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On the Holomorphic Extension of CR Functions from Nongeneric CR Submanifolds of $\mathbb{C}^n$: The Positive Defect Case

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1. Introduction

1.1. Statement of Results

A real submanifold M of $\mathbb{C}^n$ is said to be CR if the dimension of $T_pM \cap iT_pM$ does not depend on p. Here M is a generic submanifold of $\mathbb{C}^n$ if, for any $p \in M$, $T_pM + JT_pM = T_p\mathbb{C}^n$. We say that a vector v in $\mathbb{C}^n$ is *complex transversal* to M at $p \in M$ if $v \notin \operatorname{span}_{\mathbb{C}} T_pM$. The question we address in this paper is the holomorphic extension of continuous CR functions on nongeneric CR submanifolds of $\mathbb{C}^n$ to wedges whose directions are complex transversal. In [4], we proved that any decomposable CR distribution admits a holomorphic extension to a complex transversal wedge. In this paper, we shall consider the case where CR functions (or distributions) are not decomposable. Define $\mathcal{O}_p^{\mathrm{CR}}$ to be the Sussmann manifold through p (the union of the CR orbits through p), which by [9] is a CR submanifold of M of *same CR dimension*. Assume the CR dimension (the complex dimension of $T_pM \cap iT_pM$) of M is k. Then the *defect of M at p* is said to be ℓ if the real dimension of $\mathcal{O}_p^{\mathrm{CR}}$ is $2k + \ell$. Our main result is the following theorem.

Theorem 1. *Let M be a smooth ($\mathcal{C}^\infty$) nongeneric CR submanifold of $\mathbb{C}^n$ of positive defect at some p. Then, for any v complex transversal to M at p and $\mathcal{U}$ a neighborhood of p, there exists a wedge $\mathcal{W}$ of direction v and with edge a neighborhood $\mathcal{U}' \subset \mathcal{U}$ of p in M such that any continuous CR function on $\mathcal{U}$ extends holomorphically to $\mathcal{W}$.*

1.2. Background

Theorem 1 generalizes results by Nagel and Rudin that imply a version of Theorem 1 in the totally real case. Let N be a smooth submanifold of the boundary of Ω, a strictly pseudoconvex domain in $\mathbb{C}^n$. If N is complex tangential ($TN \subset (T(\partial\Omega) \cap iT(\partial\Omega))$) then N is a peak interpolating set (see e.g. [7] or [8]). Given N, a totally real nongeneric submanifold of $\mathbb{C}^n$, one can easily construct Ω as just described and deduce Theorem 1 in the totally real case. As pointed out earlier, we first obtained Theorem 1 for the class of CR functions that are decomposable [4],

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which of course implies Theorem 1 for the case of maximum defect (minimality in the sense of Tumanov).

One should note that the question of complex transversal holomorphic extension can be viewed as a Cauchy problem with Cauchy data on a characteristic set M. In [4] we constructed an example of an abstract CR structure where there is no CR extension property and hence no holomorphic extension.

There is a long history of studying the holomorphic extension of CR functions on generic submanifolds of $\mathbb{C}^n$. We recommend the survey paper of Trépreau [10] for those interested in this subject; for general background on CR geometry, we recommend the books by Baouendi, Ebenfelt, and Rothschild [1], Boggess [3], and Jacobowitz [6].

1.3. Outline of the Paper

By assumption, the Sussmann manifold CR is not a complex manifold. Hence, by Tumanov's theorem, there exists a complex wedge (that is not an open set in $\mathbb{C}^n$) where CR functions admit some holomorphic extension. In this wedge, we deform the CR orbits. We then use an elliptic theory developed by Baouendi and Treves and solve a Dirichlet problem with boundary value the CR function that we seek to extend. Thanks to our deformation, the solution to the Dirichlet problem is holomorphic.

2. Local Coordinates and Reductions

2.1. Local Coordinates

Let M be a nongeneric submanifold of $\mathbb{C}^n$, and let m be the codimension of $T_pM + iT_pM$ in $\mathbb{C}^n$. We consider a trivial (in the CR sense) "generic" version of M: $N = M \times \mathbb{R}^m$, so N is generic with the same CR structure as M. A nongeneric CR submanifold of $\mathbb{C}^n$ is, locally, the CR graph of a generic submanifold of $\mathbb{C}^{n-m}$ (see e.g. [3]); that is to say, $M = \{(z, w, h(z, w)) : (z, w) \in \mathfrak{N}, h\colon \mathfrak{N} \to \mathbb{C}^m\}$, where $\mathfrak{N}$ is a generic submanifold of $\mathbb{C}^{n-m}$ and h is a smooth CR map of $\mathfrak{N}$ into $\mathbb{C}^m$. Hence the CR structure of M is uniquely determined by $\mathfrak{N}$. We call $\mathfrak{N}$ the *graphing manifold* of M. We thus have

$$M = \{(Z, h(Z)) : Z \in \mathfrak{N},\ h\colon \mathfrak{N} \to \mathbb{C}^m\}, \qquad N = \{(Z, u''' + h(Z))\}, \quad (2.1)$$

where $\mathfrak{N}$ is generic in $\mathbb{C}^{n-m}$ and h is a CR map from $\mathfrak{N}$ into $\mathbb{C}^m$. We will prove Theorem 1 in the case where $m = 1$. That is, M is the graph of a function h. The proof in the general case follows in the same manner (see Remark 5).

