Michigan Math. J. 47 (2000)

Local Hulls of Unions of Totally Real Graphs Lying in Real Analytic Hypersurfaces

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I. Introduction

Let Y be a compact subset of $\mathbf{C}^n$ and let $\hat{Y}$ denote the polynomial convex hull of Y, that is,

$$\hat{Y} = \Bigl\{ z \in \mathbf{C}^n : |Q(z)| \leq \max_Y |Q| \text{ for every polynomial } Q \text{ on } \mathbf{C}^n \Bigr\}.$$

We say that Y is polynomially convex if $\hat{Y} = Y$. A closed subset F of $\mathbf{C}^2$ is called *locally polynomially convex* (LPC for short) at $a \in F$ if there exists $r > 0$ such that the intersection $\bar{\mathbf{B}}(a, r) \cap F$ is polynomially convex. Since the intersection of two polynomially convex sets is again polynomially convex, a polynomially convex set is LPC everywhere, but the converse is known to be false. Indeed, by classic work of Wermer, if X is a $\mathcal{C}^1$ totally real submanifold of $\mathbf{C}^n$ (i.e., if the real tangent space $T_a X$ contains no complex line for every $a \in X$) then X is LPC everywhere. But Wermer constructed a totally real embedded disk that bounds an analytic disk and hence is not (globally) polynomially convex (see e.g. [FS]).

To begin to consider nonsmooth varieties, it seems natural to study the local polynomial convexity of the union of two totally real submanifolds in $\mathbf{C}^2$. In the case where the manifolds are totally real planes in $\mathbf{C}^n$, a complete picture was obtained by Weinstock (see [W]).

Here we examine the case where the manifolds in question are graphs whose tangent planes at the origin meet only at one point. More precisely, let M_1 and M_2 be two totally real graphs in $\mathbf{C}^2$ such that $T_0 M_1 \cap T_0 M_2 = \{0\}$. We ask under what conditions the union is LPC at 0, and if it is not LPC then we will try to describe the hull near the origin. It should be observed that if $T_0 M_1 \cup T_0 M_2$ is LPC at 0 then, by an implicit result of Forstneric and Stout [FS], the union $M_1 \cup M_2$ is LPC at 0. After a linear change of coordinates, the case where $T_0 M_1 \cup T_0 M_2$ is *not* LPC at 0 reduces to

$$M_1 = \{(z, \bar{z} + \varphi_1(z))\} \quad \text{and} \quad M_2 = \{(z, \lambda \bar{z} + \varphi_2(z))\}, \qquad (*)$$

where $\lambda > 0$, $\lambda \neq 1$, and φ_i $(i = 1, 2)$ are functions of class $\mathcal{C}^1$ in a neighborhood of 0 that satisfy $\varphi_i(0) = \partial \varphi_i / \partial z(0) = \partial \varphi_i / \partial \bar{z}(0) = 0$.

Received October 6, 1999. Revision received April 12, 2000.
This work was done under the French–Vietnamese PICS program coordinated by Professors Nguyen Thanh Van and Frederic Pham.

For every $r > 0$, the additional polynomial convex hull of $(T_0 M_1 \cup T_0 M_2) \cap \bar{\mathbf{B}}(0, r)$ is a union of analytic annuli with boundary in $T_0 M_1 \cup T_0 M_2$. By an *analytic annulus* we mean the image of a continuous mapping f from an annulus $\{z : 1 \leq |z| \leq r\}$ into $\mathbf{C}^2$ such that f is holomorphic inside this annulus. Moreover, these annuli are contained in the set $\{(z, w) : \operatorname{Im}(zw) = 0\}$, which away from 0 is a smooth Levi-flat real analytic hypersurface. This phenomenon motivates the current paper. Notice that if there existed a *smooth* hypersurface S containing M_1 and M_2 then the tangent hyperplane to S at 0 would contain the planes $w = \bar{z}$ and $w = \lambda \bar{z}$, but this is impossible. Hence, in order to tackle the problem, our setup must consider hypersurfaces with singularities.

The outline of this paper is as follows. In Section II we are interested in finding necessary conditions for a real analytic hypersurface (possibly with singularities) to contain a set and its local nontrivial hull. The main result of this section is Proposition 2.3, which states roughly that if the polynomial hull of a compact set is contained in a real analytic hypersurface then either (a) its additional hull is small or (b) the hypersurface is Levi-flat away from its singular locus. In Section III an analogous (and stronger) result is obtained in the special case where our compact set is the union of two graphs of the form $(*)$ (Proposition 3.5). We also prove in Theorem 3.2 that, if $M_1 \cup M_2$ is contained in a real analytic hypersurface defined by the vanishing of the imaginary part of a holomorphic function, then $M_1 \cup M_2$ is not LPC at 0 and its local hull is a union of analytic annuli with boundary in the two manifolds. Proposition 3.6 gives an example of the non–polynomially convex situation, where (unlike in the linear case) this hull *cannot* be contained in any real analytic hypersurface defined by the vanishing of the imaginary part of a holomorphic function. Another main result of this section is Proposition 3.4, which generalizes an example of Weinstock [W, p. 137] about the union of two totally real disks with trivial (local) hull while the union of their tangent spaces has nontrivial (local) hull. Finally, in the case where S is “nondegenerate”, Proposition 4.2 gives a partial converse to Theorem 3.2 in Section III; namely, we prove that if a union of two graphs of the form $(*)$ has a nontrivial hull contained in a irreducible real analytic hypersurface, then this hypersurface must be Levi-flat away from its singular locus. The paper ends with the proofs of some lemmas.

II. Singular Real Analytic Hypersurfaces

We recall from [BG] that a subset S in $\mathbf{C}^2$ is called a *real analytic hypersurface* (possibly with singularities) if there exist a neighborhood U of S and a real analytic real-valued function f on U such that S coincides with the zero set of f, which is a real analytic set of codimension 1. Furthermore, S can be decomposed into the regular locus $\operatorname{Reg}(S)$, which is a smooth real analytic hypersurface in $\mathbf{C}^2$, and the (possibly empty) singular locus $\operatorname{Sing}(S)$, which is contained in a real analytic variety of dimension not exceeding 2. By the Lojaciewicz structure theorem (see [KP, Chap. 5]), if we consider S as a real analytic hypersurface in $\mathbf{R}^4$ then $\operatorname{Sing}(S)$ can be stratified into a union of (possibly empty) real analytic submanifolds of dimension not exceeding 2.

DEFINITION. Let S be a real analytic hypersurface (possibly with singularities) in $\mathbf{C}^2$ defined by $f = 0$, where f is real analytic real-valued in a neighborhood U of S. We call S *weakly Levi-flat* if $\mathrm{Reg}(S)$ is Levi-flat and *strongly Levi-flat* if the function f can be chosen to be the imaginary part of a holomorphic function in U. In the case $0 \in S$, we also call S weakly (resp. strongly) Levi-flat near 0 if the intersection of S and a small ball about 0 is weakly (resp. strongly) Levi-flat.

NOTATION. For a compact set Y in $\mathbf{C}^2$, we let $\mathrm{ad}(Y)$ be its additional hull; that is, $\mathrm{ad}(Y) = \hat{Y} \backslash Y$. Coordinates in $\mathbf{C}^2$ and $\mathbf{C}^4$ will be denoted by (z, w) and (z, w, u, v), respectively. For any set X in $\mathbf{C}^2$ that includes 0, we denote by X^r the intersection of X and the closed ball $\bar{\mathbf{B}}(0, r)$. We write π for the projection onto the first coordinate in $\mathbf{C}^2$. We denote by $\mathcal{C}^1_*(\{0\})$ the set of $\mathcal{C}^1$ functions in a neighborhood of $0 \in \mathbf{C}$ satisfying $\varphi(0) = \partial\varphi/\partial z(0) = \partial\varphi/\partial\bar{z}(0) = 0$. By an abuse of notation, for a given real analytic hypersurface S in $\mathbf{C}^2$, $0 \in S$, we will denote the intersection of S and the *open* ball $\mathbf{B}(0, r)$ by S^r.

