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Weil–Petersson Geometry for Families of Hyperbolic Conical Riemann Surfaces

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1. Introduction

Hyperbolic structures on weighted punctured Riemann surfaces have recently gained major attention. Hyperbolic metrics on weighted punctured Riemann surfaces have, by definition, conical singularities at the punctures; the cone angles are between 0 and 2π, corresponding to weights between 1 and 0. Conical metrics of constant negative curvature (with fixed weights) induce new structures on the Teichmüller spaces of punctured Riemann surfaces. Tan, Wong, and Zhang [28] showed the existence of corresponding Fenchel–Nielsen coordinates, proved a McShane identity for this case, and investigated the induced symplectic structure. In this way they generalized results of Mirzakani [18] to this situation (cf. [5]). Conical metrics on punctured spheres were studied by Takhtajan and Zograf in [27], who introduced Kähler structures on the moduli spaces depending on cone angles in the context of Liouville actions. From the algebraic geometry point of view, Hassett [8] introduced a hierarchy of compactifications of the moduli space of punctured Riemann surfaces according to the assigned weights of the punctures. These spaces interpolate between the classical Deligne–Mumford compactifications of the moduli spaces of Riemann surfaces with and without punctures. Conical hyperbolic metrics had been studied by Heins [9] and constructed by McOwen [16] and Troyanov [30] using the method of Kazdhan and Warner [13].

By definition, a weighted punctured Riemann surface $(X, \mathbf{a})$ is a compact Riemann surface X together with an $\mathbb{R}$-divisor $\mathbf{a} = \sum_{j=1}^{n} a_j p_j$ with weights $0 < a_j \leq 1$ at the punctures p_j. The necessary and sufficient condition for the existence of a hyperbolic conical metric according to [16; 30] is that the statement of the Gauss–Bonnet theorem holds—in other words, the degree of $K_X + \mathbf{a}$ is positive, where K_X denotes the canonical divisor of X. In this case the cone angles are $2\pi(1 - a_j)$.

Our aim is to study the Weil–Petersson geometry in the conical case and develop a theory parallel to the classical one. We show the existence of a Weil–Petersson Kähler form of class C^∞ that descends to the moduli space. Let $\mathcal{X} \to S$ be the universal family or any other holomorphic family of weighted punctured Riemann surfaces. It turns out that the classical Wolpert formula [32, Cor. 5.11] holds in our case as well; that is, the Weil–Petersson form is the push-forward of the form

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$2\pi^2 c_1({K_{\mathcal{X}/S}}^{-1}, g_{\mathbf{a}})$, where $({K_{\mathcal{X}/S}}^{-1}, g_{\mathbf{a}})$ is the relative anticanonical line bundle equipped with the family of hyperbolic conical metrics on the fibers. From this we derive the Kähler property of the Weil–Petersson metric.

For rational weights the bundle $K_{\mathcal{X}/S} + \mathbf{a}$ defines a determinant line bundle on the base space S, which carries a Quillen metric—according to the theorems of Quillen [20], Takhdajan and Zograf [26], and Bismut, Gillet, and Soulé [4]—once smooth metrics are chosen on $K_{\mathcal{X}/S} + \mathbf{a}$. We show that the conical metrics on the fibers induce a C^∞ metric on the determinant line bundle; such a metric descends to the moduli space. As in the classical case, its curvature is the generalized Weil–Petersson form.

We also prove the formula for the curvature tensor of the Weil–Petersson metric for Riemann surfaces with conical singularities. In the classical case the curvature was computed in [6; 21; 32]. Our formula holds for the case of weights $> 1/2$, which is also the range within which Fenchel–Nielsen coordinates exist. It includes also the case of orbifold singularities of degree $m > 2$.

Although hyperbolic conical metrics are well understood from the standpoint of hyperbolic geometry, the dependence upon holomorphic parameters poses essential difficulties. For this reason it was necessary in our previous paper [24] to introduce an ad hoc definition of harmonic Beltrami differentials on which a Weil–Petersson inner product could be based. Our present results are valid with no restrictions on the weights; in particular they include the interesting cases of weights between $1/2$ and 1, which arise in finite group quotients. Most results are known for cusps (i.e., punctures with zero cone angle), but our approach seems to be suitable only for positive cone angles so that we avoid mixed cases.

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2. Hyperbolic Conical Metrics

Let X be a compact Riemann surface with n punctures $p_1, \ldots, p_n$ and weights $0 < a_j \leq 1$ for $j = 1, \ldots, n$. We denote by $\mathbf{a} = \sum_j a_j p_j$ the corresponding $\mathbb{R}$-divisor and by $(X, \mathbf{a})$ the weighted punctured Riemann surface. We say that a hermitian metric of class C^∞ on the punctured Riemann surface has a cone singularity of weight $\mathbf{a}$ if, in a holomorphic local coordinate system centered at p_j, the metric is of the form $(\rho(z)/|z|^{2a_j})|dz|^2$ if $0 < a_j < 1$ and is of the form $(\rho(z)/|z|^2 \log^2(|1/z|^2))|dz|^2$ if $a_j = 1$. Here ρ is continuous at the puncture and positive. The cone angle is $2\pi(1 - a_j)$, including the complete case with angle zero. Let K_X be the canonical divisor of X; the weighted punctured Riemann surface $(X, \mathbf{a})$ is called *stable* if the the degree of the divisor $K_X + \mathbf{a}$ is positive. In this case, by a result of McOwen and Troyanov [16; 17; 31], in the given conformal class, there exists a unique conical metric $g_{\mathbf{a}}$ on X that has constant curvature -1 and prescribed cone angles. Moreover, $\mathrm{Vol}(X, g_{\mathbf{a}})/\pi = \deg(K_X + \mathbf{a}) = -\chi(X, \mathbf{a})$, where $\chi(X, \mathbf{a}) = \chi(X) - \sum a_j$ is by definition the Euler–Poincaré characteristic of the weighted punctured Riemann surface $(X, \mathbf{a})$.

At a noncomplete conical puncture we consider an emanating geodesic and see that, on a neighborhood of the puncture, the hyperbolic metric is isometric to a classical cone metric as obtained from the unit disk by removing a sector and identifying the resulting edges. So a posteriori a conical metric satisfies a somewhat stronger regularity condition than predicted in terms of the partial differential equation for hyperbolicity.

REMARK 2.1. Let $(X, \mathbf{a})$ be a weighted Riemann surface and p_j a puncture with $0 < a_j < 1$ for all $1 \le j \le n$. Then there exists a local coordinate function z near p_j such that $g_{\mathbf{a}} = (\rho(z)/|z|^{2a_j})|dz|^2$, where $\rho(z) = \eta(|z|^{2(1-a_j)})$ for some positive, real analytic function η.

The dependence of the hyperbolic cone metrics on the weights is characterized as follows.

PROPOSITION 2.2. *Let $a_j(k)$ be an increasing sequence of weights with $\mathbb{R}$-divisors $\mathbf{a}(k)$ on X. Suppose that* $\deg(K_X + \mathbf{a}(k)) > 0$ *for all $k \in \mathbb{N}$ and that $a_j(k) \to a_j$ as $k \to \infty$. Then $g_{\mathbf{a}(k)}$ converges to $g_{\mathbf{a}}$ uniformly on compact sets away from the punctures. Moreover, the sequence of functions $g_{\mathbf{a}(k)}/g_{\mathbf{a}}$ converges to the constant function* 1 *in $L^1(X, g_{\mathbf{a}})$.*

Proof. In [24, Prop. 2.4] we defined $\Psi_k = g_{\mathbf{a}(k)}/g_{\mathbf{a}}$; then $0 < \Psi_k \le \Psi_{k+1} \le 1$, as we proved there, and $-\log(\Psi_k)$ is a decreasing sequence of subharmonic functions on the complement of the punctures. Therefore $-\log(\Psi_k)$ converges pointwise to a subharmonic function $\delta \ge 0$ on the complement of the punctures. By [24, Prop. 2.5], the function δ is identically equal to 0 in a neighborhood of each puncture p_j with $a_j < 1$. Moreover, if $a_j < 1$ for all j then $\delta \equiv 0$ and the convergence is uniform on compact sets by Dini's lemma. (Observe that the argument in the proof [24, Prop. 2.5] is local.) Suppose that $a_{j_0} = 1$ for some j_0, and consider the functions $\delta_k = -\log(\Psi_k) + (1 - a(k)_{j_0}) \log(|z|^2)$ on an open neighborhood $\mathcal{U}_{j_0}$ of p_{j_0}. By the local expression of each function Ψ_k near p_{j_0}, we have that the functions δ_k are subharmonic and uniformly bounded from above, so each function δ_k extends to a subharmonic function on $\mathcal{U}_{j_0}$; moreover, the function δ', which is the upper semicontinuous envelope of $\limsup \delta_k$, is also subharmonic on $\mathcal{U}_{j_0}$ (cf. [14]). Hence $\delta = \delta'$ on $\mathcal{U}_{j_0} \backslash \{0\}$. In other words, the function δ extends to a subharmonic function on all of X, and therefore $\delta \equiv c$ is constant. By the dominated convergence theorem, the sequence Ψ_k converges to e^{-c} in $L^1(X, g_{\mathbf{a}})$. Since $\mathrm{Vol}(g_{\mathbf{a}(k)})$ converges to $\mathrm{Vol}(g_{\mathbf{a}})$, we have $e^{-c} = 1$. □

We consider the classical Teichmüller space $\mathcal{T}_{\gamma,n}$ of (marked) Riemann surfaces of genus γ with punctures $p_1, \ldots, p_n$. We denote by $\Pi \colon \mathcal{X}_{\gamma,n} \to \mathcal{T}_{\gamma,n}$ the universal family. The punctures on the fibers $\mathcal{X}_s = \Pi^{-1}(s)$ are given by n holomorphic sections $\sigma_1(s), \ldots, \sigma_n(s)$, $s \in \mathcal{T}_{\gamma,n}$, where for all s the values are pairwise distinct. Constant weights $0 < a_j \le 1$ are assigned to the $\sigma_j(s)$, and the corresponding real divisors are denoted by $\mathbf{a}(s) = \sum_{j=1}^{n} a_j \sigma_j(s)$. The resulting family of weighted punctured surfaces is denoted by $\Pi \colon (\mathcal{X}_{\gamma,n}, \mathbf{a}) \to \mathcal{T}_{\gamma,n}$. We assume that the fibers are stable and endowed with the hyperbolic conical metrics $g_{\mathbf{a}}(s)$. The complete case of weights 1 is well understood and, since the essential arguments will be local, we may assume that $0 < a_j < 1$ holds for all weights.

We will show that the conical hyperbolic metrics define new Kähler structures on the Teichmüller and moduli spaces of punctured Riemann surfaces depending on the assigned weights.

For short we will write $\Pi\colon \mathcal{X} \to S$ for any holomorphic family of punctured Riemann surfaces over a complex manifold S with holomorphic sections $\sigma_i(s)$. Our arguments will be local with respect to the base. When considering the variation of conical metrics and defining the induced hermitian structure on the Teichmüller space, we may assume that $S = \{s \in \mathbb{C} : |s| < 1\}$.

Denote by X the central fiber $\mathcal{X}_0$. In order to introduce Sobolev spaces and to use the theory of elliptic equations depending upon parameters [2], we need to fix a differentiable trivialization of the family. Our method of choice is as follows.

After shrinking S if necessary, on neighborhoods of each holomorphic section σ_j in $\mathcal{X}$ we take holomorphic coordinates $W_j \equiv \mathcal{U}_j \times S = \{(z,s)\}$ such that $\sigma_j(s) \equiv 0$. Assuming that these coordinates also exist on slightly larger neighborhoods, we can use a differentiable trivialization $\Psi\colon \mathcal{X} \to X \times S$, which is holomorphic on W_j and respects the coordinates just described. The map Ψ defines a differentiable lift

$$V_0 = \frac{\partial}{\partial s} + b_1(z,s)\frac{\partial}{\partial z} + b_2(z,s)\frac{\partial}{\partial \bar z}$$

of the vector field $\frac{\partial}{\partial s}$ on S such that $V_0|_{W_j} = \frac{\partial}{\partial s}$. We introduce Sobolev spaces $H^p_k(\mathcal{X}_s)$ defined with respect to the measure induced by a smooth family $g_0(s)$ of differentiable background metrics. We identify $H^p_k(\mathcal{X}_s)$ with $H^p_k(X)$ by the preceding differentiable trivialization.

Set

$$g_{\mathbf{a}} = e^u g_0,$$

where $g_{\mathbf{a}}(s) = g_{\mathbf{a}}(s,z)|dz|^2$ and $g_0(s) = g_0(s,z)|dz|^2$ in local coordinates. The functions u carry the singularities.

