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A Remark on Frobenius Descent for Vector Bundles

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Dedicated to Mel Hochster on the occasion of his 65th birthday

1. Introduction

Let X be a smooth projective variety defined over an algebraically closed field of characteristic $p > 0$ with a fixed very ample line bundle $\mathcal{O}_X(1)$. We denote by F the absolute Frobenius morphism $F: X \to X$, which is the identity on the topological space underlying X and the pth power map on the structure sheaf $\mathcal{O}_X$. A vector bundle $\mathcal{E}$ on X descends under F if there exists a vector bundle $\mathcal{F}$ such that $\mathcal{E} \cong F^*(\mathcal{F})$. This paper is inspired by the preprint of Joshi [6]. In the relative situation, where a morphism $\mathcal{X} \to \operatorname{Spec} R$ with generic fiber $X := \mathcal{X}_0$ is given and R is a $\mathbb{Z}$-domain of finite type, Joshi asked the following question: Assume X is a smooth projective variety and suppose V is a vector bundle that descends under Frobenius modulo an infinite set of primes; then is it true that V is semistable (with respect to any ample line bundle on X)?" He gives a positive answer to this question for rank-2 vector bundles under the additional assumption that $\operatorname{Pic}(X) = \mathbb{Z}$.

In Section 2 we provide a class of examples that give a negative answer to this question in general. We show that, on the relative Fermat curve

$$C = V_+(X^d + Y^d + Z^d) \to \operatorname{Spec} \mathbb{Z}$$

with $d \geq 5$ odd, there exists a vector bundle $\mathcal{E}$ of rank 2 such that for infinitely many prime numbers p the reduction $\mathcal{E}_p = \mathcal{E}|_{C_p}$ modulo p has a Frobenius descent, but $\mathcal{E}_0 = \mathcal{E}|_{C_0}$ is not semistable on the fiber over the generic point. In Section 3 we give an affirmative answer to this question under the assumption that, for every closed point $\mathfrak{m} \in \operatorname{Spec} R$, every semistable vector bundle on the fiber $\mathcal{X}_{\mathfrak{m}}$ is strongly semistable. We recall that a semistable vector bundle $\mathcal{E}$ is strongly semistable if $F^{e*}(\mathcal{E})$ is semistable for all integers $e \geq 0$. This provides further examples of varieties with $\operatorname{Pic}(X) \neq \mathbb{Z}$ (e.g., abelian varieties) for which the question of Joshi still has a positive answer.

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2. A Counterexample for Vector Bundles on Curves

In this section we give an example of a rank-2 vector bundle on a generically smooth projective relative curve over Spec $\mathbb{Z}$ such that infinitely many prime reductions have a Frobenius descent but the bundle is not semistable on the generic fiber in characteristic 0.

Our example will use the syzygy bundle $\mathrm{Syz}(X^2, Y^2, Z^2)(m)$ on Fermat curves $C = V_+(X^d + Y^d + Z^d) \subset \mathbb{P}^2$ defined over a field K. This vector bundle is defined by the short exact sequence

$$0 \longrightarrow \mathrm{Syz}(X^2, Y^2, Z^2)(m) \longrightarrow \mathcal{O}_C(m-2)^3 \longrightarrow \mathcal{O}_C(m) \longrightarrow 0,$$

where the penultimate mapping is given by $(s_1, s_2, s_3) \mapsto s_1X^2 + s_2Y^2 + s_3Z^2$. The bundle $\mathrm{Syz}(X^2, Y^2, Z^2)(m)$ is semistable for $d \geq 5$ by [2, Prop. 6.2]. In positive characteristic $p > 0$, since the presenting sequence involves only locally free sheaves it is easy to see that the Frobenius pull-back $F^*(\mathrm{Syz}(X^2, Y^2, Z^2)(m)) \cong \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(mp)$.

LEMMA 2.1. *Let $d = 2\ell + 1$ with $\ell \geq 2$, and let*

$$C := \mathrm{Proj}\, K[X, Y, Z]/(X^d + Y^d + Z^d)$$

be the Fermat curve of degree d defined over a field K of characteristic $p \equiv \ell \bmod d$. Then the Frobenius pull-back of $\mathrm{Syz}(X^2, Y^2, Z^2)(3)$ sits inside the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(\ell - 1) \longrightarrow \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{O}_C(-\ell + 1) \longrightarrow 0.$$

In particular, the Frobenius pull-back is not semistable and this sequence constitutes its Harder–Narasimhan filtration.