LEMMA 2.0. *Let S be an irreducible real analytic hypersurface in $\mathbf{C}^2$, $0 \in \mathrm{Sing}(S)$, defined by $f = 0$. Assume that f is irreducible at* 0. *Then, for $r > 0$ small enough*:

(a) $\{p \in S^r : df(p) = 0\} = \mathrm{Sing}(S^r)$;

(b) *S^r is weakly Levi-flat if and only if*

$$\mathcal{L} := \frac{\partial^2 f}{\partial z \partial \bar{z}} \left| \frac{\partial f}{\partial w} \right|^2 - 2\,\mathrm{Re}\left(\frac{\partial^2 f}{\partial z \partial \bar{w}} \frac{\partial f}{\partial w} \frac{\partial f}{\partial \bar{z}} \right) + \frac{\partial^2 f}{\partial w \partial \bar{w}} \left| \frac{\partial f}{\partial z} \right|^2 \equiv 0 \ \text{on } \mathrm{Reg}(S^r).$$

By an abuse of language, we call $\mathcal{L}$ the Levi form of $\mathcal{S}$.

Proof. It is clear that (b) follows from (a). To prove (a), we let $\tilde{f}$ be the complexified function of f and set $\tilde{S} = \{\tilde{f} = 0\}$. By Lemma 2.1 in [BG], the complexified function $\tilde{f}$ of f is irreducible near $0 \in \mathbf{C}^4$. Next, we choose a small neighborhood U about $0 \in \mathbf{C}^4$ such that the smooth locus V of $U \cap \tilde{S}$ is connected and such that any holomorphic function that vanishes on V is divisible by $\tilde{f}$ on U. We claim that $d\tilde{f}$ cannot vanish on an open set of V. Otherwise, $\tilde{f}$ would divide all its first derivatives with respect to z, w, u, v. It is then easy to prove by induction that, near $0 \in \mathbf{C}^4$, $\tilde{f}$ divides all its partial derivatives with order larger than 1. This implies that all these derivatives vanish near 0. Thus $f \equiv 0$, a contradiction. Hence the set $\{p \in V : d\tilde{f}(p) = 0\}$ is nowhere dense in V. This implies that $\tilde{f}$ is a *minimal* defining function for $\tilde{S}$ near $0 \in \mathbf{C}^4$ (see [Ch, Chap. 1]) and, for $r > 0$ sufficiently small, $\mathrm{Sing}(\tilde{S}^r) = \{p \in \tilde{S}^r : d\tilde{f}(p) = 0\}$. Therefore, $\{p \in S^r : df(p) = 0\} = \mathrm{Sing}(S^r)$. □

We need also the following lemma.

LEMMA 2.1. *Let S be a real analytic hypersurface defined by $f = 0$. Assume that f is irreducible at* 0. *Let K be a compact subset of S satisfying $\hat{K} \subset \{f \le 0\}$. Then $\mathcal{L}(q) \le 0$ for every $q \in \mathrm{ad}(K) \cap S$.*

Proof. Assume that there exists $q \in \operatorname{ad}(K) \cap S$ such that $\mathcal{L}(q) > 0$. Then S is a strictly pseudoconvex hypersurface near q. It is then possible to find a peaking polynomial at q in a small neighborhood of q in $\{f \le 0\}$. Since $\hat{K} \subset \{f \le 0\}$, we obtain a contradiction to the local maximum modulus principle [AW; St]. □

The following fact gives us relations between strongly and weakly Levi-flat real analytic hypersurfaces.

PROPOSITION 2.2. *Let S be a real analytic hypersurface in $\mathbf{C}^2$ with $0 \in S$. Consider the following assertions*:

(i) *S is strongly Levi-flat near* 0;
(ii) *S is LPC at* 0;
(iii) *S is weakly Levi-flat near* 0.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and these implications are strict.*

Proof. First, by [Hö, Thm. 4.3.4], we have (i) $\Rightarrow$ (ii). Next, (ii) $\Rightarrow$ (iii); otherwise there would exist a point $q \in S$ with $\mathcal{L}(q) \neq 0$. Then, near q, there exists a sequence of analytic disks not contained in S whose boundaries are contained in S. By the maximum modulus principle, we get a contradiction with (ii). To show that the implication (i) $\Rightarrow$ (ii) is strict, we take $S = \{(z,w) : \operatorname{Im}^2 z + \operatorname{Re}^3 z = 0\}$; then it is clear that S is LPC at 0, although S is not strongly Levi-flat as was shown in [BG, Prop. 5.4]. For the other, it suffices to let S be the complex cone $|w| = |z|$. □

PROPOSITION 2.3. *Let S be a real analytic hypersurface in $\mathbf{C}^2$, $0 \in S$, defined by $f = 0$. Assume that f is irreducible near 0. Let K be a compact subset of S with $0 \in K$. Suppose that $\hat{K}$ is contained in S and*

$$\overline{\lim_{r \to 0}} \frac{\mathcal{H}_2(\operatorname{ad}(K^r))}{r^{\varepsilon}} = \infty \quad \forall \varepsilon > 0, \tag{1}$$

where $\mathcal{H}_2$ denotes the 2-dimensional Hausdorff measure. Then S is weakly Levi-flat near 0.

Proof. By Lemma 2.1, we have $\mathcal{L} \equiv 0$ in $\operatorname{ad}(K^r) \cap S$ for $r > 0$ small enough. Consider the germ $V = f^{-1}(0) \cap \mathcal{L}^{-1}(0)$. It is clear that V is a real analytic germ with dimension ≥ 2 and, moreover, that $\operatorname{ad}(K^r) \subset V$. It follows from [BDM, Lemma 5.5] and (1) that $\dim V = 3$. In other words, $\mathcal{L} \equiv 0$ in S^r for $r > 0$ small enough. We conclude by Lemma 2.0 that S is weakly Levi-flat. □

III. Local Hulls of Graphs in Hypersurfaces

For $0 < a < b$ we define by $\mathbf{D}(a,b)$ the annulus $\{z : a < |z| < b\}$. We call a bounded function φ (defined on a deleted neighborhood of $0 \in \mathbf{C}$) *of order* k if k is the largest integer satisfying $\varphi(z) = O(|z|^k)$. If such k does not exist, then φ is called *of infinite order*. Unless otherwise stated, in this section we always take graphs M_1 and M_2 of the form $(*)$. Recall also that $M_j^r := M_j \cap \bar{\mathbf{B}}(0,r)$, $j = 1, 2$.

