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Jensen Measures and Approximation of Plurisubharmonic Functions

Nguyen Quang Dieu & Frank Wikström

1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$, and let $\mathcal{PSH}(\Omega)$ denote the cone of plurisubharmonic functions on Ω. Recall that a function $u\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ is said to be plurisubharmonic if it is upper semicontinuous and if the restriction of u to every complex line is subharmonic. (We regard the function that is identically $-\infty$ as (pluri-)subharmonic.) We will use $\mathcal{PSH}^c(\Omega)$ for the cone of continuous functions $\bar{\Omega} \to \mathbb{R} \cup \{-\infty\}$ whose restrictions to Ω are plurisubharmonic. Furthermore, if u is an arbitrary upper bounded function on Ω then we define $u^*\colon \bar{\Omega} \to \mathbb{R} \cup \{-\infty\}$ as the *upper semicontinuous regularization* of u; that is, if $z \in \bar{\Omega}$ then

$$u^*(z) = \varlimsup_{\Omega \ni \zeta \to z} u(\zeta).$$

Clearly, if u is plurisubharmonic on Ω then $u^* = u$ on Ω, and it is reasonable to view the restriction of u^* to $\partial\Omega$ as the boundary values of u. For a point $z \in \bar{\Omega}$, we define two classes of *Jensen measures,*

$$\mathcal{J}_z(\bar{\Omega}) = \left\{ \mu \in \mathcal{B}(\bar{\Omega}) : u(z) \le \int_{\bar{\Omega}} u^*\, d\mu \ \ \forall u \in \mathcal{PSH}(\Omega),\ \sup u < \infty \right\},$$

$$\mathcal{J}_z^c(\bar{\Omega}) = \left\{ \mu \in \mathcal{B}(\bar{\Omega}) : u(z) \le \int_{\bar{\Omega}} u^*\, d\mu \ \ \forall u \in \mathcal{PSH}^c(\Omega) \right\},$$

where $\mathcal{B}(\bar{\Omega})$ is the set of Borel probability measures with support on $\bar{\Omega}$. Our main motivation for studying these measures is a duality theorem by Edwards [8], which allows us to express upper envelopes of plurisubharmonic functions as lower envelopes of integrals with respect to Jensen measures. The traditional method of constructing interesting plurisubharmonic functions is by taking upper envelopes over some class of plurisubharmonic functions, and thus Edwards's theorem provides an alternative way of studying these constructions. Clearly, $\mathcal{J}_z \subset \mathcal{J}_z^c$ for every $z \in \bar{\Omega}$. Two natural questions arise: First, for which domains is it true that $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \bar{\Omega}$ or every $z \in \Omega$? Second, if the inclusion is proper, then what properties does the set $\{z : \mathcal{J}_z \ne \mathcal{J}_z^c\}$ have? For example, is it always "small"? These questions are intimately connected to the possibility and impossibility of approximating upper bounded plurisubharmonic functions by continuous plurisubharmonic functions.

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Previous results can be found in Wikström [17], where it was shown that (a) if Ω is B-regular (or a polydisk) then $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \bar{\Omega}$ and (b) if Ω is (strongly) star-shaped then $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \Omega$. The first result of this paper (Theorem 3.1) gives a characterization of the domains for which $\mathcal{J}_z = \mathcal{J}_z^c$ in terms of an approximation property for upper bounded plurisubharmonic functions, and this characterization allows us to give a weaker necessary condition for a domain Ω to satisfy $\mathcal{J}_z = \mathcal{J}_z^c$ than the one found in [17]. Our second main result (Theorem 4.3) gives a partial answer to the other question. Roughly speaking, we show that if the set $\{z : \mathcal{J}_z = \mathcal{J}_z^c\}$ contains a sufficiently large portion near the boundary, then it is essentially the whole domain.

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2. Necessary Background Facts

Let us introduce some notation and quickly review a few facts that we will make heavy use of later in the paper. If $\mu \in \mathcal{J}_z(\bar{\Omega})$ (resp. $\mathcal{J}_z^c(\bar{\Omega})$) and $\operatorname{supp}\mu \subset K$, we write $\mu \in \mathcal{J}_z(K)$ (resp. $\mu \in \mathcal{J}_z^c(K)$). If there is no risk of confusion, we sometimes write $\mathcal{J}_z$ and $\mathcal{J}_z^c$ instead of $\mathcal{J}_z(\bar{\Omega})$ and $\mathcal{J}_z^c(\bar{\Omega})$. It is straightforward to verify that $\mathcal{J}_z(K)$ and $\mathcal{J}_z^c(K)$ are convex, weak-* compact subsets of the set of probability measures on K.

For ϕ an arbitrary (extended real-valued) function on $\bar{\Omega}$, we define two kinds of upper envelopes of ϕ. More precisely: if $z \in \bar{\Omega}$, we let

$$S^c\phi(z) = \sup\{u(z) : u \in \mathcal{PSH}^c(\Omega),\ u \le \phi\},$$

$$S\phi(z) = \sup\{u(z) : u \in \mathcal{USC}(\bar{\Omega}) \cap \mathcal{PSH}(\Omega),\ u \le \phi\}.$$

Edwards's duality theorem [8] (see [17] for details) implies the following connection between $S^c\phi$, $S\phi$ and the Jensen measures.

PROPOSITION 2.1. *If ϕ is a lower semicontinuous function on $\bar{\Omega}$, then*:

$$S^c\phi(z) = \inf\left\{\int_{\bar{\Omega}} \phi\, d\mu : \mu \in \mathcal{J}_z^c\right\} \quad \textit{for all } z \in \bar{\Omega};$$

$$S\phi(z) = \inf\left\{\int_{\bar{\Omega}} \phi\, d\mu : \mu \in \mathcal{J}_z\right\} \quad \textit{for all } z \in \Omega.$$

We should point out that there is a minor gap in the statement of Corollary 2.2 in [17], since the set $\{u^* : u \in \mathcal{PSH}(\Omega),\ \sup u < \infty\}$ is not a cone. (In general,

$(u+v)^*$ is smaller than u^*+v^* on $\partial\Omega$.) As a consequence, the first identity in Corollary 2.2 of [17] holds only for interior points (which explains why the second equality in the foregoing proposition holds for $z \in \Omega$, not $z \in \bar{\Omega}$), and this second equality is obtained by applying Edwards's theorem to the set $\mathcal{PSH}(\Omega) \cap \mathcal{USC}(\bar{\Omega})$, which is truly a cone.

We will also need to recall some elements of pluripotential theory. A set $P \subset \mathbb{C}^n$ is called *pluripolar* if, for every $z \in P$, there is a connected neighborhood U of z and a $u \in \mathcal{PSH}(U)$ such that $u \not\equiv -\infty$ and $u = -\infty$ on $U \cap P$. A deep theorem by Josefson [11] shows that the function u can be chosen to be plurisubharmonic on the whole of $\mathbb{C}^n$. In particular, u can be made negative on any fixed bounded neighborhood of P.

The following basic theorem of Bedford and Taylor will be used frequently.

THEOREM 2.2 [1, Thm. 7.1]. *Let $\{u_\alpha\}_{\alpha \in A}$ be a family of plurisubharmonic functions on Ω that is locally uniformly bounded from above. Define*

$$u(z) = \sup\{u_\alpha(z) : \alpha \in A\}.$$

Then (u^* is plurisubharmonic and) the set $\{z \in \Omega : u(z) < u^*(z)\}$ is pluripolar.