We may choose local coordinates on M such that p is the origin and $\mathfrak{N}$ is parameterized in $\mathbb{C}^{k+r} = \mathbb{C}^k_z \times \mathbb{C}^r$ $(k + r = n - 1)$ by

$$\mathfrak{N} = \{(z, (w', w'')) \in \mathbb{C}^k \times \mathbb{C}^r : \\ \operatorname{Im}(w', w'') = a(z, \operatorname{Re}(w', w'')),\ a(0) = da(0) = 0\}. \quad (2.2)$$

If the defect of M at the origin is maximal (i.e., M is minimal at the origin and $\mathcal{O}_0^{\mathrm{CR}} = M$ near 0) then Theorem 1 is proved by [4]. We will therefore assume that the defect is not maximal—that is, M is not minimal at 0 and so $\mathfrak{N}$ is not minimal at 0. We will henceforth use the notation $(w', w'') \in \mathbb{C}^s_{w'} \times \mathbb{C}^{r-s}_{w''}$ because $\mathcal{O}_0^{\mathrm{CR}}$, the union of the CR orbits at the origin, is a CR submanifold of M that plays an important part in the proof of Theorem 1 and has a graphing manifold $\mathfrak{M}$ of CR dimension k that is generic in $\mathbb{C}^{k+s}$.

After a linear change of variables, we may assume that the complex transverse vector v is given by

$$v = (0, 1) \in \mathbb{C}^{n-1} \times \mathbb{C}. \tag{2.3}$$

We will denote by $(u', u'') = \mathrm{Re}(w', w'') \in \mathbb{R}^r$. We thus have

$$\begin{aligned} \mathfrak{N} &= \{(z, u' + ia'(z, u', u''), u'' + ia''(z, u', u''))\} \\ &= \{(z, w'(z, u', u''), w''(z, u', u''))\} \subset \mathbb{C}^k \times \mathbb{C}^r, \\ T_0\mathfrak{N} &= \mathbb{C}^k \times \mathbb{R}^r. \end{aligned} \tag{2.4}$$

Define $\mathbb{C}T_p\mathfrak{N} = T_p\mathfrak{N} \otimes \mathbb{C}$ and $T_p^{0,1}\mathfrak{N} = T_p^{0,1}\mathbb{C}^{k+m} \cap \mathbb{C}T_p\mathfrak{N}$. To say that $\mathfrak{N}$ is a CR manifold simply means that $\dim_{\mathbb{C}} T_p^{0,1}\mathfrak{N}$ does not depend on p. The CR vector fields of $\mathfrak{N}$ are vector fields L on $\mathfrak{N}$ such that for any $p \in \mathfrak{N}$ we have $L_p \in T_p^{0,1}\mathfrak{N}$. One can choose a basis $\mathcal{L}$ of $T^{0,1}\mathfrak{N}$ near the origin consisting of vector fields L_j of the form

$$L_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{\ell=1}^{r} F_{j\ell} \frac{\partial}{\partial u_\ell}. \tag{2.5}$$

We have assumed that the local defect of M (hence of $\mathfrak{N}$) at the origin is positive but not maximal. In other words, $\mathcal{O}_0^{\mathrm{CR}}$ is a proper CR submanifold of M of same CR dimension that is not a complex submanifold. Note that $\mathcal{O}_0^{\mathrm{CR}}$ is minimal at the origin. Hence we have, after a linear change of variables,

$$\mathcal{O}_0^{\mathrm{CR}} = \{(z, w'(z, u', 0), g(z, u', 0), h(z, u', 0)\} \subset M; \tag{2.6}$$

here $w'(z, u', 0) = u' + ia'(z, u', 0) \in \mathbb{C}^s$, $s < r$, and g is CR on the graphing manifold $\mathfrak{M}$ of $\mathcal{O}_0^{\mathrm{CR}}$, where $\mathfrak{M}$ is given by

$$\mathfrak{M} = \{(z, w'(z, u', 0))\} \subset \mathbb{C}^k \times \mathbb{C}^s. \tag{2.7}$$

By construction, $\mathcal{O}_0^{\mathrm{CR}}$ is minimal at the origin; hence so is $\mathfrak{M}$. Therefore, by Tumanov's theorem [12], there exists a wedge W in $\mathbb{C}^{k+s}$ with edge a neighborhood of the origin in $\mathfrak{M}$ on which we have holomorphic extension of continuous CR functions near the origin. Thus, for some $\eta > 0$ and V_η, a ball centered at the origin in $\mathbb{C}^{k+s}$ of radius η, we have

$$W = (\mathcal{U} + i\Gamma),$$

where $\mathcal{U} = V_\eta \cap \mathfrak{M}$ and Γ is a conic neighborhood of some $\mu \in \mathbb{R}^s \setminus \{0\}$ and where continuous CR functions extend holomorphically to W.