Proof. We write $2p = dk + 2\ell$ with k even. The pull-back $\mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})$ of $\mathrm{Syz}(X^2, Y^2, Z^2)$ has a nontrivial global section in total degree $d(k + 1 + k/2)$ by [3, proof of Prop. 1.2]. From the presenting sequence of the pull-back one reads off the degree as follows:

$$\begin{aligned}
\deg(\mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(d(k+1+k/2)) &= d(2d(k+1+k/2) - 6p) \\
&= d(2d(k+1+k/2) - 3(dk + 2\ell)) \\
&= d(2d - 6\ell) \\
&= d(-2\ell + 2) < 0.
\end{aligned}$$

Because a semistable vector bundle of negative degree cannot have nontrivial global sections, the Frobenius pull-back $\mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})$ is not semistable. We obtain a nontrivial mapping $\mathcal{O}_C(\ell - 1) \to \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$. We want to show that this mapping constitutes the Harder–Narasimhan filtration of the pull-back, meaning that the mapping has no zeros. Hence, assume that we have a factorization

$$\mathcal{O}_C(\ell - 1) \longrightarrow \mathcal{L} \longrightarrow \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p),$$

where $\mathcal{L}$ is a subbundle of the syzygy bundle and has degree $\deg(\mathcal{L}) := \alpha \geq (\ell - 1)d$. We have the short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{L}' \longrightarrow 0,$$

where $\mathcal{L}'$ is a line bundle of degree $-\alpha$. By [15, Cor. 2^p] (or [16, Thm. 3.1]), the inequality

$$\mu_{\max}(\mathcal{S}) - \mu_{\min}(\mathcal{S}) = \alpha - (-\alpha) = 2\alpha \leq 2g - 2$$

holds, where $\mathcal{S} := \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$ and g denotes the genus of C. The genus formula for plane curves yields

$$2g - 2 = (d - 1)(d - 2) - 2 = d(d - 3) = 2d(\ell - 1).$$

Thus we obtain $\alpha = d(\ell - 1)$. Hence, $\mathcal{O}_C(\ell - 1) \cong \mathcal{L}$ and the Harder–Narasimhan filtration is indeed $0 \subset \mathcal{O}_C(\ell - 1) \subset \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$. □

REMARK 2.2. Using Hilbert–Kunz theory and its geometric interpretation developed in [4] and [17], one can give an alternative (but more complicated) proof that the line bundle $\mathcal{O}_C(\ell - 1)$ is the maximal destabilizing subbundle of the syzygy bundle $\mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$. We recall that, for a rank-2 vector bundle, the Harder–Narasimhan filtration is already strong in the sense of [9, para. 2.6]. By the formula given in [4, Thm. 3.6] we can use the short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{L}' \longrightarrow 0$$

to compute the Hilbert–Kunz multiplicity $e_{\mathrm{HK}}(I)$ (see [12]) of the ideal $I = (X^2, Y^2, Z^2)$ in the homogeneous coordinate ring

$$R := K[X, Y, Z]/(X^d + Y^d + Z^d)$$

of the curve C and so obtain $e_{\mathrm{HK}}(I) = 3d + \alpha^2/dp^2$. But by [13, Thm. 2.3], the Hilbert–Kunz multiplicity of I equals

$$e_{\mathrm{HK}}(I) = 3d + \frac{d}{4}\frac{(d - 3)^2}{p^2},$$

which implies that $\alpha = d(\ell - 1)$.

REMARK 2.3. We briefly comment on the situation for $\ell = 0, 1$. For $\ell = 0$ (and $p \neq 2$) we have $\mathrm{Syz}(X^2, Y^2, Z^2)(3) \cong \mathcal{O}^2_{\mathbb{P}^1}$, and this is also true for its Frobenius pull-back. For $\ell = 1$ we get the Fermat cubic, which is an elliptic curve. In this case we have an exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathrm{Syz}(X^2, Y^2, Z^2)(3) \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

where the (only) global nontrivial section is given by the curve equation. Hence the syzygy bundle is F_2 in Atiyah's classification [1] and is semistable but not stable. Its Frobenius pull-back is either F_2 (for $p \equiv 1 \bmod 3$; i.e., Hasse invariant 1) or $\mathcal{O}^2_C$ (for $p \equiv 2 \bmod 3$; i.e., Hasse invariant 0).

