Michigan Math. J. 56 (2008)

Sobolev Peano Cubes

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In memoriam: Juha M. Heinonen (1960–2007)

1. Introduction

A classical theorem of Hahn [8] and Mazurkiewicz [19] states that X is a locally connected continuum if and only if there exists a continuous surjection $f: [0,1] \to X$. Since any cube $[0,1]^n$ is a continuous image of $[0,1]$, an equivalent statement is: X is a locally connected continuum if and only if there exists a continuous surjection $f: [0,1]^n \to X$.

The purpose of this paper is to generalize the Hahn–Mazurkiewicz theorem to differentiable and weakly differentiable mappings. Not surprisingly, our assumptions on X will be stronger.

Following Kirchheim [15], we say that a map $f: \Omega \to X$ from an open set $\Omega \subset \mathbb{R}^n$ to a metric space X is *metrically differentiable* at $x \in \Omega$ if there is a seminorm $\|\cdot\|_x$ on $\mathbb{R}^n$ such that

$$d(f(x), f(y)) - \|y - x\|_x = o(|y - x|) \quad \text{for } y \in \Omega. \tag{1.1}$$

The seminorm assumption means that $\|a + b\|_x \le \|a\|_x + \|b\|_x$ and $\|ta\|_x = |t|\|a\|_x$ but $\|\cdot\|_x$ can vanish on a linear subspace on $\mathbb{R}^n$, and (1.1) means that

$$\lim_{y \to x} \frac{d(f(x), f(y)) - \|y - x\|_x}{|y - x|} = 0.$$

Clearly, if $f: \Omega \to \mathbb{R}^k$ is (classically) differentiable at $x \in \Omega$, then it is metrically differentiable with $\|u\|_x = |Df(x)(u)|$. It is also easy to see that $f: (a,b) \to X$ is metrically differentiable at $x \in (a,b)$ if and only if the limit

$$\lim_{h \to 0} \frac{d(f(x+h), f(x))}{|h|}$$

exists and is finite.

A classical theorem of Rademacher [6] says that Lipschitz continuous functions $f: \Omega \to \mathbb{R}^k$ are differentiable a.e. Kirchheim [15] generalized this theorem to the case of metric space–valued mappings as follows.

Theorem 1.1 (Kirchheim). *A Lipschitz continuous map $f: \Omega \to X$ from an open set $\Omega \subset \mathbb{R}^n$ to a metric space X is metrically differentiable a.e.*

Received November 19, 2007. Revision received February 4, 2008.
P.H. supported by NSF grant DMS 0500966. J.T.T. supported by NSF grant DMS 0555869.

A function $f\colon \Omega \to \mathbb{R}$ belongs to the Sobolev space $W^{1,p}(\Omega)$, $1 \le p < \infty$, if $f \in L^p(\Omega)$ and the weak gradient $|\nabla f|$ belongs to $L^p(\Omega)$. The following definition for the Sobolev space of mappings with values in a metric space was introduced by Reshetnyak [22] and elaborated in [24] and [13].

DEFINITION 1.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and let (X,d) be a compact metric space. A map $f\colon \Omega \to X$ belongs to the *Reshetnyak–Sobolev* space $R^{1,p}(\Omega, X)$ if there is a nonnegative function $g \in L^p(\Omega)$ such that for every Lipschitz continuous $\varphi\colon X \to \mathbb{R}$ we have $\varphi \circ f \in W^{1,p}(\Omega)$ and $|\nabla(\varphi \circ f)| \le \mathrm{Lip}(\varphi) g$ a.e. Here $\mathrm{Lip}(\varphi)$ stands for the Lipschitz constant of φ.

Definition 1.2 can be extended to arbitrary open sets $\Omega \subset \mathbb{R}^n$ and metric spaces (X,d), but in this case one must also take into account the L^p-integrability of the mapping. This issue and other equivalent approaches to the definition of Sobolev mappings with values in metric spaces are carefully discussed in Section 2.

Our aim in this paper is to investigate a class of compact metric spaces X for which there exists a continuous surjection $f\colon [0,1]^n \to X$ that either is metrically differentiable a.e. or belongs to a Sobolev space.

Let $f\colon [0,1]^n \to X$ be a continuous map with

$$f|_{(0,1)^n} \in R^{1,p}((0,1)^n, X); \tag{1.2}$$

henceforth we write $f \in R^{1,p}([0,1]^n, X)$ if (1.2) is satisfied. Because Sobolev mappings are absolutely continuous on almost all lines (Lemma 2.13), the image of almost every segment in $[0,1]^n$ parallel to one of the coordinate axes is a rectifiable curve. Thus f must be constant if there are no nonconstant rectifiable curves in X. In the statement of Theorem 1.3 we assume that the class of rectifiable curves in X is rich in a certain qualitative way.

Suppose that any two points $x, y \in X$ can be connected by a curve of finite length. Then $d_\ell(x,y)$, defined as the infimum of lengths of curves connecting x to y, is a metric. We call it the *length metric.* Since $d_\ell(x,y) \ge d(x,y)$, it easily follows that if X is compact with respect to d_ℓ then X is compact with respect to d. We say that X is *length compact* if (X, d_ℓ) is compact. Our main result reads as follows.

THEOREM 1.3. *Let X be a length compact metric space. If $n \ge 2$, then there is a continuous and a.e. metrically differentiable surjection f in the Sobolev class $R^{1,n}([0,1]^n, X)$. Moreover, there is a continuous and a.e. metrically differentiable surjection $f\colon [0,1] \to X$.*

In fact, the maps that we construct will be locally Lipschitz continuous on the complement of sets of Hausdorff dimension 0 and hence metrically differentiable a.e. by the theorem of Kirchheim.

Observe that if $f \in R^{1,n}$, then $f \in R^{1,p}$ for all $1 \le p < n$. However, we cannot replace $R^{1,n}$ by $R^{1,p}$ ($p > n \ge 2$) in the theorem, because Sobolev embedding into the space of Hölder continuous functions [13, Thm. 6.2] yields an upper bound for the Hausdorff dimension of $f([0,1]^n)$ whereas the space X in Theorem 1.3 may have infinite dimension (for example, X may be the Hilbert cube).

On the other hand, the absolute continuity of Sobolev maps on one-dimensional intervals implies that $f([0,1])$ is rectifiable whenever $f \in R^{1,1}([0,1], X)$. Hence in the one-dimensional case we cannot always obtain Sobolev regularity for f.

Although the length compactness condition is quite strong, it covers, from the topological point of view, the class of all locally connected continua. This follows because Bing [2; 3] and Moise [20] proved that every locally connected continuum admits a topologically equivalent convex metric (d is *convex* if $d = d_\ell$), and with such a metric Theorem 1.3 applies.

