Michigan Math. J. 58 (2009)

Block Source Algebras in p-Solvable Groups

Lluis Puig

Dedicated to the memory of Donald G. Higman

1. Introduction

1.1. The purpose of this paper is to fill a gap that has remained open since 1979, when in the Santa Cruz conference we announced the main results on the so-called local structure of the blocks of finite p-solvable groups [6], which were mainly obtained from a suitable translation to algebras of Fong's reduction [4]. At that time, the term *local structure* referred to the paper by Alperin and Broué [2], but since that meeting it has become clear that, when studying a block of a finite group, the structure to describe is its *source algebra*.

1.2. As a matter of fact, in [6] we already described the source algebra of a *nilpotent block* in a finite p-solvable group, and one of the reasons for delaying the publication of our work on the blocks of p-solvable groups was that, after Santa Cruz, we concentrated our effort on determining the structure of the source algebra of nilpotent blocks in *any* finite group [10].

1.3. Another reason for delaying this publication was that, although the translation to algebras of Fong's reduction does indeed allow one to determine the structure of the source algebra of a block in finite p-solvable groups, this structure involves a *Dade P-algebra,* where P is a defect p-subgroup of the block, and only many years later did we find a way to prove its uniqueness. A last remark: although, for the sake of simplicity, we deal only with the source algebra of a block in characteristic $p > 0$, the interested reader will see that [10, Lemma 7.8] and [11, Cor. 3.7] allow one to determine the source algebra over a complete discrete valuation ring of characteristic 0.

2. Notation and Quoted Results

2.1. We fix a prime number p and an algebraically closed field of characteristic p. It is well known that Fong's reduction involves a central extension of the finite group we start with; precisely, it involves a central extension by a finite subgroup of k^*, and a handy way to unify our setting is to consider from the beginning a central extension $\hat{G}$ of a finite group G by k^*. This is not more general since, nevertheless, $\hat{G}$ always contains a *finite* subgroup G' covering G.

Received May 24, 2007. Revision received September 16, 2007.

2.2. Explicitly, we call k^*-*group* a group X endowed with an injective group homomorphism $\theta\colon k^* \to Z(X)$ (cf. [9, Sec. 5]) and call k^*-*quotient* of (X,θ) the group $X/\theta(k^*)$; we denote by X° the k^*-group formed by X and by the composition of θ with the automorphism $k^* \cong k^*$ mapping $\lambda \in k^*$ on λ^{-1}. We say that a k^*-group is *finite* whenever its k^*-quotient is finite. Usually, we denote by $\hat{G}$ a k^*-group and by G its k^*-quotient, and we write $\lambda \cdot \hat{x}$ for the product of $\hat{x} \in \hat{G}$ by the image of $\lambda \in k^*$ in $\hat{G}$.

2.3. If $\hat{G}'$ is a second k^*-group, we denote by $\hat{G} \mathbin{\hat{\times}} \hat{G}'$ the quotient of the direct product $\hat{G} \times \hat{G}'$ by the image in $\hat{G} \times \hat{G}'$ of the *inverse* diagonal of $k^* \times k^*$, which has the obvious structure of k^*-group with k^*-quotient $G \times G'$; moreover, if $G = G'$ then we denote by $\hat{G} * \hat{G}'$ the k^*-group obtained from the inverse image of $\Delta(G) \subset G \times G$ in $\hat{G} \mathbin{\hat{\times}} \hat{G}'$, which is nothing but the so-called *sum* of both central extensions of G by k^*. In particular, we have a canonical k^*-group isomorphism

$$\hat{G} * \hat{G}^\circ \cong k^* \times G. \tag{2.3.1}$$

A k^*-group homomorphism $\varphi\colon \hat{G} \to \hat{G}'$ is a group homomorphism that preserves the k^*-multiplication.

2.4. Note that for any k-algebra A of finite dimension—just called *algebra* in the sequel—the group A^* of invertible elements has a canonical k^*-group structure. If S is a simple algebra then $\mathrm{Aut}_k(S)$ coincides with the k^*-quotient of S^*; in particular, any finite group G acting on S determines—by *pull-back*—a k^*-group $\hat{G}$ of k^*-quotient G together with a k^*-group homomorphism

$$\rho\colon \hat{G} \to S^* \tag{2.4.1}$$

(cf. [9, 5.7]).

2.5. If $\hat{G}$ is a finite k^*-group, we call $\hat{G}$-*interior algebra* any algebra A endowed with a k^*-group homomorphism

$$\rho\colon \hat{G} \to A^*; \tag{2.5.1}$$

as usual, we write $\hat{x} \cdot a$ and $a \cdot \hat{x}$ instead of $\rho(\hat{x})a$ and $a\rho(\hat{x})$ for any $\hat{x} \in \hat{G}$ and any $a \in A$. Then, a $\hat{G}$-interior algebra homomorphism from A to another $\hat{G}$-interior algebra A' is a *not necessarily unitary* algebra homomorphism $f\colon A \to A'$ fulfilling $f(\hat{x} \cdot a) = \hat{x} \cdot f(a)$ and $f(a \cdot \hat{x}) = f(a) \cdot \hat{x}$; we say that f is an *embedding* whenever $\mathrm{Ker}(f) = \{0\}$ and $\mathrm{Im}(f) = f(1)A'f(1)$. For a k^*-group homomorphism $\varphi\colon \hat{G}' \to \hat{G}$, we denote by $\mathrm{Res}_\varphi(A)$ the $\hat{G}'$-interior algebra defined by $\rho \circ \varphi$. Note that the conjugation induces an action of the k^*-quotient G of $\hat{G}$ on A, so that A becomes an ordinary G-algebra; thus, all the pointed group language developed in [7] applies to $\hat{G}$-interior algebras.

2.6. For any k^*-subgroup $\hat{H}$ of $\hat{G}$, a *point* α of $\hat{H}$ on A is an $(A^H)^*$-conjugacy class of primitive idempotents of A^H and the pair $\hat{H}_\alpha$ is a *pointed* k^*-*group* on A; we denote by $\mathcal{P}_A(\hat{H})$ the set of points of $\hat{H}$ on A. For any $i \in \alpha$, iAi has the evident structure of an $\hat{H}$-interior algebra mapping $\hat{x} \in \hat{H}$ on $\hat{x} \cdot i = i \cdot \hat{x}$, and we denote by A_α one of these mutually $(A^H)^*$-conjugate $\hat{H}$-interior algebras.

2.7. A second pointed k^*-group $\hat{K}_\beta$ on A is *contained* in $\hat{H}_\alpha$ if $\hat{K}$ is a k^*-subgroup of $\hat{H}$ and if, for any $i \in \alpha$, there is a $j \in \beta$ such that $ij = j = ji$. Then it is quite clear that the $(A^K)^*$-conjugation induces $\hat{K}$-interior algebra embeddings

$$f_\beta^\alpha\colon A_\beta \to \mathrm{Res}_{\hat{K}}^{\hat{H}}(A_\alpha). \tag{2.7.1}$$

More generally, we say that an injective k^*-group homomorphism $\varphi\colon \hat{K} \to \hat{H}$ is an *A-fusion from $\hat{K}_\beta$ to $\hat{H}_\alpha$* whenever there is a $\hat{K}$-interior algebra embedding

$$f_\varphi\colon A_\beta \to \mathrm{Res}_{\hat{K}}^{\hat{H}}(A_\alpha) \tag{2.7.2}$$

such that the inclusion $A_\beta \subset A$ and the composition of f_φ with the inclusion $A_\alpha \subset A$ are A^*-conjugate. We denote by $F_A(\hat{K}_\beta, \hat{H}_\alpha)$ the set of such fusions (cf. [8, Def. 2.5]) and by $\tilde{F}_A(\hat{K}_\beta, \hat{H}_\alpha)$ its quotient by the action of H, whereas we denote by $E_G(\hat{K}_\beta, \hat{H}_\alpha)$ and $\tilde{E}_G(\hat{K}_\beta, \hat{H}_\alpha)$ the respective subsets of fusions determined by elements of G; we set $F_A(\hat{H}_\alpha) = F_A(\hat{H}_\alpha, \hat{H}_\alpha)$ and so forth.

