Variation Formulas for L_1-Principal Functions and Application to the Simultaneous Uniformization Problem

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1. Introduction

Let $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$ and $\widetilde{\mathcal{R}} = \bigcup_{t \in B}(t, \tilde{R}(t))$ be unramified sheeted domains over $B \times \mathbb{C}_z$, where $B = \{|t| < \rho\}$ is a disk in $\mathbb{C}_t$ and $R(t) \Subset \tilde{R}(t)$ for $t \in B$. We set $\partial \mathcal{R} = \bigcup_{t \in B}(t, \partial R(t))$ in $\widetilde{\mathcal{R}}$. In this paper, we assume that

$$\mathcal{R} \colon t \in B \to R(t)$$

is a C^ω smooth variation of domains $R(t)$ with C^ω smooth boundaries in $\tilde{R}(t)$. Namely, we can choose a real-analytic defining function $\varphi(t, z)$ of $\partial \mathcal{R}$ such that $\frac{\partial \varphi}{\partial z} \neq 0$ on $\partial \mathcal{R}$. We denote by $C_j(t)$ ($j = 0, 1, \ldots, \nu$), where $\nu \geq 0$ is independent of $t \in B$, the boundary contours of $R(t)$ in $\tilde{R}(t)$ with the orientation $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$. Assume that the total space $\mathcal{R}$ contains $B \times \{0\}$. Precisely, there exists at least one constant section $\mathbf{O}$ of $\mathcal{R}$ over $B \times \{0\}$. For each $t \in B$, we conventionally write 0 for the point $\mathbf{O} \cap R(t)$.

Let $t \in B$ be fixed. In the theory of one complex variable, it is known that there exists a unique real-valued function $u(t, z)$ on $R(t) \setminus \{0\}$ satisfying the following four conditions:

(1) $u(t, z)$ is harmonic on $R(t) \setminus \{0\}$ and is continuous on $\overline{R(t)}$;
(2) $u(t, z) - \log \frac{1}{|z|}$ is harmonic at $z = 0$;
(3) $u(t, z) = 0$ on $C_0(t)$;
(4) for each $i = 1, \ldots, \nu$, we have
    (i) $u(t, z) = a_i(t)$: constant on $C_i(t)$ and
    (ii) $\int_{C_i(t)} * du(t, z) = 0$.

We note that $u(t, z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \tilde{R}(t)$. By (2), we find a neighborhood $U_0(t)$ of $z = 0$ such that

$$u(t, z) = \log \frac{1}{|z|} + \gamma(t) + h(t, z) \quad \text{on } U_0(t), \tag{1.1}$$

where $\gamma(t)$ is the constant term and $h(t, z)$ is harmonic for z on $U_0(t)$ such that

$$h(t, 0) = 0, \quad t \in B. \tag{1.2}$$

The function $u(t, z)$ is called the L_1-*principal function* on $R(t)$ with logarithmic pole at 0 with respect to $C_0(t)$, and $\gamma(t)$ is called the L_1-*constant* for $(R(t), 0)$ with respect to $C_0(t)$ (cf. [7]). In this paper, we simply call $u(t, z)$ the L_1-principal function for $(R(t), 0, C_0(t))$ and call $\gamma(t)$ the L_1-constant for $(R(t), 0, C_0(t))$. We note that $u(t, z) > 0$ in $R(t) \setminus \{0\}$ and that $a_i(t) > 0$ $(i = 1, \ldots, \nu)$.

Then we have the following variation formula for the L_1-constant $\gamma(t)$ for $(R(t), 0, C_0(t))$.

LEMMA 1.1.   *It holds for $t \in B$ that*

$$\frac{\partial^2 \gamma(t)}{\partial t \partial \bar{t}} = -\frac{1}{\pi} \int_{\partial R(t)} k_2(t, z) \left| \frac{\partial u(t, z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 u(t, z)}{\partial \bar{t} \partial z} \right|^2 dx \, dy,$$

where

$$k_2(t, z) = \left(\frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \left| \frac{\partial \varphi}{\partial z} \right|^2 - 2 \operatorname{Re} \left\{ \frac{\partial^2 \varphi}{\partial \bar{t} \partial z} \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \right) \Big/ \left| \frac{\partial \varphi}{\partial z} \right|^3$$

on $\partial \mathcal{R}$ and does not depend on the choice of defining functions $\varphi(t, z)$ of $\partial \mathcal{R}$ and where ds_z is the arc length element of $\partial R(t)$ at z.

The function $k_2(t, z)$ on $\partial \mathcal{R}$ is due to Maitani and Yamaguchi in [4], which is based on [3]. This variation formula is formally the same as that for the Robin constant $\lambda(t)$ (induced by the Green function $g(t, z)$ on $R(t)$ with logarithmic pole at $z = 0$) in [4, Thm. 3.1]. The essential difference in the proofs for $\gamma(t)$ and $\lambda(t)$ is because, unlike the Green function $g(t, z)$, $u(t, z)$ is not a defining function of $\partial \mathcal{R}$.

THEOREM 1.2.   *Under the same conditions as in Lemma 1.1, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$ then $\gamma(t)$ is a C^ω superharmonic function on B.*

REMARK 1.   For Lemma 1.1, we assumed that $\mathcal{R}$ is unramified over $B \times \mathbb{C}_z$. However, Lemma 1.1 (and hence Theorem 1.2) holds even if each $R(t)$, $t \in B$, has a finite number of branch points $\zeta_k(t)$ $(k = 1, \ldots, m)$ such that, for $t \in B$, $\zeta_k(t)$ is a holomorphic function on B with $\zeta_k(t) \neq \zeta_l(t)$ $(k \neq l)$. The reason is that this case can be reduced to Lemma 1.1 via the standard method by using Nishimura's theorem [5].

In the special case when $R(t)$ is a planar Riemann surface, the L_1-principal function $u(t, z)$ induces a circular slit mapping $f(t, z)$. That is, if we choose a branch $u^*(t, z)$ of a harmonic conjugate function of $u(t, z)$ on $R(t)$, $t \in B$, such that

$$f(t, z) = e^{\gamma(t) - (u(t, z) + i u^*(t, z))}$$

is of the form

$$w = f(t, z) = z + \sum_{j=2}^{\infty} b_j(t) z^j \quad \text{on } U_0(t),$$

then $f(t, z)$ conformally maps $R(t)$ onto a circular slit domain $\{|w| < e^{\gamma(t)}\} \setminus \left(\bigcup_{i=1}^{\nu} \ell_i \right)$, where $\ell_i(t) = f(t, C_i(t))$ (an arc of the circle $\{|w| = e^{\gamma(t) - a_i(t)}\}$). If $\mathcal{R}$

is pseudoconvex over $B \times \mathbb{C}_z$ then $e^{\gamma(t)}$ is logarithmic superharmonic on B, so the total space $\bigcup_{t \in B}\{|w| < e^{\gamma(t)}\}$ is a Hartogs domain in $B \times \mathbb{C}_w$.

REMARK 2. We note that the same property of radial slit mapping does not hold for the radius $r_0(t)$. Indeed, as will be shown in the next section, there exist pseudo-convex domains $\mathcal{R}$ in $B \times \mathbb{C}_z$ such that the $\log r_0(t)$ are neither superharmonic nor subharmonic on B.

Under the same conditions as for the unramified domain $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$ in $\widetilde{\mathcal{R}}$ over $B \times \mathbb{C}_z$ and for $\partial R(t) = \sum_{j=0}^{\nu} C_j(t)$, we assume that there exist two holo-morphic sections,

$$\Xi_0 : z = 0 \quad \text{and} \quad \Xi_1 : z = \xi(t),$$

of $\mathcal{R}$ over B such that $\Xi_0 \cap \Xi_1 = \emptyset$. Let $t \in B$ be fixed. In the theory of one com-plex variable, there exists a unique real-valued function $p(t, z)$ on $R(t) \setminus \{0, \xi(t)\}$ satisfying the following four conditions:

(I) $p(t, z)$ is harmonic on $R(t) \setminus \{0, \xi(t)\}$ and continuous on $\overline{R(t)}$;
(II) $p(t, z) - \log \frac{1}{|z|}$ is harmonic at $z = 0$ and

$$\lim_{z \to 0} \left(p(t, z) - \log \frac{1}{|z|} \right) = 0;$$

(III) $p(t, z) - \log|z - \xi(t)|$ is harmonic at $z = \xi(t)$;
(IV) for each $j = 0, 1, \ldots, \nu$, we have
   (i) $p(t, z) = a_j(t)$: constant on $C_j(t)$ and
   (ii) $\int_{C_j(t)} * dp(t, z) = 0$.

We note that $p(t, z)$ extends harmonically across $\partial R(t)$ as a harmonic function on $V(t)$ such that $\partial R(t) \Subset V(t) \Subset \tilde{R}(t)$, $-\infty < p(t, z) < +\infty$, and $-\infty < a_j(t) < +\infty$.