As in [24, Sec. 4], for $1 \le j \le n$ we introduce a function $\Psi_j(z,s)$ that is smooth on the complement of the punctures and is of the form $\Psi_j = -\log(|z|^2|)$ on $\mathcal{U}_j$. (Here we use our assumption that $\sigma_j(s) \equiv 0$.) Let us define

$$w(z,s) = u - \sum_j a_j \Psi_j.$$

Let $\Delta = \frac{1}{g_0}\frac{\partial}{\partial z \partial \bar z}$ denote the Laplacian with respect to the smooth background metric g_0. Then the equation for hyperbolicity reads

$$\Delta u - e^u = K_{g_0}, \tag{1}$$

where K_{g_0} is the Ricci curvature of g_0; that is,

$$K_{g_0}(s,z) = -\frac{1}{g_0(s,z)} \cdot \frac{\partial^2 \log(g_0)}{\partial z \partial \bar z}.$$

Now equation (1) reads

$$\Delta w - \left(\exp\left(\sum_i a_i \Psi_i\right)\right) e^w = K - \Delta\left(\sum_i a_i \Psi_i\right),$$

and on $\mathcal{U}_j \setminus \{z = 0\}$ it is of the form

$$\Delta w - e^{M(z)} \frac{e^w}{|z|^{2a_j}} = K,$$

where the function $M(z) = \sum_{i \neq j} a_i \Psi_i$ is smooth and bounded on $\mathcal{U}_j$.

It follows that $w(s) \in H_2^p(\mathcal{X}_s)$ for all $1 \leq p < \min(1/a_j)$ (cf. [16]), and by standard regularity theory the solutions are of class C^∞ on the complement of the punctures.

Our aim is to show that the conical metrics depend differentiably on the parameters in a suitable sense. Given a family $(\mathcal{X}, \mathbf{a}) \to S$, we write the hyperbolic metrics as

$$g_{\mathbf{a}} = \exp(a_1\Psi_1 + \cdots + a_n\Psi_n + w) g_0$$

and fix a differentiable trivialization $\mathcal{X} \to X \times S$ in the above sense.

THEOREM 2.3. *Fix a real number $1 \leq p < \min(1/a_j)$. Then the assignment $s \mapsto w(s)$ defines a map $w\colon S \to H_2^p(X)$ that is of class C^∞; that is, all higher derivatives of w with respect to V_0 and $\bar{V}_0$ exist in $H_2^p(X)$ and depend in a C^∞ way on s. In particular, since $H_2^p(X) \subseteq C^0(X)$, it follows that for any fixed $z \in X$ the function $s \mapsto w(z, s)$ is of class C^∞.*

Proof. Since the argument is local, we may assume $n = 1$ for simplicity. We define a C^1 map $\Phi\colon S \times H_2^p(X) \to L^p(X)$ by

$$\Phi(s, w) = \Delta_{g_0(s)}(w) - e^{a\Psi(s)} e^w - K_{g_0(s)} + a\Delta_{g_0(s)}(\Psi(s)).$$

It is important to note that the given trivialization is holomorphic in a neighborhood of the punctures and that $\Psi(z, s) = -\log(|z|^2)$ does not depend on s. Therefore, the map Φ is of class C^1. We now indicate how to compute $(D_1\Phi)(s_0, w_0) \in L^p(X)$. We have

$$\begin{aligned}(D_1\Phi)(s_0, w_0) = {} & \frac{-\partial \log g_0(s_0)}{\partial s} \Delta_{g_0(s_0)}(w_0) - a \frac{\partial \Psi(s_0)}{\partial s} e^{a\Psi(s_0)} e^{w_0} \\ & - \frac{\partial K_{g_0}(s_0)}{\partial s} + \frac{\partial}{\partial s}\big(\Delta_{g_0(s_0)}(\Psi(s, -))\big).\end{aligned}$$

Note that this function belongs to $L^p(X)$ because $\Delta_{g(s_0)}(w_0) \in L^p(X)$ and $\frac{\partial \Psi}{\partial s} = \Delta_{g(s_0)}(\Psi) \equiv 0$ near the puncture for all $s \in S$. Moreover, both of the functions $\frac{\partial \log g_0}{\partial s}\Big|_{s_0}$ and $\frac{\partial K_{g_0}}{\partial s}\Big|_{s_0}$ are bounded. Now

$$(D_2\Phi)(s_0, w_0)(W)\colon H_2^p(X) \to L^p(X)$$

is given by

$$(D_2\Phi)(s_0, w_0)(W) = \Delta_{g_0(s_0)}(W) - e^{a\Psi(s_0)} e^{w_0} W.$$

By [24, Lemma 2.1], the implicit function theorem is applicable. Since all derivatives of $e^{a\Psi}$ with respect to s and $\bar{s}$ are in $L^p(X)$, it is possible to repeat the argument and thereby demonstrate the rest of the statement. □

REMARK 2.4. The methods just described can also be used to show that an analogous statement is true for the dependence of conical metrics on the weights—provided these are less than 1. For $\mathbf{a} = \sum p_j$ we have the statement of Proposition 2.2.

3. The Generalized Weil–Petersson Metric

The classical Weil–Petersson metric is defined as the L^2 inner product of harmonic Beltrami differentials with respect to the hyperbolic metrics on the fibers. For reasons that will become apparent later, we first introduce the Weil–Petersson metric on the cotangent space.

Let $(X, \mathbf{a})$ be a weighted punctured Riemann surface with $\mathbf{a} = \sum a_j p_j$. We set $D = \sum p_j$ and denote by

$$H^0(X, \Omega^2{}_{(X,\mathbf{a})}) = H^0(X, \Omega^2_X(D))$$

the space of holomorphic quadratic differentials with at most simple poles at the punctures, which is identified with the cotangent space of the corresponding Teichmüller space of punctured Riemann surfaces at the given point.

DEFINITION 3.1. The Weil–Petersson inner product

$$G^*_{\mathrm{WP},\mathbf{a}} \text{ on } H^0(X, \Omega^2{}_{(X,\mathbf{a})})$$

is given by

$$\langle \phi, \psi \rangle_{\mathrm{WP},\mathbf{a}} = \int_X \frac{\phi \bar{\psi}}{g_{\mathbf{a}}^2}\, dA_{\mathbf{a}},$$

where $g_{\mathbf{a}}$ is the hyperbolic conical metric with surface element $dA_{\mathbf{a}}$.

Observe that the integrals in this definition are finite because $0 \le a_j \le 1$ for all i.

The Weil–Petersson inner products depend continuously on the weights if these weights are less than 1 (cf. Remark 2.4). Under the hypotheses of Proposition 2.2, we have the following statement.

COROLLARY 3.2. *Let*

$$\phi \in H^0(X, \Omega^2{}_{(X,\mathbf{a})});$$

then

$$\lim_k |\phi|^2_{\mathrm{WP}, g_{\mathbf{a}}(k)} = |\phi|^2_{\mathrm{WP}, g_{\mathbf{a}}}.$$

Proof. Fix a reference smooth metric g_0 on X. Then $|\phi|^2/g_{\mathbf{a}(k)}$ is a decreasing sequence of g_0 integrable positive functions converging to $|\phi|^2/g_{\mathbf{a}}$. □

Observe that harmonicity of Beltrami differentials in the first place means that a certain partial differential equation holds. In the case of compact Riemann surfaces (and punctured surfaces equipped with complete hyperbolic metrics), L^2-theory implies that any Beltrami differential has a unique harmonic representative, which is the quotient of a conjugate holomorphic quadratic differential by the metric tensor.

We use an ad hoc definition of the space of *harmonic Beltrami differentials* for $(X, \mathbf{a})$ with respect to the hyperbolic conical metric $g_{\mathbf{a}}$; it coincides with the usual definition in the classical case of weights 1. Let $X' = X \setminus \{p_1, \ldots, p_n\}$.

DEFINITION 3.3. Let $g_{\mathbf{a}} = g_{\mathbf{a}}(z)\, dz\, \overline{dz}$ be the hyperbolic conical metric on $(X, \mathbf{a})$. If $\phi = \phi(z)\, dz^2 \in H^0(X, \Omega^2{}_{(X,\mathbf{a})})$ is a quadratic holomorphic differential, we call the Beltrami differential

$$\mu = \mu(z) \frac{\partial}{\partial z}\, \overline{dz} = \frac{\overline{\phi(z)}}{g_{\mathbf{a}}(z)} \frac{\partial}{\partial z}\, \overline{dz}$$

on X' *harmonic* on $(X, \mathbf{a})$ and denote the vector space of all such differentials by $H^1(X, \mathbf{a})$.

PROPOSITION 3.4. *For* $0 < a_j < 1$, *the space of harmonic Beltrami differentials* $H^1(X, \mathbf{a})$ *on* $(X, \mathbf{a})$ *can be identified with the cohomology* $H^1(X, \Theta_X(-D))$, *where* Θ_X *is the sheaf of holomorphic vector fields on X and* $D = \sum_j p_j$.

Proof. It is sufficient to verify that a duality

$$\Phi\colon H^0(X, \Omega^2{}_{(X,\mathbf{a})}) \times H^1(X, \mathbf{a}) \to \mathbb{C}$$

is defined by

$$\Phi\left(\phi(z)\, dz^2, \mu(z) \frac{\partial}{\partial z}\, \overline{dz}\right) = \int_X \phi(z) \mu(z)\, dz\, d\bar{z}. \qquad \square$$

The Weil–Petersson metric on the cotangent space to $\mathcal{T}_{\gamma,n}$, together with the duality just described, defines a Weil–Petersson metric $G_{\mathrm{WP},\mathbf{a}}$ on the tangent space identified with $H^1(X, \mathbf{a})$.

Let μ_1 and μ_2 be in $H^1(X, \mathbf{a})$. Then

$$\langle \mu_1, \mu_2 \rangle_{\mathrm{WP},\mathbf{a}} = \int_X \mu_1 \overline{\mu_2}\, dA_{\mathbf{a}}$$

(cf. [24, Lemma 3.4]).

If $1/2 \le a_j \le 1$ then the Fenchel–Nielsen coordinates can be defined [28]; it is shown in [3] that, in this case, the Fenchel–Nielsen symplectic form coincides with the Weil–Petersson Kähler form. The generalized Weil–Petersson metric can be defined on the Teichmüller space $\mathcal{T}_{\gamma,n}$ of surfaces of genus γ with n punctures. From [24, Prop. 2.4] we know that if $\mathbf{a} \le \mathbf{b}$ then $g_{\mathbf{a}} \le g_{\mathbf{b}}$; hence $G^*_{\mathrm{WP},\mathbf{b}} \le G^*_{\mathrm{WP},\mathbf{a}}$ and, for the metrics on the dual spaces, we have $G_{\mathrm{WP},\mathbf{a}} \le G_{\mathrm{WP},\mathbf{b}}$. Therefore, if $\mathbf{a} \le \mathbf{b}$ then the identity map from $(\mathcal{T}_{\gamma,n}, G_{\mathrm{WP},\mathbf{b}})$ to $(\mathcal{T}_{\gamma,n}, G_{\mathrm{WP},\mathbf{a}})$ is distance decreasing.

Since the conical metrics are intrinsically defined on the fibers, the classical mapping class group $\Gamma_{\gamma,n}$ acts on Teichmüller spaces as a group of isometries for both the classical and the generalized Weil–Petersson metrics; hence also the generalized Weil–Petersson metric descends to $\mathcal{M}_{\gamma,n}$. Let us define $\overline{\mathcal{M}}_{\gamma,\mathbf{a}}$ as the completion of the moduli space $\mathcal{M}_{\gamma,n}$ with respect to the distance defined by the generalized metric. Therefore the identity map descends to a distance-decreasing map of the moduli spaces, and such a map extends to a continuous map

$$j_{\mathbf{b},\mathbf{a}}\colon \overline{\mathcal{M}}_{\gamma,\mathbf{b}} \to \overline{\mathcal{M}}_{\gamma,\mathbf{a}}.$$

Moreover, let $\mathbf{b} = (\mathbf{b}', \mathbf{b}'')$ and $\mathbf{b}^* = (\mathbf{b}', 0)$, where $\mathbf{b}' \in [0,1]^m$. Denote by $F\colon \mathcal{T}_{\gamma,n} \to \mathcal{T}_{\gamma,m}$ the holomorphic map that forgets the punctures $\mathbf{b}''$. Then, by [24, Thm. 3.5], $G_{\mathrm{WP},\mathbf{b}^*}$ coincides with the degenerate metric $F^*(G_{\mathrm{WP},\mathbf{b}'})$. The map

$$F\colon (\mathcal{M}_{\gamma,n}, F^*(G_{\mathrm{WP},\mathbf{b}'})) \to (\mathcal{M}_{\gamma,m}, G_{\mathrm{WP},\mathbf{b}'})$$

is also obviously (pseudo)distance decreasing; and, since $\mathbf{b} \geq \mathbf{b}^*$, so is the map $F = F \circ \mathrm{id}\colon (\mathcal{M}_{\gamma,n}, G_{\mathrm{WP},\mathbf{b}})) \to (\mathcal{M}_{\gamma,m}, G_{\mathrm{WP},\mathbf{b}'})$.