2.8. Note that any p-subgroup P of $\hat{G}$ can be identified with its image in G and determines the k^*-subgroup $k^* \cdot P \cong k^* \times P$ of $\hat{G}$; as usual, we consider the quotient and the algebra homomorphism

$$\mathrm{Br}_P\colon A^P \to A(P) = A^P\big/\textstyle\sum_Q A_Q^P, \tag{2.8.1}$$

where Q runs over the set of proper subgroups of P, and we call *local* any point γ of P on A not contained in $\mathrm{Ker}(\mathrm{Br}_P)$. We denote by $\mathcal{LP}_A(P)$ the set of local points of P on A. More generally, we denote by $\mathcal{L}_A$ the *local category of* A, where the objects are the *local pointed groups* on A and the morphisms are the A-fusions between them with the usual composition (cf. 2.6 and Definition 2.15 in [8]). Recall that the maximal *local pointed groups* P_γ contained in $\hat{H}_\alpha$—called *defect pointed groups* of $\hat{H}_\alpha$—are all mutually H-conjugate (cf. [7, Thm. 1.2]).

2.9. It is clear that the inclusion $k^* \subset k$ determines a k-algebra homomorphism to k from the group algebra kk^* of the group k^*, so that k becomes a kk^*-algebra. For any finite k^*-group $\hat{G}$, it is clear that the group algebra $k\hat{G}$ of the group $\hat{G}$ is also a kk^*-algebra, and then we call *k^*-group algebra* of $\hat{G}$ the algebra

$$k_*\hat{G} = k \otimes_{kk^*} k\hat{G}; \tag{2.9.1}$$

note that the dimension of $k_*\hat{G}$ coincides with $|G|$. Coherently, a *block* of $\hat{G}$ is a primitive idempotent b of the center $Z(k_*\hat{G})$, so that $\alpha = \{b\}$ is a point of $\hat{G}$ on $k_*\hat{G}$. If P_γ is a defect pointed group of $\hat{G}_\alpha$ then we call *source algebra of the block* b the P-interior algebra $(k_*\hat{G})_\gamma = (k_*\hat{G}b)_\gamma$. Recall that, for any p-subgroup P of $\hat{G}$, we have

$$(k_*\hat{G})(P) \cong k_*C_{\hat{G}}(P) \tag{2.9.2}$$

(cf. 2.9.2 and Proposition 5.15 in [9]); moreover, recall that a local pointed group Q_δ on $k_*\hat{G}$ is *self-centralizing* if $C_P(Q) = Z(Q)$ for any local pointed group P_γ on $k_*\hat{G}$ containing Q_δ.

2.10. If $\hat{G}$ is a finite k^*-group, A a $\hat{G}$-interior algebra, and $\hat{H}$ a k^*-subgroup of $\hat{G}$, then as usual we denote by $\mathrm{Res}_{\hat{H}}^{\hat{G}}(A)$ the corresponding $\hat{H}$-interior algebra. Conversely, for any $\hat{H}$-interior algebra B, we consider the *induced $\hat{G}$-interior algebra*

$$\mathrm{Ind}_{\hat{H}}^{\hat{G}}(B) = k_*\hat{G} \otimes_{k_*\hat{H}} B \otimes_{k_*\hat{H}} k_*\hat{G}, \tag{2.10.1}$$

where the distributive product is defined by the formula

$$(\hat{x} \otimes b \otimes \hat{y})(\hat{x}' \otimes b' \otimes \hat{y}') = \begin{cases} \hat{x} \otimes b.\hat{y}\hat{x}'.b' \otimes \hat{y}' & \text{if } \hat{y}\hat{x}' \in \hat{H}, \\ 0 & \text{otherwise} \end{cases} \tag{2.10.2}$$

for any $\hat{x}, \hat{y}, \hat{x}', \hat{y}' \in \hat{G}$ and any $b, b' \in B$ and where we map the element $\hat{x} \in \hat{G}$ on $\sum_{\hat{y}} \hat{x}\hat{y} \otimes 1_B \otimes \hat{y}^{-1}$, with $\hat{y} \in \hat{G}$ running over a set of representatives for $\hat{G}/\hat{H}$.

2.11. As mentioned in the Introduction, the source algebras we are looking for involve Dade P-algebras; precisely, for a finite p-group P, we call *Dade P-algebra* a simple algebra S endowed with an action of P that stabilizes a basis of S containing 1_S. Actually, the action of P on S can be lifted to a unique group homomorphism $P \to S^*$, and usually we consider S a P-interior algebra. As we shall see, this situation appears quite naturally when dealing with finite p-solvable groups and, as a matter of fact, it was Dade's motivation for introducing them in 1978 [3].

3. Fong Reduction for $\hat{G}$-Interior Algebras

3.1. In [4] Fong developed a reduction method for the characters of a finite group from the choice of a normal p'-subgroup. In fact, for a k^*-group $\hat{G}$ with finite k^*-quotient G, Fong's arguments can be extended to $\hat{G}$-interior algebras in the following way. Let A be a $\hat{G}$-interior algebra and S a G-stable semisimple unitary subalgebra of A such that G acts transitively on the set I of primitive idempotents of the center $Z(S)$ of S; let i be an element of I and denote by $\hat{H}$ the stabilizer of i in $\hat{G}$. Then the k^*-quotient H of $\hat{H}$ acts on the simple algebra Si determining a k^*-group $\hat{}H$, together with a k^*-group homomorphism $\rho\colon \hat{}H \to (Si)^*$ (cf. 2.4), and we set (cf. 2.3)

$$H\hat{} = \hat{H} * (\hat{}H)^\circ. \tag{3.1.1}$$

PROPOSITION 3.2. *With the preceding assumptions, there exists an $H\hat{}$-interior algebra B, unique up to isomorphisms, such that we have a $\hat{G}$-interior algebra isomorphism*

$$A \cong \mathrm{Ind}_{\hat{H}}^{\hat{G}}(Si \otimes_k B) \tag{3.2.1}$$

mapping $s \in Si$ on $1 \otimes (s \otimes 1_B) \otimes 1$. In particular, A and B are Morita equivalent.

Proof. The multiplication by i determines an $\hat{H}$-interior algebra structure on iAi and, since G acts transitively on I, it is easily checked that we have a $\hat{G}$-interior algebra isomorphism $A \cong \mathrm{Ind}_{\hat{H}}^{\hat{G}}(iAi)$ mapping $a \in iAi$ on $1 \otimes a \otimes 1$ (cf. [9, 2.14.2]). Now, since Si is a unitary simple subalgebra of iAi, the multiplication in this algebra induces an algebra isomorphism

$$Si \otimes_k B \cong iAi, \tag{3.2.2}$$

where B is the centralizer of Si in iAi (cf. [7, Prop. 2.1]).