Recall that a domain Ω is called *regular* (in the real sense) if every continuous function on $\partial\Omega$ can be extended continuously to a harmonic function on Ω. One analogous property in pluripotential theory is the notion of B-regularity (see Sibony [15] and Diederich–Fornæss [6]). A bounded domain $\Omega \subset \mathbb{C}^n$ is called *B-regular* if every continuous function on $\partial\Omega$ can be extended continuously to a plurisubharmonic function on Ω. Note that every strictly pseudoconvex domain is B-regular. There is a characterization of B-regularity in terms of Jensen measures—namely, Ω is B-regular if and only if $\mathcal{J}_z^c = \{\delta_z\}$ for every $z \in \partial\Omega$ (see [15, Thm. 2.1] and [2, Thm. 1.7; 17, Cor. 3.8]).

Finally, we will also need the following simple fact, whose proof is left to the reader.

LEMMA 2.3. *Let Ω be a bounded domain in $\mathbb{C}^n$, and let $\{\phi_j\}$ be a sequence of continuous functions on Ω that decreases to an upper semicontinuous function ϕ on Ω. Then, for every sequence $\{a_j\} \subset \Omega$ with $a_j \to a \in \bar{\Omega}$, we have*

$$\phi^*(a) \geq \overline{\lim_{j\to\infty}}\, \phi_j(a_j),$$

where ϕ^* is the upper semicontinuous regularization of ϕ.

3. Equality of $\mathcal{J}_z$ and $\mathcal{J}_z^c$

The first result of this section establishes the equivalence between the equality $\mathcal{J}_z = \mathcal{J}_z^c$ and a weak version of the *bounded approximation property* introduced by Wikström in [17].

THEOREM 3.1. *Let Ω be a bounded domain in $\mathbb{C}^n$. Then the following statements are equivalent.*

(a) $\mathcal{J}_z = \mathcal{J}_z^c$ *for all* $z \in \Omega$.
(b) *For every upper bounded plurisubharmonic function* $u \in \mathcal{PSH}(\Omega)$, *there exists a uniformly upper bounded sequence* $\{u_j\}_{j\geq 1}$ *with* $u_j \in \mathcal{PSH}^c(\Omega)$ *such that* $u_j \to u$ *pointwise on* Ω *and* $\overline{\lim}_{j\to\infty} u_j \leq u^*$ *on* $\partial\Omega$. *Moreover, if* u *is continuous on* Ω *then the sequence* $\{u_j\}$ *can be chosen to converge to* u *locally uniformly.*

Proof. (a) $\Rightarrow$ (b): Let $u \in \mathcal{PSH}(\Omega)$ be upper bounded. Choose a sequence $\{\phi_j\}$ with $\phi_j \in C(\bar{\Omega})$ such that $\phi_j \searrow u^*$ on $\bar{\Omega}$. For each j we define $S\phi_j$ and $S^c\phi_j$ as before. By Edwards's theorem and the assumption $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \Omega$, we have that $S^c\phi_j = S\phi_j$ on Ω.

Since $\phi_j \in C(\bar{\Omega})$, it follows that $(S\phi_j)^* \leq \phi_j$ and so $(S\phi_j)^* = S\phi_j$ on Ω. Hence $S\phi_j \in \mathcal{PSH}(\Omega)$, and consequently we also have that $S^c\phi_j \in \mathcal{PSH}(\Omega)$. Thus $S^c\phi_j$ is upper semicontinuous on Ω, but since $S^c\phi_j$ is the supremum of continuous functions, $S^c\phi_j$ is automatically lower semicontinuous. Summing up, we have proved that $S\phi_j = S^c\phi_j \in \mathcal{PSH}(\Omega) \cap C(\Omega)$.

Furthermore, it is clear that the sequence $\{S\phi_j\}_j$ is decreasing and, since $u \leq S\phi_j \leq \phi_j$ for all j, it follows that $S\phi_j \searrow u$ on Ω. It is also clear that $\overline{\lim}_j (S\phi_j)^* \leq u^*$ on $\partial\Omega$.

Let $\{\Omega_j\}_j$ be a sequence of relatively compact subdomains of Ω with $\Omega_j \subset\subset \Omega_{j+1}$ and $\bigcup_j \Omega_j = \Omega$. For every j, by Choquet's lemma we can find a sequence $\{u_{l,j}\}_l$ in $\mathcal{PSH}^c(\Omega)$ increasing to $S^c\phi_j$ on $\bar{\Omega}$. Since $S^c\phi_j \in C(\bar{\Omega}_j)$, $u_{l,j}$ converges uniformly to $S\phi_j = S^c\phi_j$ on $\bar{\Omega}_j$ as $l \to \infty$, by Dini's theorem. Hence we can choose l_j such that $u_{l_j,j} \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$ and

$$|S\phi_j - u_{l_j,j}| \leq j^{-1} \text{ on } \bar{\Omega}_j \quad \text{and} \quad u_{l_j,j} \leq \phi_j \text{ on } \partial\Omega.$$

To finish this half of the proof, it is straightforward to verify that the sequence $\{u_{l_j,j}\}_j$ converges to u pointwise on Ω and that $\overline{\lim}\, u^*_{l_j,j} \leq u^*$ on $\partial\Omega$. Furthermore, if u is continuous on Ω then it follows from Dini's theorem that $S\phi_j \to u$ uniformly on every compact subset of Ω.

(b) $\Rightarrow$ (a): Let $z \in \Omega$ and take $\mu \in \mathcal{J}_z^c$. Let u be an upper bounded plurisubharmonic function on Ω. We have to show that

$$u(z) \leq \int_{\bar{\Omega}} u^* \, d\mu.$$

Let $\{u_j\}_{j\geq 1}$ be a uniformly upper bounded sequence in $\mathcal{PSH}^c(\Omega)$ that converges pointwise to u on Ω and such that $\overline{\lim}_{j\to\infty} u_j \leq u^*$ on $\partial\Omega$. Since $\mu \in \mathcal{J}_z^c$, we have that

$$u_j(z) \leq \int_{\bar{\Omega}} u_j \, d\mu.$$

Letting $j \to \infty$ and using Fatou's lemma, we are done. □

It is natural to ask whether there exists a bounded domain Ω in $\mathbb{C}^n$ such that, for some $z \in \Omega$, we have $\mathcal{J}_z \neq \mathcal{J}_z^c$. We do not know if such a domain can be found in $\mathbb{C}$, but in higher dimensions we can construct them using known examples of domains for which extension of (or approximation by) plurisubharmonic functions fails. We thank Professor Cegrell for pointing out the following example, which is essentially Example 2 in [5]. Let $B = \{z \in \mathbb{C}^n : |z| < 3\}$ and let $\ell = \{\alpha_j\}$ be a countable dense subset of $\mathbb{C}^n$ containing the origin. Let V be the (countable) set of all complex lines connecting pairwise distinct points in ℓ. Then V is pluripolar, since it is a countable union of pluripolar sets. Choose a function $\psi \in \mathcal{PSH}(\mathbb{C}^n)$ such that $\psi \not\equiv -\infty$ and $\psi|_V = -\infty$. Define $P = \{z \in \mathbb{C}^n : |z| = 2, \psi(z) \geq \sup_{|\zeta|<1} \psi(\zeta)\}$. Then the domain $\Omega_1 = B \setminus P$ has the interesting property that every continuous function on B which is plurisubharmonic on Ω_1 is in fact plurisubharmonic on B (see [5, Thm. 4.2]). Nevertheless, there is an upper bounded plurisubharmonic function $u \in \mathcal{PSH}(\Omega_1)$ that does not extend through P. Now, if we assume $\mathcal{J}_z^c = \mathcal{J}_z$ for every $z \in \Omega_1$, then by Theorem 3.1 there would exist a sequence $\{u_j\}$ in $\mathcal{PSH}^c(\Omega_1)$ converging pointwise to u on Ω_1. Since each u_j is in fact continuous on B, the function $\tilde{u} = \left(\overline{\lim}_{j\to\infty} u_j\right)^*$ is an extension of u to B. Hence there must be some $z \in \Omega_1$ for which $\mathcal{J}_z^c \neq \mathcal{J}_z$.