LEMMA 3.1. *Let $a > 0$ be sufficiently large. Then, for every $r > 0$ small enough, there exists a domain D_a in $\mathbf{C}^2$ satisfying the following properties.*

(i) *D_a is a non–smoothly bounded strictly pseudoconvex domain whose closure is a polynomially convex set; in particular, D_a is Runge.*
(ii) *$0 \in \partial D_a$ and $(M_1^r \cup M_2^r)\hat{} \subset \overline{D_a}$.*
(iii) *For any $a' > a > 0$, there exists $r > 0$ such that $\overline{D_a^r} \setminus \{0\} \subset D_{a'}$.*
(iv) *For any $(z, w) \in (M_1^r \cup M_2^r)\hat{} \setminus \{(0, 0)\}$ we have $\alpha(a)|z| < |w| < \beta(a)|z|$, where $\alpha(a)$ and $\beta(a)$ are two roots of the equation $t^2 - at + 1 = 0$.*

Proof. For each $a > 0$ we define the domains

$$D_a = \{(z, w) : \max(|z|, |w|) < 1/a,\\ f_a(z, w) := |w|^2 + |z|^2 - a\,\mathrm{Re}(zw)\\ + \log^2(1 - a^2|z|^2) + \log^2(1 - a^2|w|^2) < 0\}.$$

For $a > \max(4, 1 + \lambda^2)$, it is elementary to show that, for $r > 0$ small enough, the function f_a is (a) strictly plurisubharmonic in the bi-disk $\{(z, w) : \max(|z|, |w|) < 1/a\}$ and (b) nonpositive on $M_1^r \cup M_2^r$. This implies (i) and (ii). Finally, (iii) and (iv) are trivial. □

REMARK. As a direct consequence of the lemma we have that, for every $r > 0$ small enough, the hull $(M_1^r \cup M_2^r)\hat{}$ cannot contain a neighborhood of the origin in $\mathbf{C}^2$.

The main result of this section is the following theorem, which gives a complete description of the hull $(M_1^r \cup M_2^r)\hat{}$ for the pair M_1, M_2 contained in a strongly Levi-flat hypersurface.

THEOREM 3.2. *Let S be a strongly Levi-flat hypersurface containing a pair of totally real manifolds M_1, M_2 of the form $(*)$. Then the union $M_1 \cup M_2$ is not LPC at 0. More precisely, for $r > 0$ small enough, $\mathrm{ad}(M_1^r \cup M_2^r)$ is a foliation of disjoint analytic annuli with boundary in $M_1^r \cup M_2^r$.*

We need the following lemma about the general form of strongly Levi-flat hypersurfaces that contain $M_1 \cup M_2$ locally at 0. This lemma might be of independent interest.

LEMMA 3.3. *Let S be a strongly Levi-flat hypersurface defined in a neighborhood of $0 \in \mathbf{C}^2$ containing $M_1^r \cup M_2^r$ for some $r > 0$. Then there exists a defining function f for S of the form $f = \mathrm{Im}\, g$, where g is holomorphic in a neighborhood of $0 \in \mathbf{C}^2$ with the representation*

$$g(z, w) = (zw)^m + \sum_{k \geq 2m+1} P_k(z, w); \tag{2}$$

here the P_k are homogeneous polynomials of degree k.

Conversely, for any strongly Levi-flat real analytic hypersurface in $\mathbf{C}^2$ *defined by* $\operatorname{Im} g = 0$, *where* g *is a holomorphic function in a neighborhood of* $0 \in \mathbf{C}^2$ *with the expansion* (2), *one can find* M_1, M_2 *such that* $M_1^r \cup M_2^r \subset S$ *for some* $r > 0$.

Proof. Let f be a defining function of S. Then near the origin we may write

$$f(z, w) = \operatorname{Im}\biggl(\sum_{k \geq n} P_k(z, w)\biggr),$$

where P_k are homogeneous polynomials of degree k and $P_n \not\equiv 0$ for $n \geq 1$. Since $M_2 \subset S$, we obtain $\operatorname{Im} f(z, \lambda\bar{z} + \varphi_2(z)) \equiv 0$ z small enough. As $\varphi_2 \in \mathcal{C}^1_*(\{0\})$, we deduce that $\operatorname{Im} P_n(z, \lambda\bar{z}) \equiv 0$ and similarly that $\operatorname{Im} P_n(z, \bar{z}) \equiv 0$. By expanding P_n in the form $P_n(z, w) = \sum_{j=0}^n a_j z^j w^{n-j}$, we find

$$\sum_{j=0}^{n} \operatorname{Im}(a_j \lambda^{n-j} z^j \bar{z}^{n-j}) = 0.$$

It follows that $\sum_{j=0}^n (a_{n-j}\lambda^j - \bar{a}_j \lambda^{n-j}) \bar{z}^j z^{n-j} \equiv 0$. Hence $a_{n-j}\lambda^j = \bar{a}_j \lambda^{n-j}$. Since $\operatorname{Im} P_n(z, \bar{z}) \equiv 0$, by reasoning as before we have $a_{n-j} = \bar{a}_j$. Therefore, n is even (i.e., $n = 2m$) and $P_n(z, w) \equiv a_m(zw)^m$. Clearly we may put f in the desired form.

For the converse, suppose S is a strongly Levi-flat real analytic hypersurface defined by $f = \operatorname{Im} g = 0$, where g is a holomorphic function in a neighborhood of $0 \in \mathbf{C}^2$ having the expansion (2). It suffices to find $\varphi_2 \in \mathcal{C}^1_*(\{0\})$ such that M_2^r is contained in S for some $r > 0$ (i.e., $\operatorname{Im} g(z, \lambda\bar{z} + \varphi_2(z)) \equiv 0$ for z small enough); finding φ_1 is achieved by a similar method.

In the case $g(z, w) \equiv (zw)^m$, one may take simply $\varphi_2 \equiv 0$. Otherwise, we denote by s the smallest number greater than $2m$ such that $P_s \not\equiv 0$; we also let $h(z)$ be an arbitrary real-valued function of class $\mathcal{C}^1$ defined in a neighborhood of $0 \in \mathbf{C}$ satisfying $h(z) = \lambda^m |z|^{2m} + O(|z|^s)$. It is enough to find φ_2 such that

$$g(z, \lambda\bar{z} + \varphi_2(z)) - h(z) = 0 \quad \text{for } z \text{ small enough}. \tag{3}$$

We set $G(z, u) = g(z, \lambda\bar{z} + u) - h(z)$. Evidently G is well-defined in a neighborhood of $0 \in \mathbf{C}^2$ and holomorphic in u. Now, using the Taylor expansion of g, (3) amounts to finding $\varphi_2(z)$ satisfying

$$(g_0(z) + \varphi_2(z) g_1(z) - h(z)) + \sum_{k \geq 2} g_k(z) \varphi_2^k(z) = 0 \quad \text{for } z \text{ small enough},$$

where

$$g_k(z) = \frac{1}{k!} \frac{\partial^k g}{\partial^k w}(z, \lambda\bar{z}).$$

Observe that

$$g_0(z) - \lambda^m |z|^{2m} = O(|z|^s), \qquad g_1(z) = m\lambda^{m-1} z |z|^{2m-2} + O(|z|^{2m}).$$

Hence there exists $a > 0$ such that, for $z \neq 0$ small enough, we have

$$|g_0(z) + u g_1(z) - h(z)| > \biggl|\sum_{k \geq 2} g_k(z) u^k\biggr|$$

when u belongs to the circle $|u| = a|z|^{s-2m+1}$. Since $g_0(z) + ug_1(z) - h(z) = 0$ has a unique root inside this circle (if a is big enough), by the Rouché theorem the equation $G(z,u) = (g_0(z) + ug_1(z) - h(z)) + \sum_{k\geq 2} g_k(z)u^k = 0$ has a unique solution also for a fixed $z \neq 0$ near the origin; this solution is denoted by $\varphi_2(z)$. Moreover, by the Cauchy integral formula it is easy to see that φ_2 is of class $\mathcal{C}^1$ in a punctured neighborhood of $0 \in \mathbf{C}$. Finally, by putting $\varphi_2(0) = 0$, φ_2 extends as a function of class $\mathcal{C}^1$ near 0. $\square$

REMARK. It is not true in general that for any S of the form (2) we can find φ_2 real analytic in a neighborhood of $0 \in \mathbf{C}$ such that $M_2^r \subset S$ for $r > 0$ sufficiently small. Indeed, let $S = \{(z,w) : \operatorname{Im}((zw)^2 + z^5 + w^5) = 0\}$. If there exists φ_2 real analytic in a neighborhood of the origin such that $M_2^r \subset S$ for $r > 0$ small enough, then near $0 \in \mathbf{C}^2$ we would have $\operatorname{Im}\big((\lambda|z|^2 + z\varphi_2(z))^2 + z^5 + (\lambda\bar{z} + \varphi_2(z))^5\big) \equiv 0$. Dividing both sides by $|z|^5$ and letting z go to 0, we obtain $\operatorname{Im}(\lambda z|z|^2\varphi^*(z) + z^5 + \lambda^5\bar{z}^5) \equiv 0$, where φ^* is a homogeneous polynomial in $z, \bar{z}$ of degree 2. Since $\lambda \neq 1$, this is impossible.

