On the Argument Oscillation of Conformal Maps

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1. Introduction

Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disk and $\mathbf{T}$ its boundary. We shall consider (injective) conformal maps f of $\mathbb{D}$ into $\mathbb{C}$. For $\zeta \in \mathbf{T}$ we denote by $f(\zeta)$ the angular (= radial) limit if it exists and is finite. This holds for almost all $\zeta \in \mathbf{T}$; even the exceptional set has zero logarithmic capacity, by the well-known Beurling theorem (see [Be; Po2, p. 215]). Furthermore, the set $\{ \zeta \in \mathbf{T} : f(\zeta) = a \}$ has zero capacity for every $a \in \mathbb{C}$ [Du; Po2, p. 219]. A stronger condition is that f is continuous at ζ; that is

$$f(z) \to f(\zeta) \quad \text{as } z \to \zeta,\ z \in \mathbb{D}. \tag{1.1}$$

Suppose now that the angular limit $f(\zeta) \neq \infty$ exists at $\zeta \in \mathbf{T}$. The function

$$g_\zeta(z) = \log[f(z) - f(\zeta)], \quad z \in \mathbb{D}, \tag{1.2}$$

is analytic and univalent in $\mathbb{D}$ for any branch of the logarithm. It therefore has finite angular limits at all points except a set of zero capacity. For convenience, throughout this paper we will write

$$E(\zeta) = \{ \zeta' \in \mathbf{T} : f(\zeta') \text{ and } g_\zeta(\zeta') \text{ exist and are finite} \}.$$

Thus $\mathbf{T} \setminus E(\zeta)$ has zero capacity. Then we define the argument by

$$\arg[f(z) - f(\zeta)] = \begin{cases} \operatorname{Im} g_\zeta(z) & \text{for } z \in \mathbb{D}, \\ \lim_{r\to 1} \operatorname{Im} g_\zeta(re^{it}) & \text{for } z = e^{it} \in E(\zeta). \end{cases} \tag{1.3}$$

We also consider the analytic function

$$h(z) = \log \frac{f(z) - f(\zeta)}{z - \zeta} = g_\zeta(z) - \log(z - \zeta), \quad z \in \mathbb{D} \cup E(\zeta). \tag{1.4}$$

In the unit disk, we always use the branch of the logarithm determined by

$$\theta + \pi/2 < \arg(z - \zeta) < \theta + 3\pi/2, \quad z \in \mathbb{D} \cup E(\zeta),\ \zeta = e^{i\theta}.$$

The function h is a Bloch function [Po2, p. 173]. We will make frequent use of the harmonic function

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$$\arg \frac{f(z)-f(\zeta)}{z-\zeta} = \operatorname{Im} h(z)$$
$$= \arg[f(z)-f(\zeta)] - \arg(z-\zeta), \quad z \in \mathbb{D} \cup E(\zeta). \quad (1.5)$$

The McMillam twist theorem [Mc; Po2, p. 142] shows that, for almost all $\zeta \in \mathbf{T}$, there are only two alternatives:

(i) $\log \frac{f(z)-f(\zeta)}{z-\zeta}$ has a finite nonzero angular limit as $z \to \zeta$;
(ii) $\arg \frac{f(z)-f(\zeta)}{z-\zeta}$ oscillates between $-\infty$ and $+\infty$ along any curve in $\mathbb{D}$ ending at ζ.

This gives the definitive answer to the problem of argument oscillation almost everywhere as far as angular approach is concerned. See [CaPo] for a discussion of the behavior that is possible on subsets of $\mathbf{T}$ of zero measure.

In this paper we study the oscillation of the argument for unrestricted approach. It turns out that the exceptional sets tend to be either countable or of zero measure.

2. Results

Generically we define
$$\text{osc} = \limsup - \liminf,$$
provided that both limits are finite; otherwise, we define osc $= +\infty$.

THEOREM 1. *Let f map $\mathbb{D}$ conformally into $\mathbb{C}$. Except possibly for countably many $\zeta \in \partial\mathbf{T}$, we have: If the angular limit $f(\zeta)$ exists and if*
$$\operatorname*{osc}_{z\to\zeta,\, z\in\mathbb{D}} \arg[f(z)-f(\zeta)] < 2\pi, \tag{2.1}$$
then f is continuous at ζ. The constant 2π is best possible.