Therefore we also have the continuous map-forgetting punctures

$$F_{\mathbf{b},\mathbf{b}'}\colon \overline{\mathcal{M}}_{\gamma,\mathbf{b}} \to \overline{\mathcal{M}}_{\gamma,\mathbf{b}'}.$$

COROLLARY 3.5. *The space $\overline{\mathcal{M}}_{\gamma,\mathbf{a}}$ is a compactification of the moduli space $\mathcal{M}_{\gamma,n}$. In particular, the generalized Weil–Petersson metric is not complete.*

Proof. The usual Deligne–Mumford compactification of $\mathcal{M}_{\gamma,n}$ is the quotient by the mapping class group of the Weil–Petersson metric completion of Teichmüller space (see e.g. [15; 33]); hence it is the completion of $\mathcal{M}_{\gamma,n}$. Therefore, if $\mathbf{1} = (1,\dots,1)$ then $j_{\mathbf{1},\mathbf{a}}(\overline{\mathcal{M}}_{\gamma,\mathbf{1}}) \subseteq \overline{\mathcal{M}}_{\gamma,\mathbf{a}}$ is compact and dense, so that the map $j_{\mathbf{1},\mathbf{a}}$ is onto and $\overline{\mathcal{M}}_{\gamma,\mathbf{a}}$ is compact.

4. The Kodaira–Spencer Map and Conical Metrics

First, we briefly describe the close relationship of variations of hyperbolic metrics and harmonic Beltrami differentials in the classical case of holomorphic families of compact Riemann surfaces (cf. [22]).

Let $f\colon \mathcal{X} \to S$ be such a family. Let $s_0 \in S$ be a distinguished point and $X = f^{-1}(s_0)$ its fiber. The map induces a short exact sequence involving the sheaf $\mathcal{T}_{\mathcal{X}/S}$ of holomorphic vector fields in fiber direction, the sheaf of holomorphic vector fields $\mathcal{T}_{\mathcal{X}}$ on the total space, and the corresponding pull-back:

$$0 \to \mathcal{T}_{\mathcal{X}/S} \to \mathcal{T}_{\mathcal{X}} \to f^*\mathcal{T}_S \to 0.$$

The connecting homomorphism

$$\rho\colon T_{s_0} \to H^1(X, \mathcal{T}_X)$$

is the Kodaira–Spencer map, which in fact assigns to a tangent vector the cohomology class of the corresponding Beltrami differential.

In terms of Dolbeault cohomology, this map can be described as follows. Let $\partial/\partial s$ stand for a tangent vector on the base at s_0. Let V be any differentiable lift of the tangent vector to the total space $\mathcal{X}$ (along the fiber X).

PROPOSITION 4.1. *The restriction $\bar\partial V|_X$ is $\bar\partial$-closed and represents $\rho\big(\frac{\partial}{\partial s}\big|_{s_0}\big)$.*

Now the fibers $\mathcal{X}_s$ of the family are equipped with the hyperbolic metrics $g(z,s)|dz|^2$, which depend in a differentiable way on the parameter s. The collection of these metrics is considered a relative volume form on $\mathcal{X}$; that is, for a metric on the relative tangent bundle of $\mathcal{X}$, its dual is a hermitian metric on the relative canonical bundle $\mathcal{K}_{\mathcal{X}/S}$. Let

$$\omega_{\mathcal{X}} = \frac{\sqrt{-1}}{2}\partial_{\mathcal{X}}\bar\partial_{\mathcal{X}} \log(g(z,s))$$

be its curvature form.

LEMMA 4.2. *The restrictions of $\omega_{\mathcal{X}}$ to the fibers $\mathcal{X}_s$ equal the Kähler forms $\omega_{\mathcal{X}_s} = \frac{\sqrt{-1}}{2} g(z,s)\, dz \wedge \overline{dz}$.*

In particular, the real $(1,1)$-form $\omega_{\mathcal{X}}$ is positive definite along the fibers. So the *horizontal lift* V_{hor} of $\frac{\partial}{\partial s}$, which by definition consists of tangent vectors that are perpendicular to the fibers and project to the given tangent vector, is well-defined as follows.

LEMMA 4.3.

$$V_{\mathrm{hor}} = \frac{\partial}{\partial s}\bigg|_{s_0} + a^z \frac{\partial}{\partial z},$$

where

$$a^z = -\frac{1}{g}\frac{\partial^2 \log g(z,s_0)}{\partial s \partial \bar z}.$$

The lemma follows immediately from the computation of the inner product of V_{hor} and $\frac{\partial}{\partial z}$ with respect to $\omega_{\mathcal{X}}$.

So far, general theory implies the following.

PROPOSITION 4.4. *The harmonic Beltrami differential corresponding to the tangent vector $\frac{\partial}{\partial s}\big|_{s_0}$ is induced by the horizontal lift. It equals*

$$\mu = \mu(z)\frac{\partial}{\partial z}\, d\bar z = \frac{\partial a^z}{\partial \bar z}\frac{\partial}{\partial z}\, d\bar z = -\frac{\partial}{\partial \bar z}\left(\frac{1}{g}\frac{\partial^2 \log g(z,s_0)}{\partial s \partial \bar z}\right)\frac{\partial}{\partial z}\, d\bar z.$$

In fact, a straightforward verification shows that $g(z,s_0)\overline{\mu(z)}$ is a holomorphic quadratic differential; that is, μ is *harmonic* with respect to the hyperbolic metric on X.

Now let $(\mathcal{X},\mathbf{a}) \to S$ be a holomorphic family of weighted Riemann surfaces with $0 < a_j < 1$ and with central fiber $X = \mathcal{X}_{s_0}$, $s_0 \in S$. This section is concerned with how to recover the Kodaira–Spencer map $\rho\colon T_{s_0}(S) \to H^1(X,\mathbf{a})$ from the family of conical hyperbolic metrics $g_{\mathbf{a}}$.

In the case of conical hyperbolic metrics, we define the Beltrami differential given by

$$\mu_{\mathbf{a}}\left(\frac{\partial}{\partial s}\right) = -\frac{\partial}{\partial \bar z}\left(\frac{1}{g_{\mathbf{a}}}\frac{\partial^2 \log g_{\mathbf{a}}}{\partial \bar z \partial s}\right)\frac{\partial}{\partial z}\, d\bar z \tag{2}$$

and the quadratic differential $\phi_{\mathbf{a}}\big(\frac{\partial}{\partial s}\big) = g_{\mathbf{a}}\overline{\mu_{\mathbf{a}}}\big(\frac{\partial}{\partial s}\big)$.

In order to prove that the Beltrami differential $\mu_{\mathbf{a}}\left(\frac{\partial}{\partial s}\right)$ in (2) is harmonic in the sense of Definition 3.3, it is sufficient to show the following.

LEMMA 4.5. *$\phi_{\mathbf{a}}\left(\frac{\partial}{\partial s}\right)$ is in $L^1(X)$.*

Proof. Again we use the special coordinates for the family near the punctures. For simplicity we assume $n = 1$ and set $0 < a = a_1 < 1$ and $g_a = g_{\mathbf{a}}$. We have

$$\phi_{\mathbf{a}}\left(\frac{\partial}{\partial s}\right) = \frac{\partial \log g_a}{\partial z} \cdot \frac{\partial^2 \log g_a}{\partial z \partial \bar{s}} - \frac{\partial^3 \log g_a}{\partial z^2 \partial \bar{s}}.$$

Moreover, in local coordinates the following equation holds:

$$\log(g_a) = \log(g_0) + w - a \log(|z|^2). \tag{3}$$

Now, by Theorem 2.3, for $1 \le p < 1/a$ we have that

$$\frac{\partial w}{\partial z}, \frac{\partial^2 w}{\partial \bar{s} \partial z} \in H_1^p(\mathcal{U}_1)$$

whereas

$$\frac{\partial^3 w}{\partial \bar{s} \partial z^2} \in L^p(\mathcal{U}_1).$$

Therefore, by equation (3),

$$\frac{\partial^3 \log g_{\mathbf{a}}}{\partial z^2 \partial \bar{s}} \in L^1(\mathcal{U}_1).$$

Moreover, $1/z \in L^q(\mathcal{U}_1)$ and so

$$\frac{\partial \log g_{\mathbf{a}}}{\partial z} \in L^q(\mathcal{U}_1) \quad \text{for } 1 \le q < 2.$$

By the Sobolev embedding theorem, $H_1^p(\mathcal{U}_1) \subseteq L^h(\mathcal{U}_1)$ for all $h < p'$, where $p' = \frac{2p}{2-p}$ for $1 \le p < 2$ and $p' = \infty$ for $p \ge 2$. It follows that

$$\frac{\partial^2 \log g_{\mathbf{a}}}{\partial z \partial \bar{s}} \in L^h(\mathcal{U}_1) \quad \text{for } 1 \le h < \infty \text{ if } 0 < a \le 1/2$$

and

$$\frac{\partial^2 \log g_{\mathbf{a}}}{\partial z \partial \bar{s}} \in L^h(\mathcal{U}_1) \quad \text{for } 1 \le h < \frac{1}{a - 1/2} > 2 \text{ if } 1/2 < a < 1.$$

Hence, for $0 < a < 1$,

$$\frac{\partial \log g_{\mathbf{a}}}{\partial z} \cdot \frac{\partial^2 \log g_{\mathbf{a}}}{\partial z \partial \bar{s}} \in L^1(\mathcal{U}_1). \qquad \square$$

So far we only showed that, on one hand, $H^1(X, \mathbf{a})$ is the space of infinitesimal deformations and that, on the other hand, the variation of hyperbolic conical metrics gives rise to element of this space according to (2). If this assignment is injective for effective families, then we have recovered the Kodaira–Spencer map.

THEOREM 4.6. *The Kodaira–Spencer map* $\rho\colon T_{s_0}S \to H^1(X,\mathbf{a})$ *is given by*

$$\rho\left(\frac{\partial}{\partial s}\right) = \mu_{\mathbf{a}}\left(\frac{\partial}{\partial s}\right) = -\frac{\partial}{\partial \bar{z}}\left(\frac{1}{g_{\mathbf{a}}}\frac{\partial^2 \log(g_{\mathbf{a}}(z,s))}{\partial\bar{z}\partial s}\right)\bigg|_{s=s_0}\frac{\partial}{\partial z}\,d\bar{z},$$

where $\frac{\partial}{\partial s}$ *stands for a tangent vector.*

Proof. We may assume that S is a disk and that we only have one puncture. If $0 < a < 1/2$ then the proof of the theorem is given in [24, Thm. 5.4], so we suppose $1/2 \le a < 1$. Let $\mu_{\mathbf{a}}\left(\frac{\partial}{\partial s}\right) \equiv 0$. Then the locally defined quantity $\frac{1}{g_a}\frac{\partial^2 \log(g_a(z,s))}{\partial\bar{z}\partial s}\Big|_{s=s_0}$ is holomorphic outside the punctures, and the vector field

$$W_{s_0} = \frac{\partial}{\partial s} + \gamma(z)\frac{\partial}{\partial z} = \frac{\partial}{\partial s} - \left(\frac{1}{g_a}\frac{\partial^2 \log(g_a(z,s))}{\partial\bar{z}\partial s}\bigg|_{s=s_0}\right)\frac{\partial}{\partial z}$$

is a lift of the tangent vector $\frac{\partial}{\partial s}$ that is holomorphic outside the punctures. We know from the proof of Lemma 4.5 that $\frac{\partial^2 \log(g_a(z,s))}{\partial\bar{z}\partial s}\Big|_{s=s_0}$ is in $H_1^p(\mathcal{U}_1) \subseteq L^2(\mathcal{U}_1)$ for some $p > 1$. Since $\frac{1}{g_a}$ is bounded, the function $\frac{1}{g_a}\frac{\partial^2 \log(g_a(z,s))}{\partial\bar{z}\partial s}\Big|_{s=s_0}$ is also in $L^2(\mathcal{U}_1)$; hence the vector field is holomorphic on the compact surface. So the holomorphic structure of the corresponding compact Riemann surfaces is infinitesimally constant. However, the puncture need not be kept fixed. Given the choice of local coordinates, we need to show that the vector field W_{s_0} equals $\frac{\partial}{\partial s}$ at $z = 0$. We have already observed that $\frac{\partial^2 \log(g_a(z,s))}{\partial\bar{z}\partial s}\Big|_{s=s_0} = \frac{\rho(z)}{|z|^{2a}}\gamma(z)$ is in $L^2(\mathcal{U}_1)$. However, for $1/2 \le a < 1$, the function $\frac{1}{|z|^{2a}}$ is not in $L^2(\mathcal{U}_1)$; hence $\gamma(s_0) = 0$. □

5. Horizontal Lifts of Tangent Vectors

Let $f\colon (\mathcal{X},\mathbf{a}) \to S$ be the universal holomorphic family of weighted Riemann surfaces over the Teichmüller space or, for computational purposes, a family over the disk. Observe that—as in the classical case—the family of conical metrics will give rise to a C^∞ closed, real $(1,1)$-form

$$\omega_{\mathcal{X}} = \frac{\sqrt{-1}}{2}\partial_{\mathcal{X}}\bar{\partial}_{\mathcal{X}}\log(g_{\mathbf{a}})$$

on the complement of the punctures, which is positive when restricted to the fibers.