2.2. *Deformations of* $\mathfrak{M}$

Denote by B_ε the ball of radius ε centered at the origin in $\mathbb{R}^s$ and by $\mathcal{B}_\varepsilon$ the unit ball centered at the origin in $\mathbb{C}^k$. Let $d \in \mathbb{R}$ be such that, for η as given in the definition of W, we have

$$0 < d < \frac{\eta}{\sqrt{s}}. \tag{2.8}$$

Let $\varepsilon > 0$ be given, and let $\{b_j\}_{j=1}^{s}$ be $\mathcal{C}^\infty$ functions such that

$$\begin{aligned} b_j|_{\mathcal{B}_\varepsilon \times B_\varepsilon} &= a_j'(\cdot,\cdot,0)|_{\mathcal{B}_\varepsilon \times B_\varepsilon}, \\ b_j &= d \quad \text{if } u \in \mathbb{R}^s \setminus B_{2\varepsilon}. \end{aligned} \tag{2.9}$$

Define $\widetilde{\mathfrak{M}}$, a generic submanifold of $\mathbb{C}^k \times \mathbb{C}^s$, as follows:

$$\widetilde{\mathfrak{M}} = \{(z, u' + ib(z,u')) : (z,u') \in \mathbb{C}^k \times \mathbb{R}^s\} =: \{(z, \tilde{w}'(z,u'))\}. \tag{2.10}$$

We have, on some neighborhood of the origin,

$$\widetilde{\mathfrak{M}} = \mathfrak{M}.$$

PROPOSITION 1. *For ε small enough, there exist b_j as in* (2.9) *such that any continuous CR function of $\mathfrak{M}$ extends to a continuous CR function of $\widetilde{\mathfrak{M}}$ in an η neighborhood of the origin.*

Proof. The main tool is the following lemma.

LEMMA 1. *There exist b_j such that, for ε small enough, we have*

$$(\widetilde{\mathfrak{M}} \cap V_\eta) \subset \overline{W}.$$

Proof. By a linear change of variables, we may identify μ with $(1, 1, \ldots, 1)$ in $\mathbb{R}^s$. Hence, we can assume that W is of the form

$$W = (\mathcal{U} + i\Gamma),$$

where $\mathcal{U}$ is a neighborhood of the origin in $\mathfrak{M}$ and Γ is a conic neighborhood of $(1, 1, \ldots, 1)$ in $\mathbb{R}^s$. By (2.2) we have

$$\|a_j'(\cdot,\cdot,0)\|_{L^\infty(\mathcal{B}_{3\varepsilon} \times B_{3\varepsilon})} < C\varepsilon^2. \tag{2.11}$$

Let $\vartheta_j = \vartheta_j(z,u')$ be smooth ($\mathcal{C}^\infty(\mathbb{R}^s)$) real-valued functions such that

$$\begin{aligned} &\vartheta_j \geq 0; \\ &\vartheta_j = 1 \quad \text{if } u' \in \mathbb{R}^s \setminus B_{2\varepsilon}, \\ &\vartheta_j = 0 \quad \text{if } u' \in B_\varepsilon. \end{aligned} \tag{2.12}$$

We will now perturb the functions a_j' in the w_j' as follows:

$$b_j(z,u') = d\vartheta_j(z,u') + (1 - \vartheta_j(z,u'))a_j'(z,u'), \tag{2.13}$$

where d is as given by (2.8). We see that $b_j = a_j$ if $u' \in B_\varepsilon$ and that $b_j = d$ if $u' \in B_{3\varepsilon} \setminus B_{2\varepsilon}$.

Fix real constants σ_0, σ_1 such that $1 < \sigma_0 < \sigma_1 < 2$. Let $\xi = \xi(u') \in \mathcal{C}^\infty$ be such that

$$\begin{aligned} &0 \leq \xi \leq 1; \\ &\xi(u') = 1 \quad \text{if } u' \in B_{\sigma_0\varepsilon}, \\ &\xi(u') = 0 \quad \text{if } u' \in B_{\sigma_1\varepsilon} \setminus B_{\sigma_0\varepsilon}. \end{aligned} \tag{2.14}$$

Choose $\vartheta_1 = \vartheta_1(u')$ verifying (2.12) and such that

$$\vartheta_1 \equiv 1 \quad \text{if } u' \in \mathbb{R}^s \setminus B_{\sigma_0 \varepsilon}. \tag{2.15}$$

We now define the remaining ϑ_j for $j > 1$ as follows:

$$\vartheta_j(z, u') = \xi(u')\vartheta_1(u') \frac{d - a_1(z, u', 0)}{d - a_j(z, u', 0)} + (1 - \xi(u')). \tag{2.16}$$

By (2.11), we can choose ε small enough that $d - a_j(z, u', 0) \neq 0$ when $(z, u') \in \mathcal{B}_{3\varepsilon} \times B_{3\varepsilon}$, in which case $\vartheta_j \in \mathcal{C}^\infty$.

By (2.13) we now have $(z, s + ib(z, u')) = (z, u' + ia(z, u', 0)) + (0, iv(z, u'))$, where $v = (\vartheta_1(d - a_1), \vartheta_2(d - a_2), \ldots, \vartheta_m(d - a_m)) \in \mathbb{R}^s$. Hence to prove the lemma it suffices to show that, if ε is small enough, then $v \in \Gamma$. By (2.13)–(2.16), v is given by

$$v = \vartheta_1(d - a_1)(1, \ldots, 1) \quad \text{if } u' \in B_{\sigma_0 \varepsilon}.$$

Furthermore, we have

$$v(z, u') = \begin{cases} (d - a_1(z, u', 0), v_2'(z, u'), \ldots, v_m'(z, u')) & \text{if } u' \in B_{\sigma_1 \varepsilon} \setminus B_{\sigma_0 \varepsilon}, \\ (d - a_1(z, u', 0), d - a_2(z, u', 0), \ldots, \\ \qquad d - a_m(z, u', 0)) & \text{if } u' \in B_{2\varepsilon} \setminus B_{\sigma_1 \varepsilon}, \end{cases} \tag{2.17}$$

where

$$v_j'(z, u') = d + a_j(z, u', 0)(\xi(u') - 1) - a_1(z, u', 0)\xi(u'). \tag{2.18}$$

