such that $q\mathbf{y} + \mathbf{z}$ is in the box PB. Thus the variables $y_1, \ldots, y_r$ run over independent intervals whose lengths are much less than P/q, since q is small compared with P. For any integer point $\mathbf{y}$ and any differentiable function $f(\boldsymbol{\eta})$, we have

$$f(\mathbf{y}) = \int_{|\boldsymbol{\eta} - \mathbf{y}| < 1/2} f(\boldsymbol{\eta})\, d\boldsymbol{\eta} + O\left(\max\left|\frac{\partial f}{\partial \eta_j}\right|\right), \tag{15}$$

where the maximum is taken over j and over $\boldsymbol{\eta}$ in the cube of integration.

When $f(\boldsymbol{\eta}) = \exp\{2\pi i\beta\phi(q\boldsymbol{\eta} + \boldsymbol{\zeta})\}$, we have

$$\max\left|\frac{\partial f}{\partial \eta_j}\right| \ll q|\beta||q\boldsymbol{\eta} + \boldsymbol{\zeta}|^{k-1} \ll q|\beta|P^{k-1}.$$

Now applying (15) to each integer point $\mathbf{y}$ in the foregoing inner sum, we obtain an integral extended over a union of unit cubes that differs from the box of summation by at most 1 in each dimension. The discrepancy in the volume is $\ll (P/q)^{r-1}$. Hence

$$\sum_{\mathbf{y}} e(\beta\phi(q\mathbf{y} + \mathbf{z})) = \int e(\beta\phi(q\boldsymbol{\eta} + \boldsymbol{\zeta}))\, d\boldsymbol{\eta} + O(q|\beta|P^{k-1}(P/q)^r) + O((P/q)^{r-1}),$$

where the integration is over those $\boldsymbol{\eta}$ for which $q\boldsymbol{\eta} + \boldsymbol{\zeta}$ lies in PB.

In this equation, if we change from the variable $\boldsymbol{\eta}$ to $\boldsymbol{\xi} = q\boldsymbol{\eta} + \boldsymbol{\zeta}$ then the right-hand side becomes

$$q^{-r} \int_{PB} e(\beta\phi(\boldsymbol{\xi}))\, d\boldsymbol{\xi} + O(P^{r+k-1} q^{1-r} |\beta|) + O(P^{r-1} q^{1-r}).$$

Substituting in the double sum, we obtain

$$q^{-r} S(a,q) I(\beta) + O(P^{r+k-1} q|\beta|) + O(P^{r-1} q)$$

and now (13) follows from the definition of $\mathfrak{M}_{a,q}$. □

LEMMA 10. *Suppose that $h(F) \ge (k-1)2^k \sigma(k) + 1$. Then the number $\mathcal{N}(P)$ of solutions of $F(\mathbf{x}) = N$ with $\mathbf{x}$ in $PB \cap \mathbb{Z}^r$ satisfies*

$$\mathcal{N}(P) = P^{r-k}J(P)(\mathfrak{S} + O(P^{-\Delta/(k-1)2^{k-1}\sigma(k)+1}) + O(P^{r-k-1+5\Delta}),$$

where

$$\mathfrak{S} = \sum_{q=1}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-r} S(a,q)$$

and

$$J(P) = \int_{-P^{\Delta}}^{P^{\Delta}} d\gamma \int_B e(\gamma P^{-k}(F(P\mathbf{x}) - N))\, d\mathbf{x}. \tag{16}$$

Proof. The number of integer points $\mathbf{x}$ in PB with $F(\mathbf{x}) = N$ is equal to

$$\int_0^1 S(\alpha)\, d\alpha$$

by the definition of $S(\alpha)$ in (15) with $\phi(\mathbf{x}) = F(\mathbf{x}) - N$. We split the interval of integration into the various intervals $\mathfrak{M}_{a,q}$ and the set $\mathfrak{m}$. By Lemma 8, the contribution of $\mathfrak{m}$ is $O(P^{r-k-\Delta/(k-1)2^{k-1}\sigma(k)+1})$. By Lemma 9, the contribution of the intervals $\mathfrak{M}_{a,q}$ is

$$\sum_{q \le cP^{\Delta}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{M}_{a,q}} S(\alpha)\, d\alpha = \sum_{q \le cP^{\Delta}} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-r} S(a,q) \int_{|\beta| < P^{-k+\Delta}} I(\beta)\, d\beta$$
$$+ O\Bigl(\sum_{q \in cP^{\Delta}} qP^{r-1+2\Delta}P^{-k+\Delta}\Bigr).$$

