Michigan Math. J. 54 (2006)

One-Dimensional Metric Foliations on Compact Lie Groups

Marius Munteanu

A *k-dimensional metric foliation* on a Riemannian manifold M is a decomposition of M into locally equidistant (immersed) k-dimensional submanifolds called *leaves.* The homogeneous foliations—that is, foliations whose leaves are locally orbits of an isometric group action—are the primary source of metric foliations. As is well known, not all metric foliations are homogeneous. Nevertheless, it would be interesting to determine the spaces on which the homogeneity property does hold. One has complete results in this direction if the leaves are one-dimensional and the sectional curvature of the space is constant. Indeed, all one-dimensional metric foliations on spaces of constant nonnegative sectional curvature are homogeneous, whereas spaces of negative sectional curvature admit an abundance of nonhomogeneous one-dimensional metric foliations [1]. However, less is known if the constant curvature assumption is dropped. Among the few manifolds with nonconstant curvature on which it is known that the homogeneity property holds for one-dimensional foliations, we can mention $S^2 \times \mathbb{R}$ [2] and the Heisenberg group [3]. Our main purpose is to show that the class of manifolds just described also includes the compact Lie groups equipped with bi-invariant metrics and their quotients by finite groups acting freely and isometrically. It is also interesting to remark that the result is not valid without compactness. A counterexample on $\mathrm{SL}_2(\mathbb{R})$ is given in [7].

1. Introduction

In this section we introduce some notation and recall some well-known facts about compact Lie groups. Our main concern is the form of the Jacobi vector fields. It is this form that will play a key role in our investigations of one-dimensional metric foliations.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, and consider an inner product on $\mathfrak{g}$ such that $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ is orthogonal for each $g \in G$. If T is a maximal torus, then we have the following orthogonal decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = V_0 \oplus \sum_{r=1}^{M} V_r,$$

where V_0 is the Lie algebra of T and each V_r is a two-dimensional subspace, $1 \le r \le M$. Moreover, for each $x \in V_0$, $\mathrm{ad}_x = [x, \cdot] : \mathfrak{g} \to \mathfrak{g}$ acts trivially on V_0 and

Received June 29, 2004. Revision received October 19, 2005.

leaves each V_r invariant. In fact, given an orthogonal basis of V_r, the matrix of $\mathrm{ad}_x|_{V_r}$ in this basis is

$$2\pi\begin{pmatrix} 0 & -\theta_r(x) \\ \theta_r(x) & 0 \end{pmatrix},$$

where $\{\theta_r\}_{1\le r\le N}$ is a set of roots.

It is easy to see that, for $x \in V_0$, ad_x^2 has eigenvalues 0 and $-4\pi^2\theta_r^2(x)$ with $1 \le r \le M$ (some values may be repeated). Denote these eigenvalues by $0 = \lambda_0 > \lambda_1 > \cdots > \lambda_N$ and the corresponding eigenspaces by $W_x(\lambda_0), \ldots, W_x(\lambda_N)$, where N is a positive integer. Note that, for each $1 \le i \le N$, we have

$$W_x(\lambda_i) = \bigoplus_{r\in A_i} V_r, \quad \text{where } A_i = \{r \mid -4\pi^2|\theta_r(x)|^2 = \lambda_i,\, 1 \le r \le M\}.$$

If we extend the inner product on $\mathfrak{g}$ to a bi-invariant metric on G, then the sectional curvature of the 2-plane spanned by $x \in V_0$ and $y \in W_i$ is $k_i := \frac{1}{4}|[x, y]|^2 = -\frac{1}{4}\lambda_i$, where $|x| = |y| = 1$.

Now let γ be a unit-speed geodesic on G endowed with a bi-invariant metric such that $\gamma(0) = e$ and $\dot\gamma(0) = x$. It can easily be shown that if J is a Jacobi vector field along γ then

$$J(t) = E_0 + tF_0 + \sum_{i=1}^{N}\big(\cos\sqrt{k_i}tE_i + \sin\sqrt{k_i}tF_i\big), \tag{1}$$

where $E_i, F_i \in W_x(\lambda_i)$ for $0 \le i \le N$. Here we identify E_i and F_i with their parallel translates along γ. In fact, Jacobi vector fields have the same form along any geodesic. If γ doesn't start at the identity (say, $\gamma(0) = p$) then equation (1) holds with x replaced by $L_{p^{-1}*}x$, where $L_{p^{-1}}$ is the left translation by p^{-1}.