The domain Ω_1 here does not have a smooth boundary. For an example with smooth boundary, we can proceed as follows. Let Ω_2 be the domain constructed in [9, p. 260]. This is a smoothly bounded Hartogs domain in $\mathbb{C}^2$. Recall that being a Hartogs domain means that $(z, w) \in \Omega_2$ implies that $(e^{i\theta}z, w) \in \Omega_2$ for all $\theta \in \mathbb{R}$. Fornæss and Wiegerinck constructed a continuous plurisubharmonic function f on Ω_2 such that f cannot be approximated uniformly on some compact subset $K \subset\subset \Omega_2$ by plurisubharmonic functions that are defined on neighborhoods of $\bar{\Omega}$. Moreover, from the rather explicit description of f, it is easy to see that f is upper bounded on Ω_2. Now, if $\mathcal{J}_z^c = \mathcal{J}_z$ for all $z \in \Omega_2$ then Theorem 3.1 implies that, for any compact $K \subset\subset \Omega_2$, f can be uniformly approximated by functions in $\mathcal{PSH}^c(\Omega_2)$. In fact, by [9, Thm. 1] we can choose the approximating functions to be plurisubharmonic on neighborhoods of $\bar{\Omega}$, which is a contradiction.

Remark. The two preceding examples are *not* pseudoconvex. This raises the following natural question: *Does every smoothly bounded pseudoconvex domain have the weak bounded approximation property?* We will later see a pseudoconvex counterexample with nonsmooth boundary.

Furthermore, we do not know whether the approximating sequence in Theorem 3.1 can be chosen to be decreasing. If Ω is B-regular, this is possible (see [17, Thm. 4.1]).

Before formulating the main theorem of this section, we will need to introduce some terminology.

Definition 3.2. Let Ω be a domain in $\mathbb{C}^n$. By an *isotopy family of biholomorphic mappings* on Ω we mean a continuous map $\Phi\colon [0,1] \times \bar{\Omega} \to \mathbb{C}^n$ such that the following statements hold.

(a) For each $t \in [0,1]$, $\Phi_t(\cdot) = \Phi(t,\cdot)$ is a homeomorphism between $\bar{\Omega}$ and $\overline{\Phi_t(\Omega)}$; moreover, Φ_t maps Ω biholomorphically onto $\Phi_t(\Omega)$.
(b) For all $z \in \Omega$, $t \mapsto \Phi_t^{-1}(z)$ is real-analytic on $[0,1]$.
(c) Φ_t^{-1} converges uniformly to $\Phi_0^{-1} = \mathrm{Id}$ on $\bar{\Omega}$ as $t \to 0$.

DEFINITION 3.3. If Φ_t is an isotopy family of biholomorphic mappings on Ω, we define the *boundary cluster set* of Φ_t as the set of limit points of sequences of elements in $\bar{\Omega} \cap \Phi_t(\partial\Omega)$ as $t \to 0$.

DEFINITION 3.4. Let Ω be a domain in $\mathbb{C}^n$ and let $E \subset \bar{\Omega}$ be pluripolar. We define the (relative) pluripolar hull of E, $\operatorname{pph}(E)$, as

$$\operatorname{pph}(E) = \{z \in \Omega : g(z) = -\infty \text{ for all } g \in \mathcal{PSH}(\bar{\Omega}) \text{ with } g|_E = -\infty\}.$$

Here, $\mathcal{PSH}(\bar{\Omega})$ is the set of functions that are plurisubharmonic on some neighborhood of $\bar{\Omega}$.

THEOREM 3.5. *Let Ω be a bounded domain in $\mathbb{C}^n$, let Φ_t be an isotopy family of biholomorphic maps on Ω, and let X be the boundary cluster set of Φ_t. Assume that $\mathcal{J}_z^c = \{\delta_z\}$ for all $z \in X \setminus X'$, where X' is pluripolar. Then $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \Omega \setminus \operatorname{pph}(X')$.*

For the proof of this result, we need the following lemma.

LEMMA 3.6. *Under the assumptions of Theorem 3.5, for every $\phi \in C(\partial\Omega)$ there exist two increasing sequences $\{\phi_j\}$ and $\{v_j\}$ in $\mathcal{PSH}^c(\Omega)$ such that:*

(a) *$\phi_j \le \phi$ on $\partial\Omega$, and $\phi_j \nearrow \phi$ on $X \setminus X'$;*
(b) *$v_j < 0$ on Ω, $\lim_{j\to\infty} v_j < 0$, and $v_j \nearrow 0$ on $X \setminus X'$.*

Proof. Let K be a closed ball contained in Ω. Define

$$\Phi(z) = \sup\{u(z) : u \in \mathcal{PSH}^c(\Omega),\ u \le \phi \text{ on } \Omega\},$$

$$V(z) = \sup\{u(z) : u \in \mathcal{PSH}^c(\Omega),\ u \le -\chi_K \text{ on } \bar{\Omega}\}.$$

By Choquet's topological lemma, we can find a sequence $\{v_j\} \subset \mathcal{PSH}^c(\Omega)$ that increases to V on $\bar{\Omega}$. Note that $-\chi_K$ is lower semicontinuous on $\bar{\Omega}$ and so, by Proposition 2.1 and the assumption that $\mathcal{J}_z^c = \{\delta_z\}$ for all $z \in X \setminus X'$, we deduce that $V = 0$ on $X \setminus X'$. Furthermore, $V^* \in \mathcal{PSH}(\Omega)$ and $V^* \le -1$ on K, so by the maximum principle $V \le V^* < 0$ on Ω. This proves assertion (b).

For (a), we extend ϕ to a lower semicontinuous function on $\bar{\Omega}$ by defining $\phi \equiv \infty$ on Ω. Again using Proposition 2.1 and the assumption that $\mathcal{J}_z^c = \{\delta_z\}$ for all $z \in X \setminus X'$, it follows that $\Phi = \phi$ on X. The desired sequence $\{\phi_j\}$ is obtained by applying Choquet's lemma. □

Proof of Theorem 3.5. Let $z_0 \in \Omega \setminus \operatorname{pph}(X')$ and take an arbitrary $\mu \in \mathcal{J}_{z_0}^c$. We must show that

$$u(z_0) \le \int_{\bar{\Omega}} u^* \, d\mu$$

for every upper bounded $u \in \mathcal{PSH}(\Omega)$. Fix such a u and let $\{\varphi_k\}_{k\ge1}$ be a sequence of continuous functions on $\partial\Omega$ that decreases to $u^*|_{\partial\Omega}$. For each $t \in [0,1]$ we define $u_t = u \circ \Phi_t^{-1}$. Clearly, $u_t \in \mathcal{PSH}(\Phi_t(\Omega))$. Set

$$\Omega_j = \{z \in \Omega : \mathrm{dist}(z, \partial\Omega) > 1/j\}.$$

Choose a negative function $g \in \mathcal{PSH}(\bar{\Omega})$ with $g|_{X'} = -\infty$ and $g(z_0) > -\infty$.