Assume that $1 < a_j < 1$ for $1 \le j \le n$. Let $S = \{s \in \mathbb{C} : |s| < 1\}$ and denote by $X = \mathcal{X}_0$ the central fiber. As in Section 2, we use a differentiable trivialization of the family so that the Sobolev spaces of the fibers can be identified.

We will denote the coefficients of $\omega_{\mathcal{X}}$ by

$$g_{\mathbf{a}_{s\bar{s}}} = \frac{\partial^2 \log g_{\mathbf{a}}(z,s)}{\partial s\overline{\partial s}}, \tag{4}$$

$$g_{\mathbf{a}_{s\bar{z}}} = \frac{\partial^2 \log g_{\mathbf{a}}(z,s)}{\partial s\overline{\partial z}}, \tag{5}$$

$$g_{\mathbf{a}_{z\bar{s}}} = \frac{\partial^2 \log g_{\mathbf{a}}(z,s)}{\partial z \partial \overline{s}}, \tag{6}$$

$$g_{\mathbf{a}_{z\bar{z}}} = \frac{\partial^2 \log g_{\mathbf{a}}(z,s)}{\partial z \partial \overline{z}}. \tag{7}$$

As already pointed out, hyperbolicity translates into

$$g_{\mathbf{a}_{z\bar{z}}} = g_{\mathbf{a}}. \tag{8}$$

As in Lemma 4.3 we have that the horizontal lift of $\frac{\partial}{\partial s}$ is given by

$$V = \frac{\partial}{\partial s} + a^z(z)\frac{\partial}{\partial z}$$

with

$$a^z = \frac{-1}{g_{\mathbf{a}}} g_{\mathbf{a}_{s\bar{z}}}. \tag{9}$$

The function

$$\chi = g_{\mathbf{a}_{s\bar{s}}} - \frac{1}{g_{\mathbf{a}}} g_{\mathbf{a}_{s\bar{z}}} g_{\mathbf{a}_{z\bar{s}}} = g_{\mathbf{a}_{s\bar{s}}} - g_{\mathbf{a}} a^z(z)\overline{a^z(z)} \tag{10}$$

has various geometric meanings, as described in the following proposition.

PROPOSITION 5.1. *Let $\mu_{\mathbf{a}} \in H^1(\mathcal{X}_{s_0}, \mathbf{a})$ be the harmonic Beltrami differential according to* (2). *Then*

$$\chi = \|V\|^2_{\omega_{\mathcal{X}}}, \tag{11}$$

$$\omega^2_{\mathcal{X}} = \left(\frac{\sqrt{-1}}{2}\right)^2 \chi(z,s) g_{\mathbf{a}}(z,s)\, dz \wedge d\bar{z} \wedge ds \wedge d\bar{s}, \tag{12}$$

$$|\mu_{\mathbf{a}}|^2 = (-\Delta_{g_{\mathbf{a}}} + \mathrm{id})\chi. \tag{13}$$

Proof. For simplicity we will drop the index **a** and will set $\partial_s = \partial/\partial s$, $\partial_z = \partial/\partial z$, and so forth. The first claim follows from

$$\|V\|^2_{\omega_{\mathcal{X}}} = \langle \partial_s + a^z \partial_z, \partial_s + a^z \partial_z \rangle = g_{s\bar{s}} + a^z g_{z\bar{s}} + \overline{a^z} g_{s\bar{z}} + a^z \overline{a^z} g_{z\bar{z}}$$

by (9) and (8). Equation (12) follows from

$$\chi \cdot g = \chi \cdot g_{z\bar{z}} = \det\begin{pmatrix} g_{s\bar{s}} & g_{s\bar{z}} \\ g_{\bar{z}s} & g_{z\bar{z}} \end{pmatrix}.$$

The proof of (13) will require the following preparations.

In order to compute integrals over the fibers involving certain tensors, we will use covariant differentiation with respect to the hyperbolic metrics on the fibers and use the semicolon notation. For derivatives in the s direction we will use the flat connection.

First, we note that

$$\begin{aligned} g^2 \cdot g_{s\bar{s}} &= g^2 \cdot (\log g)_{;s\bar{s}} = g \cdot g_{;s\bar{s}} - g_{;s} g_{;\bar{s}} = g \cdot g_{;s\bar{s}} - g_{z\bar{z};s} g_{z\bar{z};\bar{s}} \\ &= g \cdot g_{;s\bar{s}} - g_{s\bar{z};z} g_{z\bar{s};\bar{z}} = g \cdot g_{;s\bar{s}} - g^2 \cdot a^z{}_{;z} \overline{a^z}{}_{;\bar{z}}. \end{aligned}$$

In short,

$$\frac{1}{g} g_{;s\bar{s}} = g_{s\bar{s}} + a^z{}_{;z} \overline{a^z}_{;\bar{z}}.$$

We combine this with

$$g_{s\bar{s};z\bar{z}} = (\log g)_{;s\bar{s}z\bar{z}} = (\log g)_{;z\bar{z}s\bar{s}} = g_{;s\bar{s}}$$

and get

$$\begin{aligned}\Delta_g(\chi) &= \frac{1}{g}(g_{s\bar{s}} - g \cdot a^z \overline{a^z})_{;z\bar{z}} = \frac{1}{g} g_{;s\bar{s}} - (a^z \overline{a^z})_{;z\bar{z}} \\ &= g_{s\bar{s}} - a^z{}_{;\bar{z}} \overline{a^z}_{;z} - a^z{}_{z\bar{z}} \overline{a^z} - a^z \overline{a^z}_{;z\bar{z}}.\end{aligned}$$

We know that

$$\mu(z) = a^z{}_{;\bar{z}};$$

hence

$$\overline{a^z}_{;z\bar{z}} = \overline{\mu(z)}_{;\bar{z}} = \left(\frac{\varphi(z)}{g}\right)_{;\bar{z}} = 0,$$

where φ is some holomorphic quadratic differential. Furthermore, in terms of the curvature tensor $R^z{}_{zz\bar{z}}$ and the Ricci tensor $R_{z\bar{z}} = -g$, we have

$$a^z{}_{z\bar{z}} = a^z{}_{\bar{z}z} + a^z R^z{}_{zz\bar{z}} = \bar{\mu}_{;z} + a^z(-R_{z\bar{z}}) = g \cdot a^z.$$

So

$$\Delta_g(\chi) = \chi - |\mu|^2,$$

which ends the proof of Proposition 5.1. □

The equations have so far been established on the complement of the punctures.

LEMMA 5.2. *Let* $h_0 = \min_j\left(\frac{1}{1-a_j}\right)$ *and* $q_0 = \min\left(\min_j\left(\frac{1}{a_j}\right), \min_j\left(\frac{1}{1-a_j}\right)\right)$. *Then*:

(i) $\frac{|\mu|^2 g_{\mathbf{a}}}{g_0} \in L^h(\mathcal{X}_{s_0})$ *for* $1 \le h < h_0$;

(ii) $\chi \in H_2^q(\mathcal{X}_{s_0})$ *for* $1 \le q < q_0$;

(iii) *the functions* $s \mapsto \frac{|\mu|^2 g_{\mathbf{a}}}{g_0} \in L^h(\mathcal{X}_s) \equiv L^h(X)$ *and* $s \mapsto \chi \in H_2^q(\mathcal{X}_s) \equiv H_2^q(X)$ *are both of class* C^∞; *and*

(iv) *for the coefficient of the harmonic Beltrami differential, we have that* $\mu(z) \in H_1^p$ *for every* $p < h_0$.

Proof. By Lemma 4.5, the expression $|\mu|^2 \frac{g_{\mathbf{a}}}{g_0}$ in local coordinates near the puncture p_j behaves like $\frac{1}{|z|^{2(1-a_j)}}$; hence (i) follows. Now we write equation (13) as

$$-\Delta_{g_0}\chi + \frac{g_{\mathbf{a}}}{g_0}\chi = \frac{g_{\mathbf{a}}}{g_0}|\mu|^2.$$

However, near the puncture p_j, the function $\frac{g_{\mathbf{a}}}{g_0}$ is in L^p for $1 \le p < \frac{1}{a_j}$ and so, by [24, Lemma 2.1] together with (i), claim (ii) follows. To prove (iii) we apply Theorem 2.3 together with the smooth dependence on parameters of the solution of elliptic equations. In order to see (iv), we express μ in terms of a quadratic holomorphic differential and apply Remark 2.1. □

PROPOSITION 5.3. *For every point $s_0 \in S$,*

$$\left\| \frac{\partial}{\partial s}\bigg|_{s_0} \right\|^2_{\mathrm{WP},\mathbf{a}} = \int_{\mathcal{X}_{s_0}} \chi \, dA_{g_{\mathbf{a}}}.$$

Proof. We have

$$\int_X \Delta_{g_{\mathbf{a}}} \chi \, dA_{g_{\mathbf{a}}} = \sqrt{-1} \int_X \partial\bar{\partial}\chi = 0$$

because $\chi \in H_2^q(X)$ for some $q > 1$, X is compact, and the space of smooth functions is dense in $H_2^q(X)$. Now, by equation (13),

$$\int_X |\mu|^2 \, dA_{g_{\mathbf{a}}} = \int_X \chi \, dA_{g_{\mathbf{a}}}. \qquad \square$$

Assume now that S is arbitrary and that $f : \mathcal{X} \to S$ is a holomorphic family of weighted punctured Riemann surfaces. We denote by ω_S^{WP} the real $(1,1)$-form, which is determined as follows by the Weil–Petersson inner product of tangent vectors on S. Given a tangent vector $u \in T_{S,s_0}$, we denote by $\rho_{S,s_0}(u) = \mu_{\mathbf{a}}(u) \in H^1(X, \mathbf{a})$ the corresponding harmonic Beltrami differential in the sense of Theorem 4.6.

At this point, we introduce the notion of *fiber integrals* of differential forms for a holomorphic family $f : \mathcal{X} \to S$ of compact complex manifolds of dimension n, say. Let η be a differential form of a certain degree $(k+n, k+n)$. Let

$$\begin{tikzcd} X \times S \arrow[r, "\phi"] \arrow[dr, "pr"'] & \mathcal{X} \arrow[d, "f"] \\ & S \end{tikzcd}$$

be a differentiable trivialization. Then

$$\int_{\mathcal{X}/S} \eta := \int_{X \times S/S} \phi^* \eta$$

denotes a differential form of degree (k,k), where the latter integral is defined in terms of the components of $\phi^*\eta$ that have total degree $2n$ in the fiber direction and degree $2k$ in the S direction. The exterior derivative of a fiber integral can be computed in different ways. Primarily,

$$d\left(\int_{\mathcal{X}/S} \eta \right) = \int_{\mathcal{X}/S} d\eta.$$

The latter integral can be evaluated in terms of ϕ. Since a differentiable trivialization determines a lift v of tangent vectors $\frac{\partial}{\partial x}$ of the base, any partial derivative

$$\frac{\partial}{\partial x} \int_{\mathcal{X}/S} \eta = \int_{\mathcal{X}/S} L_v(\eta),$$

where L_v denotes the Lie derivative of the differential form η with respect to v. One can verify that this is also true for differentiable lifts of complex tangent vectors, which need not arise from differentiable trivializations.

These results lead to our next theorem.