From now on, in order to avoid cumbersome notation, if x is tangent to G at p we will use λ_i, N, and $W_x(\lambda_i)$ to denote the eigenvalues, number of distinct eigenvalues, and eigenspaces corresponding to $\mathrm{ad}^2_{L_{p^{-1}*}x}$, respectively. We will also use $W_x(\lambda_i)$ for the left translation at p of the eigenspace of $\mathrm{ad}^2_{L_{p^{-1}*}x}$ corresponding to λ_i.

2. Riemannian Submersions, Metric Foliations, and Homogeneity

We next recall some important properties of Riemannian submersions. Most of these properties can easily be extended for metric foliations. For a detailed treatment of metric foliations and Riemannian submersions the reader is referred to [4], [5], and [6].

Let M, B be differentiable manifolds and let $\pi : M \to B$ be a submersion; that is, π is a surjective differentiable map of maximal rank. For any $b \in B$, $\pi^{-1}(b)$ is a submanifold of M of dimension $\dim(M) - \dim(B)$. Consequently, in the presence of a Riemannian metric on M, for each $m \in M$ one has a decomposition of the tangent space M_m into a vertical subspace $\mathcal{V}_m$ tangent to $\pi^{-1}(\pi(m))$ and a horizontal space $\mathcal{H}_m = \mathcal{V}_m^\perp$.

If M and B are Riemannian manifolds, then a map $\pi: M \to B$ is called a *Riemannian submersion* if π is a submersion and if π_* preserves the length of horizontal vectors; that is, if $|\pi_* x| = |x|$ for all $m \in M$ and $x \in \mathcal{H}_m$.

One can easily check that every Riemannian submersion $\pi: M \to B$ determines a metric foliation whose leaves are given by the preimages of points in B. The converse is also true locally. Hence the following definitions and remarks, which are formulated in the language of Riemannian submersions, can be extended to metric foliations.

As noted in [4], the crucial factors for understanding a Riemannian submersion are the integrability tensor A and the second fundamental form S:

$$A\colon \mathcal{H} \times \mathcal{H} \to \mathcal{V}, \quad A_X Y = (\nabla_X Y)^v;$$

$$S\colon \mathcal{H} \times \mathcal{V} \to \mathcal{V}, \quad S_X V = -(\nabla_V X)^v.$$

The mean curvature form κ is the horizontal 1-form defined by $\kappa(E) = \operatorname{tr}(S_{E^h})$. If the leaves are one-dimensional then $\kappa(X) = \langle S_X V, V\rangle$, where $X \in \mathcal{H}$ and $V \in \mathcal{V}$ with $|V| = 1$.

A horizontal vector field X on M is called *basic* if $\pi_* X = \tilde{X} \circ \pi$, where $\tilde{X}$ is a vector field on B. If X is a horizontal vector field along $\pi^{-1}(b)$ for $b \in B$, then X will still be called basic if $\pi_{*m} X_m = \pi_{*m'} X_{m'}$ for all $m, m' \in \pi^{-1}(b)$. Finally, a horizontal 1-form on M is called basic if its dual vector field is basic.

Let c be a geodesic in B with $c(0) = b$ and $\dot{c}(0) = x$, and let X be the unique basic vector field along $\pi^{-1}(b)$ with $\pi_* X = x$. For each $m \in \pi^{-1}(b)$, consider the geodesic γ_m of M starting at m in direction X_m. In this way we can define a diffeomorphism $h_c^t\colon \pi^{-1}(c(0)) \to \pi^{-1}(c(t))$ given by $h_c^t(m) = \gamma_m(t)$ and known as the *holonomy displacement map*. Observe that every curve ϕ in $\pi^{-1}(\phi(0))$ gives rise to a geodesic variation $H_{c,\phi}$ of $\gamma := \gamma_{\phi(0)}$ given by $H_{c,\phi}(t,s) := h_c^t(\phi(s))$. The corresponding Jacobi vector field $J = (H_{c,\phi})_*\big(\frac{\partial}{\partial s}\big)\big|_{(t,0)}$ along γ is vertical. Moreover,

$$J' = J'^v + J'^h = -S_{\dot\gamma} J - A^*_{\dot\gamma} J, \tag{2}$$

where $A^*_{\dot\gamma}$ is the adjoint of $A_{\dot\gamma}$.