Proof of Theorem 3.2. According to Lemma 3.3, there exists a defining function f for S of the form $f = \operatorname{Im} g$, where g is holomorphic in a neighborhood of $0 \in \mathbf{C}^2$ having the expansion (2). For each $r, t > 0$ we define

$$A_t^r = g^{-1}(t) \cap M_1^r, \qquad B_t^r = g^{-1}(t) \cap M_2^r.$$

Since $M_1 \cup M_2 \subset S$, we see that g sends both these manifolds into $\mathbf{R}$. Then, as is well known,

$$(M_1^r \cup M_2^r)\hat{} = \bigcup_{t\in\mathbf{R}} (A_t^r \cup B_t^r)\hat{}$$

for every $r > 0$.

We claim that there exist positive numbers r_0, t_0, c_1, c_2 such that for any $0 < t < t_0$ the sets $\pi(A_t^{r_0})$ and $\pi(B_t^{r_0})$ are disjoint, smooth curves lying inside the annulus $\mathbf{D}(c_1 t^{1/2m}, c_2 t^{1/2m})$. Indeed, from (2) we deduce that both functions $g(z, \bar{z} + \varphi_1(z))$ and $g(z, \lambda\bar{z} + \varphi_2(z))$ are asymptotically $|z|^{2m}$ near the origin; we deduce also that there exists $r_0 > 0$ such that, on each ray $\{\arg z = \theta,\ |z| < r_0,\ \theta \in [0, 2\pi]\}$, both functions are increasing. If t_0 is chosen small enough then it is easy to see that for any $0 < t < t_0$ the two sets $\pi(A_t^{r_0})$ and $\pi(B_t^{r_0})$ are disjoint, smooth curves lying inside the annulus $\mathbf{D}(c_1 t^{1/2m}, c_2 t^{1/2m})$, where c_1 and c_2 are constant. The claim is proved. Furthermore, in view of Lemma 3.1(iv) and the fact that $(A_t^r \cup B_t^r)\hat{}$ is contained in $g^{-1}(t)$, by expanding the annulus $\mathbf{D}(c_1 t^{1/2m}, c_2 t^{1/2m})$ one may assume that $\pi((A_t^r \cup B_t^r)\hat{})$ is contained in this annulus.

Next we claim that there exists $c_3 > 0$ such that, for every $t > 0$ small enough, the sector

$$S(z) = \left\{ w : |w| < c_3 t^{1/2m},\ |\arg(zw)| < \frac{2\pi}{3m} \right\}$$

contains exactly one root of the equation $g(z,w) = t$, which is a holomorphic function of z for z in $\mathbf{D}(c_1 t^{1/2m}, c_2 t^{1/2m})$. This is essentially done by the Rouché theorem. Explicitly, we notice that in $S(z)$ the equation $(zw)^m = t$ has only one root $w = t^{1/m}/z$. Now, on the circle $|w| = c_3 t^{1/2m}$ we have

$$|(zw)^m - t| > \Big| \sum_{k \geq 2m+1} P_k(z, w) \Big| \tag{4}$$

for c_3 big enough and t small enough, because the left-hand side is greater than $t(c_1^m c_3^m - 1)$ and the other side is dominated by $c_4 t^{1+1/2m}$ $(c_4 > 0)$. It remains to check the inequality (4) on each ray $|\arg(zw)| = 2\pi/3m$. Indeed, on these rays we always have $\arg(zw)^m = \pm 2\pi/3$. This implies that $|(zw)^m - t| \geq t \sin(\pi/3)$ and, by the Rouché theorem, the sector $S(z)$ contains only one root of the equation $g(z, w) = t$, which is denoted by $h_t(z)$. The Cauchy integral formula implies that h_t depends continuously on z and thus it is, in fact, a holomorphic function of z. The claim follows. This implies that, for r, t small enough, for each point $(z, w) \in A_t^r \cup B_t^r$ we have $w = h_t(z)$.

Now we claim that

$$(A_t^r \cup B_t^r)\hat{} = \{(z, w) : z \in C_t^r,\ w = h_t(z)\},$$

where C_t^r is the closed set bounded by the two curves $\pi(A_t^r)$ and $\pi(B_t^r)$. Indeed, by the maximum modulus principle we see that the set on the right is contained in the one on the left. Since $(A_t^r \cup B_t^r)\hat{}$ is contained in the graph of h_t over the annulus $\mathbf{D}(c_1 t^{1/2m}, c_2 t^{1/2m})$, to prove the reverse inclusion it suffices to show that $\pi((A_t^r \cup B_t^r)\hat{}) = C_t^r$. Otherwise, there would exist $p = (z_0, w_0) \in (A_t^r \cup B_t^r)\hat{}$ such that $z_0 \notin C_t^r$. Since $z_0 \in \mathbf{D}(c_1 t^{1/2m}, c_2 t^{1/2m})$, by the Runge approximation theorem we can find a holomorphic function q in $\mathbf{D}(c_1 t^{1/2m}, c_2 t^{1/2m})$ satisfying $q(z_0) > |q(z)|$ for every $z \in \pi(A_t^r) \cup \pi(B_t^r)$. It follows that q cannot be approximated uniformly on $(A_t^r \cup B_t^r)\hat{}$ by polynomials in z, w. Now we define

$$D_t^r = \{(z, w) : z \in \mathbf{D}(c_1 t^{1/2m}, c_2 t^{1/2m}),\ w = h_t(z)\}.$$

Notice that on D_t^r we have

$$\frac{1}{z} = \frac{w}{(t - (g(z, w) - z^m w^m))^{1/m}}.$$

Since $|g(z, w) - z^m w^m| \leq C \max(|z|, |w|)^{2m+1}$, the right-hand side can be expanded as a power series that is convergent for $|z|, |w| < C|t|^{1/(2m+1)}$ for some $C > 0$. We deduce that $1/z$ can be approximated uniformly by polynomials in z, w on D_t^r. It follows that $q(z)$ is also approximable uniformly on D_t^r and in particular on $(A_t^r \cup B_t^r)\hat{}$ by polynomials in z, w. Thus we arrive at a contradiction, and the claim is proved.

Therefore, $\operatorname{ad}(A_t^r \cup B_t^r)$ is nonempty exactly when it is an analytic annulus that is the graph of h_t over the annulus $\operatorname{Int} C_t^r$. Hence $\operatorname{ad}(M_1^r \cup M_2^r)$ is union of disjoint analytic annuli with boundary in $M_1^r \cup M_2^r$. To see that in fact we get a foliation of analytic annuli, it suffices to verify $dg(z, w) \neq 0$ on $\operatorname{ad}(M_1^r \cup M_2^r)$. However, this readily follows from (2) and Lemma 3.1(iv). □

REMARK. It should be noticed that if $\lambda = 1$ then the union $M_1 \cup M_2$ may be LPC at 0 even when $M_1 \cup M_2$ is contained in a strongly Levi-flat hypersurface (see [N, Cor. 2.6]).