We shall derive this theorem in Section 7 from two results of a topological nature. The condition that f be continuous at ζ is important for the following reason. There are two causes that may contribute to the oscillation of $\arg[f(z)-f(\zeta)]$, namely:

(i) nearby oscillation—that is, for $f(z)-f(\zeta)$ small;
(ii) faraway oscillation—that is, for $|f(z)-f(\zeta)| > c > 0$ (e.g., if $f(z)$ winds around the entire domain).

If f is continuous at ζ then only case (i) is possible.

Now we consider the oscillation of the difference quotient. If $\zeta \in \mathbf{T}$ and $f(\zeta) \neq \infty$ exists, then we define
$$\Delta(\zeta) = \operatorname*{osc}_{z\to\zeta,\, z\in\mathbb{D}} \arg \frac{f(z)-f(\zeta)}{z-\zeta}. \tag{2.2}$$

The quantity $\Delta(\zeta)$ is closely related to tangency properties of the boundary of $G = f(\mathbb{D})$. We say that ∂G has a tangent at the prime end corresponding to $\zeta = e^{i\theta}$ if
$$\arg[f(e^{it})-f(\zeta)] \to \begin{cases} \beta & \text{as } t \to \theta^+,\ e^{it} \in E(\zeta), \\ \beta+\pi & \text{as } t \to \theta^-,\ e^{it} \in E(\zeta). \end{cases} \tag{2.3}$$

Observe that this definition is completely different, in the case of an arbitrary conformal map, from the usual concept (see [Fa, p. 31]) that the planar set $\partial f(\mathbb{D})$ has a geometric tangent at $f(\zeta)$. However, if ∂G is a Jordan curve then the definitions are equivalent.

We have the following result, which should be compared to the classical Lindelöf theorem (see [Po2, p. 51]) and to the theorem of Wolff [Wo]. Its proof will be an application of Theorem 3.

THEOREM 2. *Let f map $\mathbb{D}$ conformally into $\mathbb{C}$. If $f(\zeta) \neq \infty$ exists at $\zeta \in \mathbf{T}$, then $\partial f(\mathbb{D})$ has a tangent at $f(\zeta)$ if and only if $\Delta(\zeta) = 0$.*

THEOREM 3. *Let f map $\mathbb{D}$ conformally into $\mathbb{C}$. If $f(\zeta) \neq \infty$ exists at $\zeta \in \mathbf{T}$, then*

$$\Delta(\zeta) = \operatorname*{osc}_{\eta \to \zeta,\, \eta \in E(\zeta)} \arg \frac{f(\eta) - f(\zeta)}{\eta - \zeta}. \tag{2.4}$$

If f is continuous at ζ, then

$$\operatorname*{osc}_{r \to 1} \arg[f(r\zeta) - f(\zeta)] \leq \Delta(\zeta) \leq 3\pi + \operatorname*{osc}_{z \to \zeta,\, z \in C} \arg[f(z) - f(\zeta)], \tag{2.5}$$

where C is any curve in $\mathbb{D} \cup \{\zeta\}$ ending at ζ. The constant 3π is best possible.

Note that the right-hand side of (2.4) is essentially a geometric quantity (see Section 3 for details). The radial oscillation in (2.5) has been studied previously (see e.g. [CaPo]).

THEOREM 4. *Let f map $\mathbb{D}$ conformally into $\mathbb{C}$. Then the sets $\{\zeta \in \mathbf{T} : 0 < \Delta(\zeta) < \pi\}$ and $\{\zeta \in \mathbf{T} : f$ is continuous at $\zeta,\ 2\pi < \Delta(\zeta) < \infty\}$ have zero Lebesgue measure on $\mathbf{T}$.*

If $\partial f(\mathbb{D})$ is locally connected (see the definition in [Po2, p. 19]) then it follows from Theorem 4 that

$$\Delta(\zeta) = 0 \quad \text{or} \quad \pi \leq \Delta(\zeta) \leq 2\pi \quad \text{or} \quad \Delta(\zeta) = +\infty$$

for almost all $\zeta \in \mathbf{T}$. However "linear measure zero" cannot be replaced by "Hausdorff dimension < 1", as the following theorem shows.

