Michigan Math. J. 51 (2003)

The Index of a Farey Sequence

R. R. Hall & P. Shiu

1. Introduction and Statement of Results

Let $\mathcal{F}_N = \{x_i, i = 1, 2, \dots, R\}$ denote the Farey sequence of order N; here $1/N = x_1 < x_2 < \dots < x_R = 1$ and

$$R = R_N = \sum_{1 \le a \le N} \phi(a) = \frac{3N^2}{\pi^2} + O(N \log N). \tag{1.1}$$

The sequence (x_i) may be extended onto $\mathbb{Z}$ by defining $x_{i+R} = x_i + 1$ for all i. We suppose that $x_i = b/s$ and that the adjacent fractions are

$$x_{i-1} = \frac{a}{r} \quad \text{and} \quad x_{i+1} = \frac{c}{t};$$

we write $r = r(x_i)$, $s = s(x_i)$, and $t = t(x_i)$.

Definition. We define the *index* of the fraction x_i as

$$\nu(x_i) := \frac{r+t}{s} = \frac{a+c}{b}. \tag{1.2}$$

Thus $\nu(x_i)$ is an integer because $br - as = cs - bt = 1$. In particular we have $\nu(x_1) = 1$ and $\nu(x_R) = 2N$. We are interested in some properties of the index, which is a periodic function on the extended Farey sequence $\{x_i : i \in \mathbb{Z}\}$.

There are two formulae for the index: expressing it as a function of N, s, and r, or of N, s, and b. For the first formula we recall from Hall and Tenenbaum [5] that

$$t = s\left[\frac{N+r}{s}\right] - r. \tag{1.3}$$

We remark that Boca, Cobeli, and Zaharescu [1] have made some very interesting applications of (1.3). It yields immediately

$$\nu(x_i) = \left[\frac{N+r}{s}\right] \tag{1.4}$$

and, since $r > N - s$, we see that

$$\left[\frac{2N+1}{s}\right] - 1 \le \nu(x_i) \le \left[\frac{2N}{s}\right]. \tag{1.5}$$

Received January 3, 2002. Revision received October 24, 2002.

It follows that if $s|2N+1$ then $\nu(x_i) = [2N/s]$; otherwise, the index may take the two values $[2N/s]$ and $[2N/s]-1$. We refer to these as the upper and lower values of the index. As an example, we give a table for the indices of $\mathcal{F}_9$; here $R = 28$ and the index is symmetric (i.e., $\nu(x_{R-i}) = \nu(x_i)$) so that we need only give the first 15 terms:

$$\begin{aligned} x_i &= \tfrac{1}{9}, \tfrac{1}{8}, \tfrac{1}{7}, \tfrac{1}{6}, \tfrac{1}{5}, \tfrac{2}{9}, \tfrac{1}{4}, \tfrac{2}{7}, \tfrac{1}{3}, \tfrac{3}{8}, \tfrac{2}{5}, \tfrac{3}{7}, \tfrac{4}{9}, \tfrac{1}{2}, \tfrac{5}{9}; \\ \nu(x_i) &= \underline{1}, 2, 2, \underline{2}, 3, \underline{1}, 4, \underline{1}, \underline{5}, \underline{1}, 3, 2, \underline{1}, 9, \underline{1}. \end{aligned} \tag{1.6}$$

The lower values have been underlined, and we remark that 1 is always a lower value (since $[2N/s] \geq 2$) and that, as in the case $s = 5$ here, the index may be single-valued when s does not divide $2N+1$. In fact, it is not difficult to show that $s \leq N$ is single-valued if and only if s satisfies one of the following conditions: (a) s is a divisor of N, $N+1$, or $2N+1$; (b) s is twice a divisor of N or $N+1$; (c) s is twice a divisor of $N+2$ or $N-1$ if these numbers are odd.

For the second formula for the index, we let $\bar{b}$ and n be such that $1 \leq \bar{b} < s$, $b\bar{b} \equiv 1 \pmod{s}$, and $0 \leq n < s$, $N \equiv n \pmod{s}$. Then $r \equiv \bar{b} \pmod{s}$, giving $r = ps + \bar{b}$ with

$$p = \left[\frac{N}{s}\right] + \left[\frac{n-\bar{b}}{s}\right].$$

Similarly, $t = qs - \bar{b}$ with

$$q = \left[\frac{N}{s}\right] + \left[\frac{n+\bar{b}}{s}\right],$$

so that

$$\nu(x_i) = p + q = 2\left[\frac{N}{s}\right] + \left[\frac{n-\bar{b}}{s}\right] + \left[\frac{n+\bar{b}}{s}\right]. \tag{1.7}$$

The second and third terms on the right of (1.7) can take the values -1, 0 and 0, 1, respectively. Their sum can take the values 0, ±1 but not both the values ±1.

