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Scalar Curvature Behavior for Finite-Time Singularity of Kähler–Ricci Flow

Zhou Zhang

1. Introduction

Ricci flow, since its debut in the famous original work [4] by Hamilton, has been one of the major driving forces for the development of geometric analysis in the past decades. Its astonishing power is best demonstrated by the breakthrough in solving the Poincaré conjecture and geometrization program. For this amazing story, we refer to [1; 7; 10] and the references therein. Meanwhile, Kähler–Ricci flow, which is Ricci flow with initial metric being Kähler, has shown some of its own characters coming from the natural relation with complex Monge–Ampère equation and many interesting algebraic geometric objects. Tian's program, as described in [14] or [15], has illustrated the direction to further improve people's understanding in many classic topics of great importance by Kähler–Ricci flow—for example, the minimal model program in algebraic geometry.

In this paper we give a very general discussion on Kähler–Ricci flows over closed manifolds. The closed manifold under consideration is denoted by X with $\dim_{\mathbb{C}} X = n$. The computation would be done for the following version of Kähler–Ricci flow:

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t, \quad \tilde{\omega}_0 = \omega_0, \tag{1.1}$$

where ω_0 is any Kähler metric on X. The special feature of this version as shown in [15] and fully discussed in [17] is rather superficial for this work.

The short time existence of the flow is known from either Hamilton's general existence result on Ricci flow in [4] or the fact that Kähler–Ricci flow is indeed parabolic when considered as a flow in a properly chosen infinite-dimensional space.

In light of the optimal existence result for Kähler–Ricci flow as in [2] or [15], we know the classic solution of (1.1) exists exactly as long as the cohomology class $[\tilde{\omega}_t]$ from formal computation remains Kähler. The actual meaning will be explained later.

Inevitably, it comes down to analyzing the behavior of the t-slice metric solution when time t approaches the (possibly infinite) singular time from cohomology consideration. In this work, we focus on the case when the flow singularity happens at some finite time. Now we state the main results.

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THEOREM 1.1. *Either the Kähler–Ricci flow* (1.1) *exists for all time or the scalar curvature blows up at some finite time* (*of singularity*); *that is,*

$$\sup_{X\times[0,T)}|R(\tilde{\omega}_t)| = +\infty,$$

where T is the finite time of singularity and $R(\cdot)$ is the scalar curvature for the corresponding metric.

The possible blow-up of scalar curvature would be from above because of the known lower bound of scalar curvature (as in [13], for example). Let's also point out that the statements of this theorem and the next theorem hold also for the other two common versions of Kähler–Ricci flow of great individual interest:

$$\frac{\partial\tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t), \quad \tilde{\omega}_0 = \omega_0;$$
$$\frac{\partial\tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) + \tilde{\omega}_t, \quad \tilde{\omega}_0 = \omega_0.$$

This can be justified by simple rescaling of time and metric to transform these flows from one to another.

REMARK 1.2. Some fundamental results regarding the finite-time blow-up of Ricci flow have been known for quite a while. More precisely, it is known that the curvature operator blows up [5] and that the Ricci curvature blows up [11]. The latter work also gives the blow-up of scalar curvature in the case of real dimension 3.

The next result, which provides some control of the blow-up rate, needs an extra assumption described in the following with natural background from algebraic geometry. The motivation is the semi-ampleness of the cohomology limit at the singular time. It is of quite some interest in algebraic geometry as explained in [15], for example.

We continue to denote the finite singular time by T and use $[\omega_T]$ to represent the cohomology limit of the flow as $t \to T$, whose meaning will be clear from the discussion in Section 2. We assume the existence of a holomorphic map

$$F\colon X \to Y,$$

where Y is an analytic variety smooth near the image $F(X)$ and there is a Kähler metric ω_M in a neighborhood of $F(X)$ such that $[\omega_T] = [F^*\omega_M]$.

The most natural way to come up with such a picture is to actually generate a map F from the class $[\omega_T]$. Of course, this would force some conditions (of algebraic geometry flavor) on $[\omega_T]$. Let's point out that it would be the case when X is an algebraic manifold and the initial class $[\omega_0] \in H^{1,1}(X,\mathbb{C}) \cap H^2(X;\mathbb{Q})$ by the classic rationality theorem (as stated in [8]).

THEOREM 1.3. *In the setting just described, for the Kähler–Ricci flow* (1.1) *with finite-time singularity at T,*

$$R(\tilde{\omega}_t) \leq \frac{C}{(T-t)^2},$$

where C is a positive constant depending on the specific flow.

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2. Proof of Theorem 1.1

The proof of such a result is usually by contradiction. Let's assume that *the scalar curvature stays uniformly bounded along the flow* (1.1) *with finite-time singularity at T.* One then makes use of Song and Tian's [13] computation for the parabolic Schwarz lemma and some basic computations on the Kähler–Ricci flow to get some uniform control of the flow metric. The contradiction then comes from the general result on the existence of Kähler–Ricci flow and the numerical characterization of a Kähler cone for closed Kähler manifolds by Demailly and Paun [3]. The rest of this section contains the detailed argument. Before that, we briefly introduce the standard machinery for reducing the Kähler–Ricci flow to the level of a scalar function flow, as in [15], and explain some statements in the Introduction.