Moreover, if $\hat{x} \in \hat{H}$ and $\hat{}x \in \hat{}H$ have the same image x in H then the element $\rho(\hat{}x)^{-1} \cdot \hat{x}$ of iAi centralizes Si, so that it belongs to B; whereas if $(\hat{y}, \hat{}y) \in \hat{H} \times \hat{}H$ is another such a pair then we have

$$(\rho(\hat{}x)^{-1} \cdot \hat{x})(\rho(\hat{}y)^{-1} \cdot \hat{y}) = \rho(\hat{}y)^{-1}(\rho(\hat{}x)^{-1} \cdot \hat{x}) \cdot \hat{y} = \rho(\hat{}x\hat{}y)^{-1} \cdot (\hat{x}\hat{y}), \tag{3.2.3}$$

so that B becomes an $H\hat{}$-interior algebra and isomorphism (3.2.2) becomes an $\hat{H}$-interior algebra isomorphism. □

Corollary 3.3. *With the preceding assumptions, assume that B has a unique H-conjugacy class of maximal local pointed groups P_γ, that P has a local point on Si, and that the actions of $P \times P$ on A and B by left and right multiplication stabilize bases where $P \times \{1\}$ and $\{1\} \times P$ act freely. Then Si is a Dade P-algebra and, for any local pointed group Q_δ on B, we have a local point $\iota(\delta)$ of Q on A such that isomorphism* (3.2.1) *induces a Q-interior algebra embedding*

$$A_{\iota(\delta)} \to \mathrm{Res}^H_Q(Si) \otimes_k B_\delta, \tag{3.3.1}$$

and this correspondence determines an equivalence of categories $\iota\colon \mathcal{L}_B \to \mathcal{L}_A$ between the local categories of B and A. In particular, A has a unique G-conjugacy class of maximal local pointed groups.

Proof. Since P stabilizes by conjugation a basis Y of B and since P has a local point on B, it fixes an element of Y (cf. [9, 2.8.4]) and therefore Si is a direct summand of iAi and A as kP-modules when P acts by conjugation. However, we are assuming that $P \times P$ stabilizes a basis of A by left and right multiplication; hence P stabilizes by conjugation a basis Z of Si and, since we are assuming that it has a local point on Si, P fixes an element of Z that can be replaced by 1_S, so that Si is a Dade P-algebra (cf. 2.11).

If R_{ε^A} is a local pointed group on A then R fixes at least one element of I having a nonzero image in $A(R)$; that is to say, up to G-conjugation, we may assume that $R \subset \hat{H}$ and $\mathrm{Br}^A_R(i) \neq 0$, so that R_{ε^A} comes from a local pointed group on $iAi \cong Si \otimes_k B$ (cf. [12, 2.11.2]), which forces

$$(Si)(R) \neq \{0\} \quad \text{and} \quad B(R) \neq \{0\} \tag{3.3.2}$$

since the k-algebra homomorphism $(Si)(R) \otimes_k B(R) \to (Si \otimes_k B)(R)$ (cf. [12, 7.9.2]) is unitary. In particular, R has local points on B and, since $\mathrm{Res}^H_R(Si)$ is a Dade R-algebra, we actually get

$$(iAi)(R) \cong (Si)(R) \otimes_k B(R) \tag{3.3.3}$$

(cf. [12, Lemma 7.10]).

Conversely, let Q_δ be a local pointed group on B and assume that $Q_\delta \subset P_\gamma$; once again, we get

$$(iAi)(Q) \cong (Si)(Q) \otimes_k B(Q) \tag{3.3.4}$$

and we know that $(Si)(Q)$ is a simple algebra (cf. [11, 1.8.1]). In particular, the $B(Q)^*$-conjugacy class $\mathrm{Br}^B_Q(\delta)$ of primitive idempotents of $B(Q)$ and the unique conjugacy class of primitive idempotents of $(Si)(Q)$ together determine a local point $\iota_i(\delta)$ of Q on iAi and therefore a local point $\iota(\delta)$ of Q on A (cf. [12, 2.11.2]) such that, for a suitable $j' \in \iota(\delta)$ fulfilling $j'i = j' = ij'$, isomorphism (3.3.4) maps $\mathrm{Br}_Q^A(j')$ on $\mathrm{Br}_Q^S(\ell) \otimes \mathrm{Br}_Q^B(j)$, where ℓ is a suitable primitive idempotent of $(Si)^Q$ and $j \in \delta$. Actually, up to an identification via isomorphism (3.2.1), we may assume that

$$j'(\ell \otimes j) = j' = (\ell \otimes j)j' \tag{3.3.5}$$

and then we obtain a Q-interior algebra embedding

$$A_{\iota(\delta)} \to \mathrm{Res}_Q^H(Si) \otimes_k B_\delta. \tag{3.3.6}$$

On the other hand, for a second local pointed group R_ε on B it follows from [8, Cor. 2.16] that

$$F_{Si \otimes_k B}(R_{\iota_i(\varepsilon)}, Q_{\iota_i(\delta)}) = F_A(R_{\iota(\varepsilon)}, Q_{\iota(\delta)}); \tag{3.3.7}$$

once again, we may assume that $R_\varepsilon \subset P_\gamma$ and then, since B has a $(P \times P)$-stable basis where $P \times \{1\}$ and $\{1\} \times P$ act freely, it follows from [5, Lemma 1.17] that

$$F_{Si \otimes_k B}(R_{\iota_i(\varepsilon)}, Q_{\iota_i(\delta)}) \subset F_B(R_\varepsilon, Q_\delta). \tag{3.3.8}$$

Moreover, since A and thus $Si \otimes_k B$ also have $(P \times P)$-stable bases where $P \times \{1\}$ and $\{1\} \times P$ act freely, the same Lemma 1.17 in [5] applies to the fusions on $(Si)^\circ \otimes_k (Si \otimes_k B)$ and therefore, since we successively have P-algebra embeddings $k \to (Si)^\circ \otimes_k Si$ (cf. 1.3.2 and 1.3.3 in [11]) and

$$B \to (Si)^\circ \otimes_k Si \otimes_k B, \tag{3.3.9}$$

we still obtain (cf. [8, Prop. 2.14])

$$F_B(R_\varepsilon, Q_\delta) \subset F_{Si \otimes_k B}(R_{\iota_i(\varepsilon)}, Q_{\iota_i(\delta)}). \tag{3.3.10}$$

Finally, we obtain the equality

$$F_B(R_\varepsilon, Q_\delta) = F_A(R_{\iota(\varepsilon)}, Q_{\iota(\delta)}), \tag{3.3.11}$$

which proves that the functor $\iota\colon \mathcal{L}_B \to \mathcal{L}_A$ is *fully faithful*. But we have already proved that this functor is *essentially surjective,* so that it is an equivalence of categories. We are done. □

3.4. The main point in our Fong reduction is that, if A is a *block algebra* $k_*\hat{G}b$ for a block b of $\hat{G}$, then i is a block of $\hat{H}$ and, moreover, if either p does not divide $\dim_k(Si)$ or we have $S = k_*\hat{K}b$ for some normal k^*-subgroup $\hat{K}$ of $\hat{G}$ having a block d of *defect zero* such that $db \neq 0$, then B is also a block algebra. Denote by V a simple Si-module, which becomes a $k_*\hat{\ }H$-module throughout ρ (cf. 3.1).