THEOREM 5.4. *The fiber integral*

$$\int_{\mathcal{X}/S} \omega_{\mathcal{X}}^2 = \omega_S^{\mathrm{WP}}$$

equals the Weil–Petersson form.

Proof. Let $\alpha \colon \tilde{S} \to S$ be a holomorphic map of complex manifolds. We consider $\tilde{\mathcal{X}} = \mathcal{X} \times_S \tilde{S}$ and the Cartesian diagram

$$\begin{tikzcd} \tilde{\mathcal{X}} \arrow[r, "\tilde{\alpha}"] \arrow[d, "\tilde{f}"'] & \mathcal{X} \arrow[d, "f"] \\ \tilde{S} \arrow[r, "\alpha"] & S \end{tikzcd}$$

Since the hyperbolic metrics on the fibers $\tilde{\mathcal{X}}_t$ are just the hyperbolic metrics on the $\mathcal{X}_{\alpha(t)}$, $t \in \tilde{S}$, it follows that the relative volume form on $\tilde{\mathcal{X}} \to \tilde{S}$ equals $\tilde{\alpha}^* g$, where g denotes the relative volume form for $\mathcal{X} \to S$. This implies that

$$\tilde{\alpha}^* \omega_{\mathcal{X}} = \tilde{\alpha}^* \big(\sqrt{-1}\partial\bar{\partial} \log g\big) = \sqrt{-1}\partial\bar{\partial} \log \tilde{\alpha}^* g = \omega_{\tilde{\mathcal{X}}}.$$

Hence the integral in Theorem 5.4 commutes with base change and, in particular, with the restriction to local analytic curves.

On the other hand, the Weil–Petersson hermitian product (i.e., the evaluation of ω^{WP} at tangent vectors) commutes with base change. For $v \in T_{\tilde{S},t_0}$, we have $\rho_{\tilde{S},t_0}(v) = \rho_{S,\alpha(t_0)}(\alpha_*(v))$. Hence

$$\begin{aligned} \omega_{\tilde{S}}^{\mathrm{WP}}(v,w) &= \langle \rho_{\tilde{S},t_0}(v), \rho_{\tilde{S},t_0}(w) \rangle_{\mathrm{WP},\mathbf{a}} \\ &= \langle \rho_{S,\alpha(t_0)}(\alpha_*(v)), \rho_{S,\alpha(t_0)}(\alpha_*(w)) \rangle_{\mathrm{WP},\mathbf{a}} = \omega_S^{\mathrm{WP}}(\alpha_*(v), \alpha_*(w)), \end{aligned}$$

and so

$$\alpha^* \omega_S^{\mathrm{WP}} = \omega_{\tilde{S}}^{\mathrm{WP}}.$$

Since both $\omega_{\mathcal{X}}$ and ω^{WP} are defined in a functorial way, it is sufficient to check the case $\dim_{\mathbb{C}} S = 1$, which follows from Proposition 5.3 and equation (12). □

THEOREM 5.5. *The Weil–Petersson form is of class C^∞ and is d-closed on the base of any holomorphic family. In particular, on the Teichmüller space, ω^{WP} is a Kähler form.*

Proof. At this point we introduce holomorphic coordinates s^i $(i = 1, \ldots, N)$ on the base space S. We consider the horizontal lifts V_i on $\mathcal{X}$ and their inner product with respect to $\omega_{\mathcal{X}}$:

$$\chi_{i\bar{j}} = \langle V_i, V_j \rangle.$$

Furthermore,

$$(\Delta_{g_{\mathbf{a}}} - \mathrm{id})\chi_{i\bar{j}} = \mu_i \mu_{\bar{j}}. \tag{14}$$

The relevant term for the fiber integral of $\omega_{\mathcal{X}}^2$ is

$$\sqrt{-1}\chi_{i\bar{j}} g_{\mathbf{a}}\, dA\, ds^i \wedge ds^{\bar{j}}.$$

In order to show the theorem, we need to prove that

$$d\int_{\mathcal{X}/S} \omega_{\mathcal{X}}^2 = \int_{\mathcal{X}/S} d(\omega_{\mathcal{X}}^2),$$

The map $S \to L^p$ (with p as before) that sends s to $\chi_{i\bar{j}} g_{\mathbf{a}}/g_0$ is of class C^∞; this follows from Theorem 2.3 and Lemma 5.2. So we apply a differerentiable local trivialization of the family. Then

$$\frac{\partial}{\partial s^k}\int_X \chi_{i\bar{j}} g_{\mathbf{a}}\, dA = \int_X F_{i\bar{j}k}\, g_{\mathbf{a}}\, dA$$

for some $F_{i\bar{j}k} \in L^p(X)$. Since L^p-convergence of a sequence implies pointwise convergence of a subsequence almost everywhere, the function $F_{i\bar{j}k}$ must be the derivative of the integrand outside a set of measure 0. This argument shows that the exterior derivative on S of the differential form given by the fiber integral of $\omega_{\mathcal{X}}^2$ equals the fiber integral of the exterior derivative of $\omega_{\mathcal{X}}^2$ on the total space $\mathcal{X}$. The latter form $d(\omega_{\mathcal{X}}^2)$ is in L^p and is equal to zero outside a set of measure 0, so the integral is identically zero. □

6. Determinant Line Bundles and Quillen Metrics in the Conical Case

Let $f\colon (\mathcal{X}, \mathbf{a}) \to S$ be any holomorphic family of weighted punctured Riemann surfaces equipped with the family $g_{\mathbf{a}}$ of conical metrics; in particular, f may denote the universal such family. In this section we consider rational weights $a_j \in \mathbb{Q}$. Let $m \in \mathbb{N}$ be a common denominator. Let

$$\mathcal{L}_m = \big((m(\mathcal{K}_{\mathcal{X}/S} + \mathbf{a})) - (m(\mathcal{K}_{\mathcal{X}/S} + \mathbf{a}))^{-1}\big)^{\otimes 2}$$

be an element of the corresponding Grothendieck group. Denote by

$$\lambda_m = \det f_! \mathcal{L}_m$$

the determinant line bundle on S. The Hirzebruch–Riemann–Roch theorem states that the Chern class of the determinant line bundle equals the degree-2 component

$$c_1(\lambda_m) = -f_*(\mathrm{ch}(\mathcal{L}_m) td(X/S))_{(2)} = 4m^2 f_*(c_1^2(\mathcal{K}_{\mathcal{X}/S} + \mathbf{a}))_{(2)}.$$

Now we equip the $\mathbb{Q}$-bundle $\mathcal{K}_{\mathcal{X}/S} + \mathbf{a}$ with a C^∞ hermitian metric of the form $\tilde{g}^{-1}$ with *positive* curvature, and we denote by

$$\tilde{\omega}_{\mathcal{X}} = \sqrt{-1}\partial\bar{\partial}\log\tilde{g} = 2\pi c_1(\mathcal{K}_{\mathcal{X}/S} + \mathbf{a}, \tilde{g}^{-1})$$

the Chern form. We denote by $\mathrm{ch}(\mathcal{L}_m, \tilde{g})$ the induced Chern character form. Only the term of degree 0 contributes to the Todd character form, and the metric on $\mathcal{X}$ need not be specified.

The theorem of Quillen [20], Takhtajan and Zograf [26], and Bismut, Gillet, and Soulé [4] posits the existence of a Quillen metric h_0^Q on λ_m such that, for the type-$(1,1)$ components, the following holds:

$$\begin{aligned} c_1(\lambda_m, h_0^Q) &= -\int_{\mathcal{X}/S} \operatorname{ch}(\mathcal{L}, \tilde{g}) td(\mathcal{X}/S)_{(1,1)} \\ &= 4m^2 \int_{\mathcal{X}/S} c_1(\mathcal{K}_{\mathcal{X}/S} + \mathbf{a}, \tilde{g}^{-1})^2_{(1,1)} \\ &= 16 m^2 \pi^2 \int_{\mathcal{X}/S} \tilde{\omega}^2_{\mathcal{X}}. \end{aligned}$$

THEOREM 6.1. *Let $f\colon (\mathcal{X}, \mathbf{a}) \to S$ be the universal holomorphic family of weighted punctured Riemann surfaces equipped with the family $g_{\mathbf{a}}$ of conical metrics. Let ω^{WP} be the generalized Weil–Petersson metric. Then the determinant line bundle λ_m possesses a hermitian metric h of class C^∞ whose Chern form is up to a numerical factor equal to the Weil–Petersson metric*:

$$c_1(\lambda_m, h) = 16 m^2 \pi^2 \omega^{\mathrm{WP}}.$$

The metric h descends to the moduli space.

Since Hilbert space methods are not available, the notion of an analytic torsion of Dirac operators is void; in particular, there is no Quillen metric in its original sense.

Proof of Theorem 6.1. We will use the notation of this section—in particular, the metric $\tilde{g}$ on $-(\mathcal{K}_{\mathcal{X}/S} + \mathbf{a})$. We can choose $\tilde{g}$ invariant under the Teichmüller modular group. Let σ_ν be the canonical sections of the line bundles on $\mathcal{X}$ given by the punctures. These can be chosen as invariant under the Teichmüller modular group. The quotient

$$\frac{\tilde{g}}{\Pi_\nu |\sigma_\nu|^{2a_\nu}}$$

is a well-defined relative metric on $\mathcal{X}$ with poles of fractional order at the punctures.

In view of Section 2 we have

$$g_{\mathbf{a}} = \frac{\tilde{g}}{\Pi_\nu |\sigma_\nu|^{2a_\nu}} e^w,$$

where the function w is globally defined on $\mathcal{X}$. Now

$$\int_{\mathcal{X}/S} (\omega^2_{\mathcal{X}/S} - \tilde{\omega}^2_{\mathcal{X}/S}) = \int_{\mathcal{X}/S} \sqrt{-1} \partial \bar{\partial} (w \cdot (\omega_{\mathcal{X}/S} + \tilde{\omega}_{\mathcal{X}/S})).$$

Let the induced relative metric be

$$\tilde{\omega}_{\mathcal{X}}|_{\mathcal{X}_s} = \tilde{\tilde{g}}(z,s)\, dA.$$

The assignment

$$s \mapsto w \cdot \left(\frac{g_{\mathbf{a}}}{\tilde{\tilde{g}}} + 1 \right) \tilde{\tilde{g}}$$

defines a C^∞ map $S \to L^p$. Now the argument of the proof of Theorem 5.4 applies literally, and

$$\int_{\mathcal{X}/S} \sqrt{-1}\partial\bar\partial(w \cdot (\omega_{\mathcal{X}/S} + \tilde\omega_{\mathcal{X}/S})) = \sqrt{-1}\partial\bar\partial \int_{\mathcal{X}/S} (w \cdot (\omega_{\mathcal{X}/S} + \tilde\omega_{\mathcal{X}/S})).$$

Here the integral on the right-hand side defines a C^∞ function on S, which is invariant under the Teichmüller modular group. □

7. Curvature of the Weil–Petersson Metric

In the classical case the Ricci and holomorphic sectional curvatures of the classical Weil–Petersson metric were proven to be negative by Ahlfors [1]. Royden [21] conjectured the precise upper bound for the holomorphic sectional curvature. The curvature tensor of the Weil–Petersson metric for Teichmüller spaces of compact (or punctured) Riemann surfaces was computed explicitly by Tromba [29] and Wolpert [32]. In this section we show the analogous result for the weighted punctured case. Our methods are different and originate from the higher-dimensional case treated in [22; 25].

We will first explain the approach and notation in the compact case. Let $f\colon \mathcal{X} \to S$ stand for the universal family, and let again (z,s) be local holomorphic coordinates on $\mathcal{X}$ with $f(z,s) = s$, where s^i $(i = 1,\dots,N)$ are holomorphic coordinates on S. We denote the coefficients of $\omega_{\mathcal{X}}$ by $g(z,s) = g_{z\bar z}(z,s)$, $g_{z\bar\jmath}$, and $g_{i\bar\jmath}$ (cf. [4; 5; 6; 7]).

We use the notation of Kähler geometry. Accordingly, the Christoffel symbols are

$$\Gamma = \Gamma^z_{zz} = \frac{\partial \log g}{\partial z}$$

and

$$\Gamma^{\bar z}_{\bar z\bar z} = \bar\Gamma.$$

The curvature tensor is

$$R^z{}_{z\bar z z} = -g_{z\bar z}.$$

Our computations require covariant derivatives with respect to the hyperbolic metrics $g = g(z,s)$ on the fibers $\mathcal{X}_s$, although we can use ordinary derivatives for parameters. We use the semicolon notation of the derivative of any tensor b for both:

$$\nabla_z b = b_{;z} \quad \text{and} \quad \nabla_i b = \partial_i b = b_{;i};$$

here the index i stands for the coordinate s^i, so that $\partial_i = \frac{\partial}{\partial s^i}$.