Define $\omega_t := \omega_\infty + e^{-t}(\omega_0 - \omega_\infty)$ where $[\omega_\infty] = K_X$, the canonical class of X. In practice, one chooses

$$\omega_\infty = -\mathrm{Ric}(\Omega) := \sqrt{-1}\partial\bar{\partial}\log\frac{\Omega}{\mathrm{Vol}_E}$$

for some smooth volume form Ω over X, where Vol_E is the Euclidean volume form with respect to local holomorphic coordinate system of X. Obviously, the choice of coordinates won't affect the form, $-\mathrm{Ric}(\Omega)$.

Formally, it is clear that $[\omega_t] = [\tilde{\omega}_t]$. Now set $\tilde{\omega}_t = \omega_t + \sqrt{-1}\partial\bar{\partial}u$; then one has the following parabolic evolution equation for u over space-time:

$$\frac{\partial u}{\partial t} = \log\frac{(\omega_t + \sqrt{-1}\partial\bar{\partial}u)^n}{\Omega} - u, \quad u(\cdot, 0) = 0, \tag{2.1}$$

which is equivalent to the original Kähler–Ricci flow (1.1).

It is now time to quote the following optimal existence result of Kähler–Ricci flow (as in [2] and [15]) mentioned in the Introduction.

Proposition 2.1. *Equation* (1.1) (*or, equivalently,* (2.1)) *exists as long as* $[\omega_t]$ *remains Kähler—in other words, the solution is for the time interval* $[0, T)$, *where*

$$T = \sup\{t \mid [\omega_t] \text{ is Kähler}\}.$$

The appearance of finite-time singularity means that $[\omega_T]$, which is the cohomology limit mentioned in the Introduction, is on the boundary of the (open) Kähler cone of X in $H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{R})$ and thus is no longer Kähler. Clearly it is "numerically effective", using the natural generalization of the notion from algebraic geometry. From now on, we focus on those flows existing only for some finite interval $[0,T)$.

In what follows, C denotes a positive constant that may differ from case to case. Whenever this might cause confusion, lower indices are added to distinguish different values. These constants might well depend on the specific flow—for example, the finite singular time T.

The argument is divided into three steps as follows.

2.1. Step 1: Volume Uniform Bound

With the uniform bound on scalar curvature in $[0,T)$, we can easily derive the uniform control on the volume form along the flow by using the following evolution equation of volume form:

$$\begin{aligned}\frac{\partial \tilde{\omega}_t^n}{\partial t} &= n\frac{\partial \tilde{\omega}_t}{\partial t} \wedge \tilde{\omega}_t^{n-1}\\ &= n(-\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t) \wedge \tilde{\omega}_t^{n-1}\\ &= (-R-n)\tilde{\omega}_t^n.\end{aligned}$$

Since $\tilde{\omega}_t^n = e^{\partial u/\partial t + u}\Omega$, this actually tells us that

$$\left|\frac{\partial u}{\partial t} + u\right| \leq C.$$

REMARK 2.2. Instead of the assumption on scalar curvature, one can also directly assume the existence of a positive lower bound for the volume form or, equivalently, that $\frac{\partial u}{\partial t} \geq -C$ since we are considering the finite-time singularity case. (The upper bounds for u and $\frac{\partial u}{\partial t}$ are available in general from later discussion.) This simple observation actually brings up a very intuitive analytic understanding of Theorem 1.1; namely, the flow (2.1) can be stopped at some finite time only because the term in log is tending to 0 (i.e., there is no uniform lower bound).

2.2. Step 2: Metric Estimate

We begin with the inequality from the parabolic Schwarz lemma. Throughout this paper, the Laplacian Δ without lower index is always with respect to the changing metric along the flow, $\tilde{\omega}_t$.

Set $\phi = \langle \tilde{\omega}_t, \omega_0 \rangle$, which is obviously positive for $t \in [0,T)$. Using the computation for (1.1) in [13], one has

$$\left(\frac{\partial}{\partial t} - \Delta\right)\log\phi \leq C_1\phi + 1, \tag{2.2}$$

where C_1 is a positive constant depending on the bisectional curvature of ω_0. It's quite irrelevant here that ω_0 is the initial metric for the Kähler–Ricci flow. In fact,

ω_0 doesn't even have to be a metric over X, which is an interesting part of this computation (as indicated in [13]) and is useful for the proof of Theorem 1.3.

Applying the maximum principle to (2.1) gives $u \le C$. Taking the t-derivative for (2.1) gives

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle\tilde{\omega}_t, \omega_0 - \omega_\infty\rangle - \frac{\partial u}{\partial t},$$

where $\langle\cdot,\cdot\rangle$ means taking the trace of the right term with respect to the left (metric) term. This equation can be reformulated into the following two equations:

$$\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right) = \Delta\left(e^t\frac{\partial u}{\partial t}\right) - \langle\tilde{\omega}_t, \omega_0 - \omega_\infty\rangle;$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) = \Delta\left(\frac{\partial u}{\partial t} + u\right) - n + \langle\tilde{\omega}_t, \omega_\infty\rangle.$$

Their difference gives

$$\frac{\partial}{\partial t}\left((e^t - 1)\frac{\partial u}{\partial t} - u\right) = \Delta\left((e^t - 1)\frac{\partial u}{\partial t} - u\right) + n - \langle\tilde{\omega}_t, \omega_0\rangle. \tag{2.3}$$

By the maximum principle, this gives

$$(e^t - 1)\frac{\partial u}{\partial t} - u - nt \le C,$$

which, together with the upper bound of u and the local bound for $\frac{\partial u}{\partial t}$ near $t = 0$, would provide

$$\frac{\partial u}{\partial t} \le C.$$

The upper bounds on $\frac{\partial u}{\partial t}$ and u, together with $\left|\frac{\partial u}{\partial t} + u\right| \le C$ from volume control, give the uniform (lower) bounds on $\frac{\partial u}{\partial t}$ and u.