Our investigation was initiated by one of us making a numerical observation while walking in a park. The observation led us to the following theorem.

THEOREM 1. *For all N, we have*

$$\sum_{i=1}^{R} \nu(x_i) = 3R - 1. \tag{1.8}$$

We need to consider the frequency of the upper and lower values of the index, and this leads us to another exact formula.

DEFINITION. The *deficiency* $\delta(s)$ is the number of fractions $x_i \in \mathcal{F}_N$ with denominator s such that $\nu(x_i)$ takes its lower value.

THEOREM 2. *For all N, we have*

$$\sum_{s=1}^{N} \delta(s) = N(2N+1) - R_{2N} - 2R + 1. \tag{1.9}$$

Thus, for $N = 9$, the right-hand side of (1.9) has the value $171 - 102 - 56 + 1 = 14$, which is in agreement with the table in (1.6). An immediate corollary of Theorem 2 is that the number of lower values is $\sim (2\pi^2/3 - 6)R$. The constant here is $0.57973\ldots$, so that the probability that $\nu(x_i)$ takes its lower value is rather more than $\frac{1}{2}$. Of course, there are quite a few indices taking the necessarily lower value 1.

A slightly more difficult result, which in the present treatment requires some analytic number theory, is as follows.

THEOREM 3. *For all N, we have*

$$Z(N) := \sum_{i=1}^{R} \nu(x_i)^2 = \frac{24}{\pi^2} N^2 \left(\log 2N - \frac{\zeta'(2)}{\zeta(2)} - \frac{17}{8} + 2\gamma \right) + O(N \log^2 N). \tag{1.10}$$

We may inquire about the frequency with which $\nu(x_i)$ takes the value k. We define

$$F(N, k) := \sum_{\substack{i \le R \\ \nu(x_i) = k}} 1 := L(N, k) + U(N, k), \tag{1.11}$$

where $L(N, k)$ and $U(N, k)$ count, respectively, the number of occurrences of k as a lower and upper value.

THEOREM 4. *For all N we have, uniformly for $k \in \mathbb{N}$, that*

$$\begin{aligned} L(N, k) &= \ell_k R + O\left(k + \frac{N}{k} \log N \right), \\ U(N, k) &= u_k R + O\left(k + \frac{N}{k} \log N \right), \end{aligned} \tag{1.12}$$

in which

$$\ell_k = 4\left(\frac{1}{(k+1)^2} - \frac{1}{k+1} + \frac{1}{k+2} \right), \quad k \ge 1, \tag{1.13}$$

$$u_1 = 0, \qquad u_k = 4\left(\frac{1}{k} - \frac{1}{k+1} - \frac{1}{(k+1)^2} \right), \quad k \ge 2. \tag{1.14}$$

It follows at once that

$$F(N, k) = f_k R + O\left(k + \frac{N}{k} \log N \right), \tag{1.15}$$

where

$$f_1 = \frac{1}{3}, \qquad f_k = 4\left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2} \right), \quad k \ge 2. \tag{1.16}$$

These results are useful only when $k^2 < N/\log N$, but we also have, in any case, that

$$\sum_{h \ge k} F(N, h) \le \frac{4}{k^2} \left(1 + O\left(\frac{\log N}{N} \right) \right) R \log R.$$

We next consider the partial sums of the index. One definition that seems appropriate is

$$D_j = D_j(N) := \sum_{i=0}^{j}{}^{\star}(\nu(x_i) - 3) + \frac{1}{2}, \tag{1.17}$$

where the star indicates that the end terms of the sum are each halved. For example, $D_1 = \frac{1}{2}(2N - 3) + \frac{1}{2}(1 - 3) + \frac{1}{2} = N - 2$. Notice that D_j is odd in the sense that $D_{R-j} = -D_j$. We were surprised to find in our numerical trials that $|D_j|$ seemed never to exceed $N - 2$ and apparently was much smaller than this on average. The explanation lies in the following remarkable theorem, which was communicated to us by Don Zagier.

THEOREM 5 (Zagier). *We have*

$$D_j = D(b, s) + \frac{t - r}{2s} + \frac{1}{2} - \frac{b}{s}, \tag{1.18}$$

where $x_j = b/s$ and $D(b, s)$ is 12 times Dedekind's sum; that is,

$$D(b, s) = 12 \sum_{\ell \,(\mathrm{mod}\, s)} \bar{B}_1\left(\frac{\ell}{s}\right) \bar{B}_1\left(\frac{b\ell}{s}\right), \tag{1.19}$$

with

$$\bar{B}_1(x) = \begin{cases} x - [x] - \frac{1}{2}, & x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0, & x \in \mathbb{Z}. \end{cases}$$

We give our proof, which is by induction on j; it is merely a verification of the formula (1.18) and thus does not explain how Zagier found the identity. The reader will find this secret, and much more information, in [8]. Our next set of results is concerned with the behavior of

$$\theta(x_i) = \left|\frac{r - t}{s}\right|, \quad r = r(x_i), \ s = s(x_i), \ t = t(x_i). \tag{1.20}$$

Notice that $0 \le \theta(x_i) < 1$ always because $N - s < r, t \le N$.