The error term here is $O(P^{r-k-1+5\Delta})$. Once we put $\beta = P^{-k}\gamma$, the integral with respect to β becomes

$$P^{-k} \int_{|\gamma| < P^{\Delta}} I(P^{-k}\gamma)\, d\gamma$$

and, by (14),

$$I(P^{-k}\gamma) = \int_{PB} e(P^{-k}\gamma\phi(\boldsymbol{\xi}))\, d\boldsymbol{\xi} = P^r \int_B e(P^{-k}\gamma(F(\mathbf{x}) - N))\, d\mathbf{x}.$$

Thus the integral with respect to β becomes $P^{r-k}J(P)$.

It remains to consider

$$\sum_{q \le cP^{\Delta}} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-r} S(a,q).$$

When continued to infinity, this series is absolutely convergent by Lemma 7 (since $h(F) \ge (k-1)2^k\sigma(k) + 1$) and has sum $\mathfrak{S}$. The preceding finite sum differs from $\mathfrak{S}$ by an amount

$$\ll \sum_{q > cP^{\Delta}} q \cdot q^{-r} \cdot q^{r-h(F)/(k-1)2^{k-1}\sigma(k)+\varepsilon} \ll P^{-\Delta/((k-1)2^{k-1}\sigma(k)+1)}.$$

This proves Lemma 10. $\square$

LEMMA 11. *If $s \geq 3$ and not all forms $F_i(\mathbf{x}_i)$ $(i \leq s)$ are semidefinite and of the same sign, then there exists a real nonsingular solution $(\xi_1^*, \dots, \xi_R^*)$ of $F(\mathbf{x}) = \sum_{i=1}^s F_i(\mathbf{x}_i) = 0$.*

Proof. Since not all forms $F_i(\mathbf{x}_i)$ are semidefinite and of the same sign, there exist $i, j \leq s$ and $\boldsymbol{\eta}_i, \boldsymbol{\xi}_j$ such that $F_i(\boldsymbol{\eta}_i) > 0$ and $F_j(\boldsymbol{\xi}_j) < 0$ ($i = j$ is not excluded). Because $s \geq 3$, there exists an $h \leq s$ with $h \neq i, j$. We may assume without loss of generality that $\frac{\partial F_h}{\partial x_{h1}} \neq 0$, and we let $a_0(x_{h2}, \dots, x_{hr_h})x_{h1}^d$ be the leading term of F_h with respect to x_{h1}. There exists a $\boldsymbol{\xi}_h' = [\xi_{h2}, \dots, \xi_{hr_h}]$ such that $a_0(\boldsymbol{\xi}_h') \neq 0$. In view of the symmetry between i and j, we may assume that $a_0(\boldsymbol{\xi}_h') > 0$. Let D_h be the discriminant with respect to x_{h1} of $F(\mathbf{x})$. Note that D_h contains the term $(-1)^{d(d-1)/2} d^d a_0(\mathbf{x}_h')^{d-1} F(\mathbf{x}_1, \dots, \mathbf{x}_{h-1}, 0, \mathbf{x}_h', \mathbf{x}_{h+1}, \dots, \mathbf{x}_s)^{d-1}$, which is the leading term of D_h with respect to $F(\mathbf{x}_1, \dots, \mathbf{x}_{h-1}, 0, \mathbf{x}_h', \mathbf{x}_{h+1}, \dots, \mathbf{x}_s)$. Thus, in particular, for fixed $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{h-1}, \boldsymbol{\xi}_{h+1}, \dots, \boldsymbol{\xi}_s$ and sufficiently large ζ we have $D_h(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{h-1}, \boldsymbol{\xi}_h', \boldsymbol{\xi}_{h+1}, \dots, \boldsymbol{\xi}_{j-1}, \zeta\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j+1}, \dots, \boldsymbol{\xi}_s) \neq 0$. For sufficiently large ζ we have $F(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{h-1}, 0, \boldsymbol{\xi}_h', \dots, \boldsymbol{\xi}_{j-1}, \zeta\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j+1}, \dots, \boldsymbol{\xi}_s) < 0$ and

$$\lim_{x_{h1} \to \infty} F(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{h-1}, x_{h1}, \boldsymbol{\xi}_h', \boldsymbol{\xi}_{h+1}, \dots, \boldsymbol{\xi}_{j-1}, \zeta\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j+1}, \dots, \boldsymbol{\xi}_s) = \infty,$$