Multiplying (2.3) by a large enough constant $C_2 > C_1 + 1$ and combining it with (2.2), one arrives at

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\left(\log\phi + C_2\left((e^t - 1)\frac{\partial u}{\partial t} - u\right)\right) &\le nC_2 + 1 - (C_2 - C_1)\phi \\ &\le C - \phi. \end{aligned}\tag{2.4}$$

Now we apply the maximum principle for the term $\log\phi + C_2\big((e^t - 1)\frac{\partial u}{\partial t} - u\big)$. Considering the place where it achieves maximum value, one has

$$\phi \le C$$

and so, by the bounds of u and $\frac{\partial u}{\partial t}$,

$$\log\phi + C_2\left((e^t - 1)\frac{\partial u}{\partial t} - u\right) \le C.$$

Hence we conclude that $\phi = \langle\tilde{\omega}_t, \omega_0\rangle \le C$, using again the bounds on $\frac{\partial u}{\partial t}$ and u. This trace bound, together with volume bound $\tilde{\omega}_t^n \le C\omega_0^n$, provides the uniform bound of $\tilde{\omega}_t$ as a metric:

$$C^{-1}\omega_0 \le \tilde{\omega}_t \le C\omega_0.$$

The easiest way to see this is to diagonalize $\tilde{\omega}_t$ with respect to ω_0 and then deduce the uniform control of the eigenvalues from the trace and volume bounds just described.

2.3. Step 3: Contradiction

The metric (lower) bound ensures that, for any fixed analytic variety in X, the integral of the proper power of $\tilde{\omega}_t$ is bounded away from 0 and so the limiting class $[\omega_T]$ would have positive intersection with any analytic variety by taking the cohomology limit. Thus, by [3, Thm. 4.1], we conclude that $[\omega_T]$ is actually Kähler. This contradicts our assumption of finite-time singularity at T in light of Proposition 2.1.

Hence we have finished the proof of Theorem 1.1.

REMARK 2.3. Given this numerical characterization of Kähler cone for any general closed Kähler manifold [3], the blow-up of the curvature operator or Ricci curvature in the closed Kähler manifold case is fairly obvious. The situation of scalar curvature is the first nontrivial statement.

Also, if X is an algebraic manifold, then we can apply the more classic characterization of ampleness by Kleiman [6] to draw the contradiction (see [9] for the complete story in algebraic geometry).

3. Proof of Theorem 1.3

It would be more satisfying to gain some control for the blow-up of scalar curvature at finite time of singularity. Of course, this is also important for further analysis of the singularity. That is what we shall do in this section, mainly by following the argument in [18]. This proof is also divided into three steps.

3.1. Step 1: 0th-order Estimates

That $u \le C$ follows directly from (2.1). Recall that the t-derivative of (2.1) is

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \Delta\left(\frac{\partial u}{\partial t}\right) - e^{-t}\langle\tilde{\omega}_t, \omega_0 - \omega_\infty\rangle - \frac{\partial u}{\partial t},$$

which has the following variations:

$$\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right) = \Delta\left(e^t\frac{\partial u}{\partial t}\right) - \langle\tilde{\omega}_t, \omega_0 - \omega_\infty\rangle;$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} + u\right) = \Delta\left(\frac{\partial u}{\partial t} + u\right) - n + \langle\tilde{\omega}_t, \omega_\infty\rangle. \tag{3.1}$$

A proper linear combination of these equations provides the following "finite-time version" of (3.1) (by taking $e^{-\infty} = 0$, this is exactly (3.1)):

$$\frac{\partial}{\partial t}\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u\right) = \Delta\left((1 - e^{t-T})\frac{\partial u}{\partial t} + u\right) - n + \langle\tilde{\omega}_t, \omega_T\rangle.$$

As before, the difference of the original two equations gives

$$\frac{\partial}{\partial t}\left((1-e^t)\frac{\partial u}{\partial t}+u\right)=\Delta\left((1-e^t)\frac{\partial u}{\partial t}+u\right)-n+\langle\tilde{\omega}_t,\omega_0\rangle,$$

which implies the "essential decreasing" of the metric potential along the flow; that is,

$$\frac{\partial u}{\partial t}\le\frac{nt+C}{e^t-1}.$$

Notice that this estimate depends only on the initial value of u and its upper bound along the flow. It is uniform away from the initial time.