Note that if the leaves have dimension 1 and if $|J(0)| = |\dot{c}(0)| = 1$, then

$$\langle J(0), J'(0)\rangle = -\langle J(0), S_{\dot\gamma(0)} J(0)\rangle = -\kappa(\dot\gamma(0)). \tag{3}$$

The mean curvature form κ plays a critical role in our investigations. Indeed, the homogeneity of one-dimensional metric foliations is characterized entirely in terms of the properties of κ. More precisely, we have the following results.

Theorem [1]. *A one-dimensional metric foliation $\mathcal{F}$ is homogeneous if and only if κ is closed.*

Theorem [8]. *Let $\mathcal{F}$ be a one-dimensional metric foliation on a manifold with sectional curvature bounded either from below or from above. If κ is basic then it is also closed.*

Thus, in order to prove that a one-dimensional metric foliation on a space with bounded sectional curvature is homogeneous, it is enough to show that the mean curvature form κ is basic.

In [5], O'Neill provides a detailed comparison between vector fields along a curve γ on the top manifold M and those along the projection $\pi \circ \gamma$ on the base B of a Riemannian submersion. Special considerations are given to the case when γ is a horizontal geodesic and the vector field on M is Jacobi. Next we mention some of the properties that are of interest to us. (Our notation is slightly different from that in [5] owing to the way we introduced the A- and S-tensors.)

If γ is a horizontal geodesic in M and if E is a vector field along γ, then

$$\pi_* E' = (\pi_* E)' - \pi_* A^*_{\dot\gamma} E^v. \tag{4}$$

If such an E is Jacobi then $\pi_* E$ is Jacobi along $\pi \circ \gamma$, provided that

$$E'^v + S_{\dot\gamma} E^v + A_{\dot\gamma} E^h = 0. \tag{5}$$

Moreover, if (5) is satisfied at one point then it is satisfied everywhere [5, p. 369].

We also have the following result.

PROPOSITION 1 [5]. *Let P be a vector field on $\pi \circ \gamma$ with $P(t_0) = 0$ for some real t_0. Then, given any vertical vector v at $\gamma(t_0)$, there exists a unique vector field E on γ such that $\pi_* E = P$, $E(t_0) = v$, and $D(E) := E'^v + S_{\dot\gamma} E^v + A_{\dot\gamma} E^h = 0$. Moreover, E is Jacobi if and only if P is.*

3. Preliminary Results

Let $\mathcal{F}$ be a one-dimensional metric foliation on a compact Lie group G equipped with a bi-invariant metric.

LEMMA 1. *If J is a holonomy Jacobi field along a horizontal geodesic γ, then the length of J is bounded.*

Proof. Consider $V(t) := \frac{1}{|J(t)|} J(t)$ (V is well-defined because J is never zero), and note that (2) implies $|J'|^2 = |J|^2(|S_{\dot\gamma} V|^2 + |A_{\dot\gamma} V|^2)$. Both the left side and the second factor on the right side of the equality are bounded. If the second factor vanishes at some point t_0 then $J'(t_0) = 0$ and, using (1), we can easily obtain that $|J|$ is bounded. If the second factor is never zero then the conclusion follows immediately. □

Let X and Y be two unit basic vector fields along a leaf L and let $p, \tilde p \in L$. Denote the left translations of $X_p, Y_p, X_{\tilde p}, Y_{\tilde p}$ at e by $x, y, \tilde x, \tilde y$, respectively.