By (II), we find a neighborhood $U_0(t)$ of $z = 0$ such that

$$p(t, z) = \log \frac{1}{|z|} + h_0(t, z) \text{ on } U_0(t), \tag{1.3}$$

where $h_0(t, z)$ is harmonic for z on $U_0(t)$ and

$$h_0(t, 0) = 0, \quad t \in B. \tag{1.4}$$

By (III), we find a neighborhood $U_\xi(t)$ of $z = \xi(t)$ such that

$$p(t, z) = \log|z - \xi(t)| + \alpha(t) + h_\xi(t, z) \text{ on } U_\xi(t), \tag{1.5}$$

where $\alpha(t)$ is a real constant, $h_\xi(t, z)$ is harmonic for z on $U_\xi(t)$, and

$$h_\xi(t, \xi(t)) = 0, \quad t \in B. \tag{1.6}$$

In this paper, we simply call $p(t, z)$ the *L₁-principal function* for $(R(t), 0, \xi(t))$ and call $\alpha(t)$ the *L₁-constant* for $(R(t), 0, \xi(t))$.

Under this situation, we have the following.

Lemma 1.3. *It holds for $t \in B$ that*

$$\frac{\partial^2 \alpha(t)}{\partial t \partial \bar{t}} = \frac{1}{\pi} \int_{\partial R(t)} k_2(t,z) \left| \frac{\partial p(t,z)}{\partial z} \right|^2 ds_z + \frac{4}{\pi} \iint_{R(t)} \left| \frac{\partial^2 p(t,z)}{\partial \bar{t} \partial z} \right|^2 dx\, dy.$$

Theorem 1.4. *Under the same conditions as in Lemma 1.3, if $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$ then $\alpha(t)$ is a C^ω subharmonic function on B. This is also true under the same condition for $\mathcal{R}$ as in Remark 1.*

As an application of Theorem 1.4, we demonstrate the following fact. Let B be a simply connected domain in $\mathbb{C}_t$. Let $\pi : \mathcal{S} \to B$ be a holomorphic family of compact Riemann surfaces $S(t) = \pi^{-1}(t)$ over B such that each fiber $S(t)$ is of genus ≥ 2 and nonsingular in $\mathcal{S}$. For a fixed $t \in B$, we consider the Schottky covering $\tilde{S}(t)$ of each $S(t)$ (see [1, 19F, p. 241; 2, Sec. 101, p. 266]). We denote by $\tilde{\mathcal{S}}$ the total space of the variation $t \in B \to \tilde{S}(t)$; namely, $\tilde{\mathcal{S}} = \bigcup_{t \in B}(t, \tilde{S}(t))$. Then we have our final result as follows.

Theorem 1.5. *The total space $\tilde{\mathcal{S}}$ consisting of the Schottky covering $\tilde{S}(t)$ of compact Riemann surfaces $S(t)$ with one complex parameter $t \in B$ is holomorphically uniformized to a univalent domain on $B \times \mathbb{P}^1$.*

In [4], Maitani and Yamaguchi proved that, if $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$ is an unramified pseudoconvex domain over $B \times \mathbb{C}_z$ such that each $R(t)$, $t \in B$, is planar and parabolic, then $\mathcal{R}$ is holomorphically uniformizable to a domain in $B \times \mathbb{P}^1$. Since the Schottky covering $\tilde{S}(t)$ of a compact Riemann surface $S(t)$ of genus $g \geq 2$ is planar but not parabolic, their theorem and method cannot be applied to our case. In [8], Yamaguchi discussed Theorem 1.5 and offered a rough sketch of the proof. However, his sketch had a "gap". This paper bridges that gap by establishing the variation formula for L_1-principal functions (Lemma 1.3), leading to Theorem 1.5.

2. Proof of Lemma 1.1

In the proof we put $a_0(t) \equiv 0$ on $C_0(t)$, so that we can simply write

$$u(t,z) = a_i(t) \text{ on } C_i(t), \quad i = 0, 1, \dots, \nu.$$

We divide the proof into two steps.

Step 1: For each $t \in B$ and $i = 0, 1, \dots, \nu$, it holds, along $C_i(t)$, that

$$\frac{\partial^2 u}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_z} ds_z = 2k_2(t,z) \left| \frac{\partial u}{\partial z} \right|^2 ds_z + \frac{\partial^2 a_i}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_z} ds_z$$

$$+ 4 \operatorname{Im} \left\{ \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} dz \right\} - 4 \operatorname{Im} \left\{ \frac{\partial a_i}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} dz \right\}. \qquad (2.1)$$

Proof. Since $\mathcal{R}: t \in B \to R(t)$ is a C^ω smooth variation of unramified domains $R(t)$ with C^ω smooth boundary over $\mathbb{C}_z$, we remark from the standard argument that $u(t, z)$ is real-analytically extended for (t, z) beyond $\partial\mathcal{R}$ to a neighborhood $\mathcal{V} = \bigcup_{t \in B}(t, V(t))$ of $\partial\mathcal{R}$ in $\widetilde{\mathcal{R}}$ such that, for each fixed $t \in B$, we have $V(t) \supset \partial R(t)$ and $u(t, z)$ is harmonic for $z \in R(t) \cup V(t) \setminus \{0\}$. $\big($This immediately implies that $\gamma(t)$ is also real-analytic on B, since $\gamma(t) = \frac{1}{2\pi}\int_0^{2\pi} h(t, \varepsilon e^{i\theta})\, d\theta.\big)$

It suffices to prove (2.1) for $t = 0$ and at an arbitrary fixed point ζ on $C_i(0)$. We first prove (2.1) at $(0, \zeta)$ when

$$\frac{\partial u}{\partial n_\zeta}(0, \zeta) > 0. \tag{2.2}$$

In this case we have a neighborhood $B_0 \times V_0$ of $(0, \zeta)$ such that $B_0 \Subset B$ and $V_0 \Subset V(t)$ for $t \in B_0$ and such that

$$
\begin{aligned}
u(t, z) &< a_i(t) \text{ on } R(t) \cap V_0 &&\text{for } t \in B_0,\\
u(t, z) &= a_i(t) \text{ on } \partial R(t) \cap V_0 &&\text{for } t \in B_0,\\
u(t, z) &> a_i(t) \text{ on } \overline{R(t)}^c \cap V_0 &&\text{for } t \in B_0.
\end{aligned}
$$

It follows that $u(t, z) - a_i(t)$ is a C^ω defining function of $\partial\mathcal{R} \cap [B_0 \times V_0]$. We simply set

$$u_i(t, z) = u(t, z) - a_i(t) \text{ in } B_0 \times V_0.$$

Since $k_2(t, z)$ on $\partial\mathcal{R}$ does not depend on the choice of defining functions, it follows that on $\partial\mathcal{R} \cap [B_0 \times V_0]$ we have

$$k_2(t, z) = \left(\frac{\partial^2 u_i}{\partial t \partial \bar{t}} \left| \frac{\partial u_i}{\partial z} \right|^2 - 2\,\mathrm{Re}\left\{ \frac{\partial^2 u_i}{\partial \bar{t} \partial z} \frac{\partial u_i}{\partial t} \frac{\partial u_i}{\partial \bar{z}} \right\} + \left| \frac{\partial u_i}{\partial t} \right|^2 \frac{\partial^2 u_i}{\partial z \partial \bar{z}} \right) \left| \frac{\partial u_i}{\partial z} \right|^{-3}.$$

Since $u_i(t, z)$ is harmonic for z and $a_i(t)$ is independent of z, we have

$$\frac{\partial^2 u_i}{\partial z \partial \bar{z}} = 0, \quad \frac{\partial u_i}{\partial z} = \frac{\partial u}{\partial z} \text{ and } \frac{\partial u_i}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}}, \quad \frac{\partial^2 u_i}{\partial \bar{t} \partial z} = \frac{\partial^2 u}{\partial \bar{t} \partial z}.$$

Since $u_i(t, z) = 0$ on $C_i(t)$ and $u_i(t, z) < 0$ on $R(t) \cap V_0$, we have

$$\left| \frac{\partial u}{\partial z} \right| = \left| \frac{\partial u_i}{\partial z} \right| = \frac{1}{2}\frac{\partial u_i}{\partial n_z} = \frac{1}{2}\frac{\partial u}{\partial n_z} > 0 \text{ along } C_i(t) \cap V_0.$$

It follows that, along $C_i(t) \cap V_0$,

$$
\begin{aligned}
\frac{\partial^2 u_i}{\partial t \partial \bar{t}} &= \left(k_2(t, z) \left| \frac{\partial u}{\partial z} \right|^3 + 2\,\mathrm{Re}\left\{ \frac{\partial u_i}{\partial t} \frac{\partial u}{\partial \bar{z}} \frac{\partial^2 u}{\partial \bar{t} \partial z} \right\} \right) \left| \frac{\partial u}{\partial z} \right|^{-2}\\
&= \frac{1}{2} k_2(t, z) \frac{\partial u}{\partial n_z} + 2\,\mathrm{Re}\left\{ \frac{\partial u_i}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} \Big/ \frac{\partial u}{\partial z} \right\},
\end{aligned}
$$

so that

$$\frac{\partial^2 u}{\partial t \partial \bar{t}} = \frac{\partial^2 a_i}{\partial t \partial \bar{t}} + \frac{1}{2} k_2(t, z) \frac{\partial u}{\partial n_z} + 2\,\mathrm{Re}\left\{ \frac{\partial(u - a_i)}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} \Big/ \frac{\partial u}{\partial z} \right\}.$$