Another t-derivative gives

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}\right)=\Delta\left(\frac{\partial^2 u}{\partial t^2}\right)+e^{-t}\langle\tilde{\omega}_t,\omega_0-\omega_\infty\rangle-\frac{\partial^2 u}{\partial t^2}-\left|\frac{\partial\tilde{\omega}_t}{\partial t}\right|^2_{\tilde{\omega}_t}.$$

Take the summation with the one-time t-derivative to arrive at

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\right)=\Delta\left(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\right)-\left(\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\right)-\left|\frac{\partial\tilde{\omega}_t}{\partial t}\right|^2_{\tilde{\omega}_t},$$

which gives

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}\le Ce^{-t}.$$

This implies the "essential decreasing" of volume form along the flow:

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+u\right)\le Ce^{-t}.$$

One also has $\frac{\partial}{\partial t}\left(e^t\frac{\partial u}{\partial t}\right)\le C$, which induces

$$\frac{\partial u}{\partial t}\le(Ct+C)e^{-t}.$$

After plugging in $\tilde{\omega}_t=\omega_t+\sqrt{-1}\partial\bar{\partial}u$, the metric flow equation (1.1) becomes

$$\mathrm{Ric}(\tilde{\omega}_t)=-\sqrt{-1}\partial\bar{\partial}\left(u+\frac{\partial u}{\partial t}\right)-\omega_\infty.$$

Taking the trace with respect to $\tilde{\omega}_t$ for the preceding equation and using the trivial identity $n=\langle\tilde{\omega}_t,\omega_t+\sqrt{-1}\partial\bar{\partial}u\rangle$, we have

$$R=-\Delta\left(u+\frac{\partial u}{\partial t}\right)-\langle\tilde{\omega}_t,\omega_\infty\rangle=e^{-t}\langle\tilde{\omega}_t,\omega_0-\omega_\infty\rangle-\Delta\left(\frac{\partial u}{\partial t}\right)-n,$$

where R denotes the scalar curvature of $\tilde{\omega}_t$. In light of (3.1), we also have

$$R=-n-\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+u\right),$$

so the estimate obtained previously for $\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}+u\right)$ is nothing but the well-known lower bound for scalar curvature.

Recall that we focus on the smooth solution of Kähler–Ricci flow in $X \times [0, T)$ with finite-time singularity at T.

REMARK 3.1. For Step 1, we only need that the smooth limiting background form $\omega_T \geq 0$. It is indeed equivalent to assuming that $[\omega_T]$ has a smooth nonnegative representative and is presumably weaker than the class being "semi-ample" (i.e., the existence of a map F described before the statement of Theorem 1.3 in the Introduction).

Recall the following equation derived before:

$$\frac{\partial}{\partial t}\Big((1-e^{t-T})\frac{\partial u}{\partial t}+u\Big) = \Delta\Big((1-e^{t-T})\frac{\partial u}{\partial t}+u\Big) - n + \langle \tilde{\omega}_t, \omega_T\rangle. \tag{3.2}$$

With $\omega_T \geq 0$, by the maximum principle one has

$$(1-e^{t-T})\frac{\partial u}{\partial t}+u \geq -C.$$

Taking this together with the known upper bounds, we conclude that

$$\Big|(1-e^{t-T})\frac{\partial u}{\partial t}+u\Big| \leq C.$$

3.2. Step 2: Parabolic Schwarz Estimate

Use the setup of [13] for the map F described before the statement of Theorem 1.3. Let $\varphi = \langle \tilde{\omega}_t, F^*\omega_M\rangle$, which is clearly nonnegative; then one has, over $X \times [0, T)$,

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\varphi \leq \varphi + C\varphi^2 - H,$$

where C is related to the bisectional curvature bound of ω_M near $F(X)$ and where $H \geq 0$ is described as follows. Using normal coordinates locally over X and Y with indices i, j and α, β, respectively, $\varphi = |F_i^\alpha|^2$ and $H = |F_{ij}^\alpha|^2$ with summations for all indices. Notice that the normal coordinates over X are changing along the flow with the metric. Using this inequality, one has

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)\log\varphi \leq C\varphi + 1.$$

REMARK 3.2. For application, the map F is generated by (some multiple of) the class $[\omega_T]$ with Y some projective space $\mathbb{C}P^N$; so ω_T is $F^*\omega$, where ω is (some multiple of) the Fubini–Study metric over Y.

Define

$$v := (1-e^{t-T})\frac{\partial u}{\partial t}+u.$$

From Step 1 we know that $|v| \leq C$. By (3.2), we also have

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)v = -n + \langle \tilde{\omega}_t, \omega_T\rangle = -n + \varphi.$$

After taking a large enough positive constant A, the following inequality holds:

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)(\log\varphi - Av) \leq -\varphi + C.$$

Since v is bounded, the maximum principle can be used to deduce that $\varphi \le C$; in other words,

$$\langle \tilde{\omega}_t, \omega_T \rangle \le C.$$

3.3. Step 3: Gradient and Laplacian Estimates

Here we derive gradient and Laplacian estimates for v. Recall that

$$\left(\frac{\partial}{\partial t} - \Delta\right) v = -n + \varphi, \quad \varphi = \langle \tilde{\omega}_t, \omega_T \rangle.$$

Standard computation (as in [13]) gives

$$\left(\frac{\partial}{\partial t} - \Delta\right)(|\nabla v|^2) = |\nabla v|^2 - |\nabla\nabla v|^2 - |\nabla\bar{\nabla} v|^2 + 2\,\mathrm{Re}(\nabla\varphi, \nabla v),$$
$$\left(\frac{\partial}{\partial t} - \Delta\right)(\Delta v) = \Delta v + \big(\mathrm{Ric}(\tilde{\omega}_t), \sqrt{-1}\partial\bar{\partial} v\big) + \Delta\varphi.$$

Again, all the ∇, Δ, and $(\cdot,\cdot)$ are with respect to $\tilde{\omega}_t$, and $\nabla\bar{\nabla} v$ is just $\partial\bar{\partial} v$.