and there is a ξ_{h1} such that $F(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{j-1}, \zeta\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j+1}, \dots, \boldsymbol{\xi}_s) = 0$. But then

$$\frac{\partial F}{\partial x_{h1}}(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_{j-1}, \zeta\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j+1}, \dots, \boldsymbol{\xi}_s) \neq 0,$$

proving the lemma. □

REMARK 2. In [5] it is stipulated that, in a real nonsingular solution of $F(\mathbf{x}) = 0$, all coordinates must be nonzero. In [4], however, the only coordinate that must be nonzero is the one with respect to which the partial derivative is nonzero.

LEMMA 12. *If B is a cube*

$$|\xi_j - \xi_j^*| < \varrho,$$

where $\boldsymbol{\xi}_j^*$ is a nonsingular solution of the equation $F(\mathbf{x}) = 0$ and ϱ is sufficiently small, then

$$\lim_{P \to \infty} J(P) = J_0 > 0.$$

Proof (following [3, Sec. 6]). For $\mathbf{x}$ in a fixed cube B, we have

$$e(\gamma P^{-k}(F(P\mathbf{x}) - N)) = e(\gamma F(\mathbf{x})) + O(P^{-k+\Delta})$$

if $|\gamma| < P^\Delta$. Hence, by (16),

$$J(P) = \int_{-P^\Delta}^{P^\Delta} d\gamma \int_B e(\gamma F(\mathbf{x}))\, d\mathbf{x} + O(P^{-k+2\Delta}).$$

Put $\mu = P^\Delta$. Then

$$\begin{aligned} J_0(\mu) := \int_{-\mu}^{\mu} \left(\int_B e(\gamma F(\boldsymbol{\xi}))\, d\boldsymbol{\xi} \right) d\gamma &= \int_B \frac{\sin 2\pi\mu F(\boldsymbol{\xi})}{\pi F(\boldsymbol{\xi})}\, d\boldsymbol{\xi} \\ &= \int_{-\varrho}^{\varrho} \cdots \int_{-\varrho}^{\varrho} \frac{\sin 2\pi\mu F(\boldsymbol{\xi}^* + \boldsymbol{\eta})}{\pi F(\boldsymbol{\xi}^* + \boldsymbol{\eta})}\, d\boldsymbol{\eta}, \end{aligned} \tag{17}$$

where $\boldsymbol{\xi} = \boldsymbol{\xi}^* + \boldsymbol{\eta}$.

For any $\boldsymbol{\eta}$, we have

$$F(\boldsymbol{\xi}^* + \boldsymbol{\eta}) = \sum_{i=1}^{s} \sum_{j=1}^{r_i} c_{ij}\eta_{ij} + \sum_{\kappa=2}^{k} P_\kappa(\boldsymbol{\eta}), \tag{18}$$

where the $P_\kappa(\boldsymbol{\eta})$ are forms of degree κ in $\boldsymbol{\eta}$. We have

$$c_{ij} = \frac{\partial F}{\partial x_{ij}}(\boldsymbol{\xi}^*),$$

and we may suppose without loss of generality that $c_{11} = 1$.

For $|\eta| < \varrho$ we have

$$|F(\boldsymbol{\xi}^* + \boldsymbol{\eta})| < \sigma,$$

where $\sigma = \sigma(\varrho)$ is small when ϱ is small. Put $F(\boldsymbol{\xi}^* + \boldsymbol{\eta}) = \zeta$. Now, if ϱ is sufficiently small, then we can invert the relation (18) and express η_{11} in terms of ζ and η_{ij} ($j > 1$ for $i = 1$) by means of power series. This expression will be of the form

$$\eta_{11} = \zeta - \sum_{j=2}^{r} c_{1j}\eta_{1j} - \sum_{i=2}^{s} \sum_{j=1}^{r_i} c_{ij}\eta_{ij} + P(\zeta, \eta_{ij}),$$

where P is a multiple power series beginning with terms of degree ≥ 2. Hence

$$\frac{\partial \eta_{11}}{\partial \zeta} = 1 + P_1(\zeta, \eta_{ij})$$

and, by taking ϱ sufficiently small, we can ensure that $|P_1| < \frac{1}{2}$ for $|\eta_{ij}| < \varrho$ ($j > 1$ for $i = 1$) and $|\zeta| < \sigma$.