PROPOSITION 3.5. *With the preceding assumptions, if $A \cong k_*\hat{G}b$ for a block b of $\hat{G}$, then i is a block of $\hat{H}$ that belongs to a point β of $\hat{H}$ on A and we have $i(k_*\hat{G})i = k_*\hat{H}i$. In particular, we have an equivalence of categories $\mathcal{L}_{k_*\hat{H}i} \cong \mathcal{L}_{k_*\hat{G}b}$.*

Proof. Since $i \cdot \hat{x} \cdot i = \hat{x} \cdot (i^{\hat{x}})i = 0$ for any $\hat{x} \in \hat{G} - \hat{H}$, we get $i(k_*\hat{G})i = k_*\hat{H}i$. Similarly, denoting by $\tau\colon k_*\hat{G} \to k$ the linear form vanishing on $\hat{G} - k^*.1$ and sending the unity element to 1, which clearly defines a nonsingular symmetric bilinear form, we have $i = \sum_{x\in G} \tau(i \cdot \hat{x})\hat{x}^{-1}$, where $\hat{x}$ lifts $x \in G$ to $\hat{G}$, and thus, since

$$\tau(i \cdot \hat{x}) = \tau(i \cdot \hat{x} \cdot i^{\hat{x}}) = \tau(i^{\hat{x}} i \cdot \hat{x}) = 0 \quad \text{for any } \hat{x} \in \hat{G} - \hat{H}, \tag{3.5.1}$$

i belongs to $Z(k_*\hat{H})$; moreover, since b is primitive in $Z(k_*\hat{G})$, the idempotent i must be primitive in $Z(k_*\hat{H})$ and, since $iAi = k_*\hat{H}i$, the idempotent i is primitive in A^H, too. On the other hand, assuming that $S = \bigoplus_{i\in I} k \cdot i$, it is quite clear that all the hypotheses in Corollary 3.3 hold and therefore the last statement follows from this corollary. $\square$

THEOREM 3.6. *With the preceding assumptions, assume that $A \cong k_*\hat{G}b$ for a block b of $\hat{G}$ and that p does not divide* $\dim_k(V)$. *Then we have $B \cong k_*H\hat{\ }c$ for a block c of $H\hat{\ }$, and V is a simple $k_*\hat{\ }H$-module. In particular, we have an equivalence of categories $\mathcal{L}_{k_*H\hat{\ }c} \cong \mathcal{L}_{k_*\hat{G}b}$.*

Proof. Because $Si \otimes_k B \cong i(k_*\hat{G})i = k_*\hat{H}i$, the respective images of $\hat{\ }H$ and $H\hat{\ }$ still generate Si and B; in particular, V becomes a simple $k_*\hat{\ }H$-module and, since i is primitive in $Z(k_*\hat{H})$, there is a block c of $H\hat{\ }$ such that we have a surjective $H\hat{\ }$-interior algebra homomorphism $g\colon k_*H\hat{\ }c \to B$. It remains to prove that g is also injective or, equivalently, that

$$\dim_k(k_*H\hat{\ }c) \le \dim_k(B). \tag{3.6.1}$$

Once again, since $Si \otimes_k B \cong i(k_*\hat{G})i = k_*\hat{H}i$, the structural homomorphism $\hat{H} \to Si \otimes_k k_*H\hat{\ }c$ determines a section s of the $\hat{H}$-interior algebra homomorphism

$$\mathrm{id}_{Si} \otimes g\colon Si \otimes_k k_*H\hat{\ }c \to Si \otimes_k B, \tag{3.6.2}$$

so the $k_*H\hat{\ }$-interior algebra homomorphism

$$\mathrm{id}_{(Si)^\circ\otimes_k Si} \otimes g\colon (Si)^\circ \otimes_k Si \otimes_k k_*H\hat{\ }c \to (Si)^\circ \otimes_k Si \otimes_k B \tag{3.6.3}$$

admits the section $\mathrm{id}_{(Si)^\circ} \otimes s$.

On the other hand, since we assume that p does not divide $\dim_k(Si)$, it follows that k is a direct summand of Si as kH-modules and thus we have an H-interior algebra embedding $h\colon k \to (Si)^\circ \otimes_k Si \cong \mathrm{End}_k(Si)$ (cf. [14, Ex. 4.15]). Hence the surjective $H\hat{\ }$-interior algebra homomorphism g can be embedded in homomorphism (3.6.3), determining an evident commutative diagram

$$\begin{array}{ccc} (Si)^\circ \otimes_k Si \otimes_k k_*H\hat{\ }c & \longrightarrow & (Si)^\circ \otimes_k Si \otimes_k B \\ \uparrow & & \uparrow \\ k_*H\hat{\ }c & \longrightarrow & B \end{array} \tag{3.6.4}$$

and in particular, we have

$$(\mathrm{id}_{(Si)^\circ\otimes_k Si} \otimes g)(h(1) \otimes c) = h(1) \otimes 1_B. \tag{3.6.5}$$

Consequently, since both idempotents

$$j = h(1) \otimes c \quad \text{and} \quad \ell = (\mathrm{id}_{(Si)^\circ} \otimes s)(h(1) \otimes 1_B) \tag{3.6.6}$$

lift $h(1) \otimes 1_B$ to the algebra $T = ((Si)^\circ \otimes_k Si \otimes_k k_* H\hat{\ } c)^H$ and since j is primitive, we have $j\ell^t = j = \ell^t j$ for a suitable $t \in T^*$ (cf. [14, Cor. 2.14]). However, since

$$h(1)((Si)^\circ \otimes_k Si)h(1) = k \cdot h(1) \tag{3.6.7}$$

it follows that

$$j((Si)^\circ \otimes_k Si \otimes_k k_* H\hat{\ } c)j = h(1) \otimes k_* H\hat{\ } c \cong k_* H\hat{\ } c,$$

and similarly we still have

$$(h(1) \otimes 1_B)((Si)^\circ \otimes_k Si \otimes_k B)(h(1) \otimes 1_B) = h(1) \otimes B \cong B. \tag{3.6.8}$$

Then the multiplication by j maps $(\mathrm{id}_{(Si)^\circ} \otimes s)(h(1) \otimes B)^t$, which is an $H\hat{\ }$-interior subalgebra, *onto* $h(1) \otimes k_* H\hat{\ } c$ because it maps $H\hat{\ } \cdot \ell^t$ onto $h(1) \otimes H\hat{\ } c$; this proves inequality (3.6.1).

At this point, setting $\beta = \{c\}$ and choosing a defect pointed group P_γ of $H_{\hat{\beta}}$, it is clear that the actions of $P \times P$ on A and B by left and right multiplication stabilize bases where $P \times \{1\}$ and $\{1\} \times P$ act freely (cf. [8, 3.3]); moreover, since $B(P) \neq \{0\}$ (cf. 2.8), acting by conjugation P fixes at least one element in a P-stable basis of B (cf. [9, 2.8.4]). Hence it follows from isomorphism (3.2.1) that Si is a direct summand of A as kP-modules always via the action of P by conjugation. Consequently, since P still stabilizes a basis of A, P stabilizes a basis Z of Si and moreover, since p does not divide $|Z|$, P fixes an element of Z. In other words, Si with the action of P becomes a Dade P-algebra (cf. 2.11). Now, the last statement follows from Corollary 3.3 and we are done. □

THEOREM 3.7. *With the preceding assumptions, assume that $A \cong k_*\hat{G}b$ for a block b of $\hat{G}$ and that $S = k_*\hat{K}b$ for a normal k^*-subgroup $\hat{K}$ of $\hat{G}$ having a block d of defect zero such that $db \neq 0$. Then K is a normal subgroup of $H\hat{\ }$ and we have $B \cong k_*(H\hat{\ }/K)\bar{c}$ for a block $\bar{c}$ of $H\hat{\ }/K$. In particular, we have an equivalence of categories $\mathcal{L}_{k_*(H\hat{\ }/K)\bar{c}} \cong \mathcal{L}_{k_*\hat{G}b}$.*

