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# A Uniqueness Property for $H^\infty$ on Coverings of Projective Manifolds

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## 1. Formulation of the Result

*1.1*

Let $M$ be a complex projective manifold of dimension $n \geq 2$ with a Kähler form $\omega$, and let $L$ be a positive line bundle on $M$ with canonical connection $\nabla$ and curvature $\Theta$ in a hermitian metric $h$. Let $C$ be the common zero locus of holomorphic sections $s_1, \dots, s_k$, $k < n$, of $L$ over $M$, which (in a trivialization) can be completed to a set of local coordinates at each point $C$. Then $C$ is a (possibly disconnected) $k$-dimensional submanifold of $M$, which will be referred to as an *$L$-submanifold of $M$*. Let $\pi : Y_G \to M$ be a regular covering of $M$ with a transformation group $G$, and let $X_G = \pi^{-1}(C)$. We denote the pullbacks to $Y_G$ of $\omega$ and $\Theta$ by the same letters.

Example 1.1. If $L$ is very ample, then it is the pullback of the hyperplane bundle by an embedding of $M$ into some projective space $\mathbb{CP}^N$. Further, zero loci of holomorphic sections of $L$ are hyperplane sections of $M$. By Bertini's theorem, the generic linear subspace of codimension $n-k$ ($k < n$) intersects $M$ transversely in a smooth manifold $C$ of dimension $k$, and by the Lefschetz hyperplane theorem, $C$ is connected and the induced homomorphism $\pi_1(C) \to \pi_1(M)$ of fundamental groups is surjective. Hence, in this case $X_G \subset Y_G$ is a connected submanifold.

Let $\mathrm{dist}(\cdot,\cdot)$ be the distance on $Y_G$ induced by $\omega$ and let $\delta(x) := \mathrm{dist}(x, o)$ for some fixed $o \in Y_G$. By a result of Napier [N], there is a smooth function $\tau$ on $Y_G$ such that:

(A) $c_1\delta \leq \tau \leq c_2\delta + c_3$ for some $c_1, c_2, c_3 > 0$;
(B) $d\tau$ is bounded; and
(C) $i\partial\bar\partial\tau$ is bounded.

Furthermore, by (A) and since the curvature of $Y_G$ is bounded below, there is $c > 0$ such that $e^{-c\tau}$ is integrable on $Y_G$. Then $e^{-c\tau}$ is also integrable on $X_G$. We set

$$A := \frac{cc_2}{c_1}. \tag{1.1}$$

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Let $L$ be a positive line bundle on $M$ with curvature $\Theta$ satisfying (in the sense of Nakano)

$$\Theta > i\partial\bar{\partial}(A\tau). \tag{1.2}$$

Consider the covering $X_G := \pi^{-1}(C) \subset Y_G$ of an $L$-submanifold $C \subset M$. Let $H^\infty(Y_G)$ and $H^\infty(X_G)$ be the Banach spaces of bounded holomorphic functions on $Y_G$ and $X_G$ in the corresponding supremum norms.

THEOREM 1.2. *The map $\rho\colon H^\infty(Y_G) \to H^\infty(X_G)$, $f \mapsto f|_{X_G}$, is an isometry.*

This result answers a question posed in the introduction to [L].