THEOREM 6. *For all N, we have*

$$\sum_{i=1}^{R} \theta(x_i)(1 - \theta(x_i)) = \frac{1}{6}R + O(N). \tag{1.21}$$

The same argument also shows that, for $k = 1, 2, \ldots$ and $N \to \infty$,

$$\sum_{i=1}^{R} \left(\theta(x_i) - \frac{1}{2}\right)^{2k} = \frac{R}{4^k(2k + 1)} + O_k(N).$$

This might at first suggest that θ is uniformly distributed in $(0, 1)$. However, this is not the case: it may be shown that, as $N \to \infty$,

$$\sum_{i=1}^{R} \theta(x_i) \sim CR, \qquad \sum_{i=1}^{R} \theta(x_i)^2 \sim \left(C - \frac{1}{6}\right)R, \tag{1.22}$$

where $C = 21/2 - 8\log 2 - 8\gamma = 0.33709\ldots$. The sums in (1.22) take the values $0.33523R$ and $0.17027R$ (respectively) when $N = 40$.

In an earlier version of our paper we had various conjectures that are now corollaries of Zagier's theorem.

THEOREM 7. *We have* $|D_j| \le N - 2$, *with equality if and only if* $j = 1$ *or* $j = R - 1$.

THEOREM 8. *We have*

$$\sum_{j=1}^{R} D_j^2 = \frac{5\zeta(4)}{3\zeta(3)^2} N^3 + O(N^{5/2}\log^2 N). \tag{1.23}$$

THEOREM 9. *We have*

$$\sum_{j=1}^{R} |D_j| \le 2R\log^2 N + O(R). \tag{1.24}$$

CONJECTURE 1. *There exists a distribution function*

$$F(\theta) := \lim_{N\to\infty} R^{-1}\,\mathrm{card}\{i : \theta(x_i) \le \theta\}$$

such that (*necessarily*)

$$\int_0^1 (2\theta - 1)^{2k-1} F(\theta)\, d\theta = \frac{1}{4k+2}, \quad k = 1, 2, \ldots. \tag{1.25}$$

In an earlier version of our paper, we also made the following conjecture.

CONJECTURE 2. *There exists a function* $A\colon \mathbb{N} \to \mathbb{R}^+$ *such that, for each fixed* $h \in \mathbb{N}$, *we have*

$$\sum_{i=1}^{R} \nu(x_i)\nu(x_{i+h}) \sim A(h)R, \quad N \to \infty.$$

In fact, a stronger form of Conjecture 2 has now been established in a forthcoming paper by Boca, Gologan, and Zaharescu [2].

We thank the referee for indicating to us the much simpler proof of Theorem 1 and also for suggesting that the quantity $(r - t)/s$ might have interesting properties, which led us to the results associated with $\theta(x_i)$ in (1.20). Various names for this function occurred to us, one of them suggested by the referee. After some reflection, we decided to follow the title of one of Wilkie Collins's novels: *No Name.*

2. Proofs of Theorems 1, 2, 3, and 4

Theorems 1, 2, and 4 are entirely elementary, although Theorem 1 was not proved in the park. Instead of the referee's proof given below, we had a more complicated argument that began with the formula

$$T(s) := \sum_{s(x_i)=s} \nu(x_i) = \frac{2}{s} \sum_{\substack{r=N-s+1 \\ (r,s)=1}}^{N} r = 2\sum_{d|s} \mu(d)\left[\frac{N}{d}\right] - \phi(s) + \varepsilon(s), \quad (2.1)$$

which we mention as we need it later. (In (2.1), $\varepsilon(1) = 1$ and $\varepsilon(s) = 0$ for $s > 1$.) Theorem 3 is elementary except for our estimate of the sum appearing in (2.16), for which we require contour integration and the functional equation for the Riemann zeta-function. We may have overlooked something here and we should be interested to discover an elementary treatment of this sum.