Define

$$\Psi := \frac{|\nabla v|^2}{C - v}.$$

Since v is bounded, one can easily make sure that the denominator is positive, bounded, and also away from 0. We have the following computation:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\Psi &= \left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\nabla v|^2}{C - v}\right) \\
&= \frac{1}{C - v} \cdot \frac{\partial}{\partial t}(|\nabla v|^2) + \frac{|\nabla v|^2}{(C - v)^2} \cdot \frac{\partial v}{\partial t} - \left(\frac{(|\nabla v|^2)_{\bar{i}}}{C - v} + \frac{v_{\bar{i}}|\nabla v|^2}{(C - v)^2}\right)_i \\
&= \frac{|\nabla v|^2}{(C - v)^2} \cdot \left(\frac{\partial}{\partial t} - \Delta\right) v + \frac{1}{C - v} \cdot \left(\frac{\partial}{\partial t} - \Delta\right)(|\nabla v|^2) \\
&\quad - \frac{v_i \cdot (|\nabla v|^2)_{\bar{i}}}{(C - v)^2} - v_{\bar{i}} \cdot \left(\frac{|\nabla v|^2}{(C - v)^2}\right)_i \\
&= \frac{|\nabla v|^2}{(C - v)^2} \cdot \left(\frac{\partial}{\partial t} - \Delta\right) v + \frac{1}{C - v} \cdot \left(\frac{\partial}{\partial t} - \Delta\right)(|\nabla v|^2) \\
&\quad - \frac{2\,\mathrm{Re}(\nabla v, \nabla|\nabla v|^2)}{(C - v)^2} - \frac{2|\nabla v|^4}{(C - v)^3}.
\end{aligned}$$

Plug in the equalities from before and rewrite the differential equality for Ψ as follows:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\Psi &= \frac{(-n + \varphi)|\nabla v|^2}{(C - v)^2} + \frac{|\nabla v|^2 - |\nabla\nabla v|^2 - |\nabla\bar{\nabla} v|^2}{C - v} \\
&\quad + \frac{2\,\mathrm{Re}(\nabla\varphi, \nabla v)}{C - v} - \frac{2\,\mathrm{Re}(\nabla v, \nabla|\nabla v|^2)}{(C - v)^2} - \frac{2|\nabla v|^4}{(C - v)^3}.
\end{aligned} \tag{3.3}$$

We also need the following computations:

$$
\begin{aligned}
|(\nabla v, \nabla|\nabla v|^2)| &= |v_i(v_j v_{\bar{j}})_{\bar{i}}| \\
&= |v_i v_{\bar{j}} v_{j\bar{i}} + v_i v_j v_{\bar{j}\bar{i}}| \\
&\le |\nabla v|^2(|\nabla\nabla v| + |\nabla\bar{\nabla} v|) \\
&\le \sqrt{2}|\nabla v|^2(|\nabla\nabla v|^2 + |\nabla\bar{\nabla} v|^2)^{1/2};
\end{aligned}
$$

$$
\nabla\Psi = \nabla\left(\frac{|\nabla v|^2}{C-v}\right) = \frac{\nabla(|\nabla v|^2)}{C-v} + \frac{|\nabla v|^2\nabla v}{(C-v)^2}.
$$

Given the bounds for φ and $C-v$, we may compute as follows (where ε is a small positive constant that may differ from place to place):

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\Psi \\
&\quad \le C|\nabla v|^2 + \varepsilon\cdot|\nabla\varphi|^2 - C(|\nabla\nabla v|^2 + |\nabla\bar{\nabla} v|^2) \\
&\qquad - (2-\varepsilon)\operatorname{Re}\left(\nabla\Psi, \frac{\nabla v}{C-v}\right) - \varepsilon\cdot\frac{\operatorname{Re}(\nabla v, \nabla|\nabla v|^2)}{(C-v)^2} - \varepsilon\cdot\frac{|\nabla v|^4}{(C-v)^3} \\
&\quad \le C|\nabla v|^2 + \varepsilon\cdot|\nabla\varphi|^2 - C(|\nabla\nabla v|^2 + |\nabla\bar{\nabla} v|^2) \\
&\qquad - (2-\varepsilon)\operatorname{Re}\left(\nabla\Psi, \frac{\nabla v}{C-v}\right) + \varepsilon\cdot(|\nabla\nabla v|^2 + |\nabla\bar{\nabla} v|^2) - \varepsilon\cdot|\nabla v|^4 \\
&\quad \le C|\nabla v|^2 + \varepsilon\cdot|\nabla\varphi|^2 - (2-\varepsilon)\operatorname{Re}\left(\nabla\Psi, \frac{\nabla v}{C-v}\right) - \varepsilon\cdot|\nabla v|^4.
\end{aligned}
$$

We need a few more calculations to set up our maximum principle argument. Recall that $\varphi = \langle\tilde{\omega}_t, \omega_T\rangle$ and that

$$
\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \le \varphi + C\varphi^2 - H.
$$

With φ and H as before and with the estimate for φ from Step 2 (i.e., $\varphi \le C$), we can conclude as in [13] that

$$
H \ge C|\nabla\varphi|^2.
$$

Now one arrives at

$$
\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \le C - C|\nabla\varphi|^2. \tag{3.4}
$$

We also have

$$
\left|\left(\nabla\varphi, \frac{\nabla v}{C-v}\right)\right| \le \varepsilon\cdot|\nabla\varphi|^2 + C\cdot|\nabla v|^2. \tag{3.5}
$$