Lebesgue [16, pp. 44–45] was the first to construct Peano-type space-filling curves with a.e. differentiable coordinate functions. A careful review of various classical constructions of space-filling curves—including those of Peano, Hilbert, Lebesgue, and Schoenberg—can be found in Sagan [23]. The existence of continuous and a.e. metrically differentiable surjections from $[0,1]$ to any length compact metric space X can be established by starting with a continuous surjection from the Cantor set to X and then linearly interpolating to the omitted intervals along connecting geodesics. The proof that we give is a concrete realization of this scheme. However, the primary content of Theorem 1.3 resides in the higher-dimensional ($n \ge 2$) statement, where we must work harder to guarantee membership in the Sobolev space.

A minor modification of the proof of Theorem 1.3 leads to the following result.

THEOREM 1.4. *Let $X = \mathbb{R}^n$ and let d be a metric on X that is topologically equivalent to the Euclidean metric and has the property that every two points in $\mathbb{R}^n$ can be connected by a C^∞-smooth curve of finite length with respect to d. Then there is a continuous surjection $f \in R^{1,2}(\mathbb{R}^2, X)$ that is C^∞-smooth on $\mathbb{R}^2 \setminus F$, where F is a closed set of Hausdorff dimension 0 contained in the x-axis.*

Carnot groups are nilpotent stratified Lie groups (equipped with the so-called Carnot–Carathéodory metric) that verify the assumptions of Theorem 1.4; see for example [7, Sec. 1.2B; 21]. Thus Theorem 1.4 implies the following statement.

COROLLARY 1.5. *Let $\mathbb{G}$ be a Carnot group equipped with the Carnot–Carathéodory metric. Then there is a continuous surjection $f \in R^{1,2}(\mathbb{R}^2, \mathbb{G})$ that is smooth outside a set F of Hausdorff dimension 0.*

The map that we construct in Corollary 1.5 is horizontal. In other words, its differential takes values in the horizontal distribution of $\mathbb{G}$ on $\mathbb{R}^2 \setminus F$.

The Heisenberg group $\mathbb{H}_1$ is the simplest nontrivial Carnot group. It is homeomorphic to $\mathbb{R}^3$, but as a metric space (equipped with the Carnot–Carathéodory metric d_{cc}) it has Hausdorff dimension 4. In this case Corollary 1.5 contrasts with the pure k-unrectifiability of $\mathbb{H}_1$ for $k = 2, 3, 4$: Every Lipschitz map $f\colon A \to \mathbb{H}_1$, $A \subset \mathbb{R}^k$, has $\mathcal{H}^k(f(A)) = 0$, where $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure in the metric d_{cc} (see e.g. [1]). In particular, any C^1-horizontal map $f\colon \mathbb{R}^k \to \mathbb{H}_1$ satisfies $\mathcal{H}^k(f(\mathbb{R}^k)) = 0$ for $k = 2, 3, 4$. In a forthcoming paper [11] we construct C^1-horizontal surjections from $\mathbb{R}^5$ to $\mathbb{H}_1$ as well as C^α-surjections from $\mathbb{R}^4$ to $\mathbb{H}_1$ for each $\alpha < 1$.

Although spaces that are the image of $[0,1]^n$ under a continuous Sobolev mapping must carry a rich family of rectifiable curves, it is not necessary that every two points be connected by a rectifiable curve.

EXAMPLE 1.6. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $f \in C^\infty((0,1])$, $f(0) = 0$, $f(1) = 0$, and the graph of f over $[0,\varepsilon)$ has infinite length for any $\varepsilon > 0$. Let X be the graph of f over $[0,1]$. Then $(0,0), (1,0) \in X$ are the endpoints of X. Note that $(0,0)$ cannot be connected to any other point in X by a rectifiable curve. Let $\gamma\colon [0,\infty) \to X \setminus \{(0,0)\}$ be the arc length parameterization such that $\gamma(0) = (1,0)$ and $\lim_{t\to\infty} \gamma(t) = (0,0)$. The function $f\colon \overline{B^n}(0,e^{-1}) \to X$ defined by the formula

$$f(x) = \begin{cases} \gamma(\log|\log|x||) & \text{if } x \neq 0, \\ (0,0) & \text{if } x = 0 \end{cases}$$

is continuous, belongs to $W^{1,n}(B^n(0,e^{-1}), X)$, maps $\overline{B^n}(0,e^{-1})$ onto X, and is C^∞-smooth in $B^n(0,e^{-1}) \setminus \{0\}$ (as a map to $\mathbb{R}^2$).

The paper is organized as follows. In Section 2 we compare different approaches to the theory of Sobolev mappings into metric spaces. In Section 3 we prove Theorem 1.3, and in Section 4 we prove Theorem 1.4.

ACKNOWLEDGMENTS. We are grateful to the referee for helpful suggestions that improved the paper's exposition.

Near the completion of this work, we learned the sad news that our colleague, mentor, and friend Juha Heinonen had passed away. Juha's influence on geometric analysis in metric spaces cannnot be overstated; his passion for the subject and vision for its future were instrumental in dramatically expanding its scope and visibility. Our mathematical community has suffered the grievous loss of a central leader in the prime of his career. We mourn his passing, and we dedicate this paper with deep respect to his memory.

2. Sobolev Mappings into Banach and Metric Spaces

Throughout the paper, all Banach spaces will be over the field of real numbers.

If V is a Banach space and $A \subset \mathbb{R}^n$ is (Lebesgue) measurable, then we say that $f \in L^p(A,V)$ (resp. $f \in L^p_{\mathrm{loc}}(A,V)$) if the following statements hold.

(1) f is *essentially separably valued*: $f(A \setminus Z)$ is a separable subset of V for some Lebesgue null set Z.
(2) f is *weakly measurable*: for every $v^* \in V^*$ with $\|v^*\| \le 1$, $\langle v^*, f\rangle$ is measurable.
(3) $\|f\| \in L^p(A)$ (resp. $\|f\| \in L^p_{\mathrm{loc}}(A)$).

The *Bochner integral* of an essentially separably valued and weakly measurable function $f\colon A \to V$ is defined as

$$\int_A f(x)\,dx := \lim_{i\to\infty} \int_A f_i(x)\,dx,$$

where the limit is taken over any sequence (f_i) of simple functions

$$f_i = \sum_{j=1}^{J_i} v_{ij}\chi_{A_{ij}}$$

converging pointwise a.e. to f and where $\int_A f_i(x)\,dx := \sum_{j=1}^{J_i} |A_{ij}|v_{ij}$. It is well known that the Bochner integral exists for any $f \in L^1(A, V)$ and that it satisfies

$$\left\langle v^*, \int_A f(x)\,dx \right\rangle = \int_A \langle v^*, f(x)\rangle\,dx$$

for every $v^* \in V^*$ and

$$\left\| \int_A f(x)\,dx \right\| \le \int_A \|f(x)\|\,dx.$$

For more information about vector-valued L^p spaces, see [4] and [25, Chap. 5, Secs. 4–5].