Fix real numbers $\varepsilon > 0$ and $p \geq 1$ and fix the integer $k \geq 1$. By Lemma 3.6, we can find two increasing sequences $\{\varphi_{k,j}\}_{j\geq 1}$ and $\{v_j\}$ in $\mathcal{PSH}^c(\Omega)$ such that:

(a) $\varphi_{k,j} \nearrow \psi_k$ on $\bar{\Omega}$, where $\psi_k \leq \varphi_k$ on $\partial\Omega$ and $\psi_k = \varphi_k$ on $X \setminus X'$;
(b) $v_j \nearrow V$ on $\bar{\Omega}$, where $V < 0$ on Ω and $V = 0$ on $X \setminus X'$.

Let $\rho \in C_0^\infty(\mathbb{C}^n)$ be a nonnegative radial function with support in the unit ball and with $\int_{\mathbb{C}^n} \rho\, dV = 1$, and define $\rho_\delta(z) = \delta^{-n}\rho(z/\delta)$ for $\delta > 0$.

The following claim is the main point of our argument: There exist $t_0 > 0$ and $j_0 \geq 1$ such that, for all $0 < t < t_0$ and all $j \geq j_0$,

$$(u_t * \rho_{\delta(t,j)})(z) - \varepsilon + \varepsilon(g * \rho_{\delta(t,j)})(z) \leq p v_j(z) + \varphi_{k,j}(z) \tag{3.1}$$

for all $z \in \Omega \cap \Phi_t(\partial\Omega_j)$, where $\delta(t,j) = \mathrm{dist}(\Phi_t(\partial\Omega_j), \Phi_t(\partial\Omega)) > 0$. Recall that the convolution $u_{t_m} * \rho_{\delta(t,j)}$ is defined by

$$(u_t * \rho_{\delta(t,j)})(\xi) = \int_{B(0,\delta(t,j))} u_t(\xi - w)\rho_{\delta(t,j)}(w)\, dV(w). \tag{3.2}$$

To prove the claim, let us assume that it is not true. Then we could find sequences $\{j_m\}$ and $\{t_m\}$ with $j_m \to \infty$ and $t_m \to 0$ and a sequence of points $\{\xi_m\}$ with $\xi_m \in \Omega \cap \Phi_{t_m}(\partial\Omega_{j_m})$ such that

$$(u_{t_m} * \rho_{\delta(t_m,j_m)})(\xi_m) - \varepsilon + \varepsilon(g * \rho_{\delta(t_m,j_m)})(\xi_m) > p v_{j_m}(\xi_m) + \varphi_{k,j_m}(\xi_m). \tag{3.3}$$

By passing to a subsequence if necessary and using the uniform continuity of Φ on $[0,1] \times \bar{\Omega}$, we may assume that $\{\xi_m\}$ converges to some $\xi^* \in X$ and that $\delta(t_m, j_m) \searrow 0$. Consequently, $g * \rho_{\delta(t_m,j_m)} \searrow g$ on $\bar{\Omega}$. Since $g|_{X'} = -\infty$ and since the right-hand side of (3.3) is bounded from below, it follows from Lemma 2.3 that $\xi^* \in X \setminus X'$.

Fix $\varepsilon' > 0$. Since u^* is upper semicontinuous at ξ^* and since $\Phi_t^{-1} \to \mathrm{Id}$ uniformly as $t \to 0$, we have that

$$(u \circ \Phi_{t_m}^{-1})(\xi_m - w) < u^*(\xi^*) + \varepsilon'$$

for all $w \in B(0, \delta(t_m, j_m))$ and all sufficiently large m. Hence, from (3.2), it follows that

$$(u_{t_m} * \rho_{\delta(t_m,j_m)})(\xi_m) \leq u^*(\xi^*) + \varepsilon'$$

for m sufficiently large. As a result,

$$\overline{\lim_{m\to\infty}} (u_{t_m} * \rho_{\delta(t_m,j_m)})(\xi_m) \leq u^*(\xi^*) \leq \varphi_k(\xi^*).$$

On the other hand, by Lemma 2.3 we see that

$$\lim_{m\to\infty} v_{j_m}(\xi_m) = 0 \quad \text{and} \quad \underline{\lim_{m\to\infty}}\, \varphi_{k,j_m}(\xi_m) \geq \varphi_k(\xi^*).$$

Thus, letting $m \to \infty$ in (3.3) and recalling that g is negative on a neighborhood of $\bar{\Omega}$, we obtain a contradiction. The claim is proved.

By shrinking t_0 if necessary, we may assume that $z_0 \in \Phi_t(\Omega)$ for all $0 < t < t_0$. For each t $(0 < t < t_0)$ and each $j \geq j_0$, we define

$$\psi_{t,j} = \begin{cases} \max\{(u_t * \rho_{\delta(t,j)}) - \varepsilon + \varepsilon(g * \rho_{\delta(t,j)}), pv_j + \varphi_{k,j}\}, & z \in \bar{\Omega} \cap \Phi_t(\Omega_j), \\ pv_j + \varphi_{k,j}, & z \in \bar{\Omega} \setminus \Phi_t(\Omega_j), \end{cases}$$

for $z \in \Omega$. It follows from (3.1) that $\psi_{t,j} \in \mathcal{PSH}^c(\Omega)$. Note also that $z \in \Phi_t(\Omega)$ if and only if $z \in \Phi_t(\Omega_j)$ for some j. Hence

$$\lim_{j\to\infty} \psi_{t,j}(z) = \begin{cases} \max\{u_t(z) - \varepsilon + \varepsilon g(z), pV(z) + \psi_k(z)\}, & z \in \bar{\Omega} \cap \Phi_t(\Omega), \\ pV(z) + \psi_k(z), & z \in \bar{\Omega} \setminus \Phi_t(\Omega). \end{cases} \tag{3.4}$$

In particular, $\lim_{j\to\infty} \psi_{t,j}(z) \leq \varphi_k(z)$ on $\partial\Omega$ if t is small enough. Since $\mu \in \mathcal{J}^c_{z_0}$, we have that

$$\psi_{t,j}(z_0) \leq \int_{\bar{\Omega}} \psi_{t,j}\, d\mu = \int_{\Omega} \psi_{t,j}\, d\mu + \int_{\partial\Omega} \psi_{t,j}\, d\mu. \tag{3.5}$$

Applying Fatou's lemma and using (3.4), we obtain

$$\varlimsup_{j\to\infty} \int_{\Omega} \psi_{t,j}\, d\mu \leq \int_{\Omega\cap\Phi_t(\Omega)} \max\{u_t - \varepsilon + \varepsilon g, pV + \psi_k\}\, d\mu + \int_{\Omega\setminus\Phi_t(\Omega)} (pV + \psi_k)\, d\mu.$$

But, since $\lim_{j\to\infty}\psi_{t,j}(z) \leq \varphi_k(z)$ on $\partial\Omega$, it follows that

$$\varlimsup_{j\to\infty} \int_{\partial\Omega} \psi_{t,j}\, d\mu \leq \int_{\partial\Omega} \varphi_k\, d\mu.$$

Combining all of this, we get

$$\begin{aligned} \lim_{j\to\infty} \psi_{t,j}(z_0) &= \max\{u_t(z_0) - \varepsilon + \varepsilon g(z_0), pV(z_0) + \psi_k(z_0)\} \\ &\leq \int_{\Omega\cap\Phi_t(\Omega)} \max\{u_t - \varepsilon + \varepsilon g, pV + \psi_k\}\, d\mu \\ &\quad + \int_{\Omega\setminus\Phi_t(\Omega)} (pV + \psi_k)\, d\mu + \int_{\partial\Omega} \varphi_k\, d\mu. \end{aligned}$$