In the relative situation

$$C := \mathrm{Proj}(\mathbb{Z}_d[X, Y, Z]/(X^d + Y^d + Z^d)) \longrightarrow \mathrm{Spec}\, \mathbb{Z}_d,$$

every fiber $C_p := C \times_{\operatorname{Spec}\mathbb{Z}_d} \operatorname{Spec}\mathbb{F}_p$ is a smooth projective curve—namely, the Fermat curve defined over the prime field $\mathbb{F}_p$ (and $\bar{C}_p := C \times_{\operatorname{Spec}\mathbb{Z}_d} \bar{\mathbb{F}}_p$ is a smooth projective curve over the algebraic closure of $\mathbb{F}_p$) for every prime number p such that $p \nmid d$. We recall that, by the theorem of Dirichlet (see [14, Chap. VI, Sec. 4, Thm. and Cor.]), there exist infinitely many prime numbers $p \equiv \ell \bmod d$.

LEMMA 2.4. *Let $d = 2\ell + 1$ with $\ell \geq 2$, and consider the smooth projective relative curve $C := \operatorname{Proj}(\mathbb{Z}_d[X,Y,Z]/(X^d + Y^d + Z^d)) \longrightarrow \operatorname{Spec}\mathbb{Z}_d$. Then the sequence (from Lemma 2.1)*

$$0 \longrightarrow \mathcal{O}_{C_p}(\ell - 1) \longrightarrow \operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{O}_{C_p}(-\ell + 1) \longrightarrow 0$$

does not split for almost all primes $p \equiv \ell \bmod d$.

Proof. Since $\operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \cong F^*(\operatorname{Syz}(X^2, Y^2, Z^2)(3))$ holds on every fiber C_p, the bundle $\operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$ carries an integrable connection ∇_p with p-curvature 0 by the Cartier correspondence [7, Thm. 5.1]. Assume that the sequence does split for some $p \equiv \ell \bmod d$. Then $\mathcal{O}_{C_p}(\ell - 1)$ is a direct summand of $\operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p)$. The summand $\mathcal{O}_{C_p}(\ell - 1)$ carries also a connection with the same properties. Hence, again by the Cartier correspondence it has a Frobenius descent and so its degree $d(\ell - 1)$ is divisible by p. But this can only hold for finitely many p. $\square$

EXAMPLE 2.5. As before, we consider the smooth relative curve

$$C := \operatorname{Proj}(\mathbb{Z}_d[X,Y,Z]/(X^d + Y^d + Z^d)) \longrightarrow \operatorname{Spec}\mathbb{Z}_d$$

with $d = 2\ell + 1$ for $\ell \geq 2$. The Čech cohomology class $c = Z^{d-1}/XY \in H^1(C, \mathcal{O}_C(d-3)) \cong \operatorname{Ext}^1(\mathcal{O}_C(-\ell+1), \mathcal{O}_C(\ell - 1))$ defines an extension

$$0 \longrightarrow \mathcal{O}_C(\ell - 1) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_C(-\ell + 1) \longrightarrow 0$$

with the corresponding restrictions to each fiber $C_{\mathfrak{p}}$, where $\mathfrak{p} = (0)$ or $\mathfrak{p} = (p)$ and where $p \nmid d$. Note that this extension is nontrivial on every fiber. This vector bundle $\mathcal{E}$ is our example. Since $\ell \geq 2$, the bundle $\mathcal{E}_0 = \mathcal{E}|_{C_0}$ is not semistable on C_0. By Lemma 2.1 we have, for $p \equiv \ell \bmod d$, an extension

$$0 \longrightarrow \mathcal{O}_{C_p}(\ell - 1) \longrightarrow \operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \longrightarrow \mathcal{O}_{C_p}(-\ell + 1) \longrightarrow 0$$

corresponding to $c' \in H^1(C_p, \mathcal{O}_{C_p}(2\ell - 2)) = H^1(C_p, \mathcal{O}_{C_p}(d-3))$, and by Lemma 2.4 we have $c' \neq 0$ for almost all $p \equiv \ell \bmod d$. We claim that $\mathcal{E}_p = \mathcal{E}|_{C_p} \cong F^*(\operatorname{Syz}(X^2, Y^2, Z^2)(3))$ holds for these prime numbers. Since $\omega_{C_p} = \mathcal{O}_{C_p}(d-3) = \mathcal{O}_{C_p}(2\ell - 2)$ and $h^1(C_p, \omega_{C_p}) = 1$, it follows that $c = \lambda c'$ for some $\lambda \in \mathbb{F}_p^{\times}$. Moreover, multiplication by λ induces an automorphism $\omega_{C_p} \xrightarrow{\cdot\lambda} \omega_{C_p}$ of line bundles as well as an automorphism $H^1(C_p, \omega_{C_p}) \xrightarrow{\cdot\lambda} H^1(C_p, \omega_{C_p})$ of vector spaces. We obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{C_p}(2\ell - 2) \arrow[r] \arrow[d, "\cdot\lambda"] & \operatorname{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p + \ell - 1) \arrow[r] \arrow[d] & \mathcal{O}_{C_p} \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & \mathcal{O}_{C_p}(2\ell - 2) \arrow[r] & \mathcal{E}_p(\ell - 1) \arrow[r] & \mathcal{O}_{C_p} \arrow[r] & 0,
\end{tikzcd}$$