Let the tangent vectors $\frac{\partial}{\partial s^i}\big|_s$ correspond to harmonic Beltrami differentials

$$\mu_i = \mu^z_{i\bar z}\partial_z\,\overline{dz}$$

with $\mu_{\bar\jmath} = \overline{\mu_j}$. Now the Weil–Petersson form in coordinates s^i equals

$$\omega^{\mathrm{WP}}_S = \frac{\sqrt{-1}}{2} G_{i\bar\jmath}(s)\, ds^i \wedge ds^{\bar\jmath},$$

where

$$G_{i\bar{j}}(s) = \langle \mu_i, \mu_j \rangle = \int_{\mathcal{X}_s} \mu_i \mu_{\bar{j}} g \, dA.$$

As in Lemma 4.3 and Proposition 4.4, we use the horizontal lifts

$$V_i = \partial_i + a_i^z \partial_z.$$

We set $V_{\bar{j}} = \overline{V_j}$ and $a_{\bar{j}} = \overline{a_j}$; that is, $a_{\bar{j}}^{\bar{z}} = \overline{a_k^z}$. We have

$$\mu_i = a^z_{i;\bar{z}} \partial_z \, \overline{dz}. \tag{15}$$

In order to compute derivatives ∂_k—say, of the coefficients $G_{i\bar{j}}$—in principle we need a differential trivialization of the family. Instead one can apply the Lie derivative L_{W_k} with respect to a differentiable lift W_k of the tangent vector $\frac{\partial}{\partial s^k}$ to the integrand. In this way the Lie derivative of the integrand can be separated into tensors. Also (because of the symmetry of the Christoffel symbols) we can use covariant derivatives for the computation of Lie derivatives. As usual, the metric tensor defines a transition from contravariant to covariant tensors.

As differentiable lifts we take the horizontal lifts V_k described previously. Observe that Lie derivatives are not type preserving. We will need the following identities:

$$L_{V_k}(g \, dA) = 0; \tag{16}$$

$$\chi_{i\bar{j}} := \langle V_i, V_j \rangle_{\omega_{\mathcal{X}}} = g_{i\bar{j}} - a_i^z a_{\bar{j}}^{\bar{z}} g_{z\bar{z}}; \tag{17}$$

$$\begin{aligned} L_{V_k}(\mu_{\bar{j}}) &= -(\chi_{k\bar{j}})^{;\bar{z}}{}_{;z} \partial_{\bar{z}} \, dz \\ &\quad - (\mu_k)^z{}_{\bar{z}} (\mu_{\bar{j}})^{\bar{z}}{}_z \partial_z \, dz + (\mu_k)^{\bar{z}}_{\bar{z}} (\mu_{\bar{j}})^{\bar{z}}{}_z \partial_{\bar{z}} \, \overline{dz}. \end{aligned} \tag{18}$$

Proof. We show (16) and compute the $(z, \bar{z})$-component of the Lie derivative. We have

$$(L_{V_k} g_{z\bar{z}})_{z\bar{z}} = [\partial_k + a_k^z \partial_z, g_{z\bar{z}}] = g_{z\bar{z};k} + a_k^z g_{z\bar{z};z} + a^z_{kz} g_{z\bar{z}} = g_{k\bar{z};z} + a_{k\bar{z};z} = 0.$$

The inner product of horizontal lifts in (17) with respect to $\omega_{\mathcal{X}}$ was already evaluated for $\dim S = 1$. Equation (18) follows from (15) and (17).

Proposition 7.1. *For all $s \in S$,*

$$\partial_k G_{i\bar{j}}(s) = \int_{\mathcal{X}_s} L_{V_k}(\mu_i) \mu_{\bar{j}} g \, dA \tag{19}$$

holds.

When evaluating (19), only the first component of (18) gives a contribution in the pairing with $\mu_{\bar{j}}$.

Proof of Proposition 7.1. We compute $L_{V_k}(\mu_i \mu_{\bar{j}} g \, dA)$ using (16). Now, by partial integration (for all $s \in S$),

$$\begin{aligned} \int_{\mathcal{X}_s} \mu_i L_{V_k}(\mu_{\bar{j}}) g \, dA &= -\int_{\mathcal{X}_s} (\mu_i)^z{}_{\bar{z}} (\chi_{i\bar{j}})^{;\bar{z}}{}_{;z} g \, dA \\ &= \int_{\mathcal{X}_s} (\mu_i)^z{}_{\bar{z};z} (\chi_{i\bar{j}})^{;\bar{z}} g \, dA = 0. \end{aligned} \tag{20}$$

In the last step we used the harmonicity of μ_i in the form

$$(\mu_i)^z{}_{\bar z;z} = 0. \tag{21}$$

□

LEMMA 7.2.

$$L_{V_k}(\mu_i)^z{}_{\bar z} = L_{V_i}(\mu_k)^z{}_{\bar z}. \tag{22}$$

The proof is a direct computation.

We see that Lemma 7.2 together with Proposition 7.1 also implies the Kähler property.

LEMMA 7.3. *The Lie derivatives*

$$L_{V_k}(\mu_i) = L_{V_k}(\mu_i)^z{}_{\bar z}\partial_z\, \overline{dz}$$

of the harmonic Beltrami differentials are again harmonic Beltrami differentials.

Proof. We have

$$\nabla_z L_{V_k}(\mu_i) = 0.$$

Its formal proof corresponds to $\bar\partial^* L_{V_k}(\mu_i) = 0$ in [23]. The computation is straightforward. □

It is convenient to use normal coordinates of the second kind for the components of the Weil–Petersson tensor at a given point $s_0 \in S$. Because the μ_i span the space of harmonic Beltrami differentials (for $s = s_0$), the condition

$$\partial_k G_{i\bar\jmath}(s_0) = 0$$

is equivalent (by Proposition 7.1) to saying that all derivatives $L_{V_k}(\mu_i)$ vanish at $s = s_0$ identically.

We compute the second derivative at the given point s_0. By (19),

$$\partial_{\bar\ell}\partial_k G_{i\bar\jmath} = \int_{\mathcal{X}_{s_0}} L_{V_{\bar\ell}} L_{V_k}(\mu_i)\mu_{\bar\jmath}\, g\, dA + \int_{\mathcal{X}_{s_0}} L_{V_k}(\mu_i) L_{V_{\bar\ell}}(\mu_{\bar\jmath})\, g\, dA. \tag{23}$$

LEMMA 7.4.

$$[V_{\bar\ell}, V_k] = -\chi_{k\bar\ell}^{;z}\partial_z + \chi_{k\bar\ell}^{;\bar z}\partial_{\bar z}; \tag{24}$$

$$\int_{\mathcal{X}_s} L_{[V_{\bar\ell}, V_k]}(\mu_i)\mu_{\bar\jmath}\, g\, dA = -\int_{\mathcal{X}_s} \Delta(\chi_{k\bar\ell})\mu_i\mu_{\bar\jmath}\, g\, dA. \tag{25}$$

Proof. We omit the computational proof of (24). In order to see (25), we write

$$[\chi_{k\bar\ell}^{;z}\partial_z, \mu_{i\bar z}^z\partial_z\, \overline{dz}]^z{}_{\bar z} = -\chi_{k\bar\ell}^{;z}\mu_{i\bar z;z}^z + \chi_{k\bar\ell;z}^{;z}\mu_{i\bar z}^z,$$

where the first term on the right-hand side vanishes because of the harmonicity of the Beltrami differential μ_i. So we have the right-hand side of (25). Finally,

$$[\chi_{k\bar\ell}^{;\bar z}\partial_{\bar z}, \mu_{i\bar z}^z\partial_z\, \overline{dz}]^z{}_{\bar z} = \chi_{k\bar\ell}^{;\bar z}\mu_{i\bar z;\bar z}^z + \chi_{k\bar\ell;\bar z}^{;\bar z}\mu_{i\bar z}^z = (\chi_{k\bar\ell}^{;\bar z}\mu_{i\bar z}^z)_{;\bar z},$$

so that (again by harmonicity) we have

$$[\chi_{k\bar{\ell}}^{;\bar{z}}\partial_{\bar{z}}, \mu_{i\bar{z}}^{z}\partial_z\,\overline{dz}]^z{}_{\bar{z}}\cdot\mu_{\bar{j}z}^{\bar{z}} = (\chi_{k\bar{\ell}}^{;\bar{z}}\mu_{i\bar{z}}^{z}\mu_{\bar{j}z}^{\bar{z}})_{;\bar{z}}.$$

The divergence theorem implies that the integral vanishes. □

We continue the computation of (23).

We use that $L_{V_{\bar{j}}}L_{V_k}(\mu_i) = L_{[V_{\bar{\ell}},V_k]}(\mu_i) + L_{V_k}L_{V_{\bar{j}}}(\mu_i)$ and apply Lemma 7.4. Now

$$\partial_{\bar{\ell}}\partial_k G_{i\bar{j}} = \int_{\mathcal{X}_s} L_{[V_{\bar{\ell}},V_k]}(\mu_i)\mu_{\bar{j}}g\,dA + \int_{\mathcal{X}_s} L_{V_k}L_{V_{\bar{\ell}}}(\mu_i)\mu_{\bar{j}}g\,dA$$
$$+ \int_{\mathcal{X}_s} L_{V_k}(\mu_i)L_{V_{\bar{\ell}}}(\mu_{\bar{j}})g\,dA. \tag{26}$$

The third term of (26) vanishes at s_0 because, for $s = s_0$ in normal coordinates,

$$L_{V_k}(\mu_i) = 0.$$

Now

$$\partial_{\bar{\ell}}\partial_k G_{i\bar{j}} = -\int_{\mathcal{X}_s} \Delta(\chi_{k\bar{\ell}})\mu_i\mu_{\bar{j}}g\,dA + \partial_k\int_{\mathcal{X}_s} L_{V_{\bar{\ell}}}(\mu_i)\mu_{\bar{j}}g\,dA$$
$$- \int_{\mathcal{X}_s} L_{V_{\bar{\ell}}}(\mu_i)L_{V_k}(\mu_{\bar{j}})g\,dA. \tag{27}$$

In order to treat the *first term* of (27), we use the equation

$$(-\Delta + \mathrm{id})\chi_{k\bar{\ell}} = \mu_k\mu_{\bar{\ell}} \tag{28}$$

corresponding to (13). Then

$$\begin{aligned}
&-\int_{\mathcal{X}_s} \Delta(\chi_{k\bar{\ell}})\mu_i\mu_{\bar{j}}g\,dA\\
&= \int_{\mathcal{X}_s} \Delta(-\Delta+\mathrm{id})^{-1}(\mu_k\mu_{\bar{\ell}})\cdot(\mu_i\mu_{\bar{j}})g\,dA\\
&= -\int_{\mathcal{X}_s} ((-\Delta+\mathrm{id})-\mathrm{id})(-\Delta+\mathrm{id})^{-1}(\mu_k\mu_{\bar{\ell}})\cdot(\mu_i\mu_{\bar{j}})g\,dA\\
&= -\int_{\mathcal{X}_s} (\mu_k\mu_{\bar{\ell}})\cdot(\mu_i\mu_{\bar{j}})g\,dA + \int_{\mathcal{X}_s} (-\Delta+\mathrm{id})^{-1}(\mu_k\mu_{\bar{\ell}})\cdot(\mu_i\mu_{\bar{j}})g\,dA.
\end{aligned} \tag{29}$$

By (20), the *second term* of (27) vanishes.