A change of variables from η_{11} to ζ in (17) yields

$$J_0(\mu) = \int_{-\sigma}^{\sigma} \frac{\sin 2\pi\mu\zeta}{\pi\zeta} V(\zeta)\, d\zeta, \tag{19}$$

where

$$V(\zeta) = \int_{B'} (1 + P_1(\zeta, \eta_{ij}))\, d\eta_{12} \cdots d\eta_{sr_s};$$

here B' denotes the part of the $(R-1)$-dimensional box

$$|\eta_{12}| < \varrho, \ \ldots, \ |\eta_{sr_s}| < \varrho$$

in which $|\eta_{11}| < \varrho$—that is, in which

$$\left| \zeta - \sum_{j=2}^{r_1} c_{1j}\eta_{1j} - \sum_{i=2}^{s} \sum_{j=1}^{r_i} c_{ij}\eta_{ij} + P(\zeta, \eta_{ij}) \right| < \varrho.$$

It is clear that $V(\zeta)$ is a continuous function of ζ for $|\zeta|$ sufficiently small. It can also be easily seen that $V(\zeta)$ is a function of bounded variation, since it has left and right derivatives at every value of ζ and these are bounded. Hence, by Fourier's integral theorem (see [10, Sec. 9.4]) applied to (19), we have

$$\lim_{\mu \to \infty} J_0(\mu) = V(0).$$

Finally, $V(0)$ is a positive number because the cube B' contains a sufficiently small $(R-1)$-dimensional cube centered at the origin and in such a cube we have $1+P_1>\frac{1}{2}$. This proves the lemma. $\square$

LEMMA 13. *If $s \ge 2$ and nonsingular $F_i(\mathbf{x}_i)$ $(i \le s)$ are semidefinite forms of the same sign and if N is also of the same sign for B the unit cube, then*

$$\lim_{|N|\to\infty} J(|N|^{1/k}) = J_0 > 0.$$

Proof. The forms F_i are nonsingular. Hence if they are semidefinite then they are definite, for otherwise the real points $\boldsymbol{\xi} \ne \mathbf{0}$ such that $F_i(\boldsymbol{\xi})=0$ would be singular points. Assume without loss of generality that the F_i are positive definite and that $N>0$. Put $P=N^{1/k}$. By Lemma 10, we have

$$J_0 = \int_{-\infty}^{\infty} d\gamma \int_B e(\gamma(F(\boldsymbol{\xi})-1)\, d\boldsymbol{\xi}.$$

By [9, Chap. I, Lemma 7D],

$$J_0 = \lim_{L\to\infty} L \int_B (1-L|F(\mathbf{x})-1|)\, d\mathbf{x},$$

$$|F(\mathbf{x})-1| \le \frac{1}{L}.$$

Hereafter, the inclusions written below the integrals define the domain of integration.

Let $\mathbf{x}_i = (x_{i1},\ldots,x_{ir_i})$ and perform the change of variables $x_{sj} = x_{sr_s} y_j$ $(1 \le j < r_s)$ and

$$F_1(\mathbf{x}_1)+\cdots+F_{s-1}(x_{s-1})+x_{sr_s}^k F_s(\mathbf{y},1) = 1+L^{-1}\xi.$$

We obtain

$$\begin{aligned} J_0 &= \lim_{L\to\infty} \frac{1}{k} \\ &\quad\cdot \int_{\mathbf{x}_i\in[0,1]^{r_i}} \int_{\mathbf{y}\in[0,1]^{r_s-1}} \int_{-1}^{1} (1-|\xi|) \frac{d\mathbf{x}_1\cdots d\mathbf{x}_{s-1}\, d\mathbf{y}\, d\xi}{\left(1+L^{-1}\xi-\sum_{i=1}^{s-1} F(\mathbf{x}_i)\right)^{1-r_s/k} F_s(\mathbf{y},1)^{r_s/k}} \\ &= \int_{-l}^{l} (1-|\xi|)\, d\xi \lim_{L\to\infty} K\left(\frac{\xi}{L}\right), \end{aligned}$$

where

$$K(\eta) = \int_{\mathbf{x}_i\in[0,1]^{r_i}} \int_{\mathbf{y}\in[0,1]^{r_s-1}} \frac{d\mathbf{x}_1\cdots d\mathbf{x}_{s-1}\, d\mathbf{y}}{\left(1+\eta-\sum_{i=1}^{s-1} F_i(\mathbf{x}_i)\right)^{1-r_s/k} F_s(\mathbf{y},1)^{r_s/k}},$$