Consequently, along $C_i(t) \cap V_0$ for $t \in B_0$, we have

$$\frac{\partial^2 u}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_z} ds_z$$

$$= \left[\frac{\partial^2 a_i}{\partial t \partial \bar{t}} + \frac{1}{2} k_2(t,z) \frac{\partial u}{\partial n_z} + 2 \operatorname{Re}\left\{\frac{\partial (u - a_i)}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} \bigg/ \frac{\partial u}{\partial z}\right\}\right] \frac{\partial u}{\partial n_z} ds_z.$$

Since $u(t,z) = a_i(t)$: constant on $C_i(t)$ for each $t \in B_0$, we have

$$\frac{\partial u(t,z)}{\partial n_z} ds_z = \frac{2}{i} \frac{\partial u(t,z)}{\partial z} dz \text{ along } C_i(t).$$

Since $\frac{\partial u(t,z)}{\partial n_z} ds_z$ is real it follows that, along $C_i(t) \cap V_0$ for each $t \in B_0$,

$$\frac{\partial^2 u}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_z} ds_z$$

$$= \frac{\partial^2 a_i}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_z} ds_z + \frac{1}{2} k_2(t,z) \left(\frac{\partial u}{\partial n_z}\right)^2 ds_z + 2 \operatorname{Re}\left\{\frac{2}{i} \frac{\partial (u - a_i)}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} dz\right\}$$

$$= \frac{\partial^2 a_i}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_z} ds_z + 2 k_2(t,z) \left|\frac{\partial u}{\partial z}\right|^2 ds_z + 4 \operatorname{Im}\left\{\frac{\partial (u - a_i)}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} dz\right\}.$$

In particular, putting $(t,z) = (0, \zeta)$ in this equality yields formula (2.1) at the point $\zeta \in C_i(0)$ in the case (2.2).

We next prove formula (2.1) at the point $\zeta \in C_i(0)$ when $\frac{\partial u}{\partial n_\zeta}(0, \zeta) < 0$. In fact, we put $q(t,z) := -u(t,z)$ in $\mathcal{R}$ so that $\frac{\partial q}{\partial n_\zeta}(0, \zeta) > 0$. Since $q(t,z)$ is a constant $-a_i(t)$ on $\partial C_i(t)$, we can apply our reasoning from the first case for $q(t,z)$ and obtain equation (2.1) for $q(t,z)$—that is, for $-u(t,z)$ (with $-a_i(t)$ instead of $a_i(t)$). On the other hand, the formula (2.1) for $u(t,z)$ (with $a_i(t)$) is identical to the formula (2.1) for $-u(t,z)$ (with $-a_i(t)$), so (2.1) for $u(t,z)$ holds when $\frac{\partial u}{\partial n_\zeta}(0, \zeta) < 0$.

Finally, we prove formula (2.1) at the point $\zeta \in C_i(0)$ when

$$\frac{\partial u}{\partial n_\zeta}(0, \zeta) = 0$$

or, equivalently, $\frac{\partial u}{\partial z}(0, \zeta) = 0$. Let $\varphi(t,z)$ be a C^ω defining function of $\partial \mathcal{R}$. Since $u(t,z) - a_i(t) \equiv 0$ on $\partial \mathcal{R} \cap [B_0 \times V_0]$, we find a neighborhood $B_1 \times V_1 \subset B_0 \times V_0$ of $(0, \zeta)$ and a C^ω function $f(t,z)$ on $B_1 \times V_1$ such that

$$u(t,z) - a_i(t) = f(t,z)\varphi(t,z) \text{ on } B_1 \times V_1. \tag{2.3}$$

By differentiating (2.3) with respect to z, we first have

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}\varphi + f \frac{\partial \varphi}{\partial z} \text{ on } B_1 \times V_1.$$

Since

$$\frac{\partial u}{\partial z}(0, \zeta) = 0, \quad \varphi(0, \zeta) = 0, \quad \text{and} \quad \frac{\partial \varphi}{\partial z}(0, \zeta) \neq 0,$$

we have $f(0, \zeta) = 0$. Differentiating both side of (2.3) with respect to t, we next have

$$\frac{\partial u}{\partial t} - \frac{\partial a_i}{\partial t} = \frac{\partial f}{\partial t}\varphi + f \frac{\partial \varphi}{\partial t} \text{ on } B_1 \times V_1,$$

so that

$$\frac{\partial u}{\partial t}(0, \zeta) - \frac{\partial a_i}{\partial t}(0) = \frac{\partial f}{\partial t}(0, \zeta) \cdot 0 + 0 \cdot \frac{\partial \varphi}{\partial t}(0, \zeta) = 0.$$

Together with $\frac{\partial u}{\partial n_\zeta}(0, \zeta) = 0$, this implies that both sides of (2.1) are zero at $(0, \zeta)$ and thus completes the proof of Step 1.

Step 2: Lemma 1.1 is true.

Proof. We set

$$\frac{\partial^2 u}{\partial t \partial \bar{t}}(t, 0) := \frac{\partial^2 \gamma(t)}{\partial t \partial \bar{t}}.$$

Then we see from (1.1) and (1.2) that the function $\frac{\partial^2 u}{\partial t \partial \bar{t}}(t, z)$ for each fixed $t \in B$ is harmonic for z on the whole $\overline{R(t)}$. There exists a tubular neighborhood W_0 of $\partial R(0)$ such that $u(0, z)$ is harmonic on $R(0) \cup W_0$. Then we can take a neighborhood $B_0 \subset B$ of $t = 0$ such that, for each $t \in B_0$, $W_0 \supset \partial R(t)$ and $u(t, z)$ is harmonic in $(R(0) \cup W_0) \setminus \{0, \xi(t)\}$.

For arbitrary fixed $t \in B_0$, it follows from Green's formula that

$$\int_{C_i(0) - C_i(t)} \frac{\partial u(t, z)}{\partial n_z} \, ds_z = 0.$$

On the other hand, we see from the condition (4)(ii) for the L_1-principal function $u(t, z)$ that, for each fixed $t \in B_0$,

$$\int_{C_i(0)} \frac{\partial u(t, z)}{\partial n_z} \, ds_z = 0 \quad (i = 1, \dots, \nu). \tag{2.4}$$

Differentiating both sides of (2.4) with respect to t and $\bar{t}$ then yields

$$\int_{C_i(0)} \frac{\partial \left(\frac{\partial^2 u(t, z)}{\partial t \partial \bar{t}} \right)}{\partial n_z} \, ds_z = 0 \quad (i = 1, \dots, \nu).$$

Hence,

$$\int_{\partial R(0)} u(0, z) \frac{\partial \left(\frac{\partial^2 u(t, z)}{\partial t \partial \bar{t}} \right)}{\partial n_z} \, ds_z = \sum_{i=0}^{\nu} \int_{\partial C_i(0)} a_i(0) \frac{\partial \left(\frac{\partial^2 u(t, z)}{\partial t \partial \bar{t}} \right)}{\partial n_z} \, ds_z = 0. \tag{2.5}$$

We draw a small disk $U_0^\varepsilon = \{|z| < \varepsilon\}$ such that $u(t, z)$ is harmonic for z in $R(0) \setminus U_0^\varepsilon$ and $\frac{\partial^2 u}{\partial t \partial \bar{t}}$ is harmonic for z in $\overline{R(0)}$. It follows from Green's formula that

$$\int_{\partial R(0) \setminus U_0^\varepsilon} \frac{\partial^2 u}{\partial t \partial \bar{t}}(0, z) \frac{\partial u(0, z)}{\partial n_z} \, ds_z = \int_{\partial R(0) \setminus U_0^\varepsilon} u(0, z) \frac{\partial \left(\frac{\partial^2 u}{\partial t \partial \bar{t}}(0, z) \right)}{\partial n_z} \, ds_z.$$

Letting $\varepsilon \to 0$, we see from (2.5) that

$$\int_{\partial R(0)} \frac{\partial^2 u}{\partial t \partial \bar{t}}(0, z) \frac{\partial u(0, z)}{\partial n_z} \, ds_z + 2\pi \frac{\partial^2 u}{\partial t \partial \bar{t}}(0, 0) = 0.$$

Thus,

$$\frac{\partial^2 \gamma}{\partial t \partial \bar{t}}(0) = -\frac{1}{2\pi} \int_{\partial R(0)} \frac{\partial^2 u}{\partial t \partial \bar{t}}(0, z) \frac{\partial u(0, z)}{\partial n_z} \, ds_z.$$

By (2.1) we have

$$\frac{\partial^2 \gamma}{\partial t \partial \bar{t}}(0) = -\frac{1}{2\pi} \int_{\partial R(0)} \left(2k_2(0,z) \left| \frac{\partial u}{\partial z} \right|^2 ds_z + \frac{\partial^2 a_i}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_z} ds_z \right.$$

$$\left. + 4 \operatorname{Im} \left\{ \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} dz \right\} - 4 \operatorname{Im} \left\{ \frac{\partial a_i}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} dz \right\} \right)$$

$$= -\frac{1}{\pi} \int_{\partial R(0)} k_2(0,z) \left| \frac{\partial u}{\partial z} \right|^2 ds_z - \frac{1}{2\pi} \int_{\partial R(0)} \frac{\partial^2 a_i}{\partial t \partial \bar{t}} \frac{\partial u}{\partial n_z} ds_z$$

$$- \frac{2}{\pi} \operatorname{Im} \left\{ \int_{\partial R(0)} \frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} dz \right\} + \frac{2}{\pi} \operatorname{Im} \left\{ \int_{\partial R(0)} \frac{\partial a_i}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} dz \right\}$$

$$\equiv I_1 + I_2 + I_3 + I_4.$$

Here the integrand in I_k ($k = 1, \ldots, 4$) is evaluated at $t = 0$ and $z \in \partial R(0)$.