Proof. We clearly may assume that $i = db$; then $\hat{K}$ is contained in both $\hat{H}$ and $\hat{\ }H$, which provides a canonical lifting of the k^*-quotient K to $H\hat{\ }$ (cf. 2.3.1). Up to the identification of K with its canonical image in $H\hat{\ }$, we set $\bar{H}\hat{\ } = H\hat{\ }/K$ and $\bar{H} = H/K$. On the other hand, since H fixes d, multiplying by d the direct sum decomposition

$$k_*\hat{H} = \bigoplus_{\bar{x} \in H/K} (k_*\hat{K})\hat{x}, \tag{3.7.1}$$

where $\hat{x}$ lifts $\bar{x} \in \bar{H}$ to $\hat{H}$, yields

$$\dim_k(k_*\hat{H}d) = \dim_k(k_*\hat{K}d)|H/K|. \tag{3.7.2}$$

Thus, setting $e = \mathrm{Tr}_H^G(d)$ and applying Proposition 3.2 to the $\hat{G}$-interior algebra $k_*\hat{G}e$ with the G-stable semisimple algebra $k_*\hat{K}e$, we obtain a $\hat{G}$-interior algebra isomorphism

$$k_*\hat{G}e \cong \mathrm{Ind}_{\hat{H}}^{\hat{G}}(k_*\hat{K}d \otimes_k k_*\bar{H}^\wedge); \tag{3.7.3}$$

in particular, this isomorphism induces an algebra isomorphism

$$Z(k_*\hat{G}e) \cong Z(k_*\bar{H}^\wedge) \tag{3.7.4}$$

mapping b on a block $\bar{c}$ of $H^\wedge/K$, and then it is quite clear that

$$k_*\bar{H}^\wedge\bar{c} \cong B. \tag{3.7.5}$$

Now set $\alpha = \{b\}$ and choose a defect pointed group P_γ of $\hat{G}_\alpha$. According to Proposition 3.5, we may assume that $P_\gamma \subset H_\beta$, so that P_γ comes from a local pointed group on $iAi \cong Si \otimes_k B$ (cf. [12, 2.11.2]), which forces

$$(Si)(P) \neq \{0\} \quad \text{and} \quad B(P) \neq \{0\} \tag{3.7.6}$$

because the k-algebra homomorphism $(Si)(P) \otimes_k B(P) \to (Si \otimes_k B)(P)$ (cf. [12, 7.9.2]) is unitary. In particular, since P stabilizes a basis Z of $Si = k_*\hat{K}db$, we know that P fixes an element of Z (cf. [9, 2.8.4]) and thus $\mathrm{Res}_P^H(Si)$ is a Dade P-algebra (cf. 2.11). However, d is a block of defect zero of $\hat{K}$ and so we have $(Si)(R) = \{0\}$ for any nontrivial p-subgroup R of $\hat{K}$ (cf. 2.8); thus we have $P \cap K = \{1\}$ and therefore P is isomorphic to its image $\bar{P}$ in $\bar{H}^\wedge$.

Consequently, since $A \cong k_*\hat{G}b$ and $k_*\bar{H}^\wedge\bar{c} \cong B$, the actions of $P \times P$ on A and B by left and right multiplication stabilize bases where $P \times \{1\}$ and $\{1\} \times P$ act freely (cf. [8, 3.3]). Then, since

$$(Si \otimes_k B)(P) \cong (Si)(P) \otimes_k B(P) \tag{3.7.7}$$

(cf. [9, 2.8.4]), γ determines a local point $\bar{\gamma}$ of $P \cong \bar{P}$ on B (cf. [10, Prop. 5.6]) and it follows from [8, Thm. 3.1] that

$$F_B(\bar{P}_{\bar{\gamma}}) \cong N_{\bar{H}^\wedge}(\bar{P}_{\bar{\gamma}})/C_{\bar{H}^\wedge}(\bar{P}), \tag{3.7.8}$$

so that the subgroup $F_B(\bar{P}_{\bar{\gamma}})$ of $\mathrm{Aut}(\bar{P})$ *stabilizes* the Dade P-algebra Si. At this point, it follows from [5, Lemma 1.17] and [8, Prop. 2.14] that

$$F_A(P_\gamma) \cong F_B(\bar{P}_{\bar{\gamma}}); \tag{3.7.9}$$

thus, since P_γ is a maximal local pointed group on $k_*\hat{G}b$, the Brauer First Main Theorem implies that $N_{\bar{H}^\wedge}(\bar{P}_{\bar{\gamma}})/\bar{P} \cdot C_{\bar{H}^\wedge}(\bar{P})$ is a p'-group and hence that $\bar{P}_{\bar{\gamma}}$ is a maximal local pointed group on $k_*\bar{H}^\wedge\bar{c} \cong B$. Now, the last statement follows from Corollary 3.3 and we are done. □

4. The *p*-Solvable *k**-Group Case

4.1. As before, $\hat{G}$ is a k^*-group with finite k^*-quotient G, and in this section we assume that G is p-solvable. Let b be a block of $\hat{G}$ and let S be a G-stable semi-simple unitary subalgebra of $k_*\hat{G}b$ that is maximal such that p does not divide the dimension of its simple factors; since b is primitive in $Z(k_*\hat{G}b)$, the group G acts transitively on the set I of primitive idempotents of $Z(S)$ and we borrow the notation i, $\hat{H}$, ${}^\wedge H$, ρ, and $H^\wedge$ from 3.1. According to Propositions 3.2 and 3.5 and to Theorem 3.6, i is a block of $\hat{H}$ that belongs to a point β of $\hat{H}$ on $k_*\hat{G}$ and, for a suitable block c of $H\hat{}$, we have $\hat{G}$- and $\hat{H}$-interior algebra isomorphisms

$$k_*\hat{G}b \cong \mathrm{Ind}_{\hat{H}}^{\hat{G}}(k_*\hat{H}i) \quad \text{and} \quad (k_*\hat{G})_\beta \cong k_*\hat{H}i \cong Si \otimes_k k_*H\hat{}c \tag{4.1.1}$$

as well as an equivalence of categories $\iota\colon \mathcal{L}_{k_*H\hat{}c} \to \mathcal{L}_{k_*\hat{G}b}$; in particular, there is a defect pointed group P_γ of b contained in $\hat{H}_\beta$. Denote by $\mathbf{O}_{p'}(\hat{H})$, $\mathbf{O}_{p'}(\hat{}H)$, and $\mathbf{O}_{p'}(H\hat{})$ the respective inverse images in $\hat{H}$, $\hat{}H$, and $H\hat{}$ of $\mathbf{O}_{p'}(H)$.

PROPOSITION 4.2. *Assume that G is p-solvable. Then P is a Sylow p-subgroup of H, we have $Si = k_*\mathbf{O}_{p'}(\hat{H})i$, and the inclusion $\mathbf{O}_{p'}(\hat{H})i \subset (Si)^*$ induces an H-stable k^*-group isomorphism $\sigma : k^* \times \mathbf{O}_{p'}(H) \cong \mathbf{O}_{p'}(H\hat{})$ such that*

$$c = \frac{1}{|\mathbf{O}_{p'}(H)|} \sum_{y \in \mathbf{O}_{p'}(H)} \sigma(y) \quad \textit{and} \quad k_*H\hat{}c \cong k_* \frac{H\hat{}}{\sigma(\mathbf{O}_{p'}(H))}. \tag{4.2.1}$$

Moreover, setting $Q = P \cap \mathbf{O}_{p',p}(H)$, the idempotent c is primitive in $(k_*H\hat{}c)^Q$.

Proof. If T is an H-stable semisimple unitary subalgebra of $k_*H\hat{}c$ such that p does not divide the dimension of its simple factors, then in the induced algebra $\mathrm{Ind}_{\hat{H}}^{\hat{G}}(Si \otimes_k k_*H\hat{}c)$ the direct sum $\sum_x \hat{x} \otimes (Si \otimes_k T) \otimes \hat{x}^{-1}$, where $x \in G$ runs over a set of representatives for G/H and $\hat{x} \in \hat{G}$ lifts x, determines a G-stable semisimple unitary subalgebra of $k_*\hat{G}b$ fulfilling the preceding condition and containing S. Thus, the maximality of S forces $T = k \cdot c$.