Letting $t \to 0$, applying Fatou's lemma, and using that the curve $t \mapsto \Phi_t^{-1}(z_0)$ is not pluri-thin at $t = 0$ (since we assumed that $t \mapsto \Phi_t^{-1}$ is real-analytic), we obtain

$$u(z_0) - \varepsilon + \varepsilon g(z_0) \leq \int_{\Omega} \max\{u - \varepsilon + \varepsilon g, pV + \psi_k\}\, d\mu + \int_{\partial\Omega} \varphi_k\, d\mu.$$

Finally, letting $p \to \infty$ and recalling that $V < 0$ and $g < 0$, we end up with

$$u(z_0) + \varepsilon g(z_0) \leq \int_{\Omega} u\, d\mu + \int_{\partial\Omega} \varphi_k\, d\mu;$$

now letting $k \to \infty$ and $\varepsilon \to 0$, we obtain $\mu \in \mathcal{J}_{z_0}$ because $g(z_0) \neq -\infty$. This concludes the proof. $\square$

REMARK. If Ω is B-regular, we can take $\Phi_t = \mathrm{Id}$ for all t and apply Theorem 3.5 to conclude that $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \Omega$. Similarly, if Ω is (strongly) star-shaped (i.e., $\Omega \subset\subset t\Omega$ for all $t > 1$ sufficiently close to 1) then we can take $\Phi_t(z) = (1+t)z$. In this case the boundary cluster set of Φ_t is empty and Theorem 3.5 again implies that $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \Omega$. Thus we recover Theorem 4.10 of [17].

EXAMPLE 3.7. Let Ω be the Hartogs triangle, $\Omega = \{(z,w) \in \mathbb{C}^2 : |z| < |w| < 1\}$. One can verify that $u(z,w) = |z/w|$ is an upper bounded plurisubharmonic function on Ω that cannot be approximated from above on $\bar{\Omega}$ by functions in $\mathcal{PSH}^c(\Omega)$. (For details, see [17].) Hence Ω does not have the bounded approximation property. Nevertheless, Ω *does* have the weak bounded approximation property. To see this, define $\Phi_t(z,w) = ((1+t)^2 z, (1+t)w)$. Then Φ_t is an isotopy family of biholomorphic mappings on Ω and the boundary cluster set X of Φ_t is just the origin. Since X is pluripolar and $\mathrm{pph}(X) = \{0\}$, we can apply Theorem 3.5.

COROLLARY 3.8. *Let Ω be a bounded domain in $\mathbb{C}^n$ and assume that $\Omega = \{\rho < 0\}$, where ρ is a C^1 function defined on a neighborhood of $\bar{\Omega}$. If there exists a holomorphic map $\Psi = (\psi_1, \dots, \psi_n)$ defined on a neighborhood of $\bar{\Omega}$ such that $\mathcal{J}_\xi^c = \{\delta_\xi\}$ for all $\xi \in \partial\Omega$ with*

$$\mathrm{Re}\left(\sum_{j=1}^n \psi_j(\xi) \frac{\partial \rho}{\partial z_j}(\xi)\right) \leq 0,$$

then $\mathcal{J}_z = \mathcal{J}_z^c$ for all $z \in \Omega$.

Proof. By Theorem 3.5, it suffices to find an isotopy family Φ of biholomorphic maps on Ω such that, for every point ξ in the cluster set of Φ_t, we have $\mathcal{J}_\xi^c = \{\delta_\xi\}$. By choosing $\varepsilon > 0$ small enough, the mapping $\Phi \colon [0,1] \times \bar{\Omega} \to \mathbb{C}^n$ defined by

$$\Phi_t(z) = z + t\varepsilon\Psi(z)$$

is an isotopy family of biholomorphic maps on Ω. Let us analyze the cluster set of $\bar{\Omega} \cap \Phi_t(\partial\Omega)$ as $t \to 0$. For each t $(0 < t < 1)$, let $\xi_t \in \bar{\Omega} \cap \Phi_t(\partial\Omega)$. Then $\rho(\xi_t) = \rho(\Phi_0^{-1}(\xi_t)) \leq 0$ and $\rho(\Phi_t^{-1}(\xi_t)) = 0$. Define $\tilde{\rho}_{\xi_t}(\lambda) = \rho(\Phi_\lambda^{-1}(\xi_t))$. Then there must exist λ_t, $0 < \lambda_t < t$, with $(\tilde{\rho}_{\xi_t})'(\lambda_t) \geq 0$. Write $\Phi_t^{-1} = (\tilde{\phi}_1(t,z), \dots, \tilde{\phi}_n(t,z))$. Using the chain rule, we get

$$0 \leq (\tilde{\rho}_{\xi_t})'(\lambda_t) = \frac{2}{\varepsilon}\,\mathrm{Re}\left(\sum_{j=1}^n \frac{\partial \rho}{\partial z_j}(\Phi_{\lambda_t}^{-1}(\xi_t)) \frac{\partial \tilde{\phi}_j(\lambda, z)}{\partial \lambda}(\lambda_t, \xi_t)\right). \tag{3.6}$$

However, since $\Phi_t \circ \Phi_t^{-1} = \mathrm{Id}$ it follows that

$$\lim_{\lambda \to 0} \frac{\partial \tilde{\phi}_j}{\partial t}(t,z)(\lambda, z) = -\phi_j(\xi) \quad \text{for all } \xi \in \bar{\Omega}.$$

Now, if $\xi^* = (\xi_1^*, \dots, \xi_n^*)$ is a limit point of the set $\{\xi_t\}$, then by passing to a limit in (3.6) we obtain

$$\operatorname{Re}\left(\sum_{j=1}^{n} \psi_j(\xi^*) \frac{\partial \rho}{\partial z_j}(\xi^*)\right) \le 0,$$

and we are thus in a situation where we can apply Theorem 3.5. □

EXAMPLE 3.9. Let $\varphi \in C^2(\mathbb{C})$ be such that $\lim_{z\to\infty} \varphi(z) = \infty$ and $\Delta\varphi > 0$ on $\{\varphi = -1/4\}$. Define $\Omega = \{(z,w) \in \mathbb{C}^2 : \operatorname{Re} w + |w|^2 + \varphi(z) < 0\}$. Then, clearly, Ω is a bounded domain.

Set $\psi(z,w) = (0, 1/2 + w)$. It is easy to check that the set of points $\xi \in \partial\Omega$ satisfying the inequality in the statement of Corollary 3.8 is contained in the set of strictly pseudoconvex boundary points. Hence, Corollary 3.8 implies that Ω has the weak approximation property.

4. Continuity of the Perron–Bremermann Envelope

Let $\phi \in C(\partial\Omega)$; we define the *Perron–Bremermann envelope* of ϕ as

$$U\phi(z) = \sup\{u(z) : u \in \mathcal{PSH}(\Omega),\ u^*|_{\partial\Omega} \le \phi\}.$$

(see Bremermann [3]). We will also define

$$U^c\phi(z) = \sup\{u(z) : u \in \mathcal{PSH}^c(\Omega),\ u|_{\partial\Omega} \le \phi\}.$$

It is well known that if Ω is regular (with respect to the Dirichlet problem for the Laplace operator, when viewed as a domain in $\mathbb{R}^{2n}$) then $U\phi \in \mathcal{PSH}(\Omega)$. Furthermore, if Ω is B-regular then $U\phi \in \mathcal{PSH}^c(\Omega)$ and $U\phi = \phi$ on $\partial\Omega$. It is also known that, if Ω is strictly pseudoconvex and ϕ sufficiently smooth, then $U\phi \in C^{1,1}$; this result is optimal in light of an example due to Gamelin and Sibony [10]. In this section, we will use Jensen measures to study the continuity of $U\phi$ on Ω. We should note that it is possible to find a pseudoconvex domain Ω and a $\phi \in C(\partial\Omega)$ such that $U\phi$ is not continuous. The following example can be found in Walsh [16] but is included here for completeness.