Proof of Theorem 1

We use induction on N, so that we have to establish that, in passing from $\mathcal{F}_{N-1}$ to $\mathcal{F}_N$, we add $3\phi(N)$ to the indices. Let I be the set of i such that

$$s_i + s_{i+1} = N, \quad (s_i, N) = 1, \quad (2.2)$$

so that $|I| = \phi(N)$. It suffices to show that, on inserting the new fractions with denominator N between b_i/s_i and b_{i+1}/s_{i+1}, we add exactly 3 units to the sum concerned. Indeed, the sum of the two existing relevant indices, namely

$$\frac{s_{i-1} + s_{i+1}}{s_i} + \frac{s_i + s_{i+2}}{s_{i+1}},$$

is being replaced by

$$\frac{s_{i-1} + N}{s_i} + \frac{s_i + s_{i-1}}{N} + \frac{N + s_{i+2}}{s_{i+1}},$$

so that the increase in value to the sum is simply

$$\frac{N - s_{i+1}}{s_i} + 1 + \frac{N - s_i}{s_{i+1}} = \frac{s_i}{s_i} + 1 + \frac{s_{i+1}}{s_{i+1}} = 3,$$

as required. □

Proof of Theorem 2

Let $s = s(x_i)$. Recall that $\nu(x_i)$ takes at most two values $[2N/s]$ and $[2N/s] - 1$, the latter $\delta(s)$ times. Hence the expression $T(s)$ in (2.1) is given by

$$T(s) = (\phi(s) - \delta(s))\left[\frac{2N}{s}\right] + \delta(s)\left(\left[\frac{2N}{s}\right] - 1\right) = \phi(s)\left[\frac{2N}{s}\right] - \delta(s). \quad (2.3)$$

Applying Theorem 1, we find that

$$\sum_{s \le N} \delta(s) = \sum_{s \le N} \phi(s)\left[\frac{2N}{s}\right] - 3R + 1 \quad (2.4)$$

and, since $[2N/s] = 1$ throughout the range $N < s \le 2N$, we may rewrite this as

$$\sum_{s \le N} \delta(s) = \sum_{s \le 2N} \phi(s) \left[\frac{2N}{s}\right] - R_{2N} - 2R + 1.$$

The sum on the right-hand side is

$$\sum_{s \le 2N} \phi(s) \sum_{\substack{n \le 2N \\ n \equiv 0 \ (\mathrm{mod}\, s)}} 1 = \sum_{n \le 2N} \sum_{s|n} \phi(s) = \sum_{n \le 2N} n = N(2N+1),$$

so that the required result (1.9) follows from (2.3). □

Proof of Theorem 3

It may be worth mentioning that the inductive argument used in the proof of Theorem 1 gives

$$Z(N) - Z(N-1) = 3\phi(N) + 2\sum_{i \in I} (\nu(x_i) + \nu(x_{i+1})),$$

but we are unable to evaluate the sum here to within $O(\log^2 N)$. Instead we put

$$V(s) := \sum_{s(x_i)=s} \nu(x_i)^2, \tag{2.5}$$

and we have

$$\begin{aligned} V(s) &= (\phi(s) - \delta(s)) \left[\frac{2N}{s}\right]^2 + \delta(s) \left(\left[\frac{2N}{s}\right] - 1 \right)^2 \\ &= \phi(s) \left[\frac{2N}{s}\right]^2 - \delta(s) \left(2\left[\frac{2N}{s}\right] - 1 \right). \end{aligned}$$

We write

$$X_N := \sum_{s \le N} \phi(s) \left[\frac{2N}{s}\right]^2, \tag{2.6}$$

$$Y_N := \sum_{s \le N} \delta(s) \left[\frac{2N}{s}\right], \tag{2.7}$$

so that, by Theorem 2,

$$\sum_{s \le N} V(s) = X_N - 2Y_N + N(2N+1) - R_{2N} - 2R + 1. \tag{2.8}$$

Extending the range from $1 \le s \le N$ to $1 \le s \le 2N$ as in the proof of Theorem 2, we find that

$$\begin{aligned} X_N &= \sum_{s \le 2N} \phi(s) \left[\frac{2N}{s}\right]^2 - R_{2N} + R \\ &= \sum_{s \le 2N} \phi(s) \left[\frac{2N}{s}\right] \left(\left[\frac{2N}{s}\right] + 1 \right) - N(2N+1) - R_{2N} + R. \end{aligned} \tag{2.9}$$