*1.2*

The main application of Theorem 1.2 is in the area of the corona problem. Let $X$ be a complex manifold and let $H^\infty(X)$ be the Banach algebra (in the supremum norm) of bounded holomorphic functions on $X$. Then the maximal ideal space $\mathcal{M} = \mathcal{M}(H^\infty(X))$ is the set of all nontrivial linear multiplicative functionals on $H^\infty(X)$. The norm of any $\phi \in \mathcal{M}$ is $\leq 1$ and so $\mathcal{M}$ is embedded into the unit ball of the dual space $(H^\infty(X))^*$. Thus $\mathcal{M}$ is a compact Hausdorff space in the weak-$*$ topology induced by $(H^\infty(X))^*$ (i.e., the *Gelfand topology*). Furthermore, there is a continuous map $i\colon X \to \mathcal{M}$ taking $x \in X$ to the evaluation homomorphism $f \mapsto f(x)$. This map is an embedding if $H^\infty(X)$ separates points of $X$. The complement to the closure of $i(X)$ in $\mathcal{M}$ is called the *corona*. The *corona problem* is to determine those $X$ for which the corona is empty. For example, according to Carleson's celebrated corona theorem [C], this is true if $X$ is the open unit disk $\mathbb{D} \subset \mathbb{C}$. Also there are nonplanar Riemann surfaces for which the corona is nontrivial (see e.g. [BD; G; JM] and references therein). The general problem for planar domains is still open, as is the problem in several variables for the ball and polydisk. In [L, Thm. 2.1] Lárusson discovered simplest examples of Riemann surfaces with big corona. Namely, he proved that if $Y_G \subset \mathbb{C}^n$ is a bounded domain and if $X_G \subset Y_G$ is a Riemann surface satisfying the assumptions of Theorem 1.2, then the natural map $i\colon X_G \hookrightarrow \mathcal{M}(H^\infty(X_G))$ extends to an embedding $Y_G \hookrightarrow \mathcal{M}(H^\infty(X_G))$. The next statement extends his result and produces many examples of nonplanar Riemann surfaces with big corona.

COROLLARY 1.3. *Under the assumptions of Theorem 1.2, the transpose map $\rho^*\colon \mathcal{M}(H^\infty(X_G)) \to \mathcal{M}(H^\infty(Y_G))$, $\phi \mapsto \phi \circ \rho$, is a homeomorphism.*

This follows from the fact that $\rho\colon H^\infty(Y_G) \to H^\infty(X_G)$ is an isometry of Banach algebras.

EXAMPLE 1.4. (1) (The references for this example are in [L, Sec. 4].) Let $M$ be a projective manifold covered by the unit ball $\mathbb{B} \subset \mathbb{C}^n$ with a positive line bundle $L$ of curvature $\Theta$, and let $X \subset \mathbb{B}$ be the preimage of an $L$-submanifold $C \subset M$. Let $\delta$ be the distance from the origin in the Bergman metric of $\mathbb{B}$. By $\omega$ we denote the Kähler form of the Bergman metric. It was shown in [L, Sec. 4] that there is a nonnegative function $\tau$ on $\mathbb{B}$ such that $i\partial\bar{\partial}\tau = \omega$, $d\tau$ is bounded, and

$$\sqrt{n+1}\delta \le \tau \le \sqrt{n+1}\delta + (n+1)\log 2.$$

Moreover,

$$\int_{\mathbb{B}} e^{-c\tau}\omega^n < \infty \quad \text{if and only if} \quad c > \frac{2n}{n+1}.$$

Applying Theorem 1.2 $\bigl($with $c_2 = c_1 = \sqrt{n+1}\bigr)$, we obtain that $\rho\colon H^\infty(\mathbb{B}) \to H^\infty(X)$ *is an isometry if* $\Theta > \frac{2n}{n+1}\omega$. This holds for instance if $L = K^{\otimes m}$ with $m \ge 2$, where $K$ is the canonical bundle of $M$.

(2) Let $S$ be a compact complex curve of genus $g \ge 2$ and let $\mathbb{CT}$ be a one-dimensional complex torus. Consider an $L$-curve $C$ in $M := S \times \mathbb{CT}$ with a very ample $L$ satisfying the assumptions of Theorem 1.2. Let $\pi : \mathbb{D} \times \mathbb{C} \to M$ be the universal covering. Then Theorem 1.2 is valid for the connected curve $X := \pi^{-1}(C) \subset \mathbb{D} \times \mathbb{C}$. This implies that any $f \in H^\infty(X)$ is constant on each $S_y := (\{y\} \times \mathbb{C}) \cap X$, $y \in \mathbb{D}$. Note that $S_y$ is the union of the orbits of some $z_{iy} \in X$, $i = 1, \dots, k$, under the natural action of the group $\pi_1(\mathbb{CT})$ ($\cong \mathbb{Z} \oplus \mathbb{Z}$) on $\mathbb{D} \times \mathbb{C}$.