In the *third term* of (27), all three components of (18) matter. We will use the following identity, which follows from the hyperbolicity of the metrics:

$$\chi_{k\bar{j};zz\bar{z}} = \chi_{k\bar{j};z\bar{z}z} - \chi_{k\bar{j};z}R^z{}_{zz\bar{z}} = \chi_{k\bar{j};z\bar{z}z} - g_{z\bar{z}}\chi_{k\bar{j};z}.$$

Therefore,

$$\begin{aligned}
-\int_{\mathcal{X}_s}(\chi_{i\bar{\ell}})_{;\bar{z}\bar{z}}(\chi_{k\bar{j}})_{;zz}(g^{\bar{z}z})^2g\,dA &= \int_{\mathcal{X}_s}(\chi_{i\bar{\ell}})_{;\bar{z}}(\chi_{k\bar{j}})_{;zz\bar{z}}(g^{\bar{z}z})^2g\,dA\\
&= -\int_{\mathcal{X}_s}(\chi_{i\bar{\ell}})_{;\bar{z}z}((\chi_{k\bar{j}})_{;z\bar{z}} - g_{z\bar{z}}\chi_{k\bar{j}})(g^{\bar{z}z})^2g\,dA\\
&= \int_{\mathcal{X}_s} \Delta(\chi_{i\bar{\ell}})\mu_k\mu_{\bar{j}}g\,dA.
\end{aligned}$$

The preceding argument shows that this is exactly equal to

$$-\int_{\mathcal{X}_s}(\mu_i\mu_{\bar\ell})\cdot(\mu_k\mu_{\bar\jmath})g\,dA+\int_{\mathcal{X}_s}(-\Delta+\mathrm{id})^{-1}(\mu_i\mu_{\bar\ell})\cdot(\mu_k\mu_{\bar\jmath})g\,dA.$$

Hence the third term in (27) contains the three contributions of (18); it equals

$$\begin{aligned}
-&\int_{\mathcal{X}_s}(\chi_{k\bar\jmath})_{;zz}(\chi_{i\bar\ell})_{;\bar z\bar z}(g^{\bar z z})^2 g\,dA\\
&\quad+\int_{\mathcal{X}_s}(\mu_i\mu_{\bar\jmath})(\mu_k\mu_{\bar\ell})g\,dA+\int_{\mathcal{X}_s}(\mu_i\mu_{\bar\ell})(\mu_k\mu_{\bar\jmath})g\,dA\\
&\qquad=\int_{\mathcal{X}_s}(-\Delta+\mathrm{id})^{-1}(\mu_k\mu_{\bar\jmath})(\mu_i\mu_{\bar\ell})+\int_{\mathcal{X}_s}(\mu_i\mu_{\bar\jmath})(\mu_k\mu_{\bar\ell})g\,dA. \qquad (30)
\end{aligned}$$

(Here we have gathered the Beltrami differentials in a convenient way.)

Adding all terms together yields the curvature of the Weil–Petersson metric.

THEOREM 7.5. *Let s^i be holomorphic coordinates on the Teichmüller space, and let the tangent vectors $\frac{\partial}{\partial s^i}\big|_{s_0}$ correspond to the harmonic Beltrami differentials μ_i on $X=\mathcal{X}_{s_0}$. Then*

$$\begin{aligned}
R_{i\bar\jmath k\bar\ell}(s_0)=&\int_X(\Delta-\mathrm{id})^{-1}(\mu_i\mu_{\bar\jmath})\mu_k\mu_{\bar\ell}\,g\,dA\\
&+\int_X(\Delta-\mathrm{id})^{-1}(\mu_i\mu_{\bar\ell})\mu_k\mu_{\bar\jmath}\,g\,dA. \qquad (31)
\end{aligned}$$

holds.

(We have been using the complex Laplacian with nonpositive eigenvalues, rather than the real one; this accounts for a factor of 2.)

In the case of the generalized Weil–Petersson metric for weighted Riemann surfaces, we will show that the same formula holds for weights larger that $1/2$. This is the range for which also Fenchel–Nielsen coordinates were introduced. It contains the interesting range of weights of the form $1-1/m$, $m>2$, that arise from orbifold singularities.

THEOREM 7.6. *Let $(X,\mathbf{a})$ with $1/2<a_j<1$ be a weighted punctured Riemann surface that is represented by a point s_0 in the Teichmüller space $\mathcal{T}_{\gamma,n}$. Let $s^1,\dots,s^N$ be any local holomorphic coordinates near s_0, and let $\mu_\alpha\in H^1(X,\mathbf{a})$ be harmonic representatives of the vectors $\frac{\partial}{\partial s\alpha}\big|_{s_0}$. Then the curvature tensor of the Weil–Petersson metric is given by* (31), *where the Laplacian and the area elements are replaced by $\Delta_{\mathbf{a}}$ and $dA_{g_{\mathbf{a}}}$* (*respectively*), *which are induced by the hyperbolic conical metric on the fiber.*

In all of our arguments we will assume that the antiholomorphic quadratic differentials that define harmonic Beltrami differentials have at most a pole at the given conical singularity. (The proofs are still valid in the holomorphic case.)

We first prove the statement of Proposition 7.1 in the conical case. We need to see that the integration commutes with a differentiation with respect to the parameter (after a differentiable trivialization of the family). This follows as in the proof of Theorem 5.4.

Let first $\tilde{V}_k$ be any C^∞ lift. Now the map $s \mapsto \mu_i \mu_{\bar{j}} g_{\mathbf{a}}$ again is a C^∞ map from S to $L^h(X)$ by Lemma 5.2(iii). So by our previous argument,

$$\partial_k G_{i\bar{j}}(s) = \int_{\mathcal{X}_s} L_{\tilde{V}_k}(\mu_i \mu_{\bar{j}} g \, dA)$$

holds. Next, we need that

$$V_k - \tilde{V}_k = C^z \partial_z$$

is a (global) tensor in fiber direction. Now

$$\begin{aligned}
&[C^z \partial_z, (\mu_i \mu_{\bar{j}}) g_{z\bar{z}} \sqrt{-1}\, dz \wedge \overline{dz}] \\
&\quad = C^z \partial_z((\mu_i \mu_{\bar{j}}) g_{z\bar{z}}) + \partial_z(C^z)((\mu_i \mu_{\bar{j}}) g_{z\bar{z}}) \sqrt{-1}\, dz \wedge \overline{dz} \\
&\quad = d\big(\sqrt{-1} C^z \cdot (\mu_i \mu_{\bar{j}}) g_{z\bar{z}} \overline{dz}\big) = d(C_{\bar{z}}(\mu_i \mu_{\bar{j}})).
\end{aligned}$$

Claim.

$$\int_{\mathcal{X}_s} d(C_{\bar{z}}(\mu_i \mu_{\bar{j}})) = 0.$$

Proof of Claim. We write the displayed integral as the limit of integrals over closed paths around the punctures. We first estimate the coefficient $C_{\bar{z}}$. It satisfies the same estimates as the $a_{k\bar{z}}$. Now

$$\frac{\partial a_k^z}{\partial \bar{z}} = \mu_k = \frac{\bar{\varphi}}{g}$$

for some holomorphic quadratic differential φ with at most a simple pole. By Remark 2.1 we can find a continuous $\bar{z}$-antiderivative η of the right-hand side on a punctured disk U^*. (This fact is a consequence of the more general Remark 7.8 to follow.)

The term

$$a_k^z - \eta$$

is holomorphic on U^*, and

$$a_{k\bar{z}} = g \cdot a_k^z \in H_1^p(U) \subset L^2(U)$$

for $p < 1/a$ by Theorem 2.3. In particular,

$$a_k^z - \eta \in L^2(U).$$

Hence $a_k^z - \eta$ must be holomorphic at the puncture (cf. [24, Lemma 5.3]). Now

$$|C_{\bar{z}}| \simeq |a_{k\bar{z}}| = |g \cdot a_k^z| \lesssim |z|^{-2a}$$

and

$$|C_z(\mu_i \mu_{\bar{j}})| \lesssim \frac{1}{|z|^{2(1-a)}},$$

so that

$$\lim_{\varepsilon\to 0}\int_{r=\varepsilon}\frac{r\,d\varphi}{r^{2(1-a)}}=0$$

implies the claim.

LEMMA 7.7. *Let Δ_r be the disk of radius r in $\mathbb{C}$, and let $1/2<\alpha<1$. Let $f\in C^\infty(\Delta_1\backslash\{0\},\mathbb{C})$ be a function such that $|z|^{2\alpha}f(z)$ is bounded in a neighborhood of 0. Let U be a relatively compact open subset of Δ_1 containing 0. Then the equation*

$$\frac{\partial g}{\partial\bar z}=f \tag{32}$$

has a solution that is of class C^∞ on $(U\backslash\{0\})$ and such that $|z|^{2\alpha-1}g(z)$ is bounded in a neighborhood of 0. In particular, it is contained in $L^2(U)$. Moreover, any solution of (32), that is in $L^2(U)$ has this boundedness property.

Proof. For any $0\le r<\rho<1$ and $\Delta_{r,\rho}=\{z\in\mathbb{C}: r<|z|<\rho\}$, we define

$$F(r,\rho)(z)=\frac{-1}{\pi}\int_{\Delta_{r,\rho}}\frac{f(\zeta)}{\zeta-z}i\,\frac{d\zeta\wedge d\bar\zeta}{2}.$$

From the Cauchy formula we see that, for $z\in\Delta_{r,\rho}$ with $r>0$,

$$\frac{\partial F(r,\rho)(z)}{\overline{\partial z}}=f(z)$$

holds. Let K be a compact subset of $\Delta_\rho\backslash\{0\}$. Let $r_0>0$ be chosen such that $K\subset\Delta_{r_0,\rho}$. Then, for all $0<r<r_0$, the function $|f(\zeta)/|\zeta-z|$ is uniformly bounded by some $M>0$ for $z\in K$ and $\zeta\in\Delta_{\rho_0}\backslash\{0\}$. So

$$|F(r,\rho)(z)-F(0,\rho)(z)|\le Mr^2,$$

which implies uniform convergence for $r\to 0$. The same argument holds for the derivatives with respect to z and $\bar z$. It follows that $F(0,\rho)$ is differentiable and solves (32) on $\Delta_\rho\backslash\{0\}$.

On the open set $\Delta_\rho\backslash\{0\}$ we write $f(\zeta)=|\zeta|^{-2\alpha}m(\zeta)$ with $|m(\zeta)|\le C$. Now we make the change of variables $\zeta=z\eta$. Then

$$|F(0,\rho)(z)|\le\frac{1}{\pi}C|z|^{-2\alpha+1}\int_{\mathbb{C}}\frac{1}{|\eta|^{2\alpha}|(\eta-1)|}i\,\frac{d\eta\wedge d\bar\eta}{2},$$

so we just need to show that

$$\int_{\mathbb{C}}\frac{1}{|\eta|^{2\alpha}|(\eta-1)|}i\,\frac{d\eta\wedge d\bar\eta}{2}<+\infty.$$

The convergence of the right-hand-side integral follows from

$$\int_0^1\frac{r\,dr}{r^{2\alpha}}<\infty\quad\text{and}\quad\int_2^\infty\frac{r\,dr}{r^{2\alpha+1}}<\infty.$$

Choose ρ such that $\bar U\subseteq\Delta_\rho$. Then the condition that $|z|^{2\alpha-1}g(z)$ be bounded implies that g is in $L^2(U)$. The second claim now follows because a holomorphic function in $\Delta_\rho\backslash\{0\}$ that is in $L^2(U)$ is bounded on U.

REMARK 7.8. By applying Lemma 7.7 to $f(z)/z$ we can treat the case $0 < \alpha < 1/2$. Under the same boundedness assumption, equation (32) has a solution that extends continuously to the origin.

REMARK 7.9. If $\alpha = 1/2$, then a statement as in the lemma does not hold in general.

In fact, let us choose $f(z) = \frac{1}{\bar{z}} = \frac{\partial}{\partial \bar{z}} \log(|z|^2)$. Assume there exists a bounded function g on a small punctured disk such that $g - \log(|z|^2)$ is holomorphic. Since $g - \log(|z|^2)$ is in L^2 of the disk, we would obtain that it is bounded. However, $\log(|z|^2)$ is not bounded near 0. By replacing f by $z^k f$ and g by $z^{-k} g$ for some integer k, we may prove a similar lemma for $\alpha \in \mathbb{R}$ such that $2\alpha \neq \mathbb{Z}$ and also find an example as before if $2\alpha \in \mathbb{Z}$.

We return to the discussion of the generalized Weil–Petersson metric. We know that

$$\partial_k G_{i\bar{j}}(s) = \int_{\mathcal{X}_s} L_{V_k}(\mu_i \mu_{\bar{j}} g \, dA).$$

So far the integral can be computed in terms of the (singular) horizontal lifts V_k, and we are in a position to use covariant derivatives, too, because the Lie derivatives can also be computed in terms of those.