As an application of the preceding results, we shall give a characterization of a class of functions with a certain "stability" property. First, for any natural number $n \geq 2$ we define the class

$$\mathbf{A}_n = \left\{ \varphi \in C^1_*(\{0\}) \;\Big|\; \exists p > 0 : \varphi(z) = \sum_{\substack{n \leq i+j \leq 2n-2 \\ \max(|i|,|j|) < p}} a_{i,j} z^i \bar{z}^j \right\}.$$

Note that many functions in $\mathbf{A}_n$ are not real analytic.

PROPOSITION 3.4. (a) *Let $\varphi \in \mathbf{A}_n$, and assume that $a_{l,l+1} \in \mathbf{R}$ for every l. Then there exists a function φ^* that is real analytic away from 0 and of order at least $2n-1$ such that $M_1 \cup M_2^*$ is not LPC at 0, where $M_1 = \{(z, \bar{z})\}$ and $M_2^* = \{(z, \lambda\bar{z} + \varphi(z) + \varphi^*(z))\}$.*

(b) *With the same notation, if $a_{l,l+1} \notin \mathbf{R}$ for some l then, for any function φ^* satisfying $|\varphi^*(z)| \leq C|z|^{2n-1}$ for z small enough, where C is a positive constant, we have that $M_1 \cup M_2^*$ is LPC at 0; in particular, $M_1 \cup M_2$ is LPC at 0.*

Proof. (a) Denote by k the smallest positive integer such that $|z|^{2k-2}\varphi(z)$ is real analytic in a neighborhood of $0 \in \mathbf{C}$. We define

$$\psi(z, w) = (zw)^k + \sum_{n \leq i+j \leq 2n-2} (b_{i,j} z^{k+i} w^{k+j-1} + \overline{b_{i,j}} z^{k+j-1} w^{k+i}),$$

where

$$b_{i,j} = \begin{cases} \dfrac{k\lambda^{k-1}}{\lambda^{k+i} - \lambda^{k+j-1}} a_{i,j} & \text{for } i \neq j-1, \\ 0 & \text{for } i = j-1. \end{cases}$$

By Theorem 3.2 it suffices to show that there exists φ^* real analytic away from 0 and of order at least $2n-1$ such that $M_1 \cup M_2^*$ is contained in the strongly Levi-flat real analytic hypersurface defined by $\{\operatorname{Im} \psi = 0\}$. Because $\operatorname{Im} \psi(z, \bar{z}) = 0$, it remains to look for φ^* real analytic away from 0 and of order at least $2n-1$ such that

$$\operatorname{Im} \psi(z, \lambda\bar{z} + \varphi(z) + \varphi^*(z)) \equiv 0.$$

We see that

$$\begin{aligned} \operatorname{Im}(\psi(z, \lambda\bar{z})) &= \operatorname{Im}(\lambda^k |z|^{2k}) \\ &\quad + \operatorname{Im}\left(\sum_{n \leq i+j \leq 2n-2} (\lambda^{k+j-1} b_{i,j} z^{k+i} \bar{z}^{k+j-1} + \overline{b_{i,j}} \lambda^{k+i} z^{k+j-1} \bar{z}^{k+i}) \right) \\ &= \sum_{n \leq i+j \leq 2n-2} \operatorname{Im}(b_{i,j}(\lambda^{k+j-1} - \lambda^{k+i}) z^{k+i} \bar{z}^{k+j-1}) \\ &= -k\lambda^{k-1} \sum_{i \neq j-1} \operatorname{Im}(a_{i,j} z^{k+i} \bar{z}^{k+j-1}) \end{aligned}$$

and so

$$\operatorname{Im}(\psi(z, \lambda\bar{z}) + k\lambda^{k-1}|z|^{2k-2} z\varphi(z)) = \sum \operatorname{Im}(a_{i,i+1})|z|^{2i+2k}. \tag{5}$$

Note that $a_{i,i+1} \in \mathbf{R}$ for every i. Hence, it suffices to find φ^* with the same regularity verifying

$$\psi(z, \lambda\bar{z} + \varphi(z) + \varphi^*(z)) - \psi(z, \lambda\bar{z}) - k\lambda^{k-1}|z|^{2k-2}z\varphi(z) = 0$$

in a neighborhood of $0 \in \mathbf{C}$. For this, we will regard $\varphi^*(z)$ as an independent variable; more precisely, we define

$$\Psi(z, u) = \psi(z, \lambda\bar{z} + \varphi(z) + u) - \psi(z, \lambda\bar{z}) - k\lambda^{k-1}|z|^{2k-2}z\varphi(z).$$

Expanding Ψ as a polynomial in u, we obtain $\Psi(z, u) = \sum_{l=0}^{M} c_l(z)u^l$, where the c_l are real analytic functions outside the origin and of order at least $\max(0, 2k-l)$. Moreover, it is easy to see that

$$c_1(z) = k\lambda^{k-1}|z|^{2k-2}z + O(|z|^{2k}), \qquad c_0 = O(|z|^m) \quad (m \geq 2k + 2n - 2).$$

It follows that there exists $a > 0$ large enough such that, for z small enough, the equation $c_0(z) + c_1(z)u = 0$ has a unique root inside the disk $|u| < a|z|^{m-2k+1}$. Now we claim that in this disk the equation $\Psi(z, u) = 0$ has a unique root, too. Indeed, on the boundary of this disk $|c_0(z) + c_1(z)u| \approx |z|^m$, whereas the other terms $c_l(z)u^l$ $(l \geq 2)$ are of order at least

$$l(m - 2k + 1) + 2k - l = l(m - 2k) + 2k > m - 2k + 2k = m.$$

Thus our claim follows from the Rouché theorem. Denote this root by $\varphi^*(z)$; by the Cauchy integral formula, φ^* is real analytic and of order $\geq 2n - 1$ away from 0. Setting $\varphi^*(0) = 0$, we obtain the desired function φ^*.

(b) The following result was given in [Ka] in a slightly more restrictive setting. See [St, p. 386] or [W] for a proof of the present statement.

KALLIN'S LEMMA. *Let K and L be two polynomially convex compact sets in $\mathbf{C}^n$. If we can find a polynomial p that sends K to the real line and L to a compact set meeting the real line only at the origin, and if $p^{-1}\{0\} \cap (K \cup L)$ is polynomially convex, then $K \cup L$ is polynomially convex.*

Denote by l the smallest index such that $a_{l,l+1}$ is not real. For any $\varphi^*(z) = O(|z|^{2n-1})$, from (5) it is immediate to verify that

$$\operatorname{Im}\big(\psi(z, \lambda\bar{z} + \varphi(z) + \varphi^*(z))\big) = \operatorname{Im}(a_{l,l+1})|z|^{2l+2k} + O(|z|^{2l+2k+1}).$$

This implies that, in a small neighborhood of $0 \in \mathbf{C}^2$, the polynomial ψ sends M_2^* to a compact set intersecting the real line only at the origin. Clearly $\psi(M_1)$ is contained in the real line. By Kallin's lemma, $M_1 \cup M_2^*$ is LPC at 0. □

REMARK. This result should be compared to Proposition 2.1 in [N], where a similar use of the Kallin's lemma was made.

It seems hard to formulate a satisfactory converse to Theorem 3.2. Nevertheless, we do have the following result, which improves Proposition 2.3 in our special situation.

PROPOSITION 3.5. *Let S be a real analytic hypersurface in $\mathbf{C}^2$ defined by $f = 0$, where f is real analytic and irreducible at* 0. *Assume that the following hold*:

(i) *M_1 is the plane* $\{(z, \bar{z})\}$;
(ii) $M_1 \cup M_2 \subset S$;
(iii) *for every $r > 0$ sufficiently small,* $\mathrm{ad}(M_1^r \cup M_2^r)$ *is nonempty and contained in S.*

Then S is weakly Levi-flat near 0.