LEMMA 2. *$y \in W_x(\lambda_0)$ if and only if $\tilde y \in W_{\tilde x}(\tilde\lambda_0)$.*

Proof. Since left translations are isometries, we may assume that $p = e$. Observe that, using Lemma 1, if J is a holonomy Jacobi vector field along the geodesic starting at e in direction x, then $J'(0) \perp W_x(0)$ and consequently $A_x y = 0$. But then $K_{xy} + 3|A_x y|^2 = 0$. Since (by [4]) $K_{XY} + 3|A_X Y|^2$ is constant along L it follows that $K_{X_{\tilde p} Y_{\tilde p}} = K_{\tilde x \tilde y} = \frac{1}{4}|[\tilde x, \tilde y]|^2 = 0$, which implies $\tilde y \in W_{\tilde x}(\tilde\lambda_0)$. □

Lemma 2 implies that if X is a unit basic vector field along L then the distribution $p \to W_x(\lambda_0) \cap \mathcal{H}_p$ is basic along L. Next we will show that the same holds if $W_x(\lambda_0)$ is replaced by $W_x(\lambda_i)$. However, the proof will be more involved since, a priori, N and $\tilde{N}$ as well as λ_i and $\tilde{\lambda}_i$ may be different.

PROPOSITION 2. *If y is an eigenvector of ad_x^2 with eigenvalue λ_i, then $\tilde{y}$ is an eigenvector of $\mathrm{ad}_{\tilde{x}}^2$ with the same eigenvalue.*

Proof. Let γ be the unit-speed geodesic starting at p in direction X_p, and let G_i be the parallel vector field along γ with $G_i(0) = Y_p$. The Jacobi vector field $J_i(t) = G_i \sin\bigl(\sqrt{k_i}t\bigr)$ is projectable because $D(J_i)(0) = J_i'^{v}(0) = 0$. If $\tilde{\gamma}$ is the unit-speed geodesic in direction $X_{\tilde{p}}$ then, by Proposition 1, there exists a unique Jacobi vector field $\tilde{J}_i$ along $\tilde{\gamma}$ such that $\tilde{J}_i(0) = 0$, $D(\tilde{J}_i) = 0$, and $\tilde{J}_i$ has the same projection as J_i. Since $J_i'(0) \perp W_x(\lambda_0)$ and since (by Lemma 2) $W_x(\lambda_0)$ and $W_{\tilde{x}}(\tilde{\lambda}_0)$ have the same projection, we obtain $\tilde{J}_i'(0) \perp W_{\tilde{x}}(\lambda_0)$. Consequently, $\tilde{J}_i(t) = \sum_{j=1}^{\tilde{N}} \sin\sqrt{\tilde{k}_j}t\tilde{G}_j$, where $\tilde{G}_j \in W_{\tilde{x}}(\tilde{\lambda}_j)$ for $1 \le j \le \tilde{N}$. Note that some of the $\tilde{G}_j$ may be zero.

Since $J_i\bigl(\pi/\sqrt{k_i}\,\bigr) = 0$, we obtain that $\tilde{J}_i\bigl(\pi/\sqrt{k_i}\,\bigr)$ must be vertical. Let us show that this is possible only if $\tilde{J}_i\bigl(\pi/\sqrt{k_i}\,\bigr) = 0$. By Lemma 1, the holonomy Jacobi vector field J along γ has the form

$$J(t) = E_0 + \sum_{i=1}^{N}\bigl(\cos\sqrt{k_i}tE_i + \sin\sqrt{k_i}tF_i\bigr), \tag{6}$$

where $E_0 \in W_x(\lambda_0)$ and $E_i, F_i \in W_x(\lambda_i)$ for $1 \le i \le N$. Also, if $\tilde{J}$ is the holonomy Jacobi vector field along $\tilde{\gamma}$ then

$$\tilde{J}(t) = \tilde{E}_0 + \sum_{i=1}^{\tilde{N}}\bigl(\cos\sqrt{\tilde{k}_i}t\tilde{E}_i + \sin\sqrt{\tilde{k}_i}t\tilde{F}_i\bigr), \tag{7}$$

where $\tilde{E}_0 \in W_{\tilde{x}}(\tilde{\lambda}_0)$ and $\tilde{E}_i, \tilde{F}_i \in W_{\tilde{x}}(\tilde{\lambda}_i)$ for $1 \le i \le \tilde{N}$.