We use the fact $L_{V_k}(g) = 0$ from (16), which is still pointwise true outside a set of measure 0; hence the statement of Proposition 7.1 in the conical case is reduced to showing that

$$\int_X \mu_i L_{V_k}(\mu_{\bar{j}}) g_{\mathbf{a}} \, dA = 0.$$

By (18), this integral equals

$$-\int_X \mu_i{}^z{}_{\bar{z}} \chi_{k\bar{j}}{}^{;\bar{z}}{}_{;z} g \, dA = -\int_X (\mu_i{}^z{}_{\bar{z}} \chi_{k\bar{j}}{}^{;\bar{z}})_{;z} g \, dA$$

because of the harmonicity of μ_i. This integral is, up to a numerical factor, written as

$$\int_X d\left(\mu_i \frac{\partial}{\partial z} \overline{dz} \cdot \chi_{k\bar{j};z} \, dz\right) = \int_X d(\mu_i \cdot \chi_{k\bar{j};z} \, \overline{dz}).$$

We consider the defining equation for $\chi = \chi_{k\bar{j}}$ in the form

$$\frac{\partial^2 \chi}{\partial z \partial \overline{z}} = -g \mu_k \mu_{\bar{j}} + g\chi.$$

We know that χ is continuous and for some neighborhood U of a puncture we again apply Lemma 7.7 and take a $\bar{z}$-antiderivative η of the right-hand side that satisfies (with $a > 1/2$)

$$|\eta| \lesssim r^{1-2a}.$$

Again, since $\frac{\partial \chi}{\partial z} \in H_1^p \subset L^2$, the function

$$\frac{\partial \chi}{\partial z} - \eta \in \mathcal{O}(U^*)$$

is holomorphic at the puncture. Given the estimate for μ_i, the argument of the previous claim immediately yields the vanishing of the integral.

Next, we chose normal coordinates for the Weil–Petersson metric on S of the second kind at a given point $s = s_0$. This concludes the proof of Proposition 7.1 in the conical case.

We will follow the computation of the curvature of the Weil–Petersson metric in the compact case.

LEMMA 7.10. *The Lie derivatives*

$$L_{V_k}(\mu_i)^z{}_{\bar z}\partial_z\,\overline{dz}$$

of the harmonic Beltrami differentials are again harmonic Beltrami differentials with respect to the conical structure (depending in a C^∞ way upon the parameter).

Now we can apply the argument of Proposition 7.1 literally to

$$\int_{\mathcal{X}_s} L_{V_k}(\mu_i)\mu_{\bar{\jmath}}g\,dA$$

and obtain the following statement.

COROLLARY 7.11. *Equation* (23) *holds in the conical case.*

Proof of Lemma 7.10. We have from Lemma 7.3 that

$$\nabla_z L_{V_k}(\mu_i) = 0. \tag{33}$$

We need to see that the antiholomorphic term $g \cdot L_{V_k}(\mu_i)$ is in L^1 so that it can have at most a simple pole. After the verification, we know that $L_{V_k}(\mu_i)$ is a harmonic Beltrami differential in our sense. We have

$$\begin{aligned} g \cdot L_{V_k}(\mu_i)^z{}_{\bar z} &= -g\partial_z\left(\frac{1}{g}\partial_k(g_{i\bar z})\right) - g_{i\bar z}g_{k\bar z} \\ &= -\partial_{\bar z}(\log g)(\partial_k g_{i\bar z}) + \partial_{\bar z}(\partial_k g_{i\bar z}) - g_{i\bar z}g_{k\bar z}. \end{aligned}$$

We show that all three terms are in L^1. For the first term we can use the proof of Lemma 4.5. Since $g_{i\bar z} \in H_1^p$, the second term is in L^1. Finally, both $g_{i\bar z}$ and $g_{k\bar z}$ are in $H_1^p \subset L^2$. □

We prove the statement of Lemma 7.4 in the conical case: equation (24) is pointwise and carries over. We show (25); for the required partial integration. we just need that

$$\int_{\mathcal{X}_s} d(\chi_{k\bar\ell;z}\mu_i\mu_{\bar{\jmath}}\,dz) = 0.$$

As before, we reduce this to the vanishing of limits of integrals along closed paths around the punctures.

We know from Lemma 7.15 (to follow) that $|\chi_{k\bar\ell:z}| \lesssim r^{-2a+1}$ and $\mu_i\mu_{\bar{\jmath}} \sim r^{-2+4a}$ so that $|\chi_{k\bar\ell;z}\mu_i\mu_{\bar{\jmath}}| \lesssim r^{4a-1}$. So

$$\lim_{r\to 0}(r \cdot r^{4a-1}) = 0$$

implies that the above integral vanishes, which proves (25) in the conical case.

In particular, (27) is now valid in our situation. A purely local computation (under the integral sign) implies (29).

The final step is to arrive at (30) in the conical case—that is, by applying a twofold partial integration to

$$\int_{\mathcal{X}_s} (\chi_{i\bar{\ell}})_{;\bar{z}\bar{z}}(\chi_{k\bar{j}})_{;zz}(g^{\bar{z}z})^2 g\, dA.$$

This is achieved by Lemmas 7.12 and 7.13.

LEMMA 7.12. *The following singular integral vanishes*:

$$\int_X d(\chi_{k\bar{j};z\bar{z}}\chi_{i\bar{\ell};\bar{z}}g^{\bar{z}z}\,\overline{dz}) = 0. \tag{34}$$

Proof. The integrand equals

$$d((\chi_{k\bar{j}} - \mu_k\mu_{\bar{j}})\chi_{i\bar{\ell}:\bar{z}}\,\overline{dz}).$$

We know from Lemma 7.15 (to follow) that

$$|\chi_{i\bar{\ell}:\bar{z}}| \lesssim r^{-2a+1}$$

and we have the continuity of $\chi_{k\bar{j}}$. Furthermore,

$$|\mu_k\mu_{\bar{j}}| \lesssim r^{-2+4a}$$

so that $\chi_{k\bar{j}} - \mu_k\mu_{\bar{j}}$ is continuous. We use integration along closed loops as before and see that the integral vanishes. □

LEMMA 7.13.

$$\int_X d(\chi_{k\bar{j};zz}\chi_{i\bar{\ell};\bar{z}}g^{\bar{z}z}\, dz) = 0. \tag{35}$$

We reduce the proof to the following statement, which shows that possible residues in (35) and (34) must be equal up to a sign. However, we know already that the latter integral vanishes.

LEMMA 7.14.

$$\lim_{\varepsilon\to 0}\int_{\{|z|<\varepsilon\}} \frac{\sqrt{-1}}{2}\partial\bar{\partial}(\chi_{k\bar{j};z}\chi_{i\bar{\ell};\bar{z}}g^{\bar{z}z}) = 0. \tag{36}$$

Proof of Lemma 7.13. We expand the integrand of (36) and find that

$$\begin{aligned} 0 &= \lim_{\varepsilon\to 0}\int_{|z|=\varepsilon} \partial(\chi_{k\bar{j};z}\chi_{i\bar{\ell};\bar{z}}g^{\bar{z}z}) \\ &= \lim_{\varepsilon\to 0}\int_{|z|=\varepsilon} (\chi_{k\bar{j};zz}\chi_{i\bar{\ell};\bar{z}} + \chi_{k\bar{j};z}\chi_{i\bar{\ell};\bar{z}z})g^{\bar{z}z}\, dz. \end{aligned}$$

□

LEMMA 7.15.

$$|\chi_{k\bar{j};z}| \lesssim r^{-2a+1}. \tag{37}$$

Proof. We know that

$$\partial\bar{\partial}\chi_{k\bar{j}} = (\chi_{k\bar{j}} - \mu_k\mu_{\bar{j}})g.$$

Furthermore, $\chi_{k\bar{j}} - \mu_k\mu_{\bar{j}}$ is continuous since $\chi_{k\bar{j}} \in H_2^p(X)$, and the continuity of $\mu_k\mu_{\bar{j}}$ follows since $a > 1/2$. Now the argument involving the $\bar{z}$-antiderivative again gives the claim. □

Proof of Lemma 7.14. At this point, we assume that $k = i = j = \ell$. This case is sufficient because the curvature formula will follow as usual from it by polarization. We set $\chi = \chi_{k\bar{j}}$ and $|\mu|^2 = \mu_i\mu_{\bar{j}}$ for short. The integrand equals

$$\begin{aligned}\eta &:= \frac{\sqrt{-1}}{2}\partial\bar{\partial}(g^{\bar{z}z}\chi_{;z}\chi_{;\bar{z}}) \\ &= g^{\bar{z}z}(\chi_{;z\bar{z}z}\chi_{\bar{z}} + \chi_{;z\bar{z}}\chi_{\bar{z}z} + \chi_{;zz}\chi_{;\bar{z}\bar{z}} + \chi_{;z}\chi_{;\bar{z}\bar{z}z})\frac{\sqrt{-1}}{2}\,dz\wedge\overline{dz}.\end{aligned}$$

Now we use (13) and (14) on the integrand together with the following formula:

$$\chi_{;\bar{z}\bar{z}z} = \chi_{;\bar{z}z\bar{z}} - \chi_{;\bar{z}}R^{\bar{z}}{}_{\bar{z}\bar{z}z} = \chi_{;\bar{z}z\bar{z}} - g_{z\bar{z}}\chi_{;\bar{z}} = -g_{z\bar{z}}(|\mu|^2)_{;\bar{z}}.$$

Hence

$$\begin{aligned}\eta &= \big(g_{z\bar{z}}(\chi - |\mu|^2)^2 + g^{\bar{z}z}\chi_{;zz}\chi_{;\bar{z}\bar{z}} \\ &\qquad + (\chi - |\mu|^2)_{;z}(\chi - |\mu|^2)_{;\bar{z}} - (|\mu|^2)_{;z}(|\mu|^2)_{;\bar{z}}\big)\frac{\sqrt{-1}}{2}\,dz\wedge\overline{dz} \\ &\geq -(|\mu|^2)_{;z}(|\mu|^2)_{;\bar{z}}\frac{\sqrt{-1}}{2}\,dz\wedge\overline{dz}.\end{aligned}$$

Again we realize a harmonic Beltrami differential as a quotient of an antiholomorphic quadratic differential with at most a single pole by the metric tensor, and again we use the analyticity property of Remark 2.1. This implies

$$|(|\mu|^2)_{;z}| \lesssim r^{4a-3},$$

so that

$$(|\mu|^2)_{;z}(|\mu|^2)_{;\bar{z}} \lesssim r^{8a-6}.$$

As a result, for some $c, r_0 > 0$ and all $0 < |z| \leq r_0$ we have

$$\frac{\sqrt{-1}}{2}\partial\bar{\partial}(g^{\bar{z}z}\chi_{;z}\chi_{;\bar{z}} - c\cdot r^{8a-4}) \geq 0.$$

Observe that, by our assumption, $r^{8a-4} \to 0$ with $r \to 0$. We write $\big(\sqrt{-1}/2\big)\partial\bar{\partial}\tau$ for the expression just displayed. In terms of the polar coordinates $z = r\cdot\exp\big(\sqrt{-1}\varphi\big)$ we set

$$\tilde{\tau}(r) = \int_0^{2\pi}\tau(r,\varphi)\,d\varphi.$$

Hence for all $0 < \delta < \varepsilon$ we have

$$0 \le \int_{\delta<|z|<\varepsilon} \frac{\sqrt{-1}}{2}\partial\bar{\partial}\tau = \int_\delta^\varepsilon \frac{\partial}{\partial r}\left(r \cdot \frac{\partial}{\partial r}\tilde{\tau}\right) dr = r \cdot \frac{\partial}{\partial r}\tilde{\tau}\bigg|_\delta^\varepsilon.$$

Up to a multiplicative constant, the contribution of $-c \cdot r^{8a-4}$ to the integral amounts to

$$r^{8a-4}|_\delta^\varepsilon,$$

which tends to zero as $\varepsilon, \delta \to 0$.

The monotonicity implies the existence of

$$\ell = \lim_{r\to 0} r \frac{\partial\tilde{\tau}}{\partial r} \ge -\infty.$$

If we assume $\ell \le -c' < 0$, we see immediately that

$$\tilde{\tau}(r) \ge c'' - c' \log r$$

for some real number c'' so that $\tilde{\tau} \to \infty$ with $r \to 0$. On the other hand, it follows from (37) that

$$\chi_{;z}\chi_{;\bar{z}} g^{\bar{z}z} \lesssim r^{2-2a}.$$

So

$$\lim_{r\to 0} r \frac{\partial\tilde{\tau}}{\partial r} \ge 0$$

is a finite number and

$$\begin{aligned}
\lim_{\substack{\varepsilon\to 0\\ \delta\to 0\\ \varepsilon>\delta}} \int_{\delta<|z|<\varepsilon} \frac{\sqrt{-1}}{2}\partial\bar{\partial}(\|\chi_{;z}\|^2) &= \lim_{\substack{\varepsilon\to 0\\ \delta\to 0\\ \varepsilon>\delta}} \int_\delta^\varepsilon \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot \frac{\partial\tau}{\partial r}\right) r\, dr \\
&= \lim_{\substack{\varepsilon\to 0\\ \delta\to 0\\ \varepsilon>\delta}} \left(r \cdot \frac{\partial\tilde{\tau}}{\partial r}\right)\bigg|_\delta^\varepsilon = 0. \qquad \square
\end{aligned}$$

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