Proof. By Lemma 2.1,

$$\mathcal{L}(z, w) = 0 \quad \forall (z, w) \in \mathrm{ad}(M_1^r \cup M_2^r). \tag{6}$$

We let $\tilde{f}$ and $\tilde{\mathcal{L}}$ be the complexified functions of f and $\mathcal{L}$. It is then possible to choose constants $\lambda_1, \lambda_2, \lambda_3$, such that—after the change of coordinates $z' = z$, $w' = \lambda_1 z + w$, $u' = \lambda_2 z + u$, $v' = \lambda_3 z + v$—the equations $\tilde{f}(z, w, u, v) = \tilde{\mathcal{L}}(z, w, u, v) = 0$ take the form $f_1(z', w', u', v') = \mathcal{L}_1(z', w', u', v') = 0$, where f_1 and $\mathcal{L}_1$ are Weierstrass polynomials in z'. Clearly we may assume further that $|\lambda_1 - \lambda_2| \neq |\lambda_3 + 1|$. This gives a holomorphic function $G(w', u,', v')$, which is the *resultant* of f_1 and $\mathcal{L}_1$ with respect to z' satisfying $G(w', u', v') = 0$ if $f_1(z', w', u', v') = \mathcal{L}_1(z', w', u', v') = 0$. Returning to the original coordinates and taking into account (6), we obtain

$$G(\lambda_1 z + w, \lambda_2 z + \bar{z}, \lambda_3 z + \bar{w}) = 0 \quad \forall (z, w) \in \mathrm{ad}(M_1^r \cup M_2^r). \tag{7}$$

We claim that $G \equiv 0$. Otherwise, we may write

$$G(a, b, c) = ((\lambda_3 + 1)(a - b) - (\lambda_1 - \lambda_2)c)^k H(a, b, c),$$

where H is holomorphic in a neighborhood of $0 \in \mathbf{C}^3$ and not identically 0 on the plane $(\lambda_3 + 1)(a - b) = (\lambda_1 - \lambda_2)c$. Substituting (7) into the identity just displayed yields

$$((\lambda_1 - \lambda_2)(\bar{w} - z) - (\lambda_3 + 1)(w - \bar{z}))^k H(\lambda_1 z + w, \lambda_2 z + \bar{z}, \lambda_3 z + \bar{w}) = 0 \qquad \forall (z, w) \in \mathrm{ad}(M_1^r \cup M_2^r). \tag{8}$$

Now we arbitrarily take $((0, 0) \neq)$ $p = (z_0, \bar{z}_0) \in \mathrm{closure}(\mathrm{ad}(M_1^r \cup M_2^r)) \cap M_1$. Then there exists a sequence $p_n = (z_n, w_n) \in \mathrm{ad}(M_1^r \cup M_2^r)$, $n \geq 1$, tending to p. It follows from (8) that $H(\lambda_1 z_n + w_n, \lambda_2 z_n + \bar{z}_n, \lambda_3 z_n + \bar{w}_n) = 0$ for all n. By passing to the limit, we obtain $H(\lambda_1 z_0 + \bar{z}_0, \lambda_2 z_0 + \bar{z}_0, \lambda_3 z_0 + z_0) = 0$. Defining $H^*(z, w) = H(\lambda_1 z + w, \lambda_2 z + w, \lambda_3 z + z)$ then yields $H^*(p) = 0$. Hence $\mathrm{closure}(\mathrm{ad}(M_1^r \cup M_2^r)) \cap M_1$ is contained in the zero set of H^*. Moreover, it is clear that H^* is not identically 0.

Using the Puiseux expansion theorem (see [KP, p. 84]) and noting that M_1 is real analytic, for $r > 0$ small enough we now have that the set $\{H^* = 0\} \cap M_1^r$ is either the origin alone or a contractible compact set consisting of a union of a finite number of smooth curves. Notice that, for $r > 0$ small enough, the set M_2^r is polynomially convex; hence, by Stolzenberg's theorem about approximation on curves [Sg; St, p. 404], we conclude that the union $N^r := M_2^r \cup (\{H^* = 0\} \cap M_1^r)$ is polynomially convex. On the other hand, we also have $\text{closure}\{\text{ad}(M_1^r \cup M_2^r)\} \cap (M_1^r \cup M_2^r) \subset N^r$. The local maximum modulus principle implies that, for every polynomial q in $\mathbf{C}^2$,

$$\begin{aligned}\sup\{|q(x)| : x \in \text{ad}(M_1^r \cup M_2^r)\} \\ &= \sup\{|q(x)| : x \in \text{closure}\{\text{ad}(M_1^r \cup M_2^r)\} \cap (M_1^r \cup M_2^r)\} \\ &\leq \sup\{|q(x)| : x \in N^r\}.\end{aligned}$$

It follows that $\text{ad}(M_1^r \cup M_2^r) \subset \hat{N}^r$. Since $N^r \subset M_1^r \cup M_2^r$, we deduce that $\emptyset \neq \text{ad}(M_1^r \cup M_2^r) \subset \text{ad}(N^r)$. This is a contradiction, since N^r is polynomially convex. Thus $G \equiv 0$, as claimed. It follows that f_1 divides $\mathcal{L}_1$ and hence S is weakly Levi-flat. $\square$

In our next proposition we construct a pair of M_1, M_2 such that the polynomial hull of $(M_1^r \cup M_2^r)\hat{}$ has Hausdorff dimension 3 for all $r > 0$ small enough, but unlike in Theorem 3.2 this hull cannot be contained in any strongly Levi-flat real analytic hypersurface. In fact, a bit more is true.

PROPOSITION 3.6. *For any $\lambda > 0$, $\lambda \neq 1$ there exists φ real analytic away from $0 \in \mathbf{C}$ and of infinite order such that, for $r > 0$ sufficiently small, the following assertions hold.*

(a) *M_2^r is contained in a real analytic hypersurface but is not contained in any weakly Levi-flat real analytic hypersurface, where $M_2 = \{(z, \lambda\bar{z} + \varphi(z)\}$.*

(b) *$(M_1^r \cup M_2^r)\hat{}$ is a union of disjoint analytic annuli with boundary in $M_1 \cup M_2$, where $M_1 = \{(z, \bar{z})\}$. Moreover, $(M_1^r \cup M_2^r)\hat{} \setminus \{0\}$ is contained in a strongly Levi-flat hypersurface in a Runge domain Ω such that $0 \in \partial\Omega$.*

Proof. We divide the proof into two steps.