$$1+\eta-F_s(\mathbf{y},1) \le \sum_{i=1}^{s-1} F_i(\mathbf{x}_i) \le 1+\eta.$$

Next we perform the change of variables $\mathbf{x}_i = (1+\eta)^{1/k}\mathbf{y}_i$ and obtain

$$K(\eta) = (1+\eta)^{R/k-1} \cdot \int_{\mathbf{y}_i \in [0,(1+\eta)^{-1/k}]^{r_i}} \int_{\mathbf{y} \in [0,1]^{r_s-1}} \frac{d\mathbf{y}_1 \cdots d\mathbf{y}_{s-1}\, d\mathbf{y}}{\left(1 - \sum_{i=1}^{s-1} F_i(\mathbf{y}_i)\right)^{1-r_s/k} F_s(\mathbf{y},1)^{r_s/k}},$$

$$1 - \frac{1}{1+\eta} F_s(\mathbf{y},1) \le \sum_{i=1}^{s-1} F_i(\mathbf{y}_i) \le 1.$$

When η tends to 0, the foregoing multiple integral tends to

$$\int_{\mathbf{y}_i \in [0,1]^{r_i}} \int_{\mathbf{y} \in [0,1]^{r_s-1}} \frac{d\mathbf{y}_1 \cdots d\mathbf{y}_{s-1}\, d\mathbf{y}}{\left(1 - \sum_{i=1}^{s-1} F_i(\mathbf{y}_i)\right)^{1-r_s/k} F_s(\mathbf{y},1)^{r_s/k}},$$

$$1 - F_s(\mathbf{y},1) \le \sum_{i=1}^{s-1} F_i(\mathbf{y}_i) \le 1.$$

The integrand is positive in the interior of the domain of integration, so the integral is positive provided the interior is nonempty. However, if $a = F_1(1,0,\ldots,0)$ then

$$1 - F_s(\mathbf{0},1) < F_1\left(\left(\frac{1/(1+F_s(\mathbf{0},1))}{a}\right)^{1/k}, \mathbf{0}\right) < 1,$$

which proves the lemma. □

LEMMA 14. *If* $F(\mathbf{x}) = \sum_{i=1}^{s} F_i(\mathbf{x}_i)$, *where* $F_i \in \mathbb{C}[\mathbf{x}_i] \setminus \{0\}$ *are of degree* $k > 1$ *and the* $\mathbf{x}_i$ *are disjoint* $(1 \le i \le s)$, *then* $h(F,\mathbb{C}) \ge \lceil s/2 \rceil$.

Proof. Since $F_i \ne 0$ there exist $\boldsymbol{\xi}_i \in \mathbb{C}^{r_i}$ such that $F_i(\boldsymbol{\xi}_i x_i) = x_i^k$. Therefore, it suffices to prove that

$$2h := 2h\left(\sum_{i=1}^{s} x_i^k, \mathbb{C}\right) \ge s. \tag{20}$$

If $2h < s$ and

$$\sum_{i=1}^{s} x_i^k = \sum_{i=1}^{h} G_i H_i,$$

where G_i and H_i are forms of positive degree, then there exists an $\boldsymbol{\eta} \in \mathbb{C}^s \setminus \{\mathbf{0}\}$ such that

$$G_i(\boldsymbol{\eta}) = H_i(\boldsymbol{\eta}) = 0 \quad (1 \le i \le h).$$

Taking partial derivatives at the point $\boldsymbol{\eta}$, we obtain

$$k\eta_j^{k-1} = \sum_{i=1}^{k} \left(\frac{\partial G_i}{\partial x_j}(\boldsymbol{\eta}) H_i(\boldsymbol{\eta}) + G_i(\boldsymbol{\eta}) \frac{\partial H_i}{\partial x_j}(\boldsymbol{\eta})\right) = 0;$$

hence $\boldsymbol{\eta} = \mathbf{0}$, a contradiction. □

REMARK 3. This lemma for $s = 3$ easily implies the Ehrenfeucht–Pełczyński theorem about irreducibility over $\mathbb{C}$ of $f(x)+g(y)+h(z)$, where f, g, h are nonconstant polynomials over $\mathbb{C}$.