We note that $a_i(t)$ does not depend on z and that $a_0(t) \equiv 0$ on B. By condition (4)(ii) for the L_1-principal function $u(t, z)$, we thus have

$$I_2 = -\frac{1}{2\pi} \sum_{i=0}^{\nu} \frac{\partial^2 a_i}{\partial t \partial \bar{t}}(0) \int_{C_i(0)} \frac{\partial u(0,z)}{\partial n_z} ds_z$$

$$= -\frac{1}{2\pi} \left(0 \times \int_{C_0(0)} \frac{\partial u(0,z)}{\partial n_z} ds_z + \sum_{i=1}^{\nu} \frac{\partial^2 a_i}{\partial t \partial \bar{t}}(0) \times 0 \right) = 0.$$

By (1.1) and (1.2) we see that, if we set $\frac{\partial u}{\partial t}(t, 0) := \frac{\partial \gamma(t)}{\partial t}$ for each fixed $t \in B$, then both $\frac{\partial u}{\partial t}(t, z)$ and $\frac{\partial u}{\partial t \partial \bar{t}}(t, z)$ are harmonic for z on the whole $\overline{R(t)}$. It follows from Green's formula and the harmonicity of u for z that

$$I_3 = -\frac{2}{\pi} \operatorname{Im} \iint_{R(0)} d \left(\frac{\partial u}{\partial t} \frac{\partial^2 u}{\partial \bar{t} \partial z} dz \right)$$

$$= -\frac{2}{\pi} \operatorname{Im} \iint_{R(0)} \left(\frac{\partial^2 u}{\partial t \partial \bar{z}} \frac{\partial^2 u}{\partial \bar{t} \partial z} + \frac{\partial u}{\partial t} \frac{\partial^3 u}{\partial \bar{t} \partial z \partial \bar{z}} \right) d\bar{z} \wedge dz$$

$$= -\frac{4}{\pi} \iint_{R(0)} \left| \frac{\partial^2 u}{\partial \bar{t} \partial z} \right|^2 dx \, dy.$$

Since $2 \frac{\partial u}{\partial z} dz = i \frac{\partial u}{\partial n_z} + du$ along $C_i(0)$, we have

$$2 \int_{C_i(0)} \frac{\partial u}{\partial z} dz = i \int_{C_i(0)} \frac{\partial u}{\partial n_z} ds_z + \int_{C_i(0)} du = i \int_{C_i(0)} \frac{\partial u}{\partial n_z} ds_z.$$

It follows from (2.4) that, for each $t \in B_0$,

$$\int_{C_i(0)} \frac{\partial u(t,z)}{\partial z} dz = \frac{i}{2} \int_{C_i(0)} \frac{\partial u(t,z)}{\partial n_z} ds_z = 0 \quad (i = 1, \ldots, \nu).$$

Hence, by an argument similar to that used for $I_2 = 0$, we obtain

$$I_4 = \frac{2}{\pi} \operatorname{Im} \left\{ \sum_{i=0}^{\nu} \frac{\partial a_i}{\partial t}(0) \int_{C_i(0)} \frac{\partial^2 u}{\partial \bar{t} \partial z}(0, z) \, dz \right\} = 0.$$

It turns out that

$$\frac{\partial^2 \gamma}{\partial t \partial \bar{t}}(0) = -\frac{1}{\pi} \int_{\partial R(0)} k_2(0, z) \left| \frac{\partial u(0, z)}{\partial z} \right|^2 ds_z - \frac{4}{\pi} \iint_{R(0)} \left| \frac{\partial u}{\partial \bar{t} \partial z}(0, z) \right|^2 dx\, dy,$$

which proves Step 2—namely, Lemma 1.1.

Proof of Theorem 1.2. If $\mathcal{R}$ is pseudoconvex over $B \times \mathbb{C}_z$, then $k_2(t, z) \geq 0$ on $\partial \mathcal{R}$. It follows from Lemma 1.1 that $\frac{\partial^2 \gamma}{\partial t \partial \bar{t}} \leq 0$. Thus, $\gamma(t)$ is a superharmonic function on B. □

REMARK 3.   In the theory of one complex variable, the circular slit mapping and the radial slit mapping have good correspondence. But the same result for the corresponding radius of the radial slit mapping does not hold. In fact, we have two counterexamples of pseudoconvex domains $\mathcal{R}$ in $B \times \mathbb{C}_z$.

(i) *The radius of radial slit mapping is not superharmonic on B.* Let

$$\mathcal{R} = \left\{ |t| < \tfrac{1}{2} \right\} \times \{|z| < 1\} \setminus \left\{ (t, z) : \left| z - \tfrac{1}{2} \right| \leq |t| < \tfrac{1}{2} \right\},$$

$$B = \left\{ |t| < \tfrac{1}{2} \right\}, \quad R(t) = \{|z| < 1\} \setminus \left\{ \left| z - \tfrac{1}{2} \right| \leq |t| \right\}.$$

The total space $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$ is a pseudoconvex domain in $B \times \mathbb{C}_z$.

Let $C_0 = \{|z| = 1\}$ and $C(t) = \left\{ \left| z - \tfrac{1}{2} \right| = |t| \right\}$, where $0 \leq t < \tfrac{1}{2}$; that is, $\partial R(t) = C_0 - C(t)$. There exists a unique harmonic function $q(t, z)$ on $\overline{R(t)} \setminus \{0\}$ such that

$$q(t, z) = \log|z| + \lambda(t) + h(t, z) \quad \text{at } z = 0, \text{ where } h(t, 0) = 0;$$
$$q(t, z) = 0 \qquad\qquad\qquad\quad \text{on } C_0; \text{ and}$$
$$\frac{\partial q(t, z)}{\partial n} = 0 \qquad\qquad\qquad \text{on } C(t).$$

Note that $q(0, z) = \log|z|$ on $R(0)$ and $\lambda(0) = 0$. The radial slit mapping $f_0(t, z)$ for $(R(t), 0, C_0)$ is given by $f_0(t, z) = e^{\lambda(t) - (q(t, z) + iq^*(t, z))}$ on $R(t)$, so the radius $r_0(t)$ stated in Remark 2 is in this case equal to $e^{\lambda(t)}$. Our claim is thus to show that $\lambda(t)$ is not superharmonic on B.

Let z and z^* be inverse points with respect to $C(t)$. We use $R^*(t)$ to denote the domain obtained from $R(t)$ by inversion with respect to $C(t)$. If we define $q(t, z^*) := q(t, z)$ on $R^*(t)$, then $q(t, z)$ can be continued harmonically across $C(t)$ into the domain $R^*(t)$ by a reflection principle due to Schwarz. More precisely, let $\hat{R}(t) := R(t) \cup C(t) \cup R^*(t)$ and let $C_0^*(t)$ (resp. $\alpha(t)$) be the circle (resp. point) obtained from C_0 (resp. $z = 0$) by inversion with respect to $C(t)$. Here $\partial \hat{R}(t) = C_0 - C_0^*(t)$. Then $q(t, z)$ on $R(t)$ is harmonically extended to the harmonic function $q(t, z)$ on $\hat{R}(t) \setminus \{0, \alpha(t)\}$ such that

$$q(t, z) = \log|z| + \lambda(t) + h(t, z) \quad \text{at } z = 0, \text{ where } h(t, 0) = 0;$$
$$q(t, z) - \log|z - \alpha(t)| \text{ is harmonic} \quad \text{at } z = \alpha(t); \text{ and}$$
$$q(t, z) = 0 \qquad\qquad\qquad\qquad \text{on } C_0 \cup C^*(t).$$

We note that $C_0^*(t) \subset C_0^*\left(\tfrac{1}{2}\right) = \left\{ \left| z - \tfrac{2}{3} \right| < \tfrac{1}{3} \right\} =: \delta$ for $t \in B$. We set $R_0 := \{|z| < 1\} \setminus \delta$, so that $R_0 \subset R(t)$ for $t \in B$. If we put $s(t, z) := q(t, z) - \log|z|$ and $s(t, 0) = \lambda(t)$, then $s(t, z)$ is harmonic on $\hat{R}(t)$ with pole $\log|z - \alpha(t)|$ at

$\alpha(t)$ and with boundary values 0 on C_0 and $-\log|z|$ on $C_0^*(t)$, so that $s(t,z) < \log 3$ on $C_0^*(t)$. Further, we consider the harmonic function $\tilde{s}(t,z)$ on R_0 with pole $\log|z - \alpha(t)|$ at $\alpha(t)$ and with boundary values $\log 3$ on ∂R_0. Since $\tilde{s}(t,z) - s(t,z)$ is harmonic on the whole R_0, whose boundary values ≥ 0, it follows from the maximum principle that $\tilde{s}(t,0) \geq s(t,0) = \lambda(t)$. Since $\alpha(t) \to 0$ as $|t| \to \frac{1}{2}$, we have $\tilde{s}(t,0) \to -\infty$ and hence $\lambda(t) \to -\infty$ as $|t| \to \frac{1}{2}$. Since $\lambda(0) = 0$, it follows that $\lambda(t)$ is not superharmonic on B.