The sum in (2.9) is

$$2\sum_{s\le 2N}\frac{\phi(s)}{s}\sum_{\substack{n\le 2N\\ n\equiv 0\,(\mathrm{mod}\,s)}} n = 2\sum_{n\le 2N} nf(n), \tag{2.10}$$

where

$$f(n) := \sum_{s\mid n}\frac{\phi(s)}{s}. \tag{2.11}$$

Assembling (2.8), (2.9), and (2.10), the sum $Z(N)$ in the theorem becomes

$$Z(N) = 2\sum_{n\le 2N} nf(n) - 2Y_N - 2R_{2N} - R + 1. \tag{2.12}$$

We now turn our attention to the sum Y_N in (2.7), which we are unable to evaluate exactly. We recall from (2.3) and (2.1) that

$$\delta(s) = \phi(s)\left[\frac{2N}{s}\right] - T(s) = \phi(s)\left(\left[\frac{2N}{s}\right]+1\right) - 2\sum_{d\mid s}\mu(d)\left[\frac{N}{d}\right] - \varepsilon(s)$$

so that

$$\delta(s) = \phi(s)\left(\left[\frac{2N}{s}\right]+1\right) - \frac{2N\phi(s)}{s} + O(\tau(s)), \tag{2.13}$$

where τ is the divisor function. From (2.7) and (2.13), we now have

$$Y_N = \sum_{s\le N}\phi(s)\left[\frac{2N}{s}\right]\left(\left[\frac{2N}{s}\right]+1\right) - 2N\sum_{s\le N}\phi(s)\left[\frac{2N}{s}\right] + O(N\log^2 N), \tag{2.14}$$

in which our largest error term arises. Extending the range of the sums here, we find that

$$Y_N = 2\sum_{n\le 2N} nf(n) - 2N\sum_{n\le 2N} f(n) - \frac{6N^2}{\pi^2} + O(N\log^2 N). \tag{2.15}$$

Inserting this into (2.12) yields

$$Z(N) = 2\sum_{n\le 2N}(2N-n)f(n) - \frac{15N^2}{\pi^2} + O(N\log^2 N), \tag{2.16}$$

and it remains to consider the sum here.

In the following, it will be convenient to let the letters s, σ, t, and T be the usual symbols used in the theory of the Riemann zeta-function. The Dirichlet series for $f(n)$ in (2.11) is given by

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \frac{\zeta^2(s)}{\zeta(s+1)}. \tag{2.17}$$

Employing

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^{s+1}}{s(s+1)}\,ds = \max(x-1,0), \quad x>0, \tag{2.18}$$

we find that

$$\sum_{n\le 2N}(2N-n)f(n)=\sum_{n=1}^{\infty}\max\Big(\frac{2N}{n}-1,0\Big)nf(n)$$
$$=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{(2N)^{s+1}\zeta^2(s)}{s(s+1)\zeta(s+1)}\,ds. \tag{2.19}$$

The integrand has a removable singularity at $s=0$, and we move the line of integration to the contour C comprising the five line segments $s=2+it$ ($|t|\ge T$), $s=\sigma\pm iT$ ($0<\sigma\le 2$), $s=it$ ($-T\le t\le T$). The residue of the integrand at the pole $s=1$ is given by

$$\frac{12N^2}{\pi^2}\Big(\log 2N-\frac{\zeta'(2)}{\zeta(2)}-\frac{3}{2}+2\gamma\Big), \tag{2.20}$$

and we proceed to estimate the integral along our contour C. On the segments on which $\sigma=2$, the integrand is $\ll N^3/t^2$ and the integrals are

$$\ll\int_T^{\infty}N^3\,\frac{dt}{t^2}\ll T^{-1}N^3. \tag{2.21}$$

On the line segments on which $s=\sigma\pm iT$, we have $\zeta(s)\ll T^{1/2+\varepsilon}$ and $|\zeta(s+1)|\gg 1/\log T$, so that the integrals are

$$\ll N^3T^{-1+3\varepsilon}. \tag{2.22}$$

We set $T=N^3$, so that the contributions from these integrals are $O(N^{3\varepsilon})$. On the line $\sigma=0$ we employ the functional equation. We have

$$\Big|\Gamma(1-it)\sin\frac{it\pi}{2}\Big|=\Big(\frac{|t|\pi}{2}\tanh\frac{|t|\pi}{2}\Big)^{1/2}\ll\min\big(|t|,\sqrt{|t|}\big) \tag{2.23}$$

and $|\zeta(1-it)|=|\zeta(1+it)|$, so that the integrand is

$$\ll N\frac{\min(t^2,|t|)|\zeta(1+it)|}{|t|(|t|+1)} \tag{2.24}$$

and the integral is

$$\ll N\Big(1+\int_1^T|\zeta(1+it)|\,\frac{dt}{t}\Big). \tag{2.25}$$

We apply Cauchy's inequality and the formula

$$\int_1^X|\zeta(1+it)|^2\,dt\sim\zeta(2)X \tag{2.26}$$

[7, Thm. 7.2] to see that the integral in (2.25) is $\ll\log T\ll\log N$. Hence, by (2.19), (2.20), and the estimates (2.21) and (2.22),