where the map in the middle is an isomorphism of vector bundles. Hence, $\mathcal{E}_p \cong \mathrm{Syz}(X^{2p}, Y^{2p}, Z^{2p})(3p) \cong F^*(\mathrm{Syz}(X^2, Y^2, Z^2)(3))$ and therefore $\mathcal{E}_p$ admits a Frobenius descent on every fiber C_p.

REMARK 2.6. Example 2.5 extends to all Fermat curves $C^d = V_+(X^d + Y^d + Z^d)$ where the degree d has an odd divisor $d' \geq 5$. To see this, we write $d = d'n$ and look at the cover $f : C^d \to C^{d'}$ induced by the ring map that sends each variable to its nth power. Then the pull-back under f of the vector bundles considered in Example 2.5 provide also an example on C^d with the same properties.

3. A Positive Result

Let $\mathcal{X} \to \operatorname{Spec} R$ be a smooth projective morphism of relative dimension $d \geq 1$, where R is a domain of finite type over $\mathbb{Z}$. Typical examples for the base are $\operatorname{Spec} \mathbb{Z}$ or arithmetic schemes $\operatorname{Spec} D$, where D is the ring of integers in a number field. Let $\mathcal{E}$ be a vector bundle over $\mathcal{X}$. In [6, Thm. 4.2], Joshi proved—under the assumptions $\mathrm{Pic}(X) = \mathbb{Z}$ ($X = \mathcal{X}_0$) and $\mathrm{rk}(\mathcal{E}) = 2$—that $\mathcal{E}_0 = \mathcal{E}|_X$ is semistable if, for infinitely many closed points $\mathfrak{m} \in \operatorname{Spec} R$ of arbitrarily large residue characteristic, the reduction $\mathcal{E}_{\mathfrak{m}}$ admits a Frobenius descent on the fiber $X_{\mathfrak{m}} = \mathcal{X}_{\mathfrak{m}}$. The aim of this section is to prove (using essentially the same methods) this result for vector bundles of arbitrary rank under the assumption that, for every closed point $\mathfrak{m}$, every semistable vector bundle $\mathcal{F}$ on $X_{\mathfrak{m}}$ is strongly semistable; that is, when $F^{e*}(\mathcal{F})$ is semistable for all $e \geq 0$ (it is enough to assume this for infinitely many closed points $\mathfrak{m}$ of arbitrary large residue characteristic). It is interesting to note that [6, Thm. 2.1] uses the condition $\mathrm{Pic}(Y) = \mathbb{Z}$ on a smooth projective variety Y in positive characteristic and a further hypothesis on Y to prove that every semistable rank-2 vector bundle on Y is strongly semistable.

THEOREM 3.1. *Let R be a $\mathbb{Z}$-domain of finite type. Let $f : \mathcal{X} \to \operatorname{Spec} R$ be a smooth projective morphism of relative dimension $d \geq 1$ together with a fixed f-very ample line bundle $\mathcal{O}_{\mathcal{X}}(1)$, and let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$. Further assume that every semistable vector bundle is strongly semistable* (*with respect to $\mathcal{O}_{X_{\mathfrak{m}}}(1)$*) *for every fiber $X_{\mathfrak{m}}$, where $\mathfrak{m}$ is a closed point in* $\operatorname{Spec} R$. *Then the following statement holds: If $\mathcal{E}_{\mathfrak{m}} = \mathcal{E}|_{X_{\mathfrak{m}}}$ has a Frobenius descent for infinitely many closed points $\mathfrak{m} \in$* $\operatorname{Spec} R$ *of arbitrarily large residue characteristic, then $\mathcal{E}_0$ is semistable on the generic fiber $X = X_0 = \mathcal{X}_0$.*