We now introduce a class of Banach space-valued first-order Sobolev functions.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set. The Sobolev space $W^{1,p}(\Omega, V)$, $1 \le p < \infty$, is defined as the class of all functions $f \in L^p(\Omega, V)$ such that for $i = 1, 2, \dots, n$ there exist $f_i \in L^p(\Omega, V)$ such that, for every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \frac{\partial \varphi}{\partial x_i} f = -\int_\Omega \varphi f_i,$$

where the integrals are taken in the sense of Bochner (note that the integrands are supported on compact subsets of Ω). We denote $f_i = \partial f/\partial x_i$ and call these functions *weak partial derivatives* of f. We also write $\nabla f = (\partial f/\partial x_1, \dots, \partial f/\partial x_n)$ and

$$|\nabla f| = \left(\sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|^2 \right)^{1/2}. \tag{2.1}$$

The space $W^{1,p}(\Omega, V)$ is equipped with the norm

$$\|f\|_{1,p} = \left(\int_\Omega \|f\|^p \right)^{1/p} + \left(\int_\Omega |\nabla f|^p \right)^{1/p}.$$

It is easy to prove that $W^{1,p}(\Omega, V)$ is a Banach space.

Another definition of the Sobolev space with values in a Banach space was introduced by Reshetnyak [22] and carefully developed in [13]. Definition 2.2 is equivalent to but slightly different from that introduced in [22] (cf. [13] and Proposition 2.16).

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be an open set, let V be a Banach space, and let $1 \le p < \infty$. The *Reshetnyak–Sobolev space* $R^{1,p}(\Omega, V)$ is the class of all functions $f \in L^p(\Omega, V)$ such that:

(i) for every $v^* \in V^*$, $\|v^*\| \le 1$, we have $\langle v^*, f\rangle \in W^{1,p}(\Omega)$; and
(ii) there is a nonnegative function $g \in L^p(\Omega)$ such that

$$|\nabla \langle v^*, f\rangle| \le g \text{ a.e.} \tag{2.2}$$

for every $v^* \in V^*$ with $\|v^*\| \le 1$.

It is easy to prove that $R^{1,p}(\Omega, V)$ is a Banach space with respect to the norm

$$\|f\|_{R^{1,p}} = \|f\|_p + \inf \|g\|_p,$$

where the infimum is over the class of all g that satisfy (2.2). The proof mimics the standard proof of completeness of L^p.

PROPOSITION 2.3. *If $\Omega \subset \mathbb{R}^n$ is open and V is a Banach space, then $W^{1,p}(\Omega, V) \subset R^{1,p}(\Omega, V)$ and $\|f\|_{R^{1,p}} \le \|f\|_{1,p}$ for all $f \in W^{1,p}(\Omega, V)$.*

Proof. Let $f \in W^{1,p}(\Omega, V)$. Then, for every $i = 1, 2, \dots, n$ and every $\varphi \in C_0^\infty(\Omega)$,

$$\int_\Omega \frac{\partial \varphi}{\partial x_i} f = -\int_\Omega \varphi \frac{\partial f}{\partial x_i}.$$

Hence for every $v^* \in V^*$, $\|v^*\| \le 1$, we have

$$\int_\Omega \frac{\partial \varphi}{\partial x_i} \langle v^*, f \rangle = \left\langle v^*, \int_\Omega \frac{\partial \varphi}{\partial x_i} f \right\rangle = \left\langle v^*, -\int_\Omega \varphi \frac{\partial f}{\partial x_i} \right\rangle = -\int_\Omega \varphi \left\langle v^*, \frac{\partial f}{\partial x_i} \right\rangle.$$

This proves that $\langle v^*, f \rangle \in W^{1,p}(\Omega)$ and $\frac{\partial}{\partial x_i} \langle v^*, f \rangle = \left\langle v^*, \frac{\partial f}{\partial x_i} \right\rangle$. Therefore,

$$\left| \frac{\partial}{\partial x_i} \langle v^*, f \rangle \right| = \left| \left\langle v^*, \frac{\partial f}{\partial x_i} \right\rangle \right| \le \left\| \frac{\partial f}{\partial x_i} \right\|$$

and thus

$$|\nabla \langle v^*, f \rangle| \le |\nabla f|,$$

where $|\nabla f|$ is defined by (2.1). $\square$

In order to prove the opposite inclusion $R^{1,p} \subset W^{1,p}$, we will prove that $R^{1,p}$ functions are absolutely continuous on almost all lines and that we can integrate by parts. It turns out that we can prove a suitable integration-by-parts formula only in the case when V is dual to a separable Banach space.

DEFINITION 2.4. Let V be a Banach space. We say that a function $f : [a, b] \to V$ is *absolutely continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if $[a_1, b_1], \dots, [a_n, b_n] \subset [a, b]$ are intervals with disjoint interiors such that $\sum_{i=1}^n |b_i - a_i| < \delta$, then $\sum_{i=1}^n \|f(b_i) - f(a_i)\| < \varepsilon$.

LEMMA 2.5. *If $f : [a, b] \to V$ is absolutely continuous, then it is a rectifiable curve of length*

$$\ell(f) = \sup \sum_{i=1}^n \|f(a_i) - f(a_{i-1})\| < \infty,$$

where the supremum is over all n and all partitions $a = a_0 < a_1 < \cdots < a_n = b$.

The proof is straightforward and is left to the reader. With a rectifiable curve $f : [a, b] \to V$ we can associate a length function

$$s_f : [a, b] \to [0, \ell(f)], \quad s_f(t) = \ell(f|_{[a,t]}).$$

LEMMA 2.6. *If $f\colon [a,b] \to V$ is absolutely continuous, then the length function $s_f\colon [a,b] \to [0,\ell(f)]$ is absolutely continuous.*

The proof is standard and is left to the reader. It is actually easy to see that, in the case of s_f, the choice of $\delta > 0$ for a given $\varepsilon > 0$ is the same as for the function f.

Every rectifiable curve $f\colon [a,b] \to V$ has an arc length parameterization; that is, there exists a unique function $\tilde{f}\colon [0,\ell(f)] \to V$ such that $f = \tilde{f} \circ s_f$. Moreover, $\ell(\tilde{f}|_{[0,t]}) = t$ for every $t \in [0,\ell(f)]$. In particular, $\tilde{f}\colon [0,\ell(f)] \to V$ is 1-Lipschitz; see [10, Thm. 3.2].