In particular, since the algebra $k_*\mathbf{O}_{p'}(H\hat{})$ is semisimple, we obtain

$$k_*\mathbf{O}_{p'}(H\hat{})c = k \cdot c, \tag{4.2.2}$$

which forces $\mathbf{O}_{p'}(\hat{H})i \subset Si$. Then we necessarily have $\mathbf{O}_{p'}(\hat{H})i = \rho(\mathbf{O}_{p'}(\hat{}H))$ and thus still get an H-stable k^*-group isomorphism (cf. (2.3.1))

$$\sigma : k^* \times \mathbf{O}_{p'}(H) \cong \mathbf{O}_{p'}(H\hat{}). \tag{4.2.3}$$

Therefore, setting

$$e = \frac{1}{|\mathbf{O}_{p'}(H)|} \sum_{y \in \mathbf{O}_{p'}(H)} \sigma(y) \quad \text{and} \quad L\hat{} = \frac{H\hat{}}{\sigma(\mathbf{O}_{p'}(H))}, \tag{4.2.4}$$

we have $ec = c$ and that c determines a block of $L\hat{}$; but since H is p-solvable, $C_L(\mathbf{O}_p(L)) = Z(\mathbf{O}_p(L))$ and therefore $\mathbf{O}_p(L)$ has a unique local point on $k_*L\hat{} \cong k_*H\hat{}e$ (cf. (2.9.2)) that actually has multiplicity 1 (cf. (2.9.2)). Moreover, it is easily checked that $\mathrm{Ker}(\mathrm{Br}_{\mathbf{O}_p(L)}) \subset J(k_*L\hat{})$, so the unity element is primitive in $(k_*L\hat{})^{\mathbf{O}_p(L)}$ (cf. (2.9.2)); hence c coincides with e and is primitive in $(k_*H\hat{})^R$ for any p-subgroup R of H such that $\mathbf{O}_{p',p}(H) \subset \mathbf{O}_{p'}(H) \cdot R$.

Consequently, we have

$$k_*H\hat{}c \cong k_*L\hat{} \tag{4.2.5}$$

and $\{c\}$ is the unique local point of R on $k_*H\hat{}c$; this forces P to be a Sylow p-subgroup of H because $N_H(P_{\{c\}})/P \cdot C_H(P)$ is a p'-group by the Brauer First

Main Theorem. Moreover, since $(k_*\hat{G})_\beta \cong Si \otimes_k k_*H\hat{\ }c$, R has a unique local point ε on $k_*\hat{G}$ such that $R_\varepsilon \subset \hat{H}_\beta$ (cf. Proposition 5.6 and Corollary 5.8 in [10]).

Finally, $T = k_*\mathbf{O}_{p'}(\hat{H})i$ is an H-stable semisimple unitary subalgebra of $k_*\hat{H}i$ and therefore—denoting by j a primitive idempotent of $Z(T)$, by $\hat{K}$ the stabilizer of j in $\hat{H}$, and by C the centralizer of Tj in $j(k_*\hat{H})j$—it follows from Proposition 3.2 that, for a suitable $k_*\hat{K}$-interior algebra structure on $Tj \otimes_k C$, we have a $\hat{H}$-interior algebra isomorphism

$$k_*\hat{H}i \cong \mathrm{Ind}_{\hat{K}}^{\hat{H}}(Tj \otimes_k C). \tag{4.2.6}$$

More precisely, it follows from Theorem 3.6 that C is the block algebra of a suitable k^*-group with k^*-quotient K and, since

$$\mathbf{O}_{p'}(\hat{H}) \subset \hat{K} \quad \text{and} \quad \mathbf{O}_{p'}(\hat{H})j \subset Tj, \tag{4.2.7}$$

the inverse image of $\mathbf{O}_{p'}(H)$ in this k^*-group has a trivial image in C; therefore,

$$\dim_k(C) \leq |K : \mathbf{O}_{p'}(H)|. \tag{4.2.8}$$

Furthermore, since $T \subset Si$ we have

$$|H : K| \dim_k(Tj) \leq \dim_k(Si), \tag{4.2.9}$$

the inequality being strict whenever $K \neq H$.

On the other hand, isomorphisms (4.1.1) and (4.2.6) imply that

$$\begin{aligned}\dim_k(Si)|H : \mathbf{O}_{p'}(H)| &= \dim_k(k_*\hat{H}i) \\ &= |H : K|^2 \dim_k(Tj) \dim_k(C).\end{aligned} \tag{4.2.10}$$

Hence the preceding inequalities are actually equalities, so we have $K = H$, $j = i$, and $T = Si$ as claimed. □

COROLLARY 4.3. *Assume that G is p-solvable, set $Q = P \cap \mathbf{O}_{p',p}(H)$, and denote by γ and δ the respective local points of P and Q on $k_*\hat{G}$ such that $Q_\delta \subset P_\gamma \subset H_\beta$. Then Q_δ is the unique local pointed group on $k_*\hat{G}$ that fulfills the following conditions:*

(4.3.1) *$Q_\delta \lhd P_\gamma$, $C_P(Q) = Z(Q)$, and $\mathbf{O}_p(\tilde{E}_G(Q_\delta)) = \{1\}$;*

(4.3.2) *$E_G(R_\varepsilon, P_\gamma) = E_{N_G(Q_\delta)}(R_\varepsilon, P_\gamma)$ for any local pointed group R_ε on $k_*\hat{G}$ contained in P_γ.*

Proof. With notation as in Proposition 4.2, set $\hat{L} = H\hat{\ }/\sigma(\mathbf{O}_{p'}(H))$. It follows from this proposition that the unity element is primitive in $(k_*\hat{L})^Q$ and so determines local points γ° and δ° of P and Q, respectively, on $k_*\hat{L}$; since we have $k_*\hat{L} \cong k_*H\hat{\ }c$, these local points determine local points γ and δ of P and Q, respectively, on $k_*\hat{G}$ (cf. Proposition 5.6 and Corollary. 5.8 in [10] and Proposition 2.14 in [8]). Therefore, P normalizes Q_δ and the p-solvability of H forces $C_P(Q) = Z(Q)$.

Moreover, it follows from Theorem 3.6 that, for any local pointed group R_ε on $k_*\hat{G}$ contained in P_γ, we have

$$E_G(R_\varepsilon, P_\gamma) = E_L(R_{\varepsilon^\circ}, P_{\gamma^\circ}), \tag{4.3.3}$$

where ε° denotes the corresponding local point of R on $k_*\hat{L}$. However, the uniqueness of δ forces $N_H(Q_\delta) = N_H(Q)$ and thus, by the Frattini argument, we obtain $H = \mathbf{O}_{p'}(H).N_H(Q_\delta)$. Consequently, it is easily checked that we still have

$$E_G(R_\varepsilon, P_\gamma) = E_{N_H(Q_\delta)}(R_\varepsilon, P_\gamma). \tag{4.3.4}$$

In particular, $E_G(Q_\delta) \cong N_H(Q)/C_H(Q)$ and so we still get

$$\mathbf{O}_p(\tilde{E}_G(Q_\delta)) = \{1\}. \tag{4.3.5}$$

Finally, if T_θ is a local pointed group on $k_*\hat{G}b$ fulfilling conditions (4.3.1) and (4.3.2), then we have $E_G(Q_\delta) = E_{N_G(T_\theta)}(Q_\delta)$ and therefore $E_T(Q_\delta)$ is a normal p-subgroup of $E_G(Q_\delta)$, so that $\tilde{E}_T(Q_\delta) = \{1\}$. Hence, we have

$$T \subset P \cap Q.C_G(Q) = Q \tag{4.3.6}$$

and, by symmetry, the equality follows. □

4.4. With notation as in Proposition 4.2, set $\hat{L} = H\hat{}/\sigma(\mathbf{O}_{p'}(H))$ and consider $k_*\hat{L}$ endowed with the obvious group homomorphism $P \to (k_*\hat{L})^*$ as a P-interior algebra. Then isomorphisms (4.1.1) and (4.2.1) yield a P-interior algebra embedding

$$(k_*\hat{G})_\gamma \to \mathrm{Res}_P^{\hat{H}}(Si) \otimes_k k_*\hat{L}, \tag{4.4.1}$$

which already gives a satisfactory description of a source algebra of the block b *except* that we know nothing about the uniqueness of the left tensor factor. In order to get this we need the following lemma, which we prove in a more general context.