LEMMA 15. *If $k \geq 2$ and $s \geq (k+1)2^{k+1}\sigma(k)+1$ and if $F(\mathbf{x}) = N \neq 0$ is solvable in $\mathbb{Z}_p$ for all primes p, then $\mathfrak{S} > 0$. Moreover, if all F_i are nonsingular then $\mathfrak{S} \geq \mathfrak{S}_0 > 0$, where $\mathfrak{S}_0$ is independent of N.*

Proof. If $h(F) \geq (k-1)2^k\sigma(k)+1$ then $\omega(F) > 2$ by [9, Chap. III, Thm. 6A]. Therefore, by [9, Chap. I, Lemma 6D] we have

$$\mathfrak{S} = \prod_{p \text{ prime}} \nu(p), \tag{21}$$

where

$$\nu(p) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{a=1 \\ (a,p)=1}}^{p^n} (p^n)^{-R} S(a, p^n)$$

and

$$\nu(p) = \lim_{n\to\infty} \frac{L(F, N, p^n)}{p^{n(R-1)}}. \tag{22}$$

It follows from Lemma 7 that

$$|S(a, p^n)| \ll (p^n)^{R - \frac{h(F)}{(k-1)2^{k-1}\sigma(k)} + \varepsilon},$$

and from this we deduce (since $h(F) \geq (k-1)2^k\sigma(k)+1$) that

$$|\nu(p) - 1| < p^{-\frac{(k-1)2^{k-1}\sigma(k)+1}{(k-1)2^{k-1}\sigma(k)} + \varepsilon} < p^{-\frac{(k-1)2^{k-1}\sigma(k)+2}{(k-1)2^{k-1}\sigma(k)+1}}.$$

Hence there exists a p_0 such that

$$\prod_{p > p_0} \nu(p) > \frac{1}{2},$$

yet from Lemma 2 and (22) it follows that $\nu(p) > 0$ and so, by (21), we have $\mathfrak{S} > 0$. Moreover, if $k \geq 5$ then $s \geq (k+1)2^{k+1}\sigma(k)+1 \geq (k+1)2^{k+1}\cdot 13+1 \geq 8k^3+1 > kl+1$; for $k \leq 4$ we check the relevant inequality directly. Hence, by Lemma 5 and (22),

$$\nu(p) \geq \nu_0(p) > 0 \quad \text{for all primes } p,$$

where $\nu_0(p)$ is independent of N. The second part of the lemma now follows from (21). □

LEMMA 16. *Under each set of assumptions in the Theorem there exists a positive integer s_2 such that, for $s \geq s_2$, all but finitely many integers represented by $F(\mathbf{x}) = \sum_{i=1}^{s} F(\mathbf{x}_i)$ over $\mathbb{R}$ and over $\mathbb{Z}_p$ for all primes p are represented by F over $\mathbb{Z}$.*

Proof. For $k = 1$ the choice $s_2 = 1$ is obvious. For $k = 2$ the choice $s_2 = 5$ follows from classical theorems of the theory of quadratic forms (see [2, pp. 131, 235]). For $k = 3$ the choice $s_2 = 33$ follows from Davenport and Lewis's theorem [5] and from Lemma 14 ($h(F) \geq 17$). So assume that $k \geq 4$. If the F_i are nonsingular then we take $s_2 = (k+1)2^{k+1}\sigma(k)+1$; if not all F_i are semidefinite and of the same sign and if j is the least index such that $\sum_{i=1}^{j} F_i$ is indefinite, then we take $s_2 = \max\{(k+1)2^{k+1}\sigma(k)+1, j\}$. Indeed, by Lemma 14 we have $h(F) \geq (k-1)2^k\sigma(k)+1$, and Lemmas 10, 12, 13, and 15 show that every integer sufficiently large in absolute value that is represented by $F(\mathbf{x})$ over $\mathbb{R}$ and over $\mathbb{Z}_p$ for all primes p is also represented by F over $\mathbb{Z}$. □

LEMMA 17. *With notation as in Lemma 4, let $s \geq 2kl+1$, let p be a prime, and let $\mathbf{x}_i$ be disjoint vectors of variables of length r_i $(i = 1, 2, \dots, s)$. Let $F_i \in \mathbb{Z}[\mathbf{x}_i]$ be a form of degree k such that the greatest common divisor of $F_i(\boldsymbol{\eta}_i)$ for all $\boldsymbol{\eta}_i \in \mathbb{Z}^{r_i}$ is divisible exactly by p^{δ_i}, and let $\delta_1 \leq \delta_2 \leq \cdots \leq \delta_s$. If the congruence*

$$c \equiv \sum_{i=1}^{s-1} F_i(\mathbf{x}_i) \pmod{p^{\delta_s}} \tag{23}$$

is solvable, then the equation

$$c = \sum_{i=1}^{s} F_i(\mathbf{x}_i) \tag{24}$$

is solvable in $\mathbb{Z}_p$.