(ii) *The radius of radial slit mapping is not subharmonic on B*. Let

$$\mathcal{R} = \bigcup_{t \in B} \{|z| < r(t)\} \setminus B \times (C_1 \cup C_2),$$

where $C_1 = \left[\frac{1}{2}, \frac{2}{3}\right], C_2 = \left[\frac{i}{2}, \frac{2i}{3}\right]$ in $\mathbb{C}_z$, $r(t) > 1$, and $\log r(t)$ is superharmonic on B. Thus $\mathcal{R}$ is a pseudoconvex domain in $B \times \mathbb{C}_z$. In this case the radial slit mapping $f_0(t,z)$ for $(R(t), 0, C_0(t))$, where $C_0(t) = \{|z| = r(t)\}$, is identical with z; hence $r_0(t) = r(t)$ for $t \in B$. Thus $\log r_0(t)$ is not subharmonic on B.

3. Proof of Lemma 1.3

It suffices to prove the lemma for $t = 0$. We find a neighborhood $B_0 = \{|t| < r_0\} \Subset B$ of $t = 0$, a neighborhood $V_0 \Subset \tilde{R}(0)$ of $\partial R(0)$, and a neighborhood $U_{\xi(0)}^{\rho_0} := \{|z - \xi(0)| < \rho_0\} \Subset R(0)$ of $\xi(0)$ such that:

(i) $\xi(t) \in U_{\xi(0)}^{\rho_0}, t \in B_0$;
(ii) $U_{\xi(0)}^{\rho_0} \cap \{z = 0\} = \emptyset$;
(iii) $\bigcup_{t \in B_0} (t, \partial R(t)) \subset B_0 \times V_0$; and
(iv) $p(t,z), t \in B_0$, is harmonically extended for z beyond $\partial R(t)$ onto V_0.

We see from (1.3), (1.4), and (1.5) that if we put

$$\frac{\partial^2 p}{\partial t \partial \bar{t}}(0,0) := \frac{\partial^2 h_0}{\partial t \partial \bar{t}}(0,0) = 0 \quad \text{and}$$

$$\frac{\partial^2 p}{\partial t \partial \bar{t}}(0, \xi(0)) := \frac{\partial^2 \alpha}{\partial t \partial \bar{t}}(0) + \frac{\partial^2 h_\xi}{\partial t \partial \bar{t}}(0, \xi(0))$$

then the function $\frac{\partial^2 p}{\partial t \partial \bar{t}}(0,z)$ is harmonic for z in the whole $\overline{R(0)}$. We note that, for arbitrary fixed $t \in B_0$ and $j = 0, \ldots, \nu$,

$$\int_{C_j(0)} \frac{\partial p(t,z)}{\partial n_z} ds_z = \int_{C_j(t)} \frac{\partial p(t,z)}{\partial n_z} ds_z = 0$$

by Green's formula and the condition (IV)(ii) for the L_1-principal function $p(t,z)$. Thus,

$$\int_{C_j(0)} \frac{\partial}{\partial n_z} \left(\frac{\partial^2 p(t,z)}{\partial t \partial \bar{t}} \right) ds_z = 0.$$

On each boundary component $C_j(0)$, $j = 0, 1, \ldots, \nu$, of $R(0)$, we have $p(0,z) = \text{constant} = a_j(0)$; therefore,

$$\int_{\partial R(0)} p(0,z) \frac{\partial \left(\frac{\partial^2 p}{\partial t \partial \bar{t}}(0,z) \right)}{\partial n_z} \, ds_z = \sum_{j=0}^{\nu} a_j(0) \int_{C_j(0)} \frac{\partial \left(\frac{\partial^2 p}{\partial t \partial \bar{t}}(0,z) \right)}{\partial n_z} \, ds_z = 0. \quad (3.1)$$

We draw small disks $U_0^\varepsilon = \{|z| < \varepsilon\}$ and $U_{\xi(0)}^\varepsilon = \{|z - \xi(0)| < \varepsilon\}$, where $0 < \varepsilon < \rho_0$ and $U_0^\varepsilon \cap U_{\xi(0)}^\varepsilon = \emptyset$, so that $p(t,z)$, $t \in B_0$, is harmonic for z in $\overline{R(0)} \setminus (U_0^\varepsilon \cup U_{\xi(0)}^\varepsilon)$. By Green's formula we have

$$\int_{\partial R(0) - \partial U_0^\varepsilon - \partial U_{\xi(0)}^\varepsilon} \frac{\partial^2 p}{\partial t \partial \bar{t}}(0,z) \frac{\partial p(0,z)}{\partial n_z} \, ds_z$$

$$= \int_{\partial R(0) - \partial U_0^\varepsilon - \partial U_{\xi(0)}^\varepsilon} p(0,z) \frac{\partial \left(\frac{\partial^2 p}{\partial t \partial \bar{t}}(0,z) \right)}{\partial n_z} \, ds_z.$$

Since $p(0,z)$ has singularities (II) and (III) at $z = 0$ and $z = \xi(0)$, respectively, and since $\frac{\partial^2 p}{\partial t \partial \bar{t}}(0,z)$ is harmonic on $\overline{R(0)}$, by (3.1) we have

$$\int_{\partial R(0)} \frac{\partial^2 p}{\partial t \partial \bar{t}}(0,z) \frac{\partial p(0,z)}{\partial n_z} \, ds_z - 2\pi \left(\frac{\partial^2 \alpha}{\partial t \partial \bar{t}}(0) + \frac{\partial^2 h_\xi}{\partial t \partial \bar{t}}(0, \xi(0)) \right) = 0$$

as ε tends to 0. Therefore,

$$\frac{\partial^2 \alpha}{\partial t \partial \bar{t}}(0) + \frac{\partial^2 h_\xi}{\partial t \partial \bar{t}}(0, \xi(0)) = \frac{1}{2\pi} \int_{\partial R(0)} \frac{\partial^2 p}{\partial t \partial \bar{t}}(0,z) \frac{\partial p(0,z)}{\partial n_z} \, ds_z.$$

Reasoning much as in the proof of (2.1),

$$\frac{\partial^2 \alpha}{\partial t \partial \bar{t}}(0) + \frac{\partial^2 h_\xi}{\partial t \partial \bar{t}}(0, \xi(0))$$

$$= \frac{1}{2\pi} \sum_{j=0}^{\nu} \int_{C_j(0)} \left(2k_2(0,z) \left| \frac{\partial p}{\partial z} \right|^2 \, ds_z + \frac{\partial^2 a_j}{\partial t \partial \bar{t}} \frac{\partial p}{\partial n_z} \, ds_z \right.$$

$$\left. + 4 \, \mathrm{Im} \left\{ \frac{\partial p}{\partial t} \frac{\partial^2 p}{\partial \bar{t} \partial z} \, dz \right\} - 4 \, \mathrm{Im} \left\{ \frac{\partial a_j}{\partial t} \frac{\partial^2 p}{\partial \bar{t} \partial z} \, dz \right\} \right)$$

$$= \frac{1}{\pi} \int_{\partial R(0)} k_2(0,z) \left| \frac{\partial p}{\partial z} \right|^2 \, ds_z + \frac{1}{2\pi} \sum_{j=0}^{\nu} \int_{C_j(0)} \frac{\partial^2 a_j}{\partial t \partial \bar{t}} \frac{\partial p}{\partial n_z} \, ds_z$$

$$+ \frac{2}{\pi} \, \mathrm{Im} \left\{ \int_{\partial R(0)} \frac{\partial p}{\partial t} \frac{\partial^2 p}{\partial \bar{t} \partial z} \, dz \right\} - \frac{2}{\pi} \, \mathrm{Im} \left\{ \sum_{j=0}^{\nu} \int_{C_j(0)} \frac{\partial a_j}{\partial t} \frac{\partial^2 p}{\partial \bar{t} \partial z} \, dz \right\}$$

$$\equiv J_1 + J_2 + J_3 + J_4.$$

By reasoning similar to the case $I_2 = I_4 = 0$ for $i = 1, \ldots, \nu$, we have $J_2 = J_4 = 0$ for $j = 0, 1, \ldots, \nu$. Hence for proving the lemma it suffices to show that

$$J_3 = \frac{4}{\pi} \iint_{R(0)} \left| \frac{\partial^2 p}{\partial \bar{t} \partial z} \right|^2 \, dx \, dy + \frac{\partial^2 h_\xi}{\partial t \partial \bar{t}}(0, \xi(0)). \quad (3.2)$$