LEMMA 2.7. *If $f\colon [a,b] \to V$ is absolutely continuous, then the function*

$$g(x) = \limsup_{h\to 0} \left\| \frac{f(x+h) - f(x)}{h} \right\| \le |s_f'(x)| \ \ a.e. \tag{2.3}$$

is integrable on $[a,b]$.

Proof. The inequality in (2.3) follows from the 1-Lipschitz condition for $\tilde{f}$, and integrability of g stems from the fact that $s_f' \in L^1([a,b])$ as a derivative of a real-valued absolutely continuous function. □

If $V = Y^*$ is a dual of a Banach space Y, then by the canonical embedding $Y \subset Y^{**} = V^*$ we can interpret elements of Y as functionals on V. The next lemma is similar to [1, Thm. 3.5].

LEMMA 2.8. *Let $V = Y^*$ be dual to a separable Banach space Y. If $f\colon [a,b] \to V$ is absolutely continuous, then for a.e. $x \in (a,b)$ there is a vector $f'(x) \in V$ such that $\|f'(x)\| \le |s_f'(x)|$ and*

$$\left\langle v^*, \frac{f(x+h) - f(x)}{h} \right\rangle \to \langle v^*, f'(x)\rangle \quad as\ \ h \to 0$$

*for all $v^* \in Y$. We call $f'(x)$ the w^*-*derivative *of f at x.*

REMARK 2.9. In general, f need not be differentiable in the Fréchet sense at any point unless V has the Radon–Nikodym property (cf. [17, p. 259]).

Proof of Lemma 2.8. Let $D \subset Y$ be a countable dense subset. We can assume that D is a linear space over the field of rational numbers (i.e., if $a, b \in D$ and if $s, t \in \mathbb{Q}$ then $sa + tb \in D$). For $v^* \in D$, the function

$$x \mapsto \langle v^*, f(x)\rangle$$

is a real-valued absolutely continuous function and hence is differentiable a.e. Since D is countable we conclude that, for a.e. $x \in (a,b)$ and every $v^* \in D$, there is an $f'_{v^*}(x) \in \mathbb{R}$ such that

$$\left\langle v^*, \frac{f(x+h) - f(x)}{h} \right\rangle \to f'_{v^*}(x) \quad \text{as } h \to 0.$$

Observe that $v^* \mapsto f'_{v^*}(x)$ is a linear functional defined on D. It is bounded for a.e. x because

$$|f'_{v^*}(x)| = \lim_{h\to 0}\left|\left\langle v^*, \frac{f(x+h)-f(x)}{h}\right\rangle\right| \le |s'_f(x)|\|v^*\| \tag{2.4}$$

by Lemma 2.7. Therefore, the functional $v^* \mapsto f'_{v^*}(x)$ extends to an element of $Y^* = V$. Denote this element by $f'(x) \in V$, so $\langle v^*, f'(x)\rangle = f'_{v^*}(x)$ for all $v^* \in D$. Inequality (2.4) yields $\|f'(x)\| \le |s'_f(x)|$. Now it easily follows from continuity that, for a.e. $x \in (a,b)$ and all $v^* \in Y$,

$$\left\langle v^*, \frac{f(x+h)-f(x)}{h}\right\rangle \to \langle v^*, f'(x)\rangle \quad \text{as } h \to 0.$$

This finishes the proof of Lemma 2.8. □

The following lemma is well known, but we provide a proof for the sake of completeness.

LEMMA 2.10. *If $s\colon [a,b] \to \mathbb{R}$ is absolutely continuous then, for every $a < c < d < b$,*

$$\lim_{h\to 0}\int_c^d \left|\frac{s(x+h)-s(x)}{h} - s'(x)\right| dx = 0.$$

Proof. Given $\varepsilon > 0$, let $f \in C([a,b])$ be such that

$$\int_a^b |s'(x) - f(x)|\, dx < \frac{\varepsilon}{3}.$$

The function f is uniformly continuous and hence there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/(3(d-c))$ whenever $|x-y| < \delta$. For $0 < h < \delta$ such that $d + h < b$, we have

$$\begin{aligned}
\int_c^d \left|\frac{s(x+h)-s(x)}{h} - s'(x)\right| dx &\le \frac{1}{h}\int_c^d\int_x^{x+h} |s'(\tau) - s'(x)|\, d\tau\, dx \\
&\le \frac{1}{h}\int_c^d\int_x^{x+h} |s'(\tau) - f(\tau)|\, d\tau\, dx \\
&\quad + \frac{1}{h}\int_c^d\int_x^{x+h} |s'(x) - f(x)|\, d\tau\, dx \\
&\quad + \frac{1}{h}\int_c^d\int_x^{x+h} |f(\tau) - f(x)|\, d\tau\, dx \\
&\le \int_c^{d+h} |s'(\tau) - f(\tau)|\, d\tau \\
&\quad + \int_c^d |s'(x) - f(x)|\, dx \\
&\quad + \frac{1}{h}\int_c^d\int_x^{x+h} \frac{\varepsilon}{3(d-c)}\, d\tau\, dx \\
&\le \varepsilon.
\end{aligned}$$

This proves that the limit equals 0 as $h \to 0^+$. A similar argument shows that the limit equals 0 as $h \to 0^-$. □

LEMMA 2.11. *Let $V = Y^*$ be dual to a separable Banach space Y. If $f\colon [a,b] \to V$ is absolutely continuous, then*

$$\int_a^b \varphi'(t) f(t)\, dt = -\int_a^b \varphi(t) f'(t)\, dt$$

for all $\varphi \in C_0^\infty(a,b)$, where $f'(t)$ is the w^*-derivative of f at t.

Proof. For every $v^* \in Y \subset V^*$ and $\varphi \in C_0^\infty(a,b)$, we have

$$\begin{aligned}\left\langle v^*, \int_a^b \varphi'(t) f(t)\, dt \right\rangle &= \int_a^b \varphi'(t) \langle v^*, f(t)\rangle\, dt \\ &= \lim_{h\to 0} \int_a^b \frac{\varphi(t+h) - \varphi(t)}{h} \langle v^*, f(t)\rangle\, dt \\ &= \lim_{h\to 0} -\int_a^b \varphi(t) \left\langle v^*, \frac{f(t-h) - f(t)}{-h} \right\rangle dt \\ &= -\int_a^b \varphi(t)\langle v^*, f'(t)\rangle\, dt = -\left\langle v^*, \int_a^b \varphi(t) f'(t)\, dt \right\rangle.\end{aligned}$$

We can pass the limit under the integral in view of Lemma 2.10. Indeed, we apply the lemma to $s(t) = \langle v^*, f(t)\rangle$ and $[c,d]$, $a < c < d < b$, such that $\operatorname{supp} \varphi \subset [c,d]$. Since equality holds for all $v^* \in Y$, the lemma follows. □