$$\sum_{n\le 2N}(2N-n)f(n)=\frac{12N^2}{\pi^2}\Big(\log 2N-\frac{\zeta'(2)}{\zeta(2)}-\frac{3}{2}+2\gamma\Big)+O(N\log N), \tag{2.27}$$

and the required result follows by inserting this into (2.16). $\square$

Proof of Theorem 4

It will be sufficient to consider $L(N, k)$ as the other case is similar; we already saw that $U(N, 1) = 0$. Let $s(x_i) = s$ and let $\nu(x_i) = k$ take its lower value $[2N/s] - 1$. Thus $[2N/s] = k + 1$, so that

$$\frac{2N}{k+2} < s \le \frac{2N}{k+1}; \tag{2.28}$$

moreover, from (1.4) we require that

$$\left[\frac{N+r}{s}\right] = k, \tag{2.29}$$

that is,

$$\max(N - s + 1, sk - N) \le r \le \min(N, s(k+1) - N). \tag{2.30}$$

From (2.28) this reduces to

$$N - s + 1 \le r \le s(k+1) - N \tag{2.31}$$

and, since r is prime to s, the number of choices for r in (2.31) is

$$(s(k+2) - 2N)\frac{\phi(s)}{s} + O(\tau(s)), \tag{2.32}$$

and we need to sum over the range in (2.28). We proceed by partial summation, writing

$$\Phi(s) := \sum_{m \le s} \frac{\phi(m)}{m} = \frac{6s}{\pi^2} + O(\log s) \tag{2.33}$$

and

$$y = \left[\frac{2N}{k+2}\right], \qquad z = \left[\frac{2N}{k+1}\right]. \tag{2.34}$$

Assuming that $y < z$ to begin with, we find that

$$\begin{aligned} &\sum_{y<s\le z} (s(k+2) - 2N)\frac{\phi(s)}{s} \\ &\quad = -((y+1)(k+2) - 2N)\Phi(y) \\ &\qquad - (k+2)\sum_{y<s<z} \Phi(s) + (z(k+2) - 2N)\Phi(z). \end{aligned} \tag{2.35}$$

Notice that $0 < (y+1)(k+2) - 2N \le z(k+2) - 2N \le 2N/(k+1)$, so that the end terms contribute $\ll k^{-1}N \log N$ to the error. The middle terms contribute

$$\ll (z - y - 1)(k+2)\log N \ll k^{-1}N \log N \tag{2.36}$$

to the error, since

$$z - y - 1 \le \frac{2N}{(k+1)(k+2)}. \tag{2.37}$$

The main term in (2.35) is

$$\frac{6}{\pi^2}\sum_{y<s\le z}(s(k+2)-2N)=\frac{3(z-y)((z+y+1)(k+2)-4N)}{\pi^2},\qquad(2.38)$$

and we remark that the last factor in the numerator does not exceed $2N/(k+1)$, so that the error involved in (2.38) if we remove the square brackets in (2.34) is $\ll k+N/k$. Hence the sum in (2.35) equals

$$\frac{12N^2}{\pi^2(k+1)^2(k+2)}+O\left(k+\frac{N\log N}{k}\right).\qquad(2.39)$$

It is easy to see that the error term arising from the divisor function in (2.32) is absorbed here; for example, Dirichlet's theorem gives

$$\sum_{y<s\le z}\tau(s)\ll(z-y)\log z+\sqrt{z}\ll\left(\frac{N}{k^2}+1\right)N+\sqrt{\frac{N}{k}}.\qquad(2.40)$$

Therefore, (2.39) provides a formula for $L(N,k)$ in the case $y<z$, and if $y=z$ then $L(N,k)=0$ because the range for s in (2.28) is empty and the formula remains valid. Finally we may replace $3N^2/\pi^2$ by R in (2.39) without affecting the error term, and this gives the first asymptotic formula in (1.12) together with the formula for ℓ_k. The remaining formulae can be established similarly, and (1.15) follows at once. □

3. Proofs of Theorems 5, 6, 7, 8, and 9

Proof of Theorem 5

We take as our induction hypothesis that (1.18) holds at j, and we begin by checking it when $j=1$. We already saw that $D_1=N-2$, and we have $s=N$, $b=1$, $t=N-1$, $r=1$, and $D(1,N)=N-3+2/N$, so that (1.18) is correct.