Next we look at the function $\Psi+\varphi$. By choosing $\varepsilon > 0$ small enough in the preceding computation, which also affects our choices of the Cs, we have

$$
\left(\frac{\partial}{\partial t} - \Delta\right)(\Psi+\varphi) \le C + C|\nabla v|^2 - \varepsilon\cdot|\nabla v|^4 - (2-\varepsilon)\operatorname{Re}\left(\nabla(\Psi+\varphi), \frac{\nabla v}{C-v}\right).
$$

At the maximum value point of $\Psi + \varphi$, which is either at the initial time or not, we see that $|\nabla v|^2$ cannot be too large. It's then easy to establish the upper bound for $\Psi + \varphi$ and so for Ψ. Hence we have bounded the gradient; that is,

$$|\nabla v| \leq C.$$

Now we want to do a similar thing for the Laplacian Δv. Define

$$\Phi := \frac{C - \Delta v}{C - v}.$$

Similar computation as before gives the following:

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta\right)\Phi &= \left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{C - \Delta v}{C - v}\right) \\
&= -\frac{1}{C - v} \cdot \left(\frac{\partial}{\partial t} - \Delta\right)\Delta v + \frac{C - \Delta v}{(C - v)^2} \cdot \left(\frac{\partial}{\partial t} - \Delta\right) v \\
&\quad + \frac{2\,\mathrm{Re}(\nabla v, \nabla \Delta v)}{(C - v)^2} - \frac{2|\nabla v|^2 (C - \Delta v)}{(C - v)^3} \\
&= -\frac{1}{C - v} \cdot \left(\Delta v + \left(\mathrm{Ric}(\tilde{\omega}_t), \sqrt{-1}\partial\bar{\partial} v\right) + \Delta \varphi\right) + \frac{C - \Delta v}{C - v} \\
&\quad \cdot (-n + \varphi) + \frac{2\,\mathrm{Re}(\nabla v, \nabla \Delta v)}{(C - v)^2} - \frac{2|\nabla v|^2 (C - \Delta v)}{(C - v)^3}.
\end{aligned} \tag{3.6}$$

We also have

$$\nabla\left(\frac{C - \Delta v}{C - v}\right) = \frac{(C - \Delta v)\nabla v}{(C - v)^2} - \frac{\nabla \Delta v}{C - v}.$$

It is already known that $0 \leq \varphi \leq C$. The following inequality follows from standard computation as in [13] and has actually been used to derive the inequality for the parabolic Schwarz estimate:

$$\Delta \varphi \geq (\mathrm{Ric}(\tilde{\omega}_t), \omega_T) + H - C\varphi^2.$$

Since $H \geq 0$ and $0 \leq \varphi \leq C$, we have

$$\left(\mathrm{Ric}(\tilde{\omega}_t), \sqrt{-1}\partial\bar{\partial} v\right) + \Delta \varphi \geq \left(\mathrm{Ric}(\tilde{\omega}_t), \sqrt{-1}\partial\bar{\partial} v + \omega_T\right) - C. \tag{3.7}$$

Recall that we are considering the finite-time singularity case $T < \infty$, $v = (1 - e^{t-T})\frac{\partial u}{\partial t} + u$, and $\omega_T = \omega_\infty + e^{-T}(\omega_0 - \omega_\infty)$. The following is then obvious:

$$\begin{aligned}
\mathrm{Ric}(\tilde{\omega}_t) &= -\sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t} + u\right) - \omega_\infty \\
&= -\sqrt{-1}\partial\bar{\partial} v - \omega_T - e^{t-T}\sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t}\right) + e^{-T}(\omega_0 - \omega_\infty) \\
&= -\sqrt{-1}\partial\bar{\partial} v - \omega_T - e^{t-T}\left(\sqrt{-1}\partial\bar{\partial}\left(\frac{\partial u}{\partial t}\right) - e^{-t}(\omega_0 - \omega_\infty)\right) \\
&= -\sqrt{-1}\partial\bar{\partial} v - \omega_T - e^{t-T}\frac{\partial \tilde{\omega}_t}{\partial t} \\
&= -\sqrt{-1}\partial\bar{\partial} v - \omega_T - e^{t-T}(-\mathrm{Ric}(\tilde{\omega}_t) - \tilde{\omega}_t).
\end{aligned}$$

This gives

$$(1-e^{t-T})\operatorname{Ric}(\tilde\omega_t) = -\sqrt{-1}\partial\bar\partial v - \omega_T + e^{t-T}\tilde\omega_t,$$

and so we have the following two equations:

$$\operatorname{Ric}(\tilde\omega_t) = -\frac{\sqrt{-1}\partial\bar\partial v + \omega_T}{1-e^{t-T}} + \frac{e^{t-T}\tilde\omega_t}{1-e^{t-T}};$$
$$(1-e^{t-T})R = -\Delta v - \langle\tilde\omega_t, \omega_T\rangle + ne^{t-T}.$$

Because $R \geq -C$ and $\langle\tilde\omega_t,\omega_T\rangle \geq 0$, we have $\Delta v \leq C$. So the numerator of Φ, $C-\Delta v$, is positive for large enough C.