Step 1. In this step we construct the two graphs M_1 and M_2 satisfying (a). First, let A be any $\mathcal{C}^1$ positive function on $(3, \infty)$. Define

$$V_{t,A} = \{(z, w) : z^2 + tzw + w^2 = A(t)\}.$$

Then, for any $t > 3$ we have

$$\pi(V_t \cap M_1) = \{z : z^2 + t|z|^2 + \bar{z}^2 = A(t)\} = \left\{z = re^{i\theta},\ r^2 = \frac{A(t)}{t + 2\cos(2\theta)}\right\},$$

which is a closed smooth curve around the origin in $\mathbf{C}$. Let us consider $\pi(V_t \cap M_2)$. Our goal is to find φ and A such that, for t big enough, $V_t \cap M_2$ contains the following curve, which is denoted by γ_t:

$$z = r_t(\theta)e^{i\theta}, \quad w = \lambda r_t(\theta)e^{-i\theta} + g_t(\theta)e^{i\theta},$$

where $r_t(\theta)$, $g_t(\theta)$ are real-valued functions in $\mathcal{C}^1[0, 2\pi]$ and where $r_t(0) = r_t(2\pi)$ and $g_t(0) = g_t(2\pi)$. Clearly this occurs if and only if the two functions r_t and g_t verify

$$r_t^2(\theta)e^{2i\theta} + tr_t(\theta)e^{i\theta}(\lambda r_t(\theta)e^{-i\theta} + g_t(\theta)e^{i\theta}) + (\lambda r_t(\theta)e^{-i\theta} + g_t(\theta)e^{i\theta})^2 = A(t).$$

This is equivalent to the following two equalities:

$$\cos(2\theta)\big(r_t^2(\theta)(1+\lambda^2) + tr_t(\theta)g_t(\theta) + g_t^2(\theta)\big) \\ + 2\lambda r_t(\theta)g_t(\theta) + t\lambda r_t^2(\theta) = A(t), \tag{9}$$

$$\sin(2\theta)(r_t^2(\theta)(1-\lambda^2) + tr_t(\theta)g_t(\theta) + g_t^2(\theta)) = 0. \tag{10}$$

Obviously (10) is satisfied if $g_t(\theta) = c(t)r_t(\theta)$, where

$$c(t) = (-t + (t^2 - 4(1-\lambda^2))^{1/2})/2$$

is the larger root of the equation $x^2 + tx + 1 - \lambda^2 = 0$. Observe that since $\lambda \neq 1$ we have $c(t) \neq 0$ and $|c(t)| \approx 1/t$ when t goes to infinity. Moreover, $c(t)$ extends holomorphically outside a large disk—that is, when $|t|$ is big enough. Substituting $g_t(\theta)$ into (9) yields

$$r_t(\theta)^2 = \frac{A(t)}{2\lambda^2\cos(2\theta) + 2\lambda c(t) + t\lambda};$$

equivalently, $\pi(\gamma_t)$ is given by

$$A(t) = \lambda^2(z^2 + \bar{z}^2) + (2\lambda c(t) + t\lambda)|z|^2.$$

We choose $A(t) = \frac{t}{\log t}$. This choice works because of the following lemma, whose proof is given in Section 5.

LEMMA 3.7. *There exists a positive real analytic function $t(z)$ defined in a punctured neighborhood of $0 \in \mathbf{C}$ satisfying*

$$\frac{t(z)}{\log t(z)} = \lambda^2(z^2 + \bar{z}^2) + |z|^2\big(\lambda t(z) + 2\lambda c(t(z))\big)$$

for every $z \neq 0$ close to 0. Moreover, $1/t$ is of infinite order near the origin.

Thus, by setting $\varphi(z) = zc(t(z))$, we obtain a graph $M_2 = \{(z, \lambda\bar{z} + \varphi(z))\}$ verifying $\gamma_t = M_2 \cap V_t$. Since $|c(t)| \approx 1/t$, the function $\varphi(z)$ is of infinite order and real analytic away from the origin. Notice that, since $\operatorname{Im}(\bar{z}\varphi(z)) \equiv 0$, M_2 is contained in the real analytic hypersurface $S = \{\operatorname{Im}(\bar{z}(w - \lambda\bar{z})) = 0\}$. Now assume that there exists a *weakly* Levi-flat real analytic hypersurface S' containing M_2. Let f be a real analytic defining function for S' we get $f(z, \lambda\bar{z} + \varphi(z)) \equiv 0$. Observe that $\varphi(z) = z\psi(z)$ where ψ is a real valued function and of infinite order. Define $F(t,z) = f(z, \lambda\bar{z} + zt)$ with $t \in \mathbf{R}$ and $z \in \mathbf{D}$. We can expand F in the form $F(t,z) = \sum_{k\geq 0} F_k(z)t^k$, where F_k are real analytic functions. Then, since $F(\psi(z), z) \equiv 0$, we have

$$\sum_{k\geq 0} F_k(z)\psi^k(z) \equiv 0.$$

Because ψ is of infinite order and zero-free away from 0, it is not hard to prove by induction on k that $F_k \equiv 0$ for any $k \geq 0$. It follows that $F(t,z) \equiv 0$ for every $t \in \mathbf{R}$ and $z \in \mathbf{C}$. Thus f vanishes on $S = \{\operatorname{Im}(\bar{z}(w - \lambda\bar{z})) = 0\}$ near the origin. Consequently $S \subset S'$, and since S is a smooth real analytic hypersurface away from the origin but *not* weakly Levi-flat, we deduce that S' cannot be weakly Levi-flat.

Step 2. In this step we will compute the hull $(M_1^r \cup M_2^r)\hat{}$ for $r > 0$ small enough. We denote by D_a the domain given in Lemma 3.1 and fix $a > \max(4, 1+\lambda^2)$. We require the following lemma, whose proof is given in Section 5.

LEMMA 3.8. *For any $(z, w) \in D_a$, the equation $z^2 + w^2 + zwe^{1/u} = ue^{1/u}$ has exactly one root $u = \psi(z, w)$ in the domain*

$$S(z, w)(a) := \left\{ u : \frac{1}{2}|zw| < |u| < 2|zw|,\ \frac{1}{a}|zw| < \operatorname{Re} u \right\}.$$

Moreover, ψ depends holomorphically on (z, w) and extends continuously to the boundary of D_a.

On the other hand we note that, for any point $(z, w) \in (M_1^r \cup M_2^r) \setminus \{0\}$, the corresponding u satisfying $z^2 + w^2 + e^{1/u}zw = ue^{1/u}$ lies in $S(z, w)(a)$. We claim that $\operatorname{Im} \psi(z, w) = 0$ on $(M_1^r \cup M_2^r) \setminus \{0\}$. In fact, it is clear for $(z, w) \in M_1^r \setminus \{0\}$; if $(z, w) \in M_2^r \setminus \{0\}$ then (from Lemma 3.7) we have $\psi(z, w) = 1/\log(t(z))$, which is a real number.

By invoking an approximation theorem of Henkin [HKL, p. 139], ψ can be approximated uniformly on $\overline{D_a}$ by holomorphic functions in neighborhoods of $\overline{D_a}$. Combining this with the Oka–Weil theorem [AW; Hö] gives a sequence of polynomials P_n in $\mathbf{C}^2$ that approximate ψ uniformly in $\overline{D_a}$. Take an arbitrary $(0 \neq)$ $p \in (M_1^r \cup M_2^r)\hat{}$. Then, for any n we have

$$|\operatorname{Im}(P_n(p))| \le \sup_{(z,w)\in M_1^r \cup M_2^r} |\operatorname{Im}(P_n(z, w))|.$$

Letting n go to infinity, one obtains $\operatorname{Im}(\psi(p)) = 0$. Thus $(M_1^r \cup M_2^r)\hat{} \setminus \{0\}$ is contained in the hypersurface $\operatorname{Im} \psi = 0$ in D_a. To compute the hull $(M_1^r \cup M_2^r)\hat{}$ we will use the same argument as made in Theorem 3.2. More precisely, since ψ can be approximated uniformly on $\overline{D_a}$ by polynomials, we have

$$(M_1^r \cup M_2^r)\hat{} = \bigcup (\psi^{-1}(t) \cap (M_1^r \cup M_2^r))\hat{}$$

for every $t \in \psi(M_1^r \cup M_2^r)$. By the definition of φ, each term in the union on the right-hand side is made of the two curves $V_t \cap M_1$ and γ_t. Notice that, for any z inside the annulus with boundary $\pi(V_t \cap M_1)$ and $\pi(\gamma_t)$, there exists a unique $w(z)$ satisfying $(z, w(z)) \in V_t$ and $|w(z)| < |A(t) - z^2|^{1/2}$. In fact, if $w_1(z)$ and $w_2(z)$ are two roots of $z^2 + tzw + w^2 = A(t)$, then $|w_1(z)w_2(z)| = |A(t) - z^2|$. Thus there exists one root, say $w_1(z)$, having its absolute value not greater than $|A(t) - z^2|^{1/2}$. Suppose z exists such that the two roots have the same absolute value $|A(t) - z^2|^{1/2}$; then, since $w_1(z) + w_2(z) = tz$, we infer that $|tz| \le 2|A(t) - z^2|^{1/2}$. This is absurd because, when t is big enough, $A(t) \ll \frac{1}{2}t|z|^2$. Hence $w_1(z)$ is holomorphic in z and, as in the proof of Theorem 3.2, we derive that $\psi^{-1}(t) \cap (M_1^r \cup M_2^r))\hat{}$ is an analytic annulus with boundary $\pi(\gamma_t) \cup \pi(V_t \cap M_1)$. □