Suppose now that (1.18) is true. We have

$$D_{j+1}=D_j+\frac{1}{2}(\nu(x_j)-3)+\frac{1}{2}(\nu(x_{j+1})-3)=D_j+\frac{r+t}{2s}+\frac{s+u}{2t}-3,$$

where u is the denominator of the fraction following c/t in $\mathcal{F}_N$. From (1.18),

$$D_{j+1}=D(b,s)+\frac{t}{s}+\frac{s+u}{2t}-\frac{5}{2}-\frac{b}{s};\qquad(3.1)$$

we now apply Lemma 2 of [4], which tells us that

$$D(b,s)=D(c,t)-\frac{s^2+t^2+1}{st}+3,\qquad(3.2)$$

so that from (3.1) and (3.2) we have

$$D_{j+1}=D(c,t)+\frac{u-s}{2t}+\frac{1}{2}-\frac{1}{st}-\frac{b}{s}=D(c,t)+\frac{u-s}{2t}+\frac{1}{2}-\frac{c}{t},$$

as required. This completes the induction and the proof. □

Proof of Theorem 6

We use induction on N as we did in the proof of Theorem 1. For $i \in I$, where I is defined in (2.2), we have

$$\theta(x_i) = \left|\frac{s_{i-1} - s_{i+1}}{s_i}\right| = \frac{s_{i-1} + s_i - N}{s_i},$$

$$\theta(x_{i+1}) = \left|\frac{s_i - s_{i+2}}{s_{i+1}}\right| = \frac{s_{i+1} + s_{i+2} - N}{s_{i+1}}.$$

These two values for θ are replaced by the following three new ones:

$$\alpha = \frac{N - s_{i-1}}{s_i}, \quad \beta = \frac{|s_i - s_{i+1}|}{N}, \quad \gamma = \frac{N - s_{i+2}}{s_{i+1}}.$$

We then have

$$\alpha + \beta + \gamma - \theta(x_i) - \theta(x_{i+1}) = 2\frac{N - s_{i-1}}{s_i} + 2\frac{N - s_{i+2}}{s_{i+1}} + \frac{|s_i - s_{i+1}|}{N} - 2,$$

$$\alpha^2 + \beta^2 + \gamma^2 - \theta(x_i)^2 - \theta(x_{i+1})^2 = 2\frac{N - s_{i-1}}{s_i} + 2\frac{N - s_{i+2}}{s_{i+1}} + \frac{|s_i - s_{i+1}|^2}{N^2} - 2,$$

so that the sum concerned is increased by

$$\Delta_N = \sum_{i \in I}\left(\frac{|s_i - s_{i+1}|}{N} - \frac{|s_i - s_{i+1}|^2}{N^2}\right). \tag{3.3}$$

We observe here that $\theta(x_i) - \frac{1}{2} = \frac{1}{2} - \alpha$ and $\theta(x_{i+1}) - \frac{1}{2} = \frac{1}{2} - \gamma$; furthermore, $|s_i - s_{i+1}| = |2s_i - N|$. Starting from the not quite obvious formula

$$\sum_{q=1}^{M} |2q - M|^h = \frac{1}{h+1} M^{h+1} + O(M^{h-1}), \quad M, h \in \mathbb{N},$$

we derive

$$\sum_{\substack{s=1 \\ (s,N)=1}}^{N} |2s - N|^h = \frac{1}{h+1} N^h \phi(N) + O(N^{h-1}\sigma(N)),$$

where $\sigma(N)$ is the sum of the divisors of N. This then yields $\Delta_N = \phi(N)/6 + O(\sigma(N)/N)$ in (3.3) and, moreover, copes with all the even moments of $\theta - \frac{1}{2}$. The odd moments, none of which may be expected to be zero, are more mysterious. $\square$

Proof of Theorem 7

We first prove the following lemma.

LEMMA 1. *We have*

$$|t - r| \le s - 2 + \operatorname{pip}(s),$$

where $\operatorname{pip}(s) = 1$ *if* $s | 2N + 1$ *and* $= 0$ *otherwise.*

Proof. Since $N - s + 1 \le r$ and $s \le N$, it is evident that $|t - r| \le s - 1$ with equality if and only if $\max(r, t) = N$ and $\min(r, t) = N - s + 1$, which implies $r + t = 2N + 1 - s$. Since $s | r + t$ this gives the result stated.