We remark that $\frac{\partial^2 p}{\partial \bar{t} \partial z}(0,z)$ is regular on the whole $\overline{R(0)}$ and that $\frac{\partial p}{\partial t}(0,z)$ is harmonic on $\overline{R(0)}$ except at $z = \xi(0)$. By Green's formula we have

$$\int_{\partial R(0)-\partial U^\varepsilon_{\xi(0)}} \frac{\partial p}{\partial t}\frac{\partial^2 p}{\partial \bar{t}\partial z}\,dz = \iint_{R(0)-U^\varepsilon_{\xi(0)}} d\left(\frac{\partial p}{\partial t}\frac{\partial^2 p}{\partial \bar{t}\partial z}\,dz\right)$$

$$= \iint_{R(0)-U^\varepsilon_{\xi(0)}} \left|\frac{\partial^2 p}{\partial t\partial \bar{z}}\right|^2 d\bar{z}\,dz.$$

Letting $\varepsilon \to 0$, we note that

$$\lim_{\varepsilon\to 0} \iint_{U^\varepsilon_{\xi(0)}} \left|\frac{\partial^2 p}{\partial t\partial \bar{z}}\right|^2 d\bar{z}\,dz = 0$$

and

$$\lim_{\varepsilon\to 0} \int_{\partial U^\varepsilon_{\xi(0)}} \frac{\partial p}{\partial t}\frac{\partial^2 p}{\partial \bar{t}\partial z}\,dz = -\frac{1}{2}\lim_{\varepsilon\to 0} \int_{\partial U^\varepsilon_{\xi(0)}} \frac{\xi'(0)}{z-\xi(0)}\frac{\partial^2 p}{\partial \bar{t}\partial z}\,dz$$

$$= -\pi i\xi'(0)\frac{\partial^2 p}{\partial \bar{t}\partial z}(0,\xi(0)).$$

Thus,

$$\int_{\partial R(0)} \frac{\partial p}{\partial t}\frac{\partial^2 p}{\partial \bar{t}\partial z}\,dz = 2i\iint_{R(0)} \left|\frac{\partial^2 p}{\partial t\partial \bar{z}}(0,z)\right|^2 dx\,dy - \pi i\xi'(0)\frac{\partial^2 p}{\partial \bar{t}\partial z}(0,\xi(0)).$$

On the other hand, it follows from (1.6) that

$$\frac{\partial h_\xi}{\partial t}(t,\xi(t)) + \frac{\partial h_\xi}{\partial z}(t,\xi(t))\xi'(t) \equiv 0,$$

and then

$$\frac{\partial^2 h_\xi}{\partial t\partial \bar{t}}(t,\xi(t)) + \frac{\partial^2 h_\xi}{\partial t\partial \bar{z}}(t,\xi(t))\overline{\xi'(t)}$$

$$+ \frac{\partial^2 h_\xi}{\partial z\partial \bar{t}}(t,\xi(t))\xi'(t) + \frac{\partial^2 h_\xi}{\partial z\partial \bar{z}}(t,\xi(t))|\overline{\xi'(t)}|^2 \equiv 0$$

on B. Since h_ξ is harmonic for z, we have $\frac{\partial^2 h_\xi}{\partial z\partial \bar{z}}(t,z) = 0$. Hence,

$$\frac{\partial^2 h_\xi}{\partial t\partial \bar{t}}(0,\xi(0)) + 2\,\mathrm{Re}\left\{\frac{\partial^2 h_\xi}{\partial \bar{t}\partial z}(0,\xi(0))\xi'(0)\right\} = 0.$$

Thus,

$$J_3 = \frac{2}{\pi}\,\mathrm{Im}\left\{\int_{\partial R(0)} \frac{\partial p}{\partial t}\frac{\partial^2 p}{\partial \bar{t}\partial z}\,dz\right\}$$

$$= \frac{2}{\pi}\,\mathrm{Im}\left\{\left(2i\iint_{R(0)} \left|\frac{\partial^2 p}{\partial t\partial \bar{z}}(0,z)\right|^2 dx\,dy\right) - \pi i\xi'(0)\frac{\partial^2 p}{\partial \bar{t}\partial z}(0,\xi(0))\right\}$$

$$= \frac{4}{\pi}\iint_{R(0)} \left|\frac{\partial^2 p}{\partial t\partial \bar{z}}(0,z)\right|^2 dx\,dy - 2\,\mathrm{Re}\left\{\xi'(0)\frac{\partial^2 p}{\partial \bar{t}\partial z}(0,\xi(0))\right\}$$

$$= \frac{4}{\pi}\iint_{R(0)} \left|\frac{\partial^2 p}{\partial t\partial \bar{z}}(0,z)\right|^2 dx\,dy + \frac{\partial^2 h_\xi}{\partial t\partial \bar{t}}(0,\xi(0)).$$

Therefore, we conclude (3.2), which implies Lemma 1.3.

Note that the same proof of Theorem 1.2 and Remark 1 under Lemma 1.1 implies Theorem 1.4.

4. Approximation Condition

Let R be a noncompact Riemann surface. Fix points a and b in R and let $|z| < 1$ (resp. $|w| < 1$) be a local coordinate of a (resp. b) such that a (resp. b) corresponds to $z = 0$ (resp. $w = 0$). Let $\{R_n\}_n$ be a canonical exhaustion of R. In particular, $R_n \Subset R_{n+1} \Subset R$, $R = \bigcup_{n=1}^{\infty} R_n$, and ∂R_n consists of a finite number of C^{ω} smooth dividing closed curves $\{C_n^{(i)}\}_{i=1,\ldots,v_n}$ such that each $C_n^{(i)}$ is homologous to a finite number of dividing curves $\{C_{n+1}^{(i_k)}\}_{k=1,\ldots,i_\mu}$ of ∂R_{n+1}; that is,

$$\partial R_n = \sum_{i=1}^{v_n} C_n^{(i)}, \quad C_n^{(i)} \sim \sum_{k=1}^{i_\mu} C_{n+1}^{(i_k)}. \tag{4.1}$$

We assume that $a, b \in R_1$ and $\{|z| < 1\} \cup \{|w| < 1\} \Subset R_1$. On each R_n, we uniquely have the principal function $p_n(z)$ such that:

(1) $p_n(z)$ is harmonic on $R_n \setminus \{a, b\}$ and is harmonically extended beyond ∂R_n in a neighborhood V_n of R_{n+1} such that $\partial R_n \Subset V_n \Subset R_{n+1}$;
(2) $p_n(z) - \log \frac{1}{|z|}$ is harmonic at $z = 0$;
(3) $p_n(w) - \log|w|$ is harmonic at $w = 0$ and

$$\lim_{w \to b} (p_n(w) - \log|w|) = 0;$$

(4) for each $i = 1, \ldots, v_n$, we have
   (i) $p_n(z) = $ constant a_i on $C_n^{(i)}$ and
   (ii) $\int_{C_n^{(i)}(t)} * dp_n(z) = 0$.

Thus we have

$$p_n(z) = \log \frac{1}{|z|} + \mathrm{Re}\left\{ \sum_{k=1}^{\infty} a_k^{(n)} z^k \right\} \quad \text{and} \tag{4.2}$$

$$p_n(z) = \log|w| + \alpha_n + \mathrm{Re}\left\{ \sum_{k=1}^{\infty} b_k^{(n)} w^k \right\}, \tag{4.3}$$

where α_n is a real constant.

It is known that, for $m \geq n$,

$$\lim_{n \to \infty} \| d(p_n(z) - p_m(z)) \|_{R_n}^2$$

$$:= \lim_{n \to \infty} \iint_{R_n} \left[\left(\frac{\partial(p_n - q_m)}{\partial x} \right)^2 + \left(\frac{\partial(p_n - q_m)}{\partial y} \right)^2 \right] dx \, dy = 0.$$

This implies that $\{p_n(z)\}_n$ uniformly converges in any compact set K in $R \setminus \{a, b\}$ to a harmonic function $q(z)$ on $R \setminus \{a, b\}$ such that $\{\alpha_n\}_n$ converges to a real constant α and

$$\lim_{n \to \infty} \| p_n(z) - p(z) \|_{R_n} = 0. \tag{4.4}$$

Therefore we have

$$p(z) = \log \frac{1}{|z|} + \mathrm{Re}\left\{ \sum_{k=1}^{\infty} a_k z^k \right\} \quad \text{near } z = 0$$

and

$$p(w) = \log|w| + \alpha + \mathrm{Re}\left\{\sum_{k=1}^{\infty} b_k w^k\right\} \quad \text{near } w = 0.$$

Let $(\mathcal{R}, B, \pi)$ be a two-dimensional holomorphic triple: $\mathcal{R}$ is a two-dimensional complex manifold, $B = \{|t| < \rho\}$ is a disk, and π is a holomorphic mapping from $\mathcal{R}$ onto B such that the fiber $\pi^{-1}(t) := R(t)$ over each $t \in B$ is connected and nonsingular in $\mathcal{R}$. Thus $R(t)$ is a Riemann surface. We can conventionally write

$$\mathcal{R} = \bigcup_{t \in B}(t, R(t)).$$

We assume that $\mathcal{R}$ is a C^ω topologically trivial triple in the sense that there exist a noncompact Riemann surface R and a C^ω topological mapping

$$T: (t, w) \in B \times R \mapsto (t, z) = (t, \varphi(t, w)) \in \mathcal{R};$$

here, for each $t \in B$, the mapping $w \in R \mapsto z = \varphi(t, w)$ is a homeomorphism from R onto $R(t)$.