IV. The Case Where S Has a Nonvanishing "Quadratic Part"

In this section we will investigate the case of S defined by $f = 0$, where f is a real analytic germ that is not "highly degenerate". By using the main result in

[BG] we will show that, if S is weakly Levi-flat and contains graphs M_1, M_2 near 0, then in fact S is strongly Levi-flat and is a small “perturbation” of the hypersurface $\{\mathrm{Im}(zw) = 0\}$.

PROPOSITION 4.1. *Let S be a weakly Levi-flat real analytic hypersurface in $\mathbf{C}^2$, $0 \in S$, defined by $f = 0$. Assume that the quadratic part q of f is not identically 0 and irreducible. If S contains $M_1^r \cup M_2^r$ for $r > 0$ sufficiently small, then S is strongly Levi-flat.*

Proof. Since S contains $M_1^r \cup M_2^r$ for $r > 0$ small enough, the tangent cone $\{q = 0\}$ must contain the planes $w = \bar{z}$ and $w = \lambda\bar{z}$. This implies that $\{q = 0\}$ has dimension 3, since q is quadric. Moreover, because S is weakly Levi-flat and the set $q = 0$ has dimension 3, by [BG, Prop. 2.4] and [BG, Thm. 2.3] we have that the cone $\{q = 0\}$ is also weakly Levi-flat. Furthermore, after a nonsingular linear change of coordinates, it can be put into one of the following normal forms:

(a) $Q_1 = \{\mathrm{Re}(z^2 + w^2) = 0\}$;
(b) $Q_2 = \{z^2 + 2\alpha|z|^2 + \bar{z}^2 = 0,\ 0 \le \alpha \le 1\}$;
(c) $Q_3 = \{|z| = |w|\}$;
(d) $Q_4 = \{\mathrm{Im}\, z\, \mathrm{Im}\, w = 0\}$.

Observe that (d) cannot hold, since $q = 0$ is irreducible. Suppose that (b) is true. Then there exists a nonzero linear function $h(z, w) = az + bw$ such that $h^2 + 2\alpha|h|^2 + \bar{h}^2 \equiv 0$ on $(w - \bar{z})(w - \lambda\bar{z}) = 0$. This gives $h \equiv c_1\bar{h}$ when $w = \bar{z}$ and $h \equiv c_2\bar{h}$ when $w = \lambda\bar{z}$, where c_1, c_2 are roots of the equation $c^2 + 2\alpha c + 1 = 0$. Collecting the terms $z, \bar{z}$, we have $a = c_1\bar{b} = c_2\lambda\bar{b}$ and $b = c_1\bar{a} = c_2\bar{a}/\lambda$. This implies $a = b = 0$, a contradiction. Now assume (c) is valid; then there exists $h_1(z, w) = az + bw$ and $h_2(z, w) = cz + dw$ such that $ad - bc \neq 0$ and $|h_1|^2 \equiv |h_2|^2$ on $(w - \bar{z})(w - \lambda\bar{z}) = 0$. By collecting the terms $|z|^2$ and z^2, we obtain $a\bar{b} = c\bar{d}$ and $|b| = |d|$. This gives $ad = bc$, which is absurd. Thus, there remains only the case (a), and applying [BG, Thm. 1.1] finishes the proof. $\square$

PROPOSITION 4.2. *Let $M_1 \cup M_2$ be contained in S, a real analytic hypersurface in $\mathbf{C}^2$ defined by $f = 0$. Assume that the quadratic part of f is not identically 0 and irreducible. Then the following assertions are equivalent:*

(i) *S is weakly Levi-flat near 0;*
(ii) *S is strongly Levi-flat near 0;*
(iii) *for $r > 0$ small enough, $(M_1^r \cup M_2^r)\hat{}$ is a union of disjoint analytic annuli whose boundaries are contained in $M_1 \cup M_2$;*
(iv) *for $r > 0$ small enough, $(M_1^r \cup M_2^r)\hat{}$ is nontrivial and is contained in S.*

Proof. (ii) follows from (i) by Proposition 4.1, (ii) implies (iii) by Theorem 3.2, and (iii) trivially implies (iv). Finally, by Proposition 3.5, (i) follows from (iv). $\square$

V. Appendix

Proof of Lemma 3.7. For $z, v \neq 0$, define

$$F(v, z) = ve^{1/v} - |z|^2(\lambda e^{1/v} + 2\lambda c(e^{1/v})) - \lambda^2(z^2 + \bar{z}^2).$$

On the circle $|v - \lambda|z|^2| = |z|^4$ we have

$$|ve^{1/v} - \lambda|z|^2 e^{1/v}| = |z|^4 e^{\mathrm{Re}(1/v)} > |z|^4 e^{1/8\lambda|z|^2} > \lambda^2|z^2 + \bar{z}^2| + 2\lambda|z|^2 c(e^{1/v})$$

for z close to 0. By the Rouché theorem, in the disk $|v - \lambda|z|^2| < |z|^4$ the equation $F(v, z) = 0$ has a unique root, denoted by $v(z)$. The Cauchy integral formula then implies that v is real analytic in z outside the origin. Observe that $F(\lambda|z|^2 + |z|^4, z) > 0$ while $F(\lambda|z|^2 - |z|^4, z) < 0$; thus, $v(z)$ is a positive number between $\lambda|z|^2 - |z|^4$ and $\lambda|z|^2 + |z|^4$. Evidently, $t(z) = e^{1/v(z)}$ is the desired function. □

Proof of Lemma 3.8. For $(z, w) \in D_a$ and $u \in S(z, w)(a)$, define the two functions $G(z, w, u) = e^{1/u}(u - zw) - (z^2 + w^2)$ and $G^*(z, w, u) = e^{1/u}(u - zw)$. Notice that $G^* = 0$ has a unique root $u = zw$ in $S(z, w)(a)$. On the other hand, for $u \in \partial(S(z, w)(a))$ we have

$$|G^*(z, w, u)| = |u - zw|e^{(\mathrm{Re}\, u)/|u|^2} > \frac{|zw|}{a} e^{1/4a|zw|} > |z^2 + w^2|.$$

By the Rouché theorem, $G(z, w, u) = 0$ has a unique root $u = \psi(z, w)$ in $S(z, w)(a)$; moreover, ψ is a holomorphic function of (z, w). To see that ψ can be extended continuously to the boundary of D_a, we observe in view of Lemma 3.1(iii) that, by setting $\psi(0, 0) = 0$ and shrinking D_a, we are done. □

ACKNOWLEDGMENT. I am deeply grateful to my thesis advisors Professor Pascal J. Thomas and Professor Do Duc Thai for bringing the problem to my attention as well as for useful suggestions and constant encouragement during the preparation of this work. I would like also to thank Professor Daniel Burns for sending the paper [BG] and useful discussions. Finally, it is my pleasure to thank the referee for suggesting valuable comments, in particular some ideas in the proof of Theorem 3.2.

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