## 2. Proof of Theorem 1.2

*2.1*

As in Section 1, let $X_G \subset Y_G$ be the covering of an $L$-submanifold $C \subset M$. Consider a function $\phi\colon Y_G \to \mathbb{R}$ such that $d\phi$ is bounded; that is,

$$|\phi(x) - \phi(y)| \le a \cdot \operatorname{dist}(x, y) \quad \text{for some } a > 0.$$

By $\mathcal{O}_\phi(X_G)$ we denote the vector space of holomorphic functions on $X_G$ such that $|f|^2 e^{-\phi}$ is integrable on $X_G$ with respect to the volume form of the induced Kähler metric on $X_G$. This is a Hilbert space with respect to the inner product

$$(f, g) \mapsto \int_{X_G} f\bar{g}e^{-\phi}\omega^k.$$

We define $\mathcal{O}_\phi(Y_G)$ similarly, and by $|\cdot|_{\phi, X_G}$ and $|\cdot|_{\phi, Y_G}$ we denote the corresponding norms. It was shown in [L] that the restriction determines a continuous linear map

$$\rho\colon \mathcal{O}_\phi(Y_G) \to \mathcal{O}_\phi(X_G), \quad f \mapsto f|_{X_G}.$$

The following remarkable result was proved by Lárusson [L, Thm. 1.2].

THEOREM 2.1. *Suppose*

$$\Theta \ge i\partial\bar{\partial}\phi + \varepsilon\omega$$

*for some* $\varepsilon > 0$. *Then* $\rho$ *is an isomorphism.*

Now assume that the curvature $\Theta$ of an $L$-submanifold $C \subset M$ satisfies (1.2). Then Lárusson's theorem holds for coverings $X_G := \pi^{-1}(C) \subset Y_G$ of $C$ with $\phi := A\tau$ and with $\phi := c\tau$ (because $A \ge c$).

*2.2*

We fix a fundamental compact $K$ of the action of $G$ on $Y_G$, that is, $Y_G = \bigcup_{g\in G} g(K)$. Consider finite covers $\mathcal{U} = (U_i)$ and $\mathcal{V} = (V_j)$ of $K$ by compact coordinate polydisks such that each $V_j$ belongs to the interior of some $U_{i_j}$.

LEMMA 2.2. *Let $f$ be a holomorphic function defined in an open neighborhood $O$ of $\bigcup_i U_i$. Assume that*

$$\int_O |f|^2\omega^n = B < \infty.$$

*Then there is a constant $b > 0$ (depending on $\mathcal{U}$ and $\mathcal{V}$ only) such that*

$$\max_K |f| \le b\sqrt{B}. \tag{2.1}$$

The proof of the lemma is a consequence of the following facts:

(a) after the identification of $U_{i_j}$ with the closed unit polydisk $D$ and of $V_j$ with a compact subset $D_j \subset D$, the volume form $\omega^n$ restricted to each $U_{i_j}$ is equivalent to the Euclidean volume form $do := dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$;

(b) the Bergman inequality (see [GR, Chap. 6, Thm. 1.3])

$$\max_{D_j} |f| \le \frac{(\sqrt{n})^n}{(\sqrt{\pi}d)^n} \cdot \left(\int_D |f|^2\, do\right)^{1/2},$$

where $d$ is the Euclidean distance from $D_j$ to the boundary of $D$.

We leave the details to the reader.