REMARK 4. For $k = 2$, the number $2kl+1 = 113$ can be replaced by 4.

Proof of Lemma 17. Equation (24) is solvable for $c = 0$, so let $c = p^{\delta}d$ for d a p-adic unit. We shall prove by induction on nonnegative $\kappa < \gamma$ that if $s \geq kl+\kappa l+1$ and $\delta \geq \delta_s - \kappa$ then solvability of (23) implies solvability of (24). For $\kappa = 0$ there is a residue r such that the set $S = \{i \leq s : \delta_i \equiv r \pmod{k}\}$ satisfies $|S| \geq l+1$. Let $m = \max_{i \in S} i$. By the definition of δ_i there exist $\boldsymbol{\eta}_i \in \mathbb{Z}^{r_i}$ such that $F_i(\boldsymbol{\eta}_i) = p^{\delta_i}d_i$, where d_i is a p-adic unit $(i \in S)$. By Lemma 3 there exist $\xi_i \in \mathbb{Z}_p$ $(i \in S)$ such that

$$p^{\delta-\delta_m}d = \sum_{i \in S} d_i\xi_i^k;$$

therefore,

$$c = \sum_{i \in S} F_i(p^{(\delta_m-\delta_i)/k}\xi_i\boldsymbol{\eta}_i).$$

Assume now that the implication holds for $s \geq kl+(\kappa-1)l+1$ and for the left-hand side of (23) and (24) divisible by $p^{\delta_s-\kappa+1}$ $(\kappa \geq 1)$. Let $\delta = \delta_s - \kappa$ and $s \geq kl+\kappa l+1$. If $\delta > \delta_{s-l} - \kappa$ then the implication holds by the inductive assumption with s replaced by $s-l$. If $\delta = \delta_{s-l} - \kappa$, then $\delta_i = \delta_s$ $(s-l \leq i \leq s)$. From the solvability of (23) we infer that, for certain $\boldsymbol{\zeta}_i \in \mathbb{Z}^{r_i}$,

$$c - \sum_{i=1}^{s-l-1} F_i(\boldsymbol{\zeta}_i) = p^{\delta_s}t, \quad t \in \mathbb{Z}_p.$$

By the definition of δ_i there exist $\boldsymbol{\eta}_i \in \mathbb{Z}^{r_i}$ such that $F_i(\boldsymbol{\eta}_i) = p^{\delta_i}d_i$, where d_i is a p-adic unit $(s-l \leq i \leq s)$. Now, by Lemma 3 there exist $\xi_i \in \mathbb{Z}_p$ $(s-l \leq i \leq s)$ such that

$$\sum_{i=s-l}^{s} d_i \xi_i^k = t.$$

It follows that $(\boldsymbol{\zeta}_1, \dots, \boldsymbol{\zeta}_{s-l-1}, \xi_{s-l}\boldsymbol{\eta}_{s-l}, \dots, \xi_s\boldsymbol{\eta}_s)$ is a solution of (24). The inductive proof shows that the implication holds provided $\delta \geq \delta_s - (\gamma - 1)$ and $s \geq lk + l(\gamma - 1) + 1$. For $\delta \leq \delta_s - \gamma$ the implication holds, by Lemma 1, for every s. Since $\gamma - 1 \leq \tau + 1 \leq k$ it follows that the implication holds for $s \geq 2kl + 1$, which was to be proved. □

Proof of Theorem. For each prime p let the greatest common divisor of $F_i(\boldsymbol{\eta}_i)$ for $\boldsymbol{\eta}_i \in \mathbb{Z}^{r_i}$ be divisible exactly by $p^{\delta_{pi}}$. Put

$$m_p = \min_{\substack{S \subset \mathbb{N} \\ |S| = 2kl+1}} \sum_{i \in S} \delta_{pi}$$

and let S_p be a unique set S such that $|S| = 2kl + 1$, $\sum_{i \in S} \delta_{pi} = m_p$, and $\sum_{i \in S} i$ is minimal. For all p such that $\delta_{pi} = 0$ for all $i \leq 2kl + 1$ (and thus for all but finitely many p) we have $S_p = \{1, \dots, 2kl + 1\}$. Now take