LEMMA 2.12. *Let $\Omega \subset \mathbb{R}^n$ be open and let $V = Y^*$ be dual to a separable Banach space Y. If $f \in L^p(\Omega, V)$ is absolutely continuous on compact intervals in $\ell \cap \Omega$ for almost all lines ℓ parallel to coordinate axes* (*possibly after redefinition on a set of measure* 0) *and if w^*-partial derivatives exist and satisfy $\|\partial f/\partial x_i\| \le g$ a.e. for some $g \in L^p(\Omega)$, then $f \in W^{1,p}(\Omega, V)$ and $\|f\|_{1,p} \le \|f\|_p + \sqrt{n}\|g\|_p$.*

Proof. Let ℓ be a line parallel to the x_i-axis such that f is absolutely continuous on compact intervals in $\ell \cap \Omega$. Let $\varphi \in C_0^\infty(\Omega)$. It follows from Lemma 2.11 that

$$\int_{\ell\cap\Omega} \frac{\partial \varphi}{\partial x_i} f = -\int_{\ell\cap\Omega} \varphi \frac{\partial f}{\partial x_i},$$

and hence Fubini's theorem yields

$$\int_{\Omega} \frac{\partial \varphi}{\partial x_i} f = -\int_{\Omega} \varphi \frac{\partial f}{\partial x_i}.$$

This proves that the w^*-partial derivatives $\partial f/\partial x_i$ are actually weak partial derivatives and hence $f \in W^{1,p}(\Omega, V)$. Now the inequality $\|\partial f/\partial x_i\| \le g$ easily implies that $\|f\|_{1,p} \le \|f\|_p + \sqrt{n}\|g\|_p$. □

The following result is related to [13, Thm. 3.17 and Prop. 5.4].

LEMMA 2.13. *Let $f \in R^{1,p}(\Omega, V)$, $1 \le p < \infty$. Then, for every $\nu \in S^{n-1}$ and almost every line ℓ parallel to ν, f is absolutely continuous on compact intervals in $\ell \cap \Omega$* (*after possible redefinition on a set of measure* 0). *Moreover, if $g \in L^p(\Omega)$ satisfies* (2.2) *then, for every $\nu \in S^{n-1}$ and a.e. $x \in \Omega$, the following limit exists and satisfies*

$$\lim_{h\to 0}\left\|\frac{f(x+h\nu)-f(x)}{h}\right\| \le g(x). \tag{2.5}$$

As a result, the w^**-partial derivatives exist and satisfy* $\|\partial f/\partial x_i\| \le g$ *a.e.*

Proof. The mapping $f \in R^{1,p}(\Omega, V)$ is essentially separably valued. Let $\{v_i\}_{i=1}^{\infty}$ be a dense subset in the difference set

$$f(\Omega \setminus Z) - f(\Omega \setminus Z) = \{f(x) - f(y) : x, y \in \Omega \setminus Z\},$$

and let $v_i^* \in V^*$, $\|v_i^*\| = 1$, be such that $\langle v_i^*, v_i\rangle = \|v_i\|$. Then, for almost every line ℓ parallel to $\nu \in S^{n-1}$, we have:

(a) $g \in L^p(\ell \cap \Omega)$;
(b) $\ell \cap Z$ has one-dimensional measure 0;
(c) $\langle v_i^*, f\rangle$ is absolutely continuous on compact subintervals in $\ell \cap \Omega$ for $i = 1, 2, \dots$ and, for a.e. $x \in \ell \cap \Omega$,

$$|D_\nu \langle v_i^*, f\rangle(x)| \le g(x)$$

for all $i = 1, 2, \dots$.

Properties (a) and (b) follow from the Fubini theorem, and (c) is a consequence of the fact that, for every $v^* \in V^*$ with $\|v^*\| = 1$, we have $\langle v^*, f\rangle \in W^{1,p}(\Omega)$ and hence $\langle v^*, f\rangle$ is absolutely continuous on almost all lines parallel to ν with

$$|D_\nu\langle v^*, f\rangle| = |(\nabla\langle v^*, f\rangle)\cdot \nu| \le g \ \text{ a.e.}$$

Let ℓ be a line parallel to $\nu \in S^{n-1}$ and for which conditions (a)–(c) are satisfied. Choose a compact interval $I \subset \ell \cap \Omega$. Such an interval is of the form $I = \{x_0 + t\nu : t \in [a,b]\}$. Then the inequality in (c) implies that

$$\left|\frac{d}{dt}\langle v_i^*, f(x_0 + t\nu)\rangle\right| \le g(x_0 + t\nu)$$

for all $i = 1, 2, \dots$ and a.e. $t \in [a,b]$. For almost all $s, t \in [a,b]$, for $x_0 + s\nu \notin Z$ and $x_0 + t\nu \notin Z$, and for $i = 1, 2, 3, \dots$, we have

$$\begin{aligned}\langle v_i^*, f(x_0+t\nu) - f(x_0+s\nu)\rangle &= \int_s^t \frac{d}{d\tau}\langle v_i^*, f(x_0+\tau\nu)\rangle\, d\tau \\ &\le \int_s^t g(x_0+\tau\nu)\, d\tau.\end{aligned}$$

It follows from the definition of the set $\{v_i^*\}_{i=1}^{\infty}$ that there is a sequence $v_{i_j}^*$ such that

$$\langle v_{i_j}^*, f(x_0+t\nu) - f(x_0+s\nu)\rangle \to \|f(x_0+t\nu) - f(x_0+s\nu)\| \quad \text{as} \quad j \to \infty.$$

Accordingly,

$$\|f(x_0+t\nu) - f(x_0+s\nu)\| \le \int_s^t g(x_0+\tau\nu)\, d\tau. \tag{2.6}$$

This proves the absolute continuity of f on compact intervals in $\ell \cap \Omega$ and also proves inequality (2.5). □

Now we are prepared to prove the main result of this section.

THEOREM 2.14. *If $\Omega \subset \mathbb{R}^n$ is open and $V = Y^*$ is dual to a separable Banach space Y, then*

$$W^{1,p}(\Omega, V) = R^{1,p}(\Omega, V)$$

and $\|f\|_{R^{1,p}} \le \|f\|_{1,p} \le \sqrt{n}\|f\|_{R^{1,p}}$.

Proof. According to Proposition 2.3, it suffices to prove that $R^{1,p} \subset W^{1,p}$ and $\|f\|_{1,p} \le \sqrt{n}\|f\|_{R^{1,p}}$. Let $f \in R^{1,p}(\Omega, V)$ and $g \in L^p$ be as in (2.2). Lemma 2.13 and Lemma 2.12 imply that $\|f\|_{1,p} \le \|f\|_p + \sqrt{n}\|g\|_p$, and the theorem follows upon taking the infimum over all the functions g as before. □

In the sequel we will also need the following fact.