Case 1: If $E_0 \neq 0$ then

$$\dim(G) - \dim(W_{\tilde{x}}(\tilde{\lambda}_0) \cap \mathcal{H}_{\tilde{p}}) = \dim(G) - \dim(W_x(\lambda_0) \cap \mathcal{H}_p)$$

is odd by Lemma 1 and because $\dim(G) - \dim(\{z \in \mathfrak{g} \mid [t,z] = 0\})$ is even for any $t \in \mathfrak{g}$. Consequently, we obtain $\tilde{E}_0 \neq 0$. But then $\tilde{J}_i\bigl(\pi/\sqrt{k_i}\,\bigr) = 0$ since $\tilde{J}$ and $\tilde{J}_i$ are vertical (and hence linearly dependent) at $t = \pi/\sqrt{k_i}$.

Case 2: If $E_0 = 0$ then, since the horizontal component $\tilde{J}_i^h$ of $\tilde{J}_i$ vanishes at $\pi/\sqrt{k_i}$, we can use the equality case of Cauchy–Schwartz on $\tilde{J}$ and $\tilde{J}_i$ to obtain

$$\left|\tilde{J}_i^h\!\left(\frac{\pi}{\sqrt{k_i}}\right)\right|^2 = \sum_{j=1}^{\tilde{N}}\left(|\tilde{G}_j|^2 - \frac{\bigl\langle\tilde{J}\bigl(\pi/\sqrt{k_i}\,\bigr), \tilde{G}_j\bigr\rangle^2}{\bigl|\tilde{J}\bigl(\pi/\sqrt{k_i}\,\bigr)\bigr|^2}\right)\sin^2\!\left(\frac{\pi\sqrt{\tilde{k}_j}}{\sqrt{k_i}}\right) = 0.$$

Each of the terms in the sum just displayed is nonnegative, so it follows that either:

(i) $\sin\big(\pi\sqrt{\tilde{k}_j}/\sqrt{k_i}\,\big) = 0$ for all j with $\tilde{G}_j \neq 0$; or
(ii) there exists some j_0 ($1 \leq j_0 \leq \tilde{N}$) with $\tilde{G}_{j_0} \neq 0$ such that $\sin\big(\pi\sqrt{\tilde{k}_{j_0}}/\sqrt{k_i}\,\big) \neq 0$, $\tilde{J}\big(\pi/\sqrt{k_i}\,\big)$ is a nonzero multiple of $\tilde{G}_{j_0}$, and $\sin\big(\pi\sqrt{\tilde{k}_j}/\sqrt{k_i}\,\big) = 0$ for all j with $\tilde{G}_j \neq 0$ and $j \neq j_0$.

We will show that only (i) can hold. Indeed, if we assume that (ii) holds then $\tilde{E}_l = 0$ for any $l \neq j_0$ with $\tilde{E}_l \neq 0$ and $1 \leq l \leq \tilde{N}$. Also, observe that $\tilde{J}_i^h$ vanishes at $2\pi/\sqrt{k_i}$. Consequently, if we denote $\cos\big(\pi\sqrt{\tilde{k}_{j_0}}/\sqrt{k_i}\,\big)$ by c and $\sin\big(\pi\sqrt{\tilde{k}_{j_0}}/\sqrt{k_i}\,\big)$ by s, then

$$c\tilde{E}_{j_0} + s\tilde{F}_{j_0} = a_1\tilde{G}_{j_0} \quad \text{and} \quad 2c(c\tilde{E}_{j_0} + s\tilde{F}_{j_0}) - \tilde{E}_{j_0} = a_2\tilde{G}_{j_0}$$

for some real nonzero a_1 and a_2. Thus, $(2ca_1 - a_2)\tilde{G}_{j_0} = \tilde{E}_{j_0}$. Note that $\tilde{G}_{j_0} \perp \tilde{E}_{j_0}$ since $\tilde{J}_i'(0) \perp \tilde{J}(0)$. Hence $2ca_1 = a_2$ and $\tilde{E}_{j_0} = 0$. But this is a contradiction because $\tilde{E}_{j_0} = 0$ implies that $\tilde{J}(0) = 0$, which is impossible.