EXAMPLE 4.1. Let $u \in \mathcal{SH}(\mathbb{C})$ be locally bounded and such that $\{u < 0\}$ is a bounded nonempty set. Define $\Omega = \{(z_1,z_2) \in \mathbb{C}^2 : u(z_1) + |z_2| < 0\}$ and let $\phi(z) = -|z_2|$ on $\partial\Omega$. Then Ω is pseudoconvex and $\phi \in C(\partial\Omega)$. It is not difficult to verify that $U\phi = u$. Hence, $U\phi$ need not be continuous on Ω, even though $(U\phi)^* = \phi$ on $\partial\Omega$.

To motivate the use of Jensen measures as a tool for studying the continuity of $U\phi$, let us start with the following simple observation.

PROPOSITION 4.2. *Assume that $\mathcal{J}_z(\partial\Omega) = \mathcal{J}_z^c(\partial\Omega)$ for all $z \in \Omega$ and that Ω is regular (in the real sense). Then, for every $\phi \in C(\partial\Omega)$, we have that $U\phi \in \mathcal{PSH}(\Omega) \cap C(\Omega)$.*

Proof. Extend ϕ to a lower semicontinuous function on $\bar{\Omega}$ by setting $\phi(z) = \infty$ for $z \in \Omega$. By Edwards's theorem, $U\phi = S\phi = S^c\phi$ on Ω. (Note that, since $\phi|_{\Omega} = \infty$, only Jensen measures supported on $\partial\Omega$ are relevant.) Because $S^c\phi$ is lower semicontinuous and $U\phi$ is plurisubharmonic (in particular, upper semicontinuous), we see that $U\phi \in C(\Omega)$. $\square$

Proposition 4.2 raises the natural question of how to find a characterization of the domains Ω for which $\mathcal{J}_z(\partial\Omega) = \mathcal{J}_z^c(\partial\Omega)$ for all $z \in \Omega$. Theorem 3.5 provides us with a sufficient condition for this to hold. A related question is to estimate the size of the set $A = \{z \in \Omega : \mathcal{J}_z(\partial\Omega) = \mathcal{J}_z^c(\partial\Omega)\}$. A partial result in this direction is given by the following result, which (roughly speaking) states that if A contains sufficiently many points near $\partial\Omega$ then $\Omega \setminus A$ must be small.

THEOREM 4.3. *Let K be a compact subset of $\bar{\Omega}$ with $K \supset \partial\Omega$, and let*

$$A = \{z \in \Omega : \mathcal{J}_z^c(K) = \mathcal{J}_z(K)\}.$$

Assume there exist a set $W \subset \Omega$ with Lebesgue measure 0, *a pluripolar set $P \subset \partial\Omega$, and an open neighborhood V of $\partial\Omega \setminus P$ such that $(\Omega \cap V) \setminus W \subset A$. Then $A = \Omega \setminus Q$ for some pluripolar set Q.*

Proof. For $\phi \in C(K)$, define

$$U\phi(z) = \sup\{u(z) : u \in \mathcal{PSH}(\Omega),\ u^*|_K \le \phi\},$$
$$U^c\phi(z) = \sup\{u(z) : u \in \mathcal{PSH}^c(\Omega),\ u|_K \le \phi\}.$$

The first step in the proof will be to show that, for every $\phi \in C(K)$, $U\phi = U^c\phi$ outside a pluripolar set. Indeed, since $\partial\Omega \subset K$ it follows that $U\phi$ and $U^c\phi$ are upper bounded on Ω, hence $(U\phi)^*$ and $(U^c\phi)^*$ are both plurisubharmonic on Ω. By Choquet's topological lemma, we can find a sequence $\{u_j\}_{j\ge1} \subset \mathcal{PSH}(\Omega)$ that increases to a function $\tilde{u}$ on Ω such that $u_j^* \le \phi$ on K and $\tilde{u}^* = (U\phi)^*$. By Bedford and Taylor's theorem, there is a pluripolar set $\tilde{P} \subset \Omega$ such that $\tilde{u} = U\phi$ and $U^c\phi = (U^c\phi)^*$ on $\Omega \setminus \tilde{P}$.

The set $P \cup \tilde{P}$ is pluripolar, so we can find a plurisubharmonic function v defined on a neighborhood of $\bar{\Omega}$ with $v < 0$, $v \not\equiv -\infty$, and $v|_{P\cup\tilde{P}} = -\infty$. Since $(\Omega \cap V) \setminus W \subset A$ and using once again the trick of extending ϕ to ∞ on $\bar{\Omega} \setminus K$, Edwards's theorem implies that $U\phi = U^c\phi$ on $(\Omega \cap V) \setminus W$ because only Jensen measures supported on K are relevant. Fix an $\varepsilon > 0$ and an integer $j \ge 1$. It is clear that

$$u_j + \varepsilon v \le U\phi = U^c\phi \le (U^c\phi)^* \text{ on } (\Omega \cap V) \setminus W.$$

Since W has Lebesgue measure 0 and since $(U^c\phi)^*$ is plurisubharmonic on Ω, it follows that

$$u_j + \varepsilon v \le (U^c\phi)^* \text{ on } \Omega \cap V.$$

Choose a sequence $\{\Omega_k\}_{k\ge1}$ of subdomains exhausting Ω. We claim that, for k sufficiently large,

$$u_j + \varepsilon v \le (U^c\phi)^* + \varepsilon \text{ on } \partial\Omega_k. \tag{4.1}$$

Indeed, if this were not so then we could find a sequence $\{\xi_k\}$ with $\xi_k \in \partial\Omega_k$ and

$$(U^c\phi)^*(\xi_k) + \varepsilon \le u_j(\xi_k) + \varepsilon v(\xi_k).$$

Therefore,

$$(U^c\phi)^*(\xi_k) + \varepsilon \le (U\phi)(\xi_k) + \varepsilon v(\xi_k).$$

Since $v < 0$, it follows that $\xi_k \in \partial\Omega_k \setminus (\Omega \cap V)$. Hence, passing to a subsequence if necessary, we can assume that $\xi_k \to \xi^* \in \tilde{P}$. This is clearly a contradiction, since $v(\xi^*) = -\infty$. This establishes the claim (4.1).

Fix an integer $k \geq 1$ large enough that (4.1) holds. By Choquet's lemma again, we can find a sequence $\{v_m\} \subset \mathcal{PSH}^c(\Omega)$ with $v_m \leq \phi$ on K and $v_m \nearrow U^c\phi$ on $\bar{\Omega}$. We claim that we can find $\delta > 0$ and an integer $m \geq 1$ such that

$$(u_j + \varepsilon v) * \rho_\delta \leq v_m + 2\varepsilon \quad \text{on } \partial\Omega_k.$$

Again, seeking a contradiction, if this were not so then we could find sequences $\{\delta_l\}$ and $\{m_l\}$ with $\delta_l \searrow 0$ and $m_k \nearrow \infty$ as well as $\{\xi_l\} \subset \partial\Omega_k$ such that

$$(u_j + \varepsilon v) * \rho_\delta(\xi_k) > v_m(\xi_l) + 2\varepsilon.$$

By passing to a subsequence, we may assume that $\xi_l \to \xi^* \in \partial\Omega_k$. Letting $l \to \infty$ and applying Lemma 2.3, we obtain

$$(u_j + \varepsilon v)(\xi^*) \geq (U^c\phi)(\xi^*) + 2\varepsilon.$$

This inequality implies that $v(\xi^*) \neq -\infty$, so $(U^c\phi)(\xi^*) = (U^c\phi)^*(\xi^*)$. If we combine this with (4.1), we obtain our contradiction.