LEMMA 4.5. *Let $\hat{L}$ and $\hat{L}'$ be k^*-groups with respective finite k^*-quotients L and L' fulfilling*

$$C_L(\mathbf{O}_p(L)) = Z(\mathbf{O}_p(L)) \quad \text{and} \quad C_{L'}(\mathbf{O}_p(L')) = Z(\mathbf{O}_p(L')), \tag{4.5.1}$$

and denote by P a Sylow p-subgroup of L. If $\tau\colon P \to \hat{L}'$ is an injective group homomorphism and if T is a Dade P-algebra such that there exists a P-interior algebra embedding

$$k_*\hat{L} \to T \otimes_k k_*\hat{L}', \tag{4.5.2}$$

then T is similar to k and $\hat{L}$ isomorphic to $\hat{L}'$.

Proof. Through embedding (4.5.2), any local pointed group R_ε on $k_*\hat{L}$ determines a local point ε' of R on $k_*\hat{L}'$ such that, denoting by ρ the unique local point of R on T, (4.5.2) induces an R-interior algebra embedding

$$(k_*\hat{L})_\varepsilon \to T_\rho \otimes_k (k_*\hat{L}')_{\varepsilon'} \tag{4.5.3}$$

(cf. Proposition 5.6 and Corollary 5.8 in [10] and Proposition 2.14 in [8]). Moreover, since there is a P-interior algebra embedding $k \to T^\circ \otimes_k T$ (cf. [10, 5.7]), from (4.5.3) we easily derive the embedding

$$(k_*\hat{L}')_{\varepsilon'} \to (T_\rho)^\circ \otimes_k (k_*\hat{L})_\varepsilon; \tag{4.5.4}$$

in particular, since $(T_\rho)(R) \cong k$ (cf. [10, Cor. 5.8]), if R_ε is self-centralizing (cf. 2.9) then, setting $R' = \tau(R)$, we have

$$(k_*\hat{L}')_{\varepsilon'}(R') \cong (k_*\hat{L})_\varepsilon(R) \cong kZ(R) \cong kZ(R'). \tag{4.5.5}$$

Therefore, according to [13, Lemma 2.14], $R'_{\varepsilon'}$ is self-centralizing, too.

Set $Q = \mathbf{O}_p(L)$. Actually, since $\mathrm{Ker}(\mathrm{Br}_Q) \subset J(k_*\hat{L})$ and since we assume that $C_L(Q) = Z(Q)$, the unity element is primitive in $(k_*\hat{L})^Q$ (cf. (2.9.2)) and therefore $\gamma = \{1_{k_*\hat{L}}\}$ is the unique point of P on $k_*\hat{L}$, which is maximal local. In this situation, setting $P' = \tau(P)$, denoting by γ' the corresponding local point of P' on $k_*\hat{L}'$, and considering the corresponding embeddings (4.5.3) and (4.5.4), it follows from [5, Lemma 1.17] that

$$E_L(R_\varepsilon, P_\gamma) \cong E_{L'}(R'_{\varepsilon'}, P'_{\gamma'}). \tag{4.5.6}$$

Hence, by the Brauer First Main Theorem, the maximality of P_γ implies that $\tilde{E}_L(P_\gamma)$ is a p'-group; then, since $P'_{\gamma'}$ is self-centralizing, isomorphism (4.5.6) implies that $P'_{\gamma'}$ is maximal local on k_*L' and so P' contains $\mathbf{O}_p(L')$ (cf. [1, Sec. 13, Thm. 6]). Consequently, according to our assumption on L', we have $\gamma' = \{1_{k_*\hat{L}'}\}$ and $C_{L'}(P') = Z(P')$, which implies that P' is a Sylow p-subgroup of L'.

At this point, the existence of embedding (4.5.4) for $R_\varepsilon = P_\gamma$ shows that our hypotheses are actually symmetric on $\hat{L}$ and $\hat{L}'$. On the other hand, considering the point $\delta = \gamma$ of Q on $k_*\hat{L}$, it follows easily from the isomorphism $E_L(Q_\delta) \cong E_{L'}(Q'_{\delta'})$ that

$$|L| = |Q||E_L(Q_\delta)| \le |N_{L'}(Q'_{\delta'})| \le |L'| \tag{4.5.7}$$

and thus, by symmetry, we obtain $|L| = |L'|$, $Q' = \mathbf{O}_p(L')$, and $\delta' = \gamma'$. More precisely, by isomorphisms (4.5.6) we can apply [5, Thm. 1.8] to show that L and L' are isomorphic.

Moreover, by [10, Thm. 5.3], the isomorphism $E_L(Q_\delta) \cong E_{L'}(Q'_{\delta'})$ can be lifted to a k^*-group isomorphism $\hat{E}_L(Q_\delta) \cong \hat{E}_{L'}(Q'_{\delta'})$; but in our situation it is clear that

$$\hat{E}_L(Q_\delta) \cong \hat{L}/\mathbf{O}_p(L) \quad \text{and} \quad \hat{E}_{L'}(Q'_{\delta'}) \cong \hat{L}'/\mathbf{O}_p(L'). \tag{4.5.8}$$

That is to say, from now on we may assume that $\hat{L} = \hat{L}'$ and that the unity element is primitive in T^P.

Since $(k_*\hat{L})(Q) \cong kZ(Q)$, embedding (4.5.2) induces a P/Q-algebra embedding (cf. [10, Cor. 5.8])

$$k \to T(Q), \tag{4.5.9}$$

which, since T^P covers $T(Q)^P$ (cf. [9, 2.8.4]), is actually an isomorphism. However, it is clear that embedding (4.5.2) determines an embedding between the corresponding semisimple quotients, so that setting $\mathrm{S}(k_*\hat{L}) = k_*\hat{L}/J(k_*\hat{L})$ yields a P-algebra embedding

$$\mathrm{S}(k_*\hat{L}) \to T \otimes_k \mathrm{S}(k_*\hat{L}), \tag{4.5.10}$$

which tensored by $\mathrm{S}(k_*\hat{L})^\circ$ determines another embedding,

$$h\colon \mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ \to T \otimes_k \mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ. \tag{4.5.11}$$

Furthermore, since Q acts trivially on $\mathrm{S}(k_*\hat{L})$, the image of h is contained in $T^Q \otimes_k \mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ$. Consequently, since $T(Q) \cong k$ (cf. (4.5.9)), h induces a P-algebra automorphism

$$h(Q)\colon \mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ \cong \mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ \tag{4.5.12}$$

mapping $s \in \mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ$ on $\mathrm{Br}_Q(h(s))$.