LEMMA 2.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded and open set and let $V = Y^*$ be dual to a separable Banach space Y. If $f \in W^{1,p}(\Omega)$, $f(\Omega) \subset [a,b]$, and $\gamma : [a,b] \to V$ is 1-Lipschitz, then $\gamma \circ f \in W^{1,p}(\Omega, V)$ and $\|\nabla(\gamma \circ f)\|_p \le \sqrt{n}\|\nabla f\|_p$.*

Proof. According to Theorem 2.14, it suffices to prove that $\gamma \circ f \in R^{1,p}(\Omega, V)$ and that $g = |\nabla f|$ satisfies (2.2) with f replaced by $\gamma \circ f$, because then (2.5) will imply that $\|\partial(\gamma \circ f)/\partial x_i\| \le |\nabla f|$ a.e. and hence $|\nabla(\gamma \circ f)| \le \sqrt{n}|\nabla f|$ a.e. If $v^* \in V^*$, $\|v^*\| \le 1$, then $\langle v^*, \gamma \circ f\rangle = \varphi \circ f$, where $\varphi\colon [a,b] \to \mathbb{R}$ and $\varphi(x) = \langle v^*, \gamma(x)\rangle$ is a 1-Lipschitz function. Hence $\langle v^*, \gamma \circ f\rangle \in W^{1,p}(\Omega)$ and $|\nabla\langle v^*, \gamma \circ f\rangle| = |\nabla(\varphi \circ f)| \le |\nabla f|$ a.e. □

The following result links the definition of $R^{1,p}(\Omega, V)$ to that provided in the Introduction. Compare [13, Thm. 3.17] and [22, Thm. 5.1].

PROPOSITION 2.16. *Let $\Omega \subset \mathbb{R}^n$ be open, let V be a Banach space, and let $1 \le p < \infty$. Then $f \in R^{1,p}(\Omega, V)$ if and only if $f \in L^p(\Omega, V)$ and there is a non-negative function $g \in L^p(\Omega)$ such that, for every Lipschitz continuous function $\varphi\colon V \to \mathbb{R}$, we have $\varphi \circ f \in W^{1,p}(\Omega)$ and $|\nabla(\varphi \circ f)| \le \mathrm{Lip}(\varphi) g$ a.e.*

Proof. ($\Leftarrow$) This implication is obvious because, for $v^* \in V^*$ with $\|v^*\| \le 1$, $\varphi(v) = \langle v^*, v\rangle$ is 1-Lipschitz continuous and hence $\langle v^*, f\rangle \in W^{1,p}(\Omega)$ with $|\nabla\langle v^*, f\rangle| \le g$ a.e.

($\Rightarrow$) Let $f \in R^{1,p}(\Omega, V)$ and $g \in L^p(\Omega)$ be as in the definition of $R^{1,p}(\Omega, V)$, and let $\varphi\colon V \to \mathbb{R}$ be a Lipschitz continuous function.

Let $\nu \in S^{n-1}$. We proved in Lemma 2.13 that, for almost all lines ℓ parallel to ν, the function f is absolutely continuous on compact intervals contained in $\ell \cap \Omega$ and

$$\|f(x_0 + t\nu) - f(x_0 + s\nu)\| \le \int_s^t g(x_0 + \tau\nu)\, d\tau$$

provided $\{x_0 + \tau\nu : \tau \in [s,t]\} \subset \ell \cap \Omega$; see (2.6). Hence

$$\|(\varphi \circ f)(x_0 + t\nu) - (\varphi \circ f)(x_0 + s\nu)\| \le \mathrm{Lip}(\varphi) \int_s^t g(x_0 + \tau\nu)\, d\tau.$$

This proves that $\varphi \circ f$ is absolutely continuous on almost all lines. Hence $\varphi \circ f \in W^{1,p}(\Omega)$ and $|D_\nu(\varphi \circ f)| \le \mathrm{Lip}(\varphi) g$ a.e. Taking the supremum over all ν in a countable dense subset of S^{n-1} yields $|\nabla(\varphi \circ f)| \le \mathrm{Lip}(\varphi) g$ a.e. □

Let $\Omega \subset \mathbb{R}^n$ be open and let X be a metric space. We can always assume that X is isometrically embedded into a Banach space V. Indeed, every metric space X admits an isometric embedding to $V = \ell^\infty(X)$, the Banach space of bounded functions on X. In this case we have two natural definitions of the space of Sobolev mappings from Ω to X:

$$W^{1,p}(\Omega, X) = \{f \in W^{1,p}(\Omega, V) : f(\Omega) \subset X\};$$
$$R^{1,p}(\Omega, X) = \{f \in R^{1,p}(\Omega, V) : f(\Omega) \subset X\}.$$

Since every Lipschitz function $\varphi\colon X \to \mathbb{R}$ can be extended to a Lipschitz function $\tilde{\varphi}\colon V \to \mathbb{R}$ with the same Lipschitz constant (McShane extension theorem; see e.g. [12, Thm. 6.2]), Proposition 2.16 shows that the preceding definition of $R^{1,p}(\Omega, X)$ is equivalent to that in the Introduction. In particular, the definition of $R^{1,p}(\Omega, X)$ does not depend on the isometric embedding of X to a Banach space. Moreover, Theorem 2.14 yields the following.

THEOREM 2.17. *If $\Omega \subset \mathbb{R}^n$ is open, $V = Y^*$ is dual to a separable Banach space, $1 \le p < \infty$, and $X \subset V$, then $W^{1,p}(\Omega, X) = R^{1,p}(\Omega, X)$.*

Observe that if X is separable then it admits an isometric embedding to ℓ^∞. The space ℓ^∞ is dual to a separable Banach space ($\ell^\infty = (\ell^1)^*$), so in this case $W^{1,p}(\Omega, X)$ and $R^{1,p}(\Omega, X)$ coincide.