By (2.11), (2.17), and (2.18) we see that $v(z, u')$ can be written as $v(z, u') = (d + O(\varepsilon^2), d + O(\varepsilon^2), \ldots, d + O(\varepsilon^2))$ when $u' \in B_{3\varepsilon} \setminus B_{\sigma_0 \varepsilon}$. Therefore, if ε is chosen small enough then $v(z, u') \in \Gamma$. To conclude the proof of Lemma 1, we need to make sure that ε is small enough that $(z, u' + \mathrm{id}) \in V_\eta$ when $(z, u') \in \mathcal{B}_{3\varepsilon} \times B_{3\varepsilon}$, which is indeed the case by (2.8). □

The proof of Proposition 1 is now immediate. □

2.3. Reductions

We now will make use of an analogue of the edge-of-the-wedge theorem due to Tumanov [13], although in this case it deals with CR extension instead of holomorphic extension.

THEOREM 2 [13]. *Let M be as before. Then any continuous CR function in a neighborhood of the origin CR extends locally, near the origin, to a CR manifold $\mathcal{M}$ of CR dimension $k + s$ with edge M given by*

$$\mathcal{M} = \{(z, w', \mathfrak{w}''(z, w', u''), H(z, w', u''))\},$$

where H is CR on

$$\{(z, w', \mathfrak{w}''(z, w', u''))\} := \{(z, w', u'' + i\beta(z, w', u'')) : u'' \in \mathbb{R}^{r-s},\ b \colon \mathbb{C}^k \times \mathbb{C}^s \times \mathbb{R}^{r-s} \to \mathbb{R}^{r-s}\}.$$

Furthermore, in a sufficiently small neighborhood of the origin, we have

$$\{(z, w', g(z, w'), h(z, w')) : (z, w') \in W\} \subset \mathcal{M}. \tag{2.19}$$

By (2.19) there exist $u''\mu\colon \mathbb{C}^k \times \mathbb{R}^s \times \mathbb{R}^{r-s} \to \mathbb{R}^s$ such that, if we set

$$\begin{gathered} u''\mu(z,u',u'') = \sum_{j=1}^{r-s} u''_j \mu_j(z,u',u'') \quad \text{and} \\ \tilde{w}'(z,u') + iu''\mu(z,u',u''),u'') = \tilde{w}'(z,u',u''), \end{gathered} \tag{2.20}$$

then we have

$$(z,\tilde{w}'(z,u',u''),\mathfrak{w}''(z,\tilde{w}'(z,u',u''),u''),H(z,\tilde{w}'(z,u',u''),u'')) \in \mathcal{M}. \tag{2.21}$$

We are now ready to define the deformed version of M on which we shall be working. Consider the manifold $\widetilde{M}$ given by

$$\widetilde{M} = \{(z,\tilde{w}'(z,u',u''),\mathfrak{w}''(z,\tilde{w}'(z,u',u''),u''),H(z,\tilde{w}'(z,u',u''),u''))\}. \tag{2.22}$$

This $\widetilde{M}$ verifies the following two properties.

(1) *$\widetilde{M}$ is a CR submanifold of $\mathbb{C}^n$ of CR dimension k.*
Indeed, $H(z,\tilde{w}'(z,u',u''),u'')$ is CR on

$$\{(z,\tilde{w}'(z,u',u''),\mathfrak{w}''(z,\tilde{w}'(z,u',u''),u''))\}$$

because it is CR on $\{(z,w',\mathfrak{w}''(z,w',u'')\}$ and, by (2.21),

$$\{(z,\tilde{w}'(z,u',u''),\mathfrak{w}''(z,\tilde{w}'(z,u',u''),u''))\} \subset \{(z,w',\mathfrak{w}''(z,w',u'')\}.$$

(2) *$\widetilde{M}$ is not minimal at the origin.*
We have

$$\begin{aligned} \{(z,\tilde{w}'(z,u',0),\mathfrak{w}''(z,\tilde{w}'(z,u',0),0),& \\ & H(z,\tilde{w}'(z,u',0),\mathfrak{w}''(z,\tilde{w}'(z,u',0),0)))\} \subset \mathcal{M}, \end{aligned}$$

where $\mathfrak{w}''(z,\tilde{w}'(z,u',0),0)$ and $H(z,\tilde{w}'(z,u',0),\mathfrak{w}''(z,\tilde{w}'(z,u',0),0))$ are CR on $\widetilde{\mathfrak{M}}$.

By Proposition 1, together with (2.19) and (2.21), we now obtain the following corollary.

COROLLARY 3. *Given any neighborhood of the origin $\mathcal{U}$ in M, we can choose ε small enough such that any continuous CR function on $\mathcal{U}$ extends as a continuous CR function to a neighborhood $\widetilde{\mathcal{U}}$ of the origin in $\widetilde{M}$ such that*

$$\begin{aligned} \{(z,\tilde{w}'(z,u',u''),\mathfrak{w}''(z,\tilde{w}'(z,u',u''),u''),H(z,\tilde{w}'(z,u',u''),u''))&: \\ z \in \mathcal{B}_{3\varepsilon}, u' \in B_{3\varepsilon}, u'' \in \mathcal{B}_{3\varepsilon}\} &\subset \widetilde{\mathcal{U}}. \end{aligned} \tag{2.23}$$

(ABUSE OF) NOTATION. Henceforth we shall be working exclusively on $\widetilde{M}$ and therefore drop the tilde notation.