Recall that, given $1 \leq i \leq N$, there exist $G_i \in W_x(\lambda_i)$ with G_i horizontal such that $J_i(t) = G_i \sin\big(\sqrt{k_i}t\big)$ is a projectable Jacobi vector field. For each such J_i we considered the corresponding Jacobi vector field $\tilde{J}_i(t) = \sum_{j=1}^{\tilde{N}} \sin\sqrt{\tilde{k}_j}t\tilde{G}_j$ along $\tilde{\gamma}$ having the same projection as J_i. Now let $\tilde{\Lambda}_i := \{j \mid \tilde{G}_j \neq 0\}$, and observe that $\tilde{J}_i\big(\pi/\sqrt{k_i}\,\big) = 0$ implies $\sqrt{\tilde{k}_j}/\sqrt{k_i} \in \mathbb{Z}$ for any $j \in \tilde{\Lambda}_i$. The same argument can be repeated by starting with projectable Jacobi vector fields along $\tilde{\gamma}$. Thus we obtain $\sqrt{k_l}/\sqrt{\tilde{k}_j} \in \mathbb{Z}$ for any $j \in \Lambda_l$, $1 \leq l \leq N$.

If $i = N$ and $j \in \tilde{\Lambda}_N$ then, for any $l \in \Lambda_j$, $\sqrt{k_l}/\sqrt{k_N} = \sqrt{k_l}/\sqrt{\tilde{k}_j} \cdot \sqrt{\tilde{k}_j}/\sqrt{k_N} \in \mathbb{Z}$. But $k_l < k_N$ if $l \neq N$. Thus, $l = N$ and $\tilde{\Lambda}_N$ contains a single element j for which $\tilde{k}_j = k_N$. Since a similar property is satisfied by $\tilde{k}_{\tilde{N}}$, we must have $k_N = \tilde{k}_{\tilde{N}}$. Moreover, if $J_N(t) = G_N \sin\big(\sqrt{k_N}t\big)$ for $G_N \in W_x(\lambda_N) \cap \mathcal{H}_p$, then $\tilde{J}_{\tilde{N}}(t) = \tilde{G}_{\tilde{N}} \sin\big(\sqrt{k_N}t\big)$ for some $\tilde{G}_{\tilde{N}} \in W_{\tilde{x}}(\lambda_{\tilde{N}})$. Using $D\tilde{J}_N(0) = 0$ and (4), it follows that $\tilde{G}_N$ is horizontal and projects to the same vector as G_N.

To summarize: $W_x(\lambda_N) \cap \mathcal{H}_p$ and $W_{\tilde{x}}(\lambda_{\tilde{N}}) \cap \mathcal{H}_{\tilde{p}}$ have the same projection. By induction, it follows that $N = \tilde{N}$, $\lambda_i = \tilde{\lambda}_i$, and the distribution $p \to W_x(\lambda_i) \cap \mathcal{H}_p$ is basic along the leaf for any $0 \leq i \leq N$. □

4. The Main Theorem

THEOREM 1. *One-dimensional metric foliations on compact Lie groups endowed with a bi-invariant metric are homogeneous.*

Proof. As before, let X be a basic vector field of unit length along a leaf L, and let γ and $\tilde{\gamma}$ be the geodesics with $\gamma(0) = p \in L$ and $\tilde{\gamma}(0) = \tilde{p} \in L$ in directions X_p and $X_{\tilde{p}}$, respectively. Also consider the corresponding holonomy Jacobi vector fields J and $\tilde{J}$ with $|J(0)| = |\tilde{J}(0)| = 1$. Depending on the form (6) of J, we distinguish two cases as follows.

Case (i) If there exists an i $(1 \le i \le N)$ such that F_i is not a (possibly zero) multiple of E_i, consider the Jacobi vector field $J_i(t) = G_i \sin\big(\sqrt{k_i}t\big)$, where $G_i \in W_x(\lambda_i)$, $\langle G_i, F_i\rangle \neq 0$, and G_i is horizontal at p. By the proof of Proposition 2, the corresponding Jacobi vector field $\tilde{J}_i$ along $\tilde{\gamma}$ is of the form $\tilde{G}_i \sin\big(\sqrt{k_i}t\big)$, where $\tilde{G}_i \in W_{\tilde{x}}(\lambda_i)$ is horizontal at $\tilde{p}$ and $|\tilde{G}_i| = |G_i|$. Because J_i and $\tilde{J}_i$ have the same horizontal components, using (6) and (7) yields