$$S_1 = \bigcup_{p \text{ prime}} S_p,$$

$$s_1 = \max\left\{s_2, \max_{i \in S_1} i\right\}.$$

By Lemma 16 for $s \geq s_1 \geq s_2$ only finitely many integers N exist that are represented by $\sum_{i=1}^{s} F_i(\mathbf{x}_i)$ over $\mathbb{R}$ and over $\mathbb{Z}_p$ for all primes p yet are not represented by $\sum_{i=1}^{s} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$. Let s_0 be the least integer $s \geq s_1$ for which the number of exceptions is minimal. We show that s_0 has the property asserted in the theorem. Suppose N is an integer represented by $\sum_{i=1}^{s} F_i(\mathbf{x}_i)$ over $\mathbb{R}$ and over $\mathbb{Z}_p$ for all primes p. By the choice of s_2, N is represented by $\sum_{i=1}^{s_0} F_i(\mathbf{x}_i)$ over $\mathbb{R}$. Since for $i \notin S_p$ we have $\delta_{pi} \geq \max_{j \in S_p} \delta_{pj}$, it follows from Lemma 17 that N is represented by $\sum_{i=1}^{s_0} F_i(\mathbf{x}_i)$ over $\mathbb{Z}_p$ for every prime p. If N is represented over $\mathbb{Z}$ by $\sum_{i=1}^{s} F_i(\mathbf{x}_i)$ but not by $\sum_{i=1}^{s_0} F_i(\mathbf{x}_i)$, then the number of exceptions for s is smaller than the number of exceptions for s_0, contrary to the choice of s_0. □

Proof of Corollary. Let $I_k = \{i \in \mathbb{N} : k_i = k\}$. Because the sequence k_i is bounded, almost all the I_k are empty. For each k such that I_k is infinite, the Theorem implies there are s_k such that every integer represented by $\sum_{i=1,\, i \in I_k}^{s} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$ is represented by $\sum_{i=1,\, i \in I_k}^{s_k} F_i(\mathbf{x}_i)$ over $\mathbb{Z}$. For each k such that $0 < |I_k| < \infty$, put $s_k = \max_{i \in I_k} i$ and take

$$s_0 = \max_{I_k \neq \emptyset} s_k.$$

Now s_0 has the asserted property because if $N = \sum_{i=1}^{s} F_i(\mathbf{y}_i)$ then, for each k, $\sum_{i=1,\, i \in I_k}^{s} F_i(\mathbf{y}_i) = \sum_{i=1,\, i \in I_k}^{s_0} F_i(\mathbf{x}_i)$; after summation over k, we have

$$N = \sum_{i=1}^{s_0} F_i(\mathbf{x}_i).$$

Added in proof. Suitable modifications in the proofs of Lemmas 5, 13, and 16 show that the condition $(*)$ can be replaced by a weaker one: either not all forms are semidefinite of the same sign, or at least $kl+1$ forms are nonsingular.

References

[1] Z. I. Borevič and I. R. Šafarevič, *Number theory,* Academic Press, New York, 1966.
[2] J. W. S. Cassels, *Rational quadratic forms,* London Math. Soc. Monogr. (N.S.), 13, Academic Press, New York, 1978.
[3] H. Davenport, *Cubic forms in thirty-two variables,* Philos. Trans. Royal Soc. London Ser. A 251 (1959), 193–232; reprinted in *The collected works of Harold Davenport,* vol. III, pp. 1149–1188, Academic Press, New York, 1977.
[4] ———, *Analytic methods for Diophantine equations and Diophantine inequalities,* 2nd ed., Cambridge Univ. Press, Cambridge, 2005.
[5] H. Davenport and D. J. Lewis, *Non-homogeneous cubic equations,* J. London Math. Soc. 39 (1964), 657–671; reprinted in *The collected works of Harold Davenport,* vol. III, pp. 1245–1259, Academic Press, New York, 1977.
[6] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants,* Birkhäuser, Boston, 1994.
[7] J. P. Jouanolou, *Le formalisme du résultant,* Adv. Math. 90 (1991), 117–263.
[8] O. Perron, *Algebra, I,* 3rd ed., de Gruyter, Berlin, 1951.
[9] W. M. Schmidt, *Analytische Methoden für Diophantische Gleichungen,* Birkhäuser, Basel, 1984.
[10] E. T. Whittaker and G. N. Watson, *A course in modern analysis,* Cambridge Univ. Press, Cambridge, 1963.

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