We have

$$\mathcal{O}_0^{\mathrm{CR}} \subset M \subset N \subset \mathbb{C}^n,$$

where N is a generic submanifold of $\mathbb{C}^n$ that is parameterized near the origin by

$$\begin{aligned}
N = \{(z, w'(z,u',u''), w''(z,u',u''), w'''(z,u',u'',u'''))\};\\
w'(z,u) = u' + i(b(z,u') + u''\mu(z,u',u'')),\\
w''(z,u) = g(z,u') + u'' + iu''\beta(z,u',u''),\\
w'''(z,u) = u''' + h(z,u',u'');\\
g(z,u') \text{ is CR on } \{(z, w'(z,u',0))\},\\
h(z,u',u'')) \text{ is CR on } \{(z, w'(z,u',u''), w''(z,u',u''))\}.
\end{aligned} \tag{2.24}$$

Here M is given as a subset of N by $u''' = 0$; that is,

$$M = \{(z, w'(z,u',u''), w''(z,u',u''), h(z,u',u''))\}.$$

Since $\mathcal{O}_0^{\mathrm{CR}}$ is given as a subset of N by $u'' = 0$ and $u''' = 0$, it follows that

$$\begin{aligned}
\mathcal{O}_0^{\mathrm{CR}} &= \{(z, w'(z,u',0), w''(z,u',0), h(z,u',0))\}\\
&= \{(z, w'(z,u',0), g(z,u'), h(z,u',0)\}.
\end{aligned}$$

After the deformation argument, we have

$$w'(z,u',0) = u' \quad \text{if } u' \in \mathbb{R}^s \setminus B_{2\varepsilon}. \tag{2.25}$$

We also deal with CR functions defined in a neighborhood $\mathcal{U}$ of the origin such that

$$\{(z, w'(z,u',u''), w''(z,u',u''), h(z,u',u'')) : \\ z \in \mathcal{B}_{3\varepsilon}, u' \in B_{3\varepsilon}, u'' \in B_{3\varepsilon}\} \subset \mathcal{U}. \tag{2.26}$$

3. Analytic Vectors

In this section, we apply a theory developed by Baouendi and Treves in their study of approximation of CR functions [2]. We include some proofs but claim no originality.

3.1. Elliptic Operator Δ

LEMMA 2. *If L_j are the generators of the CR vectors fields near the origin in N, then there exist $r+1$ vector fields R_j of the form*

$$R_j = \sum_{l=1}^{r+1} a_{jl}(z,u)\frac{\partial}{\partial u_l},$$

where the a_{jl} are smooth functions, such that:

(i) $R_j(w_l) = \delta_{jl}$, $j, l \in \{1, \ldots, r+1\}$;
(ii) $[R_j, R_l] = 0$;
(iii) $[L_j, R_l] = 0$;
(iv) *the set* $\{L_1, \ldots, L_k, \bar{L}_1, \ldots, \bar{L}_k, R_1, \ldots, R_{r+1}\}$ *spans the complex tangent plane to N near the origin.*

The proof of this lemma is classic. Parts (ii), (iii), and (iv) are a consequence of (i). We thus determine the R_j by solving for their coefficients in (i) (see e.g. [1, Lemma 8.7.13, p. 234]).

We shall consider an elliptic operator Δ, of degree 2 and with no constant terms, given by

$$\Delta = \sum_{j=1}^{r+1} R_j^2. \tag{3.1}$$

From Lemma 2, we immediately deduce the following.

PROPOSITION 2. *The operator $-\Delta$ is strongly elliptic of degree 2 on N with smooth ($\mathcal{C}^\infty$) coefficients and no constant terms.*

Here Δ is a differential operator in the variables u whose coefficient functions depend smoothly on the z variable. We shall study a Dirichlet problem on N. To do this, we must construct Ω—an open set in N with boundary $\partial\Omega$ parameterized by some closed submanifold of N of codimension 1.

3.2. Construction of the Open Set Ω

Let ω be a smooth ($\mathcal{C}^\infty$) bounded open set contained in $\mathbb{R}^s_{u'} \times \mathbb{R}_{u'''} \cap \{u''' > 0\}$ with boundary $\partial\omega$. Define Ω as follows:

$$\Omega = \{(z, w'(z,u), w''(z,u), w'''(z,u)) : z \in \mathcal{B}_{3\varepsilon}, u \in \omega\}. \tag{3.2}$$

If Π is the projection of $\mathbb{R}^{r+1}$ onto $\mathbb{R}^r \times \{0\}$, we choose ω such that Ω verifies the following property:

$$\begin{gathered}\{(z, w'(z,u), w''(z,u), w'''(z,u)) : z \in \mathcal{B}_{3\varepsilon},\ u \in \Pi(\omega)\} \subset \mathcal{U},\\ \{(z, w'(z,u), w''(z,u), w'''(z,u)) : z \in \mathcal{B}_{3\varepsilon},\ u' \in B_{3\varepsilon},\ u'' \in B_{3\varepsilon},\ u''' = 0\}\\ \subset \{(z, w'(z,u), w''(z,u), w'''(z,u)) : z \in \mathcal{B}_{3\varepsilon},\ u \in [\bar{\omega}\{\cap u''' = 0\}]\}.\end{gathered} \tag{3.3}$$