$$\left(|G_i|^2 - \frac{\langle G_i, F_i\rangle^2}{|J(t)|^2}\right)\sin^2\big(\sqrt{k_i}t\big) = \left(|\tilde{G}_i|^2 - \frac{\langle \tilde{G}_i, \tilde{F}_i\rangle^2}{|\tilde{J}(t)|^2}\right)\sin^2\big(\sqrt{k_i}t\big)$$

for any t. Dividing by $\sin^2\big(\sqrt{k_i}t\big)$ and using continuity arguments, it follows that $\langle G_i, F_i\rangle^2/|J(t)|^2 = \langle \tilde{G}_i, \tilde{F}_i\rangle^2/|\tilde{J}(t)|^2$ for any t. At $t = 0$ the preceding equality gives $\langle G_i, F_i\rangle^2 = \langle \tilde{G}_i, \tilde{F}_i\rangle^2$, which implies that $|J(t)| = |\tilde{J}(t)|$ for all t. The theorem follows now by (3).

Case (ii) If $F_i = \alpha_i E_i$ with $\alpha_i \in \mathbb{R}$ for all $1 \le i \le N$ then, for any i with $E_i \neq 0$, let $t_i \in \mathbb{R}$ be such that $\cos\big(\sqrt{k_i}t_i\big) + \alpha_i \sin\big(\sqrt{k_i}t_i\big) = 0$. It follows that $W_x(\lambda_i)$ is horizontal at $\gamma(t_i)$. Thus, for any $H_i \in W_x(\lambda_i)$, the Jacobi vector field $A_i(t) = H_i \sin\sqrt{k_i}(t - t_i)$ is projectable since $DA_i(t_i) = 0$. But, by Proposition 2, $W_{\tilde{x}}(\lambda_i)$ must also be horizontal at $\tilde{\gamma}(t_i)$. If $\tilde{A}_i$ denotes the corresponding Jacobi vector field along $\tilde{\gamma}$ with $\tilde{A}_i(t_i) = 0$ and $D\tilde{A}_i = 0$, then $\tilde{A}_i$ must be of the form $\tilde{A}_i(t) = \tilde{H}_i \sin\sqrt{k_i}(t - t_i)$, where $\tilde{H}_i \in W_{\tilde{x}}(\lambda_i)$ and $\tilde{H}_i$ projects to the same vector as H_i. Now it is enough to choose $H_i \in W_x(\lambda_i)$ such that $\langle H_i, E_i\rangle \neq 0$. The theorem follows by applying arguments similar to those used in Case (i). □

If Γ is a finite group acting freely and isometrically on G and if $\mathcal{F}$ is a one-dimensional metric foliation on G/Γ, then the lift of $\mathcal{F}$ is metric and thus (by Theorem 1) homogeneous. The corresponding local isometries of G induce local isometries of G/Γ whose orbits are locally the leaves of $\mathcal{F}$. This proves our last result.

THEOREM 2. *One-dimensional metric foliations on G/Γ are homogeneous.*

References

[1] D. Gromoll and K. Grove, *One-dimensional metric foliations in constant curvature spaces,* Differential geometry and complex analysis, pp. 165–168, Springer-Verlag, Berlin, 1985.

[2] D. Gromoll and K. Tapp, *Nonnegatively curved metrics on $S^2 \times \mathbb{R}$*, Geom. Dedicata 99 (2003), 127–136.

[3] M. Munteanu, *One-dimensional Riemannian foliations on the Heisenberg group,* Ph.D. dissertation, Univ. of Oklahoma, 2002.

[4] B. O'Neill, *The fundamental equations of a submersion,* Michigan Math. J. 13 (1966), 459–469.

[5] ———, *Submersions and geodesics,* Duke. Math. J. 34 (1967), 363–373.

[6] P. Tondeur, *Foliations on Riemannian manifolds,* Springer-Verlag, New York, 1988.

[7] G. Walschap, *Geometric vector fields on Lie groups,* Differential Geom. Appl. 7 (1997), 219–230.
[8] ———, *Umbilic foliations and curvature,* Illinois J. Math. 41 (1997), 122–128.

Department of Mathematics,
Computer Science, and Statistics
SUNY Oneonta
Oneonta, NY 13820

munteam@oneonta.edu