Let $\{R_n\}_n$ with $\mathcal{R} = \bigcup_{n=1}^{\infty} R_n$ be a canonical exhaustion of R. We put

$$\mathcal{R}_n = T(B \times R_n) =: \bigcup_{t \in B}(t, R_n(t))$$

so that, for each $t \in B$, $\{R_n\}_n$ with $R(t) = \bigcup_{n=1}^{\infty} R_n(t)$ is a canonical exhaustion such that each connected component C_n^i, $i = 1, \ldots, \nu_n$ (which is C^ω smooth closed dividing curve with ν_n independent of $t \in B$) of $\partial R(t)$ moves real-analytically smooth for $t \in B$.

Assume that there exist two disjoint holomorphic sections γ and α of $\mathcal{R}$ over B via π. In particular, let $\gamma: t \in B \to \gamma(t) \in R(t)$ and $\zeta: t \in B \to \zeta(t)$ be holomorphic from B into $\mathcal{R}$. Let $B \times \{|z| < \rho_0\}$ and $B \times \{|w| < \rho_1\}$ be local coordinates of neighborhoods of γ and ζ in $\mathcal{R}$ such that γ and ζ correspond to $B \times \{z = 0\}$ and $B \times \{w = 0\}$, respectively. Then, by the previous argument, for each fixed $t \in B$ we uniquely have the principal function $p(t, z)$ on $R(t)$ such that $p(t, z)$ is harmonic on $R(t) \setminus \{z = 0\} \cup \{w = 0\}$ and such that

$$p(t, z) = \log\frac{1}{|z|} + \mathrm{Re}\left\{\sum_{k=1}^{\infty} a_k(t)z^k\right\} \quad \text{near } z = 0$$

and

$$p(t, w) = \log|w| + \alpha(t) + \mathrm{Re}\left\{\sum_{k=1}^{\infty} b_k(t)w^k\right\} \quad \text{near } w = 0,$$

where $\alpha(t)$ is a constant.

We shall prove the following statement.

Lemma 4.1. *Assume that each $\mathcal{R}_n$ ($n = 1, 2, \ldots$) is pseudoconvex in $\mathcal{R}_{n+1}$. Then $-\alpha(t)$ is a subharmonic function on B.*

Proof. Let $\mathcal{R}_n$ be fixed. We consider the principal function $p_n(t, z)$ on $R_n(t)$ with respect to $z = 0$ and $w = 0$. We denote by $\alpha_n(t)$ the constant term of $p_n(t, z)$ at $z = 0$:

$$p_n(t, z) = \log \frac{1}{|z|} + \alpha_n(t) + \mathrm{Re}\left\{ \sum_{n=1}^{\infty} a_k^{(n)} z^n \right\} \quad \text{near } z = 0.$$

By Nishimura's theorem, we may assume that $\mathcal{R}_n$ is holomorphically equivalent to a unramified domain $\mathcal{R}_n$ over $B \times \mathbb{C}_\zeta$ such that $B \times \{a\}$ corresponds to $B \times \{a\}$ and $B \times \{b\}$ corresponds to a holomorphic section $\xi : z = \xi(t), t \in B$, where $\xi(t)$ is a holomorphic function on B such that $\xi(t) \neq 0$. Let's say that

$$T_n : (t, z) \in \mathcal{R}_n \mapsto (t, \zeta) = (t, f_n(t, z)) \in \mathcal{R}_n.$$

Here $\mathcal{R}_n = \bigcup_{t \in B}(t, R_n(t))$ (where $R_n(t)$ is equivalent to $D_n(t)$ as Riemann surfaces) is a pseudoconvex domain with C^ω boundary in $\mathcal{R}_{n+1}$ and each $\partial D_n(t)$ is of class C^ω in $D_{n+1}(t)$. In Section 3, for each $t \in B$ we constructed the principal function $\tilde{p}_n(t, \zeta)$ for $D_n(t)$ with respect to $\zeta = 0$ and $\zeta = \xi(t)$. As usual, we denote by $\tilde{\alpha}(t)$ the constant term of $p_n(t, \zeta)$ at $\zeta = 0$:

$$\tilde{p}_n(t, \zeta) = \log \frac{1}{|\zeta|} + \tilde{\alpha}(t) + \mathrm{Re}\left\{ \sum_{k=1}^{\infty} a_k^{(n)} \zeta^k \right\}.$$

It is not difficult to show that

$$\alpha(t) = \tilde{\alpha}(t) + \log\left| \frac{\partial f_n}{\partial z} \right|_{z=0}.$$

Since

$$\left| \frac{\partial f_n}{\partial z} \right|_{z=0} \neq 0,$$

we have

$$\frac{\partial \alpha(t)}{\partial t \partial \bar{t}} = \frac{\partial \tilde{\alpha}(t)}{\partial t \partial \bar{t}}.$$

We showed in Lemma 4.1 that, since $\mathcal{R}_n$ is pseudoconvex, $\tilde{\alpha}_n(t)$ is subharmonic on B; it follows that $\alpha_n(t)$ is subharmonic on B.

Furthermore, we showed that for each fixed $t \in B$, $\alpha_n(t)$ converges $\alpha(t)$. By its proof we can also show that

$$\lim_{n \to \infty} \alpha_n(t) = \alpha(t) \quad \text{is uniform on } B.$$

Hence $\alpha(t)$ is subharmonic on B. $\qquad\square$

5. An Example of Theorem 1.4

We begin with a simple example of our general result shown in this paper. Let $B = \{|t| < \rho\}$ be a disk in $\mathbb{C}_t$. For each $t \in B$, let $R(t)$ be a disk $\{|z| < r(t)\}$ in $\mathbb{C}_z$, where $\log r(t)$ is a superharmonic function on B. If we set the Hartogs domain of disks $\mathcal{R} = \bigcup_{t \in B}(t, R(t))$, then $\mathcal{R}$ is a pseudoconvex domain. Assume that there exists a holomorphic section $\xi : t \in B \mapsto \xi(t) \in R(t)$, where $\xi(t) \neq 0$. We consider the following function:

$$f(t, z) = -\frac{1}{\xi(t)} \cdot \frac{r(t)^2 (z - \xi(t))}{z(r(t)^2 - \bar{\xi}(t)z)} \quad \text{on } R(t).$$

Then f is a circular slit mapping on $R(t)$ with a zero at $z = \xi(t)$ and a pole at $z = 0$. The L_1-constant $\lambda(t)$ on B is written into

$$\lambda(t) = \log\left|\frac{\partial f}{\partial z}(t, \xi(t))\right| = \log\left|-\frac{1}{\xi(t)^2} \cdot \frac{r(t)^2}{r(t)^2 - |\xi(t)|^2}\right|$$

$$= -2\log|\xi(t)| + \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{|\xi(t)|}{r(t)}\right)^{2n}.$$

Since $\xi(t)$ is holomorphic on B and since $\log r(t)$ is superharmonic on B, it follows that $\log\frac{|\xi(t)|}{r(t)}$ is subharmonic on B and so is the second term on the right-hand side. Hence, $\lambda(t)$ is a subharmonic function on B.

6. Proof of Theorem 1.5

First, we take the holomorphic sections ξ_0 and ξ_1 on B of $\mathcal{S}$, and we may assume that they are constant: $z = 0$ and $z = 1$ on B. We can make a canonical exhaustion $\{\tilde{S}_n(t)\}_n$ of the Schottky covering $\tilde{S}(t)$ of $S(t)$ for each t; namely, $\tilde{S}_j(t) \Subset \tilde{S}_{j+1}(t)$ ($j = 1, 2, \ldots$) and $\bigcup_{n=1}^{\infty} \tilde{S}_n(t) = \tilde{S}(t)$ such that each $\bigcup_{t \in B}(t, \tilde{S}_n(t)) = \tilde{\mathcal{S}}_n$ is a smooth pseudoconvex domain and the projection of $\partial\tilde{S}_n$ to $S(t)$ does not contain $0, 1$ in $S(t)$. We thus have $\tilde{\mathcal{S}} = \bigcup_{n=1}^{\infty} \tilde{\mathcal{S}}_n$. Let $\hat{\mathcal{S}} = \tilde{\mathcal{S}} \setminus (B \times \{0\} \cup B \times \{1\})$. We fix two points $0, 1$ in $\tilde{S}_n(t)$ over $0, 1$ in $S(t)$. We note that $\tilde{S}_n(t)$ is a planar Riemann surface. Then, there exists an L_1-principal function $u_{1n}(t, z)$ on $\tilde{S}_n(t)$ such that

$$u_{1n}(t, z) = \begin{cases} \log\frac{1}{|z|} + h_{0n}(t, z) & \text{on } U_0, \\ \log|z - 1| + \lambda_n(t) + h_{1n}(t, z) & \text{on } U_1, \end{cases}$$