The boundary of Ω on which we shall impose the Dirichlet data is defined to be

$$\partial\Omega = \{(z, w(z,u)) : z \in \mathcal{B}_{3\varepsilon}, u \in \partial\omega\}.$$

Set

$$\partial\Omega^0 = \{(z, w(z,u)) : z \in \mathcal{B}_{3\varepsilon},\ (u', u''') \in \partial\omega,\ u'' = 0\}.$$

We then have

$$\partial\Omega^0 \subset \partial\Omega.$$

LEMMA 3. *$\partial\Omega^0$ is a smooth CR submanifold of N of same CR dimension.*

Proof. The lemma is clear for the part of $\partial\Omega^0$ that is $\mathcal{O}_0^{\mathrm{CR}}$. Out of $\mathcal{O}_0^{\mathrm{CR}} \times \mathbb{R}$, the generators of the CR vector fields L_j are equal to $\frac{\partial}{\partial \bar{z}_j}$, from which the lemma follows. □

Denote by $(\mathcal{D})$ the Dirichlet problem on Ω:

$$(\mathcal{D}) \quad \begin{cases} \Delta(u) = 0 \text{ in } \Omega, \\ u = f \text{ on } \partial\Omega. \end{cases}$$

THEOREM 4. *For $f \in \mathcal{C}^0(\bar{\Omega})$, $(\mathcal{D})$ has a unique solution.*

Let S be the solution operator for $(\mathcal{D})$. That is, for $f \in \mathcal{C}^0(\Omega)$ we have

$$\begin{cases} \Delta(S(f)) = 0 \ \text{ in } \Omega, \\ S(f) = f \ \text{ on } \partial\Omega. \end{cases} \tag{3.4}$$

3.3. Elliptic Estimates

DEFINITION. For $\alpha \in \mathbb{N}^{r+1}$, set $R^\alpha = R_1^{\alpha_1} \cdots R_{r+1}^{\alpha_{r+1}}$. We shall say that a continuous function f in ω is an *analytic vector* of the system of vector fields $\{R_1, \dots, R_{r+1}\}$ if $R^\alpha f \in \mathcal{C}^0$ for any $\alpha \in \mathbb{N}^{k+r+1}$ and if for every compact set K of ω there is a constant $\rho > 0$ such that, in K,

$$\sup_{\alpha \in \mathbb{N}^{r+1}} \left(\rho^{|\alpha|} \frac{|R^\alpha f|}{|\alpha|!} \right) < \infty. \tag{3.5}$$

For V a neighborhood of the origin in $\mathbb{C}^k$, we have

$$(z, w(z,u)) \in \Omega \Longleftrightarrow (z,u) \in V \times \omega.$$

The following proposition is a simplified version of Lemma 4.1 in [2]. Let B be a ball of center x and radius ρ in $\mathbb{R}^{r+1}$; for $\vartheta \in (0,1]$, let B_ϑ be a ball of center x and radius $\rho\vartheta$.

PROPOSITION 3. *With B as before and such that $B \Subset \omega$, there exist C_1 and C_2 depending only on ω and Δ such that, for any $\mathcal{C}^\infty$ function f in an open neighborhood of the closure of B with*

$$\Delta f = 0 \ \textit{ in } B,$$

for any z in V and every $\alpha \in \mathbb{N}^{r+1}$ we have

$$\|R^\alpha f\|_{L^2(B_\vartheta)} \leq C_1 \left(\frac{C_2}{(1-\vartheta)} \right)^{|\alpha|} |\alpha|!. \tag{3.6}$$

Here the L^2 norm can be replaced by L^∞ norm.

The proof of this proposition is found in [2, p. 403].

PROPOSITION 4 [11, Prop. II.4.1]. *Let $p \in \Omega$ so that $f \in \mathcal{C}^0$ is an analytic vector of the system of vector fields $\{R_1, \dots, R_{r+1}\}$. It is necessary and sufficient that there exist an open neighborhood $\mathcal{V}$ of p in $\mathbb{C}^{r+1}$ and a continuous function $F(z,w)$ in $\mathcal{V}$ holomorphic with respect to w and such that $f(z,u) = F(z, w(z,u))$.*

The main difficulty in the proof of Proposition 4 is showing that the function defined by

$$F(z,w) = \sum_{\alpha \in \mathbb{N}^{r+1}} \frac{R^\alpha f(z,u)}{|\alpha|!} (w - w(z,u))^\alpha$$

is equal to f for w near $w(z,u)$ in Ω if f is an analytic vector of the vector fields $\{R_1, \dots, R_{r+1}\}$.

We shall use Proposition 3 to construct analytic vectors of the vector fields $\{R_1, \dots, R_{r+1}\}$ and then apply Proposition 4 to these vectors in order to obtain a holomorphic extension in the variables w.