Now we can continue the estimation (3.7) as follows:

$$\begin{aligned}
&(\operatorname{Ric}(\tilde\omega_t), \sqrt{-1}\partial\bar\partial v) + \Delta\varphi\\
&\quad\geq (\operatorname{Ric}(\tilde\omega_t), \sqrt{-1}\partial\bar\partial v + \omega_T) - C\\
&\quad= \left(-\frac{\sqrt{-1}\partial\bar\partial v + \omega_T}{1-e^{t-T}} + \frac{e^{t-T}\tilde\omega_t}{1-e^{t-T}}, \sqrt{-1}\partial\bar\partial v + \omega_T\right) - C\\
&\quad= -\frac{\left|\sqrt{-1}\partial\bar\partial v + \omega_T\right|^2}{1-e^{t-T}} + \frac{e^{t-T}(\Delta v + \langle\tilde\omega_t,\omega_T\rangle)}{1-e^{t-T}} - C.
\end{aligned}$$

Since $C^{-1}(T-t) \leq 1-e^{t-T} \leq C(T-t)$ for $t\in[0,T)$, using $\Delta v = \Delta v - C + C$ and $0 \leq \langle\tilde\omega_t,\omega_T\rangle \leq C$ yields

$$\begin{aligned}
&(\operatorname{Ric}(\tilde\omega_t), \sqrt{-1}\partial\bar\partial v) + \Delta\varphi\\
&\quad= -\frac{|\sqrt{-1}\partial\bar\partial v + \omega_T|^2}{1-e^{t-T}} + \frac{e^{t-T}(\Delta v + \langle\tilde\omega_t,\omega_T\rangle)}{1-e^{t-T}} - C\\
&\quad\geq -\frac{1}{T-t}\big((1+\varepsilon)|\sqrt{-1}\partial\bar\partial v|^2 + C|\omega_T|^2\big) - \frac{C}{T-t}(C-\Delta v) - \frac{C}{T-t}\\
&\quad\geq -\frac{1+\varepsilon}{T-t}|\sqrt{-1}\partial\bar\partial v|^2 - \frac{C}{T-t}(C-\Delta v) - \frac{C}{T-t},
\end{aligned}$$

where $|\omega_T|^2 \leq C$, which follows from $0 \leq \langle\tilde\omega_t,\omega_T\rangle \leq C$, is applied in the last step.

Now we can continue the computation for Φ, (3.6), as follows:

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)\Phi\\
&\quad\leq \frac{C}{T-t} + \frac{C}{T-t}\cdot(C-\Delta v) + \frac{(1+\varepsilon)|\nabla\bar\nabla v|^2}{(T-t)(C-v)} - 2\operatorname{Re}\left(\nabla\Phi, \frac{\nabla v}{C-v}\right).
\end{aligned}$$

Using $\Phi = \frac{C-\Delta v}{C-v} \geq C(C-\Delta v)$, one arrives at

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta\right)((T-t)\Phi)\\
&\quad\leq C + C\cdot(C-\Delta v) + \frac{(1+\varepsilon)|\nabla\bar\nabla v|^2}{C-v} - 2\operatorname{Re}\left(\nabla((T-t)\Phi), \frac{\nabla v}{C-v}\right).
\end{aligned}$$

In light of (3.4) and (3.5), we have

$$\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \leq C - 4\operatorname{Re}\left(\nabla\varphi, \frac{\nabla v}{C - v}\right) + C|\nabla v|^2.$$

Also, (3.3) can be rewritten as

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\Psi \leq{}& \frac{(-n+\varphi)|\nabla v|^2}{(C-v)^2} + \frac{|\nabla v|^2 - |\nabla\nabla v|^2 - |\nabla\bar{\nabla} v|^2}{C - v}\\ &+ 2\operatorname{Re}\left(\nabla\varphi, \frac{\nabla v}{C - v}\right) - 2\operatorname{Re}\left(\Psi, \frac{\nabla v}{C - v}\right).\end{aligned}$$

Using the known bound for $|\nabla v|$ and choosing proper $0 < \varepsilon < 1$, we have

$$\begin{aligned}&\left(\frac{\partial}{\partial t} - \Delta\right)((T-t)\Phi + 2\Psi + 2\varphi)\\ &\quad\leq C + C\cdot(C - \Delta v)\\ &\qquad - 2\operatorname{Re}\left(\nabla((T-t)\Phi + 2\Psi + 2\varphi), \frac{\nabla v}{C - v}\right) - C|\nabla\bar{\nabla} v|^2\\ &\quad\leq C + C\cdot(C - \Delta v)\\ &\qquad - 2\operatorname{Re}\left(\nabla((T-t)\Phi + 2\Psi + 2\varphi), \frac{\nabla v}{C - v}\right) - C(C - \Delta v)^2,\end{aligned}$$

where $|\nabla\bar{\nabla} v|^2 \geq C(\Delta v)^2 \geq C(C - \Delta v)^2 - C$ is applied in the last step.

Now we apply the maximum principle. At the maximum value point of the function $(T-t)\Phi + 2\Psi + 2\varphi$, we have $C - \Delta v \leq C_1$. Using the known bounds on Ψ and φ, we arrive at

$$(T-t)\Phi + 2\Psi + 2\varphi \leq C$$

and so

$$\Phi \leq \frac{C}{T-t}; \quad \text{that is,} \;\; \Delta v \geq -\frac{C}{T-t}.$$

Finally, since $(1 - e^{t-T})R = -\Delta v - \langle\tilde{\omega}_t, \omega_T\rangle + ne^{t-T}$, we conclude that

$$R \leq \frac{C}{(T-t)^2}$$

and so finish the proof of Theorem 1.3.