Finally, define

$$\tilde{v}_m(z) = \begin{cases} \max\{v_m(z) + 2\varepsilon, (u_j + \varepsilon v) * \rho_\delta(z)\}, & z \in \Omega_k, \\ v_m(z) + 2\varepsilon, & z \in \bar{\Omega} \setminus \Omega_k. \end{cases}$$

Clearly, $\tilde{v}_m \in \mathcal{PSH}(\Omega) \cap C(\bar{\Omega})$. Moreover, if δ is small enough then $\tilde{v}_m \leq \phi + 2\varepsilon$ on $\Omega_k \cap K$. Hence, for sufficiently small δ, $\tilde{v}_m \leq U^c\phi + 2\varepsilon$ on Ω. In particular,

$$u_j + \varepsilon v \leq (u_j + \varepsilon v) * \rho_\delta \leq U^c\phi + 2\varepsilon \quad \text{on } \Omega_k.$$

If we now let $j \to \infty$ and $\varepsilon \to 0$ in this estimate, it follows that $\tilde{u} \leq U^c\phi$ on $\Omega_k \setminus v^{-1}(-\infty)$. This holds for all k large enough and so, in fact, $U\phi = U^c\phi$ outside a pluripolar set. This concludes the first step of the proof.

To finish off the proof, let $\Lambda = \{\phi_j\}$ be a countable dense subset of $C(K)$. By the first step, for each j there is a pluripolar set $P_j \subset \Omega$ such that $U\phi_j = U^c\phi_j$ on $\Omega \setminus P_j$. Let $Q = \bigcup_j P_j$. Then Q is pluripolar and $U\phi_j = U^c\phi_j$ on $\Omega \setminus Q$ for all j. If $\phi \in C(K)$ is arbitrary, we can find a subsequence $\{\phi_{j_k}\}$ of Λ such that ϕ_{j_k} converges to ϕ uniformly on K. It follows that $U\phi_{j_k} \to U\phi$ and $U^c\phi_{j_k} \to U^c\phi$ uniformly on $\bar{\Omega}$. Hence $U\phi = U^c\phi$ on $\Omega \setminus Q$ for all $\phi \in C(K)$.

Now assume that $\mathcal{J}_z \subsetneq \mathcal{J}_z^c$ for some $z \in \Omega$. By the Hahn–Banach theorem, we can find $\mu \in \mathcal{J}_z^c(K)$ and $\phi \in C(K)$ such that

$$\int_K \phi\, d\mu < 0 < \int_K \phi\, d\nu \quad \text{for all } \nu \in \mathcal{J}_z.$$

Edwards's theorem then implies that $U^c\phi(z) < U\phi(z)$, so $z \in Q$. Hence $\mathcal{J}_z^c = \mathcal{J}_z$ for all $z \in \Omega \setminus Q$. □

5. Compactly Supported Measures in $\mathcal{J}^c$

The traditional way of defining and studying Jensen measures is in terms of function algebras. In the context of plurisubharmonic functions this is equivalent to the following definition.

DEFINITION 5.1. Let Ω be a domain in $\mathbb{C}^n$ and let $z \in \Omega$. Let μ be a Borel probability measure with $\operatorname{supp} \mu \subset\subset \Omega$. We say that μ is a *traditional Jensen measure* with barycenter z if

$$u(z) \leq \int_{\Omega} u \, d\mu$$

for all $u \in \mathcal{PSH}(\Omega)$. We denote the set of such measures by $\mathcal{J}_z^0$.

Note that in this definition we require the defining inequality to hold for *all* plurisubharmonic functions. On the other hand, this means that we must assume that $z \notin \partial\Omega$ and furthermore that $\operatorname{supp} \mu \subset\subset \Omega$. A natural question is the following: If $\mu \in \mathcal{J}_z^c$ (or $\mu \in \mathcal{J}_z$) and $\operatorname{supp} \mu \subset\subset \Omega$, is it true that $\mu \in \mathcal{J}_z^0$? In general the answer is No, as shown by the following example.

EXAMPLE 5.2. Let $\Omega \subset \mathbb{C}$ be a neighborhood of the unit disc punctured at the origin. Take any $p \in \Omega$ with $|p| < 1$ and let μ be the Poisson kernel for p times the normalized Lebesgue measure on the unit circle. Then $\mu \in \mathcal{J}_p$ (and hence in $\mathcal{J}_p^c$), since every upper bounded subharmonic function on Ω extends across the puncture. However, it is easy to see that $\mu \notin \mathcal{J}_p^0$. (If $u(z) = -\log|z|$, then $u(p) > \int_{\Omega} u \, d\mu$.)

If we assume something more about the domain—for example, that it is hyperconvex—then the answer is Yes. To prove this we will need some preliminary results.

PROPOSITION 5.3. *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Assume that either* (a) *Ω is hyperconvex or* (b) *every holomorphic function on Ω can be approximated uniformly on compact sets by functions that are holomorphic on Ω and continuous up to $\partial\Omega$. Let $\delta > 0$ and define $\Omega_\delta = \{z \in \Omega : \operatorname{dist}(z, \partial\Omega) > \delta\}$.*

Then, for δ small enough, every $u \in \mathcal{PSH}(\Omega_\delta)$ is the pointwise limit of a sequence of functions in $\mathcal{PSH}^c(\Omega)$.

Proof. Choose $\delta > 0$ so small that Ω_δ is a (nonempty) connected subset of Ω. Since $-\log \operatorname{dist}(\cdot, \partial\Omega)$ is plurisubharmonic, it follows that Ω_δ is Runge in Ω. Fix $u \in \mathcal{PSH}(\Omega_\delta)$. Since Ω_δ is pseudoconvex, we can find a sequence $\{u_m\}_{m \geq 1} \subset \mathcal{PSH}(\Omega_\delta) \cap C^\infty(\Omega_\delta)$ with $u_m \searrow u$. Choose an increasing sequence $\{K_m\}$ of compact subsets of Ω_δ with $\bigcup_m K_m = \Omega_\delta$. For a fixed $m \geq 1$, by [14, Thm. 9] we can find holomorphic functions $f_{1,m}, \dots, f_{p_m,m}$ on Ω_δ and positive integers $c_{1,m}, \dots, c_{p_m,m}$ such that

$$u_m(z) - \frac{1}{m} \leq \max_{k \leq p_m} \left\{ \frac{1}{c_{k,m}} \log|f_{k,m}(z)| \right\} \leq u_m(z)$$

for all $z \in K_m$. Let $\alpha_m = \min_{K_m} u_m$ and let $d_m = 2c_{1,m} \cdots c_{p_m,m}$. For every $j \leq 1$, define

$$\psi_{j,m} = \frac{1}{2jd_m} \log(|f_{1,m}|^{2jd_m/c_{1,m}} + \dots + |f_{p_m,m}|^{2jd_m/c_{p_m,m}} + e^{2j(\alpha_m - 1/m)}).$$

We claim that $\phi_{j,m}$ is plurisubharmonic on Ω_δ. Indeed, if $\Omega \subset \mathbb{C}$ then this can be verified by a direct computation of $\Delta\psi_{j,m}$. The higher-dimensional case reduces to this by restricting $\psi_{j,m}$ to complex lines. It is straightforward to verify that $\psi_{j,m} \to \max_{k \le p_m}\{(1/c_{k,m})|f_{k,m}(z)|\}$ uniformly on K_m as $j \to \infty$. Since Ω_δ is Runge in Ω, we may assume that each $f_{k,m}$ is holomorphic on Ω. This implies that $\psi_{j,m}$ extends to a continuous plurisubharmonic function on Ω.