4. Proof of Theorem 1

Let Ω be as previously constructed, and let f be a continuous CR function on $\mathcal{U}$. Then, trivially, f CR-extends to $\mathcal{U} \times \mathbb{R} \subset N$ (since N has the same CR structure as M, any CR function on M is itself a CR function on N). Consider the function $S(f)$ given by (3.4); we have $\Delta S(f) = 0$ on Ω. By Proposition 3, $S(f)$ is an analytic vector for the vector fields R_j on Ω that extends holomorphically as a function of w to a wedge $\mathcal{W}_v$ of direction v. We shall next prove that this function is holomorphic in z. By Proposition 4, the holomorphic extension F of $S(f)$ is given by

$$F(z, w) = \sum_{\alpha \in \mathbb{N}^{r+1}} \frac{R^\alpha Sf(z, w(z, u))}{|\alpha|!} (w - w(z, u))^\alpha. \tag{4.1}$$

PROPOSITION 5. *For any $j \in [1, \dots, k]$ and $\alpha \in \mathbb{N}^{r+1}$ we have*

$$R^\alpha \left[\frac{\partial}{\partial \bar{z}_j} \right] F(z, w(z, u', 0, u''')) = 0.$$

LEMMA 4. *For F as before we have, for any generator L_j of the CR vector fields on $\mathcal{O}_0^{\mathrm{CR}}$,*

$$L_j(F)|_{\partial\Omega^0} = 0.$$

Proof. The lemma holds because $\partial\Omega^0$ is a CR submanifold of Ω^0 of same CR dimension. Hence $L_j(F)|_{\partial\Omega^0} = L|_{\partial\Omega^0}(f|_{\partial\Omega^0}) = 0$. □

Since F extends holomorphically as a function of w, we have

$$L_j(F) = \left[\frac{\partial}{\partial \bar{z}_j} F \right] (z, w(z, u)). \tag{4.2}$$

By Lemma 4 and (4.2), $\frac{\partial}{\partial \bar{z}_j} F|_{\partial\Omega^0} = 0$. By the maximum principle, since F is holomorphic in w we obtain

$$\frac{\partial}{\partial \bar{z}_j} F|_{\Omega^0} = 0. \tag{4.3}$$

Let $\alpha \in \mathbb{N}^{r+1}$, and consider $R^\alpha F$. Since $[L_j, R_s] = 0$, by Lemma 4 we have $L_j(R^\alpha F)|_{\partial\Omega^0} = 0$. Hence, since $R^\alpha F$ extends holomorphically in w,

$$L_j(R^\alpha F)|_{\partial\Omega^0} = \frac{\partial}{\partial \bar{z}_j} R^\alpha F|_{\partial\Omega^0} = 0. \tag{4.4}$$

So by the maximum principle we have

$$L_j(R^\alpha F)|_{\Omega^0} = \frac{\partial}{\partial \bar{z}_j} R^\alpha F|_{\Omega^0} = 0, \tag{4.5}$$

and since $[L_j, R_s] = 0$ it follows that

$$\frac{\partial}{\partial \bar{z}_j} R^\alpha F = L_j R^\alpha F = R^\alpha L_j F = R^\alpha \frac{\partial}{\partial \bar{z}_j} F. \tag{4.6}$$

Hence, restricting (4.6) to Ω^0 and combining with (4.5), we conclude that

$$R^\alpha \frac{\partial}{\partial \bar{z}_j} F|_{\Omega^0} = 0.$$

This finishes the proof of Proposition 5. □

Now the proof of Theorem 1 follows easily. Indeed, by (4.1) we have

$$F(z, w) = \sum_{\alpha \in \mathbb{N}^{r+1}} \frac{R^\alpha F(z, w(z, u))}{|\alpha|!} (w - w(z, u))^\alpha.$$

So

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}_j} F &= L_j \sum_{\alpha \in \mathbb{N}^{r+1}} \frac{R^\alpha F(z, w(z, u))}{|\alpha|!} (w - w(z, u))^\alpha \\
&= \sum_{\alpha \in \mathbb{N}^{r+1}} \frac{L_j R^\alpha F(z, w(z, u))}{|\alpha|!} (w - w(z, u))^\alpha \\
&= \sum_{\alpha \in \mathbb{N}^{r+1}} \frac{R^\alpha L_j F(z, w(z, u))}{|\alpha|!} (w - w(z, u))^\alpha \\
&= \sum_{\alpha \in \mathbb{N}^{r+1}} \frac{R^\alpha \frac{\partial}{\partial \bar{z}_j} F(z, w(z, u))}{|\alpha|!} (w - w(z, u))^\alpha.
\end{aligned}$$

Therefore, by choosing $(z, w(z, u))$ in Ω^0 and given Proposition 5, we see that $\frac{\partial}{\partial \bar{z}_j} F$ is null on some open subset of $\mathcal{W}_v$. Hence F is holomorphic on $\mathcal{W}_v$. This concludes the proof of Theorem 1.

REMARK 5. If the manifold M is given by $M = \{(z, w, h(z, w)) : (z, w) \in \mathfrak{N} \subset \mathbb{C}^{n-m}, h \colon \mathfrak{N} \to \mathbb{C}^m\}$ with $m > 1$, then proceed as before with $N = \{(z, w, u''' + h(z, w)), (z, w) \in \mathfrak{N}\}$. Construct Δ in the same manner and, by a linear change of variable, assume that $v = (0, v') \in \mathbb{C}^{n-m} \times \mathbb{C}^m$ and $v' = (0, \ldots, 0, 1) \in \mathbb{C}^{m-1} \times \mathbb{C}$. Define Ω as before and solve the Dirichlet problem. The $\partial_{\bar{z}}$ of the solution of the Dirichlet problem still vanishes to infinite order on $\mathcal{O}_0^{\mathrm{CR}}$ and hence is null.