3. Proof of Theorem 1.3

In the proof we will employ some ideas from [18] and [14] (cf. [9]). Without loss of generality we may assume that the diameter of X with respect to d_ℓ is 1. Since X is length compact, for every $\ell = 0, 1, 2, 3, \ldots$ there is a finite $2^{-\ell}$-net $X_\ell = \{x_i^\ell\}_{i=1}^{k_\ell}$ with respect to d_ℓ; that is, every point of X can be connected to a point in the set X_ℓ by a curve of length $\le 2^{-\ell}$. Because the length metric diameter equals 1, we may assume that $X_0 = \{x_1^0\}$. For every $i = 1, 2, \ldots, k_{\ell+1}$ there is a curve $\gamma_i^{\ell+1}$ of length $\le 2^{-\ell}$ that connects $x_i^{\ell+1}$ to a point $x_{j(i,\ell+1)}^\ell \in X_\ell$. We may assume that $\gamma_i^{\ell+1}$ is parameterized by arc length [10, Thm. 3.2]—in other words, that $\gamma_i^{\ell+1} : [0, \ell(\gamma_i^{\ell+1})] \to X$ is 1-Lipschitz with $\ell(\gamma_i^{\ell+1}) \le 2^{-\ell}$, $\gamma_i^{\ell+1}(0) = x_{j(i,\ell+1)}^\ell$, and $\gamma_i^{\ell+1}(\ell(\gamma_i^{\ell+1})) = x_i^{\ell+1}$.

The idea of the proof is to construct a sequence of Lipschitz continuous mappings $f_\ell\colon [0,1]^n \to X$ that converge uniformly and in the Sobolev norm to a surjection $f\colon [0,1]^n \to X$. The mapping f_0 is defined as a constant map that maps $[0,1]^n$ onto the point $x_1^0 \in X_0$. The mapping f_1 will be a modification of f_0 on an open set of small measure that connects x_1^0 to points in the set X_1 through the curves γ_i^1. The mapping f_2 will be a modification of f_1 on an open set of small measure that connects points of X_1 to points of X_2 through the curves γ_i^2, and so on. The lengths of the curves at each step decrease at a geometric rate, which guarantees uniform convergence. Since the limiting mapping f covers a dense subset of X, it is surjective. In each step the Sobolev norm of the difference $f_\ell - f_{\ell+1}$ does not exceed $2^{-(\ell-1)}$, which guarantees that the sequence converges in the Sobolev norm. Finally, we will modify the mappings on a rapidly decreasing family of open sets whose intersection is a compact set E of Hausdorff dimension 0. The limit mapping will be locally Lipschitz continuous in $[0,1]^n \setminus E$ and hence a.e. metrically differentiable according to the theorem of Kirchheim.

To construct the sequence f_ℓ we need auxiliary Lipschitz functions. It is well known and easy to prove that, for $n \geq 2$, $\eta(x) = \log|\log|x|| \in W^{1,n}(B(0,e^{-1}))$. Define the *truncation* of η between levels s and t, $0 < s < t < \infty$, by

$$\eta_s^t(x) = \begin{cases} t-s & \text{if } \eta(x) \geq t, \\ \eta(x) - s & \text{if } s \leq \eta(x) \leq t, \\ 0 & \text{if } \eta(x) \leq s. \end{cases}$$

Fix $\tau > 0$. For every $\varepsilon > 0$ there is a sufficiently large s such that $\eta_{\varepsilon,\tau} := \eta_s^{s+\tau}$ is a Lipschitz continuous function on $\mathbb{R}^n$ with the following properties:

$$\operatorname{supp} \eta_{\varepsilon,\tau} \subset B(0,\varepsilon);$$

$$0 \leq \eta_{\varepsilon,\tau} \leq \tau, \quad \eta_{\varepsilon,\tau} = \tau \ \text{ in a neighborhood of 0;} \tag{3.1}$$

$$\|\nabla \eta_{\varepsilon,\tau}\|_n < \varepsilon. \tag{3.2}$$

Equivalently, the existence of a function $\eta_{\varepsilon,\tau}$ with these properties follows from the fact that the capacity of a point is zero. However, the construction here is more straightforward.

Since X is compact, we may assume that X is isometrically embedded in $V = \ell^\infty$. For simplicity we will denote the norm in ℓ^∞ by $|\cdot|$, and for two continuous functions $g,h\colon [0,1]^n \to X \subset \ell^\infty$ we define $\|g-h\|_\infty = \sup_x |g(x) - h(x)|$.

Suppose $n \geq 2$. We define $f_0\colon [0,1]^n \to X$ to be a constant map, $f_0([0,1]^n) = \{x_0^1\}$. Let $y_1^1, \dots, y_{k_1}^1 \in (0,1)^n$ be distinct points and let $\varepsilon_1 > 0$ be so small that $\sqrt{n} k_1 \varepsilon_1 < 1$ and the balls $B(y_i^1, \varepsilon_1) \subset (0,1)^n$ are pairwise disjoint for $i = 1,2,\dots,k_1$. Define

$$f_1(x) = \begin{cases} f_0(x) & \text{if } x \notin \bigcup_{i=1}^{k_1} B(y_i^1, \varepsilon_1), \\ \gamma_i^1(\eta_{\varepsilon_1, \ell(\gamma_i^1)}(x - y_i^1)) & \text{if } x \in B(y_i^1, \varepsilon_1). \end{cases}$$

Clearly, f_1 is Lipschitz continuous. The mapping f_1 differs from f_0 on the set $\bigcup_{i=1}^{k_1} B(y_i^1, \varepsilon_1)$ and $\|f_1 - f_0\|_\infty \leq 1$, because

$$\begin{aligned} |\gamma_i^1(\eta_{\varepsilon_1,\ell(\gamma_i^1)}(x - y_i^1)) - x_0^1| &= |\gamma_i^1(\eta_{\varepsilon_1,\ell(\gamma_i^1)}(x - y_i^1)) - \gamma_i^1(0)| \\ &\leq |\eta_{\varepsilon_1,\ell(\gamma_i^1)}(x - y_i^1)| \\ &\leq \ell(\gamma_i^1) \leq 1 \end{aligned}$$

by (3.1) and the 1-Lipschitz continuity of γ_i^1. Since $[0,1]^n$ has unit measure, we have $\|f_0 - f_1\|_n \leq \|f_0 - f_1\|_\infty$. Moreover, f_0 is constant in each ball $B(y_i^1, \varepsilon_1)$, so its gradient equals zero there and hence

$$\begin{aligned} \|f_0 - f_1\|_{1,n} &\leq \|f_0 - f_1\|_\infty + \sum_{i=1}^{k_1} \|\nabla(\gamma_i^1 \circ \eta_{\varepsilon_1,\ell(\gamma_i^1)}(\cdot - y_i^1))\|_n \\ &\leq 1 + k_1 \sqrt{n} \varepsilon_1 < 2 \end{aligned}$$

by Lemma 2.15 and (3.2). Clearly $f_1(y_i^1) = x_i^1$ and f_1 is constant in a neighborhood of y_i^1; that is, there is an $\tilde{\varepsilon}_1 > 0$ such that

$$f_1(B(y_i^1, \tilde{\varepsilon}_1)) = \{x_i^1\} \quad \text{for } i = 1, 2, \dots, k_1.$$