where $h_{0n}(t, 0) = 0$ and $h_{1n}(t, 1) = 0$. Then $u_{1n}(t, z)$ induces the circular slit mapping $f_{1n}(t, z) = e^{u_{1n}(t, z) + iu_{1n}^*(t, z)}$ on $\tilde{S}_n(t)$ such that

$$f_{1n}(t, z) = \begin{cases} \frac{1}{z} + A_{0n}(t) + A_{1n}(t)z + \cdots & \text{on } U_0, \\ B_{1n}(t)(z - 1) + B_{2n}(t)(z - 1)^2 + \cdots & \text{on } U_1, \end{cases}$$

where $f_{1n}(t, 1) = 0$ and $f_{1n}(t, 0) = \infty$. Since $\tilde{S}_n$ is pseudoconvex, Theorem 1.4 implies that $\lambda_{1n}(t) = \log|B_{1n}(t)| = \log\left|\frac{\partial f_{1n}}{\partial z}\right|(t, 1)$ is a subharmonic function on B. As we have already shown, $u_{1n}(t, z)$ and hence $f_{1n}(t, z)$ uniformly converge (respectively) to a harmonic function $u_1(t, z)$ and to a univalent holomorphic function $f_1(t, z)$ on every compact set in $\hat{\mathcal{S}}$ such that

$$u_1(t, z) = \begin{cases} \log\frac{1}{|z|} + h_0(t, z) & \text{on } U_0, \\ \log|z - 1| + \lambda(t) + h_1(t, z) & \text{on } U_1, \end{cases}$$

where $h_0(t, 0) = 0$ and $h_1(t, 1) = 0$, and

$$f_1(t, z) = \begin{cases} \frac{1}{z} + A_0(t) + A_1(t)z + \cdots & \text{on } U_0, \\ B_1(t)(z - 1) + B_2(t)(z - 1)^2 + \cdots & \text{on } U_1. \end{cases}$$

We note that $\lim_{n\to\infty} \lambda_n(t) = \lambda(t)$ uniformly converges on B. It follows that $\lambda(t) = \log\left|\frac{\partial f}{\partial z}\right|(t,1)$ is a subharmonic function on B.

Next we shall show that $\log\left|\frac{\partial f_1(t,z)}{\partial z}\right|$ is a plurisubharmonic function on $\hat{S}$. It is enough to show that $\log\left|\frac{\partial f_1}{\partial z}(t,\xi(t))\right|$ restricted to every holomorphic section $\xi \neq 0,1$ of $\tilde{S}$ is subharmonic on B. In fact, we consider another slit mapping $f_\xi(t,z)$ on $R(t)$ such that

$$f_\xi(t,z) = \begin{cases} \frac{1}{z} + C_0(t) + C_1(t)z + \cdots & \text{on } U_0, \\ D_1(t)(z-\xi(t)) + D_2(t)(z-\xi(t))^2 + \cdots & \text{on } U_{\xi(t)}. \end{cases}$$

We consider the translation $\phi(t,z) = f_1(t,z) - f_1(t,\xi(t))$ on $R(t)$; then $\phi(t,z)$ is a holomorphic mapping on $\tilde{R}(t)$ such that

$$\phi(t,z) = \begin{cases} \frac{1}{z} + \tilde{A}_0(t) + \tilde{A}_1(t)z + \cdots & \text{on } U_0, \\ \tilde{B}_1(t)(z-\xi(t)) + \tilde{B}_2(t)(z-\xi(t))^2 + \cdots & \text{on } U_{\xi(t)}. \end{cases}$$

We set $w = f_\xi(t,z)$ on $\tilde{R}(t)$ and $\zeta = \phi(t,z)$ on $\tilde{R}(t)$. Then, $\zeta = L(t,w) := \phi(t, f_\xi^{-1}(t,w))$ is a univalent function on the domain $f_\xi(t,\tilde{R}(t))$ in $\mathbb{P}_w^1$. By Koebe's theorem concerning Schottky covering, we see that $L(t,w)$ is a fractional linear transformation on $\mathbb{P}_w^1$. Since $L(t,0) = 0$ and $L(t,w) = w + c_0(t)w^2 + \cdots$ at $w = \infty$, it follows that $L(t,w) \equiv w$; that is,

$$f_\xi(t,z) = \phi(t,z) = f(t,z) - f(t,\xi(t)), \quad z \in \tilde{R}(t).$$

Thus, we have

$$\frac{\partial f_\xi}{\partial z} = \frac{\partial(f_1 - f_1(t,\xi(t)))}{\partial z} = \frac{\partial f_1}{\partial z}.$$

Moreover, we recall that $\lambda_\xi(t) = \log\left|\frac{\partial f_\xi}{\partial z}(t,\xi(t))\right|$ is subharmonic on B, which implies the assertion.

Finally, we show that f is a holomorphic function for t and z on $\hat{S}$. In fact, if we set $p(t,z) := \log\left|\frac{\partial f_1}{\partial z}\right|$ on $\hat{S}$, then p is harmonic for z and hence we have $\frac{\partial^2 p}{\partial z\partial\bar{z}} \equiv 0$. Furthermore,

$$\det\begin{pmatrix} \dfrac{\partial^2 p}{\partial t\partial\bar{t}} & \dfrac{\partial^2 p}{\partial t\partial\bar{z}} \\[2mm] \dfrac{\partial^2 p}{\partial\bar{t}\partial z} & \dfrac{\partial^2 p}{\partial z\partial\bar{z}} \end{pmatrix} \geq 0$$

because p is a plurisubharmonic function on $\hat{S}$. Therefore, $\frac{\partial^2 p}{\partial\bar{t}\partial z} \equiv 0$; in other words, the holomorphic function $\frac{\partial p}{\partial z}$ for z is also holomorphic for t on $\hat{S}$. Namely, $\frac{\partial^2 f_1}{\partial z^2}\Big/\frac{\partial f_1}{\partial z}$ is holomorphic for (t,z).

Now we show that $\frac{\partial f_1}{\partial z}$ is holomorphic for (t,z) on $B \times \{|z| \ll 1\}$. On the neighborhood U_0 of $z = 0$, we have

$$f_1 = \frac{1}{z} + A_0(t) + \sum_{k=1}^{\infty} A_k(t)z^k,$$

$$\frac{\partial f_1}{\partial z} = -\frac{1}{z^2} + A_1(t) + \sum_{k=1}^{\infty} kA_k(t)z^{k-1},$$

$$\frac{\partial^2 f_1}{\partial z^2} = \frac{2}{z^3} + \sum_{k=2}^{\infty} k(k-1)A_k(t)z^{k-2}.$$

Thus,

$$-\frac{z}{2}\frac{\frac{\partial^2 f_1}{\partial z^2}}{\frac{\partial f_1}{\partial z}} = \frac{1 + \sum_{k=2}^{\infty}\frac{k(k-1)}{2}A_k(t)z^{k+1}}{1 - \sum_{k=2}^{\infty}kA_k(t)z^{k+1}}.$$

The left-hand side of this last equality is holomorphic for (t, z), and hence so is the right-hand side. Forming the Taylor expansion of the right-hand side on $\{|z| \ll 1\}$, we can inductively see that each $A_k(t)$ is holomorphic for t. Therefore, $\frac{\partial f_1}{\partial z}$ is holomorphic for (t, z) on $B \times \{|z| \ll 1\}$. By the identity theorem, $\frac{\partial f_1}{\partial z}$ is holomorphic for (t, z) on $\hat{S}$; hence $\int_1^z \frac{\partial f_1}{\partial z}(t, z)\, dz$ is holomorphic for (t, z) on $\hat{S}$. Since $\int_1^z \frac{\partial f_1}{\partial z}(t, z)\, dz = f_1(t, z) - f_1(t, 1) = f_1(t, z)$, we conclude that $f_1(t, z)$ is holomorphic for (t, z) and obtain a simultaneous uniformization of $\tilde{S}$ by $t = t$, $w = f_1(t, z)$.

References

[1] L. Ahlfors and L. Sario, *Riemann surfaces,* Princeton Math. Ser., 26, Princeton Univ. Press, Princeton, NJ, 1960.

[2] L. R. Ford, *Automorphic functions,* Chelsea, New York, 1972.

[3] N. Levenberg and H. Yamaguchi, *The metric induced by the Robin function,* Mem. Amer. Math. Soc. 448 (1991), 1–155.

[4] F. Maitani and H. Yamaguchi, *Variation of Bergman metrics on Riemann surfaces,* Math. Ann. 330 (2004), 477–489.

[5] Y. Nishimura, *Immersion analytique d'une famille de surfaces de Riemann ouvertes,* Publ. Res. Inst. Math. Sci. 14 (1978), 643–654.

[6] T. Nishino, *Nouvelles recherches sur les fonctions entières de plusieurs variables complexes (II). Fonctions entières qui se réduisent à celles d'une variable,* J. Math. Kyoto Univ. 9 (1969), 221–274.

[7] L. Sario and M. Nakai, *Classification theory of Riemann surfaces,* Grundlehren Math. Wiss., 164, Springer-Verlag, Berlin, 1970.

[8] H. Yamaguchi, *Variations de surfaces de Riemann,* C. R. Acad. Sci. Paris Ser. A-B 286 (1978), 1121–1124.

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