On the other hand, it is well known (cf. [14, Cor. 12.10]) that some simple $k_*\hat{L}$-module M has a dimension prime to p and, in particular, that there is a point λ of L on $\mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ$ such that

$$(\mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ)_\lambda \cong k. \tag{4.5.13}$$

It then follows from [14, Thm. 7.2] that λ is contained in a local point of P on $\mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ$ and hence, choosing $j \in \lambda$, there is a primitive idempotent j' in $(\mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ)^P$ such that (cf. [10, Prop. 5.6])

$$h(j)(1 \otimes j')^a = h(j) = (1 \otimes j')^a h(j) \tag{4.5.14}$$

for some invertible element a of $(T \otimes_k \mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ)^P$. Therefore, since $\mathrm{Br}_Q(1 \otimes j') = j'$ and j' is primitive in $(\mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ)^P$, it follows from equalities (4.5.14) that $\mathrm{Br}_Q(h(j)) = j'^{\mathrm{Br}_Q(a)}$; in particular, isomorphism (4.5.13) implies that

$$j'(\mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ)j' \cong k. \tag{4.5.15}$$

Finally, according to equalities (4.5.14), h and the conjugation by a determine a P-algebra embedding

$$k \cong j(\mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ)j \to T \otimes_k j'(\mathrm{S}(k_*\hat{L}) \otimes_k \mathrm{S}(k_*\hat{L})^\circ)j' \cong T, \tag{4.5.16}$$

which proves that T is similar to k (cf. [11, 1.7.2]). □

THEOREM 4.6. *Assume that G is p-solvable. With notation as before, denote by γ the local point of P on $k_*\hat{G}$ such that $P_\gamma \subset H_\beta$. Then there exist a k^*-group $\hat{L}$, with finite k^*-quotient L, endowed with an injective group homomorphism $\tau\colon P \to \hat{L}$, and a Dade P-algebra T both unique up to isomorphisms, that fulfill the following conditions*:

(4.6.1) $C_L(\mathbf{O}_p(L)) = Z(\mathbf{O}_p(L))$ *and the unity element is primitive in T^P*;
(4.6.2) *there is a P-interior algebra embedding*

$$(k_*\hat{G})_\gamma \to T \otimes_k k_*\hat{L}. \tag{4.6.3}$$

In particular, P has a unique local point γ' on $T \otimes_k k_*\hat{L}$, and this embedding induces a P-interior algebra isomorphism $(k_*\hat{G})_\gamma \cong (T \otimes_k k_*\hat{L})_{\gamma'}$.

Proof. From 4.4 we already know that there exist a k^*-group $\hat{L}'$, with a finite p-solvable k^*-quotient L' such that $\mathbf{O}_{p'}(L') = \{1\}$, endowed with an injective group homomorphism $\tau'\colon P \to \hat{L}'$ such that the image of P is a Sylow p-subgroup of L', together with a Dade P-algebra T' with the unity element primitive in T'^P, admitting a P-interior algebra embedding

$$(k_*\hat{G})_\gamma \to T' \otimes_k k_*\hat{L}', \tag{4.6.4}$$

which already proves the existence statement.

Moreover, since there is a P-interior algebra embedding (cf. [10, 5.7])

$$k \to T'^{\circ} \otimes_k T'$$

and since the unity element is primitive in $(k_*\hat{L}')^P$, it follows from embedding (4.6.4) that

$$k_*\hat{L}' \to T'^{\circ} \otimes_k (k_*\hat{G})_\gamma. \tag{4.6.5}$$

Consequently, for $\hat{L}$ and T as in the statement of the theorem, we have another P-interior algebra embedding

$$k_*\hat{L}' \to (T'^{\circ} \otimes_k T) \otimes_k k_*\hat{L} \tag{4.6.6}$$

and it then follows from Lemma 4.5 that $T'^{\circ} \otimes_k T$ is similar to k or, equivalently, that T is similar to T'. Because the unity elements are primitive in T^P and T'^P, T and T' are actually isomorphic. We are done. □

4.7. We now give a "constructive" description of the Dade P-algebra that appears in a source P-interior algebra of the block b. Consider the *chains* $\{Z_n\}_{n\in\mathbb{N}}$ and $\{T_n\}_{n\in\mathbb{N}}$ of G-stable semisimple unitary subalgebras of $k_*\hat{G}b$ defined recursively by

$$Z_0 = k.b, \quad T_n = \sum_{\{j\}\in\mathcal{P}(Z_n)} k_*\mathbf{O}_{p'}(\hat{G}_j)j, \quad Z_{n+1} = Z(T_n), \tag{4.7.1}$$

where $\hat{G}_j$ denotes the stabilizer of j in $\hat{G}$ for any $n \in \mathbb{N}$ and any $\{j\} \in \mathcal{P}(Z_n)$. It is clear that $Z_n \subset Z(T_n) = Z_{n+1}$ and that $T_n \subset T_{n+1}$, so the union

$$T = \bigcup_{n\in\mathbb{N}} T_n \tag{4.7.2}$$

is also a G-stable semisimple unitary subalgebra of $k_*\hat{G}b$; in fact, $T = T_n$ for some $n \in \mathbb{N}$.

4.8. Choosing a primitive idempotent j of $Z(T)$ fixed by P such that $s_\gamma(j) \neq 0$, which is possible because γ is local, and denoting by $\hat{K}$ the stabilizer of j in $\hat{G}$, we see that the maximality of T forces $k_*\mathbf{O}_{p'}(\hat{K})j = Tj$. In particular, denote by $\hat{}K$ the k^*-group determined by the action of K on Tj and, as before, set $K\hat{} = \hat{K} * (\hat{}K)^{\circ}$; then, up to suitable identifications, $\hat{}K$ and $\hat{}K$ contain $\mathbf{O}_{p'}(\hat{K})$ and $\mathbf{O}_{p'}(K)$, respectively, and it follows from Proposition 3.2 and Theorem 3.6 that we have a $\hat{G}$-interior algebra isomorphism

$$k_*\hat{G}b \cong \mathrm{Ind}_{\hat{K}}^{\hat{G}}(k_*\mathbf{O}_{p'}(\hat{K})j \otimes_k k_*(K\hat{}/\mathbf{O}_{p'}(K))) \tag{4.8.1}$$

since the unity element is the unique block of $K\hat{}/\mathbf{O}_{p'}(K)$ (cf. Proposition 4.2). Consequently, we obtain a P-interior algebra embedding

$$(k_*\hat{G})_\gamma \to k_*\mathbf{O}_{p'}(\hat{K})j \otimes_k k_*(K\hat{}/\mathbf{O}_{p'}(K)) \tag{4.8.2}$$

and then Theorem 4.6 applies.

References

[1] J. Alperin, *Local representation theory,* Cambridge Stud. Adv. Math., 11, Cambridge Univ. Press, Cambridge, 1986.

[2] J. Alperin and M. Broué, *Local methods in block theory,* Ann. of Math. (2) 110 (1979), 143–157.

[3] E. Dade, *Endo-permutation modules over p-groups I, II,* Ann. of Math. (2) 107, 108 (1978), 459–494, 317–346.

[4] P. Fong, *On the characters of p-solvable groups,* Trans. Amer. Math. Soc. 98 (1961), 263–284.

[5] B. Külshammer and L. Puig, *Extensions of nilpotent blocks,* Invent. Math. 102 (1990), 17–71.

[6] L. Puig, *Local block theory in p-solvable groups,* Santa Cruz conference on finite groups (Santa Cruz, 1979), Proc. Sympos. Pure Math., 37, pp. 385–388, Amer. Math. Soc., Providence, RI, 1980.

[7] ———, *Pointed groups and construction of characters,* Math. Z. 176 (1981), 265–292.

[8] ———, *Local fusions in block source algebras,* J. Algebra 104 (1986), 358–369.

[9] ———, *Pointed groups and construction of modules,* J. Algebra 116 (1988), 4–129.

[10] ———, *Nilpotent blocks and their source algebras,* Invent. Math. 93 (1988), 77–116.

[11] ———, *Affirmative answer to a question of Feit,* J. Algebra 131 (1990), 513–526.

[12] ———, *On the local structure of Morita and Rickard equivalences between Brauer blocks,* Progr. Math., 178, Birkhäuser, Basel, 1999.

[13] ———, *Source algebras of p-central group extensions,* J. Algebra 235 (2001), 359–398.

[14] ———, *Blocks of finite groups,* Springer Monogr. Math., Springer-Verlag, New York, 2002.

CNRS
Université Paris 7
94340 Joinville-le-Pont
France

puig@math.jussieu.fr