For $i = 1, 2, \dots, k_2$, the curve γ_i^2 connects x_i^2 with $x_{j(i,2)}^1$ for some $j(i,2) \in \{1, 2, \dots, k_1\}$. Let $y_i^2 \in B(y_{j(i,2)}^1, \tilde{\varepsilon}_1)$ be distinct points and let $\varepsilon_2 > 0$ be so small that $\sqrt{n} k_2 \varepsilon_2 < 2^{-1}$ and

$$B(y_i^2, \varepsilon_2) \subset B(y_{j(i,2)}^1, \tilde{\varepsilon}_1)$$

are pairwise disjoint. Observe that the mapping f_1 is constant on each ball $B(y_i^2, \varepsilon_2)$ and that $f_1(B(y_i^2, \varepsilon_2)) = \{x_{j(i,2)}^1\}$. We define f_2 as follows:

$$f_2(x) = \begin{cases} f_1(x) & \text{if } x \notin \bigcup_{i=1}^{k_2} B(y_i^2, \varepsilon_2), \\ \gamma_i^2(\eta_{\varepsilon_2, \ell(\gamma_i^2)}(x - y_i^2)) & \text{if } x \in B(y_i^2, \varepsilon_2). \end{cases}$$

Clearly, f_2 is Lipschitz continuous. The mapping f_2 differs from f_1 on the set $\bigcup_{i=1}^{k_2} B(y_i^2, \varepsilon_2)$ and $\|f_2 - f_1\|_\infty \le 2^{-1}$, because $f_1 \equiv x_{j(i,2)}^1$ on $B(y_i^2, \varepsilon_2)$ and hence, for $x \in B(y_i^2, \varepsilon_2)$,

$$\begin{aligned} |f_2(x) - f_1(x)| &= |\gamma_i^2(\eta_{\varepsilon_2, \ell(\gamma_i^2)}(x - y_i^2) - x_{j(i,2)}^i| \\ &= |\gamma_i^2(\eta_{\varepsilon_2, \ell(\gamma_i^2)}(x - y_i^2) - \gamma_i^2(0)| \\ &\le |\eta_{\varepsilon_2, \ell(\gamma_i^2)}(x - y_i^2)| \le \ell(\gamma_i^2) \le 2^{-1} \end{aligned}$$

by (3.1) and the 1-Lipschitz continuity of γ_i^2. Moreover,

$$\begin{aligned} \|f_1 - f_2\|_{1,n} &\le \|f_1 - f_2\|_\infty + \sum_{i=1}^{k_2} \|\nabla(\gamma_i^2 \circ \eta_{\varepsilon_2, \ell(\gamma_i^2)}(\cdot - y_i^2))\|_n \\ &\le 2^{-1} + \sqrt{n} k_2 \varepsilon_2 < 1 \end{aligned}$$

by Lemma 2.15 and (3.2). Clearly $f_2(y_i^2) = x_i^2$ and f_2 is constant in a neighborhood of y_i^2.

Similarly, we construct Lipschitz mappings $f_3, f_4, f_5, \dots$ so that $f_{\ell+1}$ differs from f_ℓ on the set

$$\bigcup_{i=1}^{k_{\ell+1}} B(y_i^{\ell+1}, \varepsilon_{\ell+1}),$$

$$\|f_{\ell+1} - f_\ell\|_\infty \le 2^{-\ell}, \tag{3.3}$$

$$\|f_{\ell+1} - f_\ell\|_{1,n} \le 2^{-(\ell-1)}, \tag{3.4}$$

and

$$f_{\ell+1}(y_i^{\ell+1}) = x_i^{\ell+1},$$

where $f_{\ell+1}$ is constant in a neighborhood of $y_i^{\ell+1}$. The mappings f_ℓ are Lipschitz continuous, and by (3.3) they converge uniformly to a continuous mapping $f\colon [0,1]^n \to X$. Since the image of f covers a dense subset $\bigcup_{n=0}^\infty X_n \subset f([0,1]^n)$, it follows that $f([0,1]^n) = X$. The sets

$$\bigcup_{i=1}^{k_\ell} B(y_i^\ell, \varepsilon_\ell)$$

form a decreasing sequence of open sets. Because the radii ε_ℓ can be taken arbitrarily small, we can guarantee that the intersection

$$E = \bigcap_{\ell=1}^{\infty} \bigcup_{i=1}^{k_\ell} B(y_i^\ell, \varepsilon_\ell)$$

is a compact set of Hausdorff dimension 0. Observe that the consecutive mappings in the sequence are obtained by modifications of preceding mappings on a decreasing sequence of sets. Hence

$$f = f_\ell \quad \text{on } [0,1]^n \setminus \bigcup_{i=1}^{k_{\ell+1}} B(y_i^{\ell+1}, \varepsilon_{\ell+1}),$$

and we conclude that f is locally Lipschitz continuous in $[0,1]^n \setminus E$. Therefore, f is a.e. metrically differentiable according to Kirchheim's theorem.

Finally, (3.4) implies that (f_ℓ) is Cauchy in the Sobolev norm $W^{1,n}$. Hence the limit mapping f belongs to $W^{1,n}([0,1]^n, X) = R^{1,n}([0,1]^n, X)$.

If $n = 1$, then the functions $\eta_{\varepsilon,\tau}$ are still Lipschitz continuous but the Sobolev norm estimate (3.2) is no longer true. We can repeat the previous construction also for $n = 1$; then the limit mapping $f\colon [0,1] \to X$ will be a surjection that is locally Lipschitz on the complement of a set of Hausdorff dimension 0 and hence metrically differentiable a.e. However, in this case f will not belong to the Sobolev space $W^{1,1} = R^{1,1}$.

4. Proof of Theorem 1.4

The functions $\eta_{\varepsilon,\tau}$ constructed in the proof of Theorem 1.3 are not smooth, but we can make them smooth via convolution approximation. Then—using smooth curves to connect points—we can follow the construction from the proof of Theorem 1.3 to obtain, for every positive integer k, a mapping $F_k\colon [-1/2, 1/2]^2 \to X$ that is C^∞ smooth outside a compact set E_k of Hausdorff dimension 0 and that satisfies $\|F_k\|_{1,2} < 2^{-k}$ and $B(0,k) \subset F_k([-1/2,1/2]^2)$, where $B(x,r)$ denotes a ball in X with respect to the metric d. Furthermore, it follows from the proof of Theorem 1.3 that the set E_k can be located on the x-axis. We can also put $F_k = 0$ near the boundary of the cube, so we can extend it by zero to a mapping from $\mathbb{R}^2$ to X. Now the mapping

$$f(x,y) = \sum_{k=1}^{\infty} F_k(x-k, y)$$

satisfies the claim of the theorem.

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