Now recall that $\operatorname{dist}(\cdot,\cdot)$ is the distance on $Y_G$ in the metric induced by $\omega$ and that $\delta(x) := \operatorname{dist}(x, o)$. Since $\omega$ is invariant with respect to the action of $G$, we also have $\operatorname{dist}(g(x), g(y)) = \operatorname{dist}(x, y)$ for any $g \in G$. From inequalities (A) for $\tau$ and the triangle inequality for the distance, we obtain

$$\begin{aligned}\tau(g(x)) \ge c_1 \operatorname{dist}(g(x), o) &\ge c_1[\operatorname{dist}(g(x), g(o)) - \operatorname{dist}(g(o), o)]\\ &= c_1[\operatorname{dist}(x, o) - \operatorname{dist}(g(o), o)]\\ &\ge (c_1/c_2)\tau(x) - (c_1c_3/c_2) - c_1\delta(g(o)).\end{aligned} \tag{2.2}$$

Further, if $x \in K$ then

$$a_1 \le \tau(x) \le a_2 \quad \text{for some } a_1, a_2 > 0. \tag{2.3}$$

By $|\cdot|_{\infty,X_G}$ and $|\cdot|_{\infty,Y_G}$ we shall denote the corresponding $H^\infty$-norms. Let $f \in H^\infty(X_G)$. Then $f \in \mathcal{O}_{A\tau}(X_G) \cap \mathcal{O}_{c\tau}(X_G)$, and there exists an $a_3 > 0$ such that

$$\max\{|f|_{A\tau,X_G}, |f|_{c\tau,X_G}\} \le a_3 \sup_{X_G}|f| := a_3|f|_{\infty,X_G}.$$

(Note that any expression of the form $\max\{|f|_{A\tau,\cdot}, |f|_{c\tau,\cdot}\}$ is in fact equal to $|f|_{c\tau,\cdot}$.) By Theorem 2.1, there is a unique $\tilde{f} \in \mathcal{O}_{A\tau}(Y_G) \cap \mathcal{O}_{c\tau}(Y_G)$ such that $\tilde{f}|_{X_G} = f$ and

$$\max\{|\tilde{f}|_{A\tau,Y_G}, |\tilde{f}|_{c\tau,Y_G}\} \le a_4 \max\{|f|_{A\tau,X_G}, |f|_{c\tau,X_G}\} \quad \text{for some } a_4 > 0.$$

Combining these inequalities with (2.3) and (2.1) yields

$$\max_K |\tilde{f}| \le a_5 |f|_{\infty, X_G},$$

with some $a_5 > 0$ depending on $X_G$, $Y_G$ only. Now, for a fixed $g \in G$ consider $(g^*f)(z) := f(g(z))$. As before, there exists a unique function $\tilde{f}_g \in \mathcal{O}_{A\tau}(Y_G) \cap \mathcal{O}_{c\tau}(Y_G)$, $\tilde{f}_g|_{X_G} = g^*f$, such that

$$\max_K |\tilde{f}_g| \le a_5 |f|_{\infty, X_G}.$$

However, according to (2.2) and (1.1), the function $(g^*\tilde{f})(z) := \tilde{f}(g(z))$ belongs to $\mathcal{O}_{A\tau}(Y_G)$ and $(g^*\tilde{f} - \tilde{f}_g)|_{X_G} \equiv 0$. Thus by Theorem 2.1 we have $\tilde{f}_g = g^*\tilde{f}$. Since $K$ is the fundamental compact, the inequality just displayed implies for each $\tilde{f}_g$ that

$$|\tilde{f}|_{\infty, Y_G} \le a_5 |f|_{\infty, X_G}. \tag{2.4}$$

We will now prove that $a_5 = 1$, which gives the required statement. Indeed, the same arguments as before show that, for any integer $n \ge 1$, the function $(\tilde{f})^n$ is the unique extension of $f^n$ satisfying (2.4):

$$|(\tilde{f})^n|_{\infty, Y_G} \le a_5 |f^n|_{\infty, X_G}.$$

Thus

$$|\tilde{f}|_{\infty, Y_G} \le \lim_{n \to \infty} (a_5)^{1/n} |f|_{\infty, X_G} = |f|_{\infty, X_G}.$$

The proof of the theorem is complete.

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