Proof. One can show by induction over $\dim R$ that there exists a bound b such that $\mu_{\max}(\mathcal{E}_{\mathfrak{m}}) \leq b$ for all closed points $\mathfrak{m} \in \operatorname{Spec} R$ (see [5, Lemma 3.1] for an explicit proof). For a closed point $\mathfrak{m} \in \operatorname{Spec} R$ with descent data $\mathcal{E}_{\mathfrak{m}} \cong F^*(\mathcal{F}_{\mathfrak{m}})$ with $\mathcal{F}_{\mathfrak{m}}$ locally free on the fiber $X_{\mathfrak{m}}$, we have

$$\mu_{\max}(\mathcal{E}_{\mathfrak{m}}) = \mathrm{char}(\kappa(\mathfrak{m}))\mu_{\max}(\mathcal{F}_{\mathfrak{m}})$$

because semistable vector bundles are strongly semistable on every fiber $X_{\mathfrak{m}}$ by assumption. Since $\mathcal{E}_{\mathfrak{m}} \cong F^*(\mathcal{F}_{\mathfrak{m}})$ for infinitely many closed points $\mathfrak{m}$ of arbitrarily large residue characteristic, this forces the similar equalities $\deg(\mathcal{E}_0) = \deg(\mathcal{E}_{\mathfrak{m}}) = \operatorname{char}(\kappa(\mathfrak{m})) \deg(\mathcal{F}_{\mathfrak{m}})$ (we take the degree always with respect to $\mathcal{O}_{X_{\mathfrak{m}}}(1)$), which implies $\deg(\mathcal{E}_{\mathfrak{m}}) = \deg(\mathcal{F}_{\mathfrak{m}}) = 0$. Assume that the restriction $\mathcal{E}_0$ to the generic fiber X is not semistable. Then, by the openness of semistability [11, Sec. 5], every restriction $\mathcal{E}_{\mathfrak{m}}$ on $X_{\mathfrak{m}}$ is not semistable. Again by our assumption, $\mathcal{F}_{\mathfrak{m}}$ is not semistable either and so $\mu_{\max}(\mathcal{F}_{\mathfrak{m}}) \geq 1/r$ for $r = \operatorname{rk}(\mathcal{E})$. This yields

$$b \geq \mu_{\max}(\mathcal{E}_{\mathfrak{m}}) = \operatorname{char}(\kappa(\mathfrak{m}))\mu_{\max}(\mathcal{F}_{\mathfrak{m}}) \geq \frac{\operatorname{char}(\kappa(\mathfrak{m}))}{r},$$

which contradicts the assumption that we have Frobenius descent at closed points $\mathfrak{m} \in \operatorname{Spec} R$ of arbitrarily large residue characteristic. □

COROLLARY 3.2. *Let R be a $\mathbb{Z}$-domain of finite type. Let $f : \mathcal{X} \to \operatorname{Spec} R$ be a smooth projective morphism of relative dimension $d \geq 1$ together with a fixed f-very ample line bundle $\mathcal{O}_{\mathcal{X}}(1)$, and let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$. Suppose that the fibers $X_{\mathfrak{m}}$, with $\mathfrak{m} \in \operatorname{Spec} R$ closed, fulfill at least one of the following (not necessarily independent) properties:*

(1) *$X_{\mathfrak{m}}$ is an abelian variety;*
(2) *$X_{\mathfrak{m}}$ is a homogenous space of the form G/P, where P is a reduced parabolic subgroup;*
(3) *the cotangent bundle $\Omega_{X_{\mathfrak{m}}}$ fulfills $\mu_{\max}(\Omega_{X_{\mathfrak{m}}}) \leq 0$.*

Then the following holds: If $\mathcal{E}_{\mathfrak{m}}$ has a Frobenius descent for infinitely many closed points $\mathfrak{m} \in \operatorname{Spec} R$ of arbitrarily large residue characteristic, then $\mathcal{E}_0$ is semistable on $X = X_0$.

Proof. That every semistable vector bundle is strongly semistable in case (3) is due to [10, Thm. 2.1], and (3) holds in particular for the varieties occurring in (1) and (2). Other proofs of this property for cases (1) and (2) are given in [15, Cor. 3^p] and for case (3) in [9, Cor. 6.3]. Hence, the assertion follows from Theorem 3.1. □

REMARK 3.3. On the one hand, it is well known that every semistable vector bundle on an elliptic curve is strongly semistable (see [18, Apx.]). So elliptic curves provide an important class of smooth projective varieties with $\operatorname{Pic}(X) \neq \mathbb{Z}$ for which Theorem 3.1 holds. On the other hand, it is also known that for every smooth projective curve of genus $g \geq 2$ there exists a semistable vector bundle $\mathcal{F}$ such that $F^*(\mathcal{F})$ is not semistable (see [8, Thm. 1]). Thus we see that Theorem 3.1 is applicable in relative dimension 1 only for elliptic curves and the projective line $\mathbb{P}^1$.

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