It will be sufficient to show that $D_j \le N - 2$ with equality if and only if $j = 1$. We have

$$D_j \le D(b, s) + \frac{s - 2 + \mathrm{pip}(s)}{2s} + \frac{1}{2} - \frac{1}{s} \tag{3.4}$$

by Theorem 5 and the lemma; there is equality in (3.3) if and only if $b = 1$. We have

$$D(b, s) \le D(1, s) = s - 3 + \frac{2}{s},$$

whence, from (3.4),

$$D_j \le s - 2 + \frac{\mathrm{pip}(s)}{2s} \le N - 2$$

with equality if and only if $s = N$. This is all we need. □

Proof of Theorem 8

By Theorem 5 and Lemma 1, we have $D_j = D(b, s) + O(1)$ and therefore

$$D_j^2 = D(b, s)^2 + O(|D_j|) + O(1).$$

Hence the sum in (1.23) is

$$\sum_{s=1}^{N} \sum_{(b,s)=1} D(b, s)^2 + O\left(\sum_{j=1}^{R} |D_j|\right) + O(R).$$

We employ a theorem of Jia [6] to evaluate the inner sum, which is

$$f_1(s)s^2 + O(s^{3/2} \log^2 s),$$

where $f_1(s)$ is defined as the coefficient in a Dirichlet series—namely,

$$\sum_{n=1}^{\infty} \frac{f_1(n)}{n^z} = 5\frac{\zeta(z + 3)}{\zeta(z + 2)^2}\zeta(z).$$

From this it follows that our sum is

$$\frac{5\zeta(4)}{3\zeta(3)^2} N^3 + O(N^{5/2} \log^2 N) + O\left(\sum_{j=1}^{R} |D_j|\right). \tag{3.5}$$

As Theorem 9 shows, the second error term in (3.5) is of a smaller order than the first; in any case, for our purpose here, Cauchy's inequality yields

$$\sum_{j=1}^{R} |D_j| \ll N^{5/2},$$

so the theorem is proved. □

Proof of Theorem 9

An alternative representation of the Dedekind sum, due to Eisenstein, is

$$D(b,s) = \frac{3}{s}\sum_{\ell=1}^{s-1}\cot\left(\frac{\pi\ell}{s}\right)\cot\left(\frac{\pi b\ell}{s}\right), \tag{3.6}$$

and it is a straightforward matter to deduce from (3.6) that

$$\sum_{(b,s)=1}|D(b,s)| < \frac{12}{\pi^2}s\log^2 s. \tag{3.7}$$

Hence

$$\begin{aligned}\sum_{j=1}^{R}|D_j| &\le \frac{12}{\pi^2}\sum_{s=1}^{N}s\log^2 s + O(R)\\ &\le \frac{6}{\pi^2}N^2\log^2 N + O(R)\\ &\le 2R\log^2 N + O(R),\end{aligned}$$

as required. □

We end with a table of values for R, $\sum_{j\le R}|D_j|$, and $\sum_{j\le R}D_j^2$. We remark that $\frac{5\zeta(4)}{3\zeta(3)^2}\approx 1.24841$, and that it appears from the table that (3.7) has a constant which is perhaps too large by a factor of about 4.

N	R	$\sum\|D_j\|$	$\sum D_j^2$	$\frac{\sum\|D_j\|}{R\log^2 N}$	$\frac{\sum D_j^2}{N^3}$
10	32	80.5	384.25	0.47447	0.38425
50	774	5672.5	104831	0.47888	0.83865
100	3044	31093.5	971927	0.48165	0.97192
500	76116	1.41210×10^6	1.44082×10^8	0.48035	1.15266
1000	304192	6.97989×10^6	1.18975×10^9	0.48086	1.18975
5000	7600458	2.66173×10^8	1.53870×10^{11}	0.48276	1.23096
10000	30397486	1.24780×10^9	1.23822×10^{12}	0.48390	1.23822

References

[1] F. P. Boca, C. Cobeli, and A. Zaharescu, *A conjecture of R. R. Hall on Farey arcs,* J. Reine Angew. Math. 535 (2001), 207–236.

[2] F. P. Boca, R. N. Gologan, and A. Zaharescu, *On the index of Farey sequences,* preprint, 2001.

[3] R. R. Hall, *On consecutive Farey arcs II,* Acta Arith. 66 (1994), 1–9.

[4] R. R. Hall and M. N. Huxley, *Dedekind sums and continued fractions,* Acta Arith. 63 (1993), 79–90.

[5] R. R. Hall and G. Tenenbaum, *On consecutive Farey arcs,* Acta Arith. 44 (1984), 397–405.

[6] C. Jia, *On the mean value of Dedekind sums,* J. Number Theory 87 (2001), 173–188.

[7] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed. (D. R. Heath-Brown, ed.), Oxford University Press, New York, 1986.
[8] D. Zagier, *Nombres de classes et fractions continues*, Journées Arithmétiques de Bordeaux (Conference, Univ. Bordeaux, 1974), Astérisque 24–25 (1975), 81–97.

R. R. Hall
Department of Mathematics
York University
Heslington
York YO10 5DD
United Kingdom

rrh1@york.ac.uk

P. Shiu
Department of Mathematical Sciences
Loughborough University
Loughborough
Leicestershire LE11 3TU
United Kingdom

P.Shiu@lboro.ac.uk