5. Corollaries and Remarks

COROLLARY 6. *Let M be a smooth ($\mathcal{C}^\infty$) nongeneric CR submanifold of $\mathbb{C}^m$ such that the defect of the graphing manifold at some point p is positive. Then, for any v complex transversal to M at $(p, h(p))$, there exists a wedge $\mathcal{W}$ of direction v and edge a neighborhood of $(p, h(p))$ in M such that any CR distribution near p extends holomorphically to $\mathcal{W}$.*

Proof. Consider the elliptic operator on M given by

$$\tilde{\Delta} = \Delta - \frac{\partial^2}{\partial u'''^2} = \sum_{j=1}^{r} R_j^2.$$

Given u, a CR distribution in a neighborhood of $(p, h(p))$, by [11] there exist an integer $k(u)$ and f a continuous CR function such that, in a neighborhood $\mathcal{V}$ of $(p, h(p))$, we have

$$u = [\tilde{\Delta}]^{k(u)} f.$$

Apply Theorem 1 to f to obtain a holomorphic extension F. Set U to be defined by

$$U = \sum_{j=1}^{r+s} \frac{\partial^2}{\partial w_j^2} F.$$

Then U is obviously holomorphic in $\mathcal{W}_v$ and has boundary value u. □

COROLLARY 7. *Let M be a $\mathcal{C}^\infty$ smooth nongeneric CR submanifold of $\mathbb{C}^n$ of positive defect at the origin. If the graphing manifold $\mathfrak{N}$ is generic in $\mathbb{C}^{n-m}$ then, for any v complex transversal to M at the origin, there exist a wedge $\mathcal{W}_v$ of direction v and g_j $(j = 1, \dots, m)$ holomorphic functions in $\mathcal{W}_v$ such that*

$$dg_1 \wedge dg_2 \wedge \cdots \wedge dg_m \neq 0 \ \text{ in } \mathcal{W}_v,$$
$$g_j|_M = 0.$$

Proof. Since M is not generic, it is given by

$$M = \{(z, w, h(z, w)) : (z, w) \in \mathfrak{N}\} \subset \mathbb{C}^{n-m} \times \mathbb{C}^m,$$

where $\mathfrak{N}$ is the graphing manifold. Note that, by Theorem 1, the functions $g_j = w_j''' - h_j(z, w', w'')$ are holomorphic in $\mathcal{W}_v$ and vanish on M. □

COROLLARY 8. *Let $N \subset M$ be two $\mathcal{C}^\infty$ smooth CR submanifolds of $\mathbb{C}^n$ such that N and M have the same CR dimension and such that the defect of N at the origin is positive. Then any continuous CR function of N admits a $\mathcal{C}^\infty$ smooth CR extension to M in a complex transversal direction. Furthermore, given $v \in T_0M \setminus [T_0N + iT_0N]$, there exists a nonconstant CR function g of M, defined in a wedge in M of direction v and edge N, such that g is null on N.*

REMARK 9. We do not know whether, in Corollary 8, there exists a CR function of M defined in a neighborhood of the origin that is nonconstant and vanishes on N. In [4] we constructed an example of an abstract CR structure in which there is no CR extension. It is even easier to construct an example where there is no nontrivial CR function vanishing on N.

EXAMPLE. Let L be a real analytic vector field (e.g., the Lewy operator) of the form

$$L = \frac{\partial}{\partial \bar{z}} + f(z, u) \frac{\partial}{\partial u}$$

that is not solvable (Hörmander's theorem [5]). Let g be a smooth function not in the image of L. Define the abstract CR structure $(M, \mathcal{L})$, where $\mathcal{L}$ is given by

$$\mathcal{L} = L + tg\frac{\partial}{\partial t}.$$

Then the equation $\mathcal{L}(tf) = 0$ is not solvable. Indeed, suppose it were; then we would have

$$t\mathcal{L}(f) + f\mathcal{L}(t) = t(\mathcal{L}(f) + fg) = 0.$$

Decomposing $f = f_0 + tf_1$, where $f_0 = f_0(z, u)$, we obtain

$$L(\mathrm{Log}(f_0)) + g = 0.$$

By real analyticity of L, we see that there are plenty of nontrivial functions in the kernel of $\mathcal{L}$.

It is also obvious that, if we require the function f to be holomorphic in a neighborhood of a nongeneric CR manifold, then it cannot vanish on M and be nonconstant. For example, let I_n be a sequence of disjointed intervals (separated by some open sets) in $\mathbb{R}$ accumulating to the origin and let f be a smooth function such that $f|_{I_n} = 1/n$. Let $\gamma = \{(s, f(u))\} \subset \mathbb{C}^2$. Suppose g is holomorphic on a neighborhood of the origin and that $g|_\gamma = 0$. Then $g(w_1, 1/n) = 0$ for all n for $|w_1|$ small enough and thus $g \equiv 0$.

Corollary 8 gives a continuous CR extension in the case where N has codimension 1 in M. In the case of codimension 1, there exist $\mathcal{C}^\infty$ nonconstant CR functions vanishing on N. Indeed, let g be a CR function obtained by Corollary 7 in M^+ (N cuts M in two). Then we may assume without loss of generality that $\mathrm{Re}(g) \geq 0$ on M^+. The desired smooth CR function is then obtained by

$$f(Z) = \begin{cases} e^{-1/g(Z)}, & Z \in M^+; \\ 0, & Z \in M \setminus M^+. \end{cases}$$

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