4. Further Remarks

Here we indicate how these results fit into the big picture. There are several closely related results also worth mentioning. Some of the following remarks should give an idea of the essential difference between finite-time and infinite-time singularities for Kähler–Ricci flow.

(1) In [12], following Perelman's idea, Sesum and Tian proved that, for X with $c_1(X) > 0$ and for any initial Kähler metric ω such that $[\omega] = c_1(X)$, the Kähler–Ricci flow

$$\frac{\partial \tilde{\omega}_t}{\partial t} = -\mathrm{Ric}(\tilde{\omega}_t) + \tilde{\omega}_t$$

has uniformly bounded scalar curvature and diameter for $t \in [0, \infty)$. Notice that this is not finite-time singularity and so does not trouble our comments after Theorem 1.1. By a simple rescaling of time and metric, one can see that, for our flow (1.1) with $[\omega_0] = c_1(X) > 0$,

$$R(\tilde{\omega}_t) \leq \frac{C}{T - t}$$

for $t \in [0, T)$, where the finite singular time $T = \log 2$, which is a better control than Theorem 1.3 in this special case. So the conclusion of Theorem 1.3, though fairly general, will not be optimal for many special cases of interest.

(2) For the infinite-time singularity case, the scalar curvature would be uniformly bounded for all time if the infinite-time limiting class $[\omega_\infty]$ provides a holomorphic fiber bundle structure for X; that is, the map F as in our setting is a smooth (holomorphic) bundle map. This is actually proved in [13] if one restricts to the smooth collapsing case.

(3) The scalar curvature would also be bounded for the infinite-time singularity case if the limiting class is "semi-ample and big"—in other words, if the (possibly singular) image of the map F is of the same dimension as X, which is usually called the global volume noncollapsing case. This result is proved in [18]. The more recent work of Zhang [16] provides a nice application.

(4) A major characteristic of Kähler–Ricci flow is the cohomology information in a finite-dimensional cohomology space. It provides a natural expectation of the behavior for the flow metric, though until now most of the behavior has remained difficult to justify. Meanwhile, scalar curvature also provides very condensed information about the metric, so it is reasonable to conjecture a close relation between cohomology data of the Kähler–Ricci flow and the behavior of scalar curvature. That is exactly what we have achieved in this paper. It gives us hope that the cohomology data would indeed provide a good prediction of Kähler–Ricci flow.

References

[1] H. Cao and X. Zhu, *A complete proof of the Poincaré and geometrization conjectures—Application of the Hamilton–Perelman theory of the Ricci flow,* Asian J. Math. 10 (2006), 165–492.

[2] P. Cascini and G. La Nave, *Kähler–Ricci flow and the minimal model program for projective varieties,* ArXiv:math/0603064 (math.AG).

[3] J.-P. Demailly and M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold,* Ann. of Math. (2) 159 (2004), 1247–1274.

[4] R. S. Hamilton, *Three-manifolds with positive Ricci curvature,* J. Differential Geom. 17 (1982), 255–306.

[5] ———, *The formation of singularities in the Ricci flow,* Surveys in differential geometry, vol. II (Cambridge, 1993), pp. 7–136, International Press, Cambridge, MA, 1995.

[6] S. L. Kleiman, *Toward a numerical theory of ampleness,* Ann. of Math. (2) 84 (1966), 293–344.

[7] B. Kleiner and J. Lott, *Notes on Perelman's papers,* Geom. Topol. 12 (2008), 2587–2855.
[8] J. Kollar and S. Mori, *Birational geometry of algebraic varieties,* with the collaboration of C. H. Clemens and A. Corti [translated from the 1998 Japanese original], Cambridge Tracts in Math., 134, Cambridge Univ. Press, Cambridge, 1998.
[9] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: Line bundles and linear series,* Ergeb. Math. Grenzgeb. (3), 48, Springer-Verlag, Berlin, 2004.
[10] J. Morgan and G. Tian, *Ricci flow and the Poincaré conjecture,* Clay Math. Monogr., 3, Amer. Math. Soc., Providence, RI, 2007.
[11] N. Sesum, *Curvature tensor under the Ricci flow,* Amer. J. Math. 127 (2005), 1315–1324.
[12] N. Sesum and G. Tian, *Bounding scalar curvature and diameter along the Kähler Ricci flow (after Perelman),* J. Inst. Math. Jussieu 7 (2008), 575–587.
[13] J. Song and G. Tian, *The Kähler–Ricci flow on surfaces of positive Kodaira dimension,* Invent. Math. 170 (2007), 609–653.
[14] G. Tian, *New results and problems on Kähler–Ricci flow,* Astérisque 322 (2008), 71–92.
[15] G. Tian and Z. Zhang, *On the Kähler–Ricci flow on projective manifolds of general type,* Chinese Ann. Math. Ser. B 27 (2006), 179–192.
[16] Y. Zhang, *Miyaoka–Yau inequality for minimal projective manifolds of general type,* Proc. Amer. Math. Soc. 137 (2009), 2749–2754.
[17] Z. Zhang, *Degenerate Monge–Ampère equations over projective manifolds,* Ph.D. thesis, MIT, 2006.
[18] ———, *Scalar curvature bound for Kähler–Ricci flows over minimal manifolds of general type,* Int. Math. Res. Not. 20 (2009), 3901–3912.

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109

zhangou@umich.edu