If Ω is hyperconvex, we can use [9, Thm. 2] to conclude that each u_m can be approximated uniformly on K_m by functions in $\mathcal{PSH}^c(\Omega)$. Let $m \to \infty$ to conclude the proof. For the other case, simply note that $f_{k,m}$ can be assumed to be continuous on $\bar{\Omega}$. □

Using Proposition 5.3 and Fatou's lemma, we immediately deduce the following.

COROLLARY 5.4. *Let Ω be as in Proposition 5.3 and let $z \in \Omega$. If $\mu \in \mathcal{J}_z^c$ and* $\operatorname{supp}\mu \subset\subset \Omega$, *then $\mu \in \mathcal{J}_z^0(\Omega_\delta)$ for $\delta > 0$ small enough (and consequently $\mu \in \mathcal{J}_z^0(\Omega)$).*

PROPOSITION 5.5. *The assumptions in Proposition 5.3 are satisfied if either* (a) Ω *is fat (i.e.,* $\operatorname{int}(\bar{\Omega}) = \Omega$) *and $\bar{\Omega}$ is holomorphically convex in some pseudoconvex* $\Omega' \supset \Omega$ *or* (b)

$$\Omega = \{(z,w) \in \mathbb{C}^2 : z \in \mathbb{D},\ |w| < e^{-\psi(z)}\}$$

for some lower bounded subharmonic function ψ on $\mathbb{D}$, where $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$.

Proof. Consider the first case, where Ω is fat and $\bar{\Omega}$ is holomorphically convex in Ω'. Then every holomorphic function on Ω can be approximated uniformly on compact subsets of Ω by functions that are holomorphic on Ω'. Thus Proposition 5.3 applies.

For the second case, when Ω is a Hartogs domain over $\mathbb{D}$ we can expand a holomorphic function f on Ω in a Hartogs series

$$f(z,w) = \sum_{k=0}^{\infty} f_k(z) w^k,$$

where each f_k is holomorphic on $\mathbb{D}$. For $j \ge 1$, let

$$f_j(z,w) = \sum_{k=0}^{\infty} f_k\left(\frac{jz}{j+1}\right) w^k.$$

It is clear that f_j converges pointwise to f on Ω. □

Our final result is inspired by Theorem 4.7 in [7]; in fact, it is an easy consequence of a theorem by Bu and Schachermayer [4] (see also [13]), which states that $\mathcal{J}_z^0$ is the weak-* closure of the measures that are push-forwards of the Lebesgue measure on the circle under closed analytic disks. Before formulating the result, let us introduce

$$\mathcal{A}_z(\Omega) = \{f_*\sigma : f \in \mathcal{O}(\bar{\mathbb{D}}, \Omega),\ f(0) = z\},$$

where σ is the normalized Lebesgue measure on $\partial\mathbb{D}$ and $\mathcal{O}(\bar{\mathbb{D}}, \Omega)$ is the set of maps that are holomorphic on some neighborhood of $\bar{\mathbb{D}}$ with values in Ω.

PROPOSITION 5.6. *Let Ω be a bounded domain in $\mathbb{C}^n$ and let $z \in \Omega$. Then $\mathcal{J}_z^c \subset W$, where W is the weak-* closure in $C(\bar{\Omega})^*$ of*

$$\{a\nu - (a-1)\delta_z : a \geq 0,\ \nu \in \mathcal{A}_z(\Omega)\}$$

and $\mathcal{J}_z$ is the weak- closure in $C(\bar{\Omega})^*$ of $\mathcal{A}_z(\Omega)$.*

Proof. For the first assertion, let

$$X = \left\{\sum_{k=1}^{j} a_k(\mu_k - \delta_z) : a_k \geq 0,\ \mu_k \in \mathcal{A}_z(\Omega)\right\}.$$

It is easy to check that $\bar{X}$ is a convex cone in $C(\bar{\Omega})^*$. Let $\mu \in \mathcal{J}_z^c$. We claim that $\mu - \delta_z \in \bar{X}$. Assume otherwise; then, by the Hahn–Banach theorem, we can find $\phi \in C(\bar{\Omega})$ such that

$$\int_{\bar{\Omega}} \phi\, d\nu < c < \int_{\bar{\Omega}} \phi\, d\mu - \phi(z) \tag{5.1}$$

for all $\nu \in X$. Since X is a cone, we can take $c = 0$. Consequently, the restriction of ϕ to the intersection of any complex line with Ω satisfies the sub-mean value inequality and hence $\phi \in \mathcal{PSH}^c(\Omega)$. This is a contradiction since $\mu \in \mathcal{J}_z^c$, so the right-hand side of (5.1) is negative. On the other hand, if $\nu \in X$ then

$$\nu = \sum_{k=1}^{j} a_k(\mu_k - \delta_z)$$

for some choices of $a_k \geq 0$ and $\mu_k \in \mathcal{A}_z(\Omega)$. Hence

$$\nu = A\left(\sum_{k=1}^{j} \frac{a_k}{A}\mu_k - \delta_z\right),$$

where $A = \sum_{k=1}^{j} a_k$. By the Bu–Schachermayer theorem [4], $\nu = A(\nu' - \delta_z)$ for some ν' in the weak-* closure of $\mathcal{A}_z(\Omega)$.

For the second assertion, we again argue by contradiction. Assume that there is a $\mu \in \mathcal{J}_z \setminus \overline{\mathcal{A}_z(\Omega)}$. From Bu–Schachermayer we know that $\overline{\mathcal{A}_z(\Omega)}$ is a closed convex set in $C(\bar{\Omega})^*$, so by the Hahn–Banach theorem we can find $\phi \in C(\bar{\Omega})$ such that

$$\begin{aligned}
\int_{\bar{\Omega}} \phi\, d\mu < 0 < \inf&\left\{\frac{1}{2\pi}\int_0^{2\pi} (\phi \circ f)(e^{i\theta})\, d\theta : f \in \mathcal{O}(\bar{\mathbb{D}}, \Omega),\ f(0) = z\right\} \\
&= \sup\{u(z) : u \in \mathcal{PSH}(\Omega),\ u \leq \phi\} \\
&= \sup\{u(z) : u \in \mathcal{PSH}(\Omega) \cap \mathcal{USC}(\bar{\Omega}),\ u \leq \phi\}.
\end{aligned}$$

In the second line we have used a theorem by Poletsky (see [12; 13]) about the plurisubharmonicity of the Poisson envelope. Edwards's theorem gives us the desired contradiction. $\square$

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N. Q. Dieu
Department of Mathematics
University of Education, Hanoi
(Dai Hoc Su Pham Hanoi)
Cau Giay, Tuliem
Hanoi
Vietnam

ngquangdieu@hn.vnn.vn

F. Wikström
Department of Mathematics
University of Michigan
530 Church St.
Ann Arbor, MI 48109

fwikstro@umich.edu