THEOREM 5. *There exists a conformal map f onto a Jordan domain such that*

$$\dim\{\zeta \in \mathbf{T} : 0 < \Delta(\zeta) < \pi\} = 1. \tag{2.6}$$

3. One-Sided Oscillation

Throughout this section we assume that f maps $\mathbb{D}$ conformally into $\mathbb{C}$, that $\zeta \in \mathbf{T}$, and that the angular limit $f(\zeta) \neq \infty$ exists. We write

$$G = f(\mathbb{D}), \quad \zeta = e^{i\theta} \ (0 \leq \theta < 2\pi), \quad w = f(\zeta).$$

We define

$$\begin{aligned}\alpha^{\pm}(\zeta) &= \liminf_{t\to\theta^{\pm},\, e^{it}\in E(\zeta)} \arg[f(e^{it}) - w],\\ \beta^{\pm}(\zeta) &= \limsup_{t\to\theta^{\pm},\, e^{it}\in E(\zeta)} \arg[f(e^{it}) - w].\end{aligned} \tag{3.1}$$

Also, if C is a Jordan arc such that $\zeta \in C$ and $C \setminus \{\zeta\} \subset \mathbb{D}$, we put

$$\begin{aligned}\alpha_C(\zeta) &= \liminf_{z\to\zeta,\, z\in C} \arg[f(z) - w],\\ \beta_C(\zeta) &= \limsup_{t\to\zeta,\, z\in C} \arg[f(z) - w].\end{aligned} \tag{3.2}$$

Note that $\alpha^{\pm}(\zeta)$ and $\beta^{\pm}(\zeta)$ can essentially be determined from the domain G, whereas one must know the function f in order to find $\alpha_C(\zeta)$ and $\beta_C(\zeta)$. When there is no possibility of confusion, we will omit the point ζ and simply write $\alpha^{\pm}$ and $\beta^{\pm}$.

Now we restate Theorem 3.

THEOREM 6. *If $f(\zeta) \neq \infty$ exists, then*

$$\Delta(\zeta) = \max(\beta^+ - \alpha^+,\ \beta^+ - \alpha^- + \pi,\ \beta^- - \alpha^+ - \pi,\ \beta^- - \alpha^-), \tag{3.3}$$

and if f is continuous at ζ then

$$\beta_R - \alpha_R \le \Delta(\zeta) \le 3\pi + \beta_C - \alpha_C, \tag{3.4}$$

where $R = [0, \zeta)$ and C is any curve in $\mathbb{D} \cup \{\zeta\}$ ending at ζ. The constant 3π is best possible.

To see that (2.4) and (3.3) are equivalent, we first observe that

$$\arg(e^{it} - \zeta) \to \begin{cases} \theta + \pi/2 & \text{as } t \to \theta^+,\\ \theta + 3\pi/2 & \text{as } t \to \theta^-.\end{cases} \tag{3.5}$$

Therefore,

$$\limsup_{\eta\to\zeta,\, \eta\in E(\zeta)} \arg\frac{f(\eta) - w}{\eta - \zeta} = \max\left(\beta^+ - \theta - \frac{\pi}{2},\ \beta^- - \theta - \frac{3\pi}{2}\right)$$

and

$$\liminf_{\eta\to\zeta,\, \eta\in E(\zeta)} \arg\frac{f(\eta) - w}{\eta - \zeta} = \min\left(\alpha^+ - \theta - \frac{\pi}{2},\ \alpha^- - \theta - \frac{3\pi}{2}\right).$$

Then the foregoing identities give the equivalence.

PROPOSITION 1. *If f is continuous at ζ, then*

$$\alpha^+(\zeta) \le \alpha_C(\zeta) \le \alpha^-(\zeta) \le \alpha^+(\zeta) + 2\pi, \tag{3.6}$$

$$\beta^-(\zeta) \ge \beta_C(\zeta) \ge \beta^+(\zeta) \ge \beta^-(\zeta) - 2\pi. \tag{3.7}$$

Proof. Let $g = g_\zeta$, where g_ζ is defined by (1.2). By (1.1) we have

$$\operatorname{Re} g(z) = \log|f(z) - w| \to -\infty \quad \text{as } z \to \zeta,\ z \in \mathbb{D}. \tag{3.8}$$

Let us consider a parameterization $p(t)$, $t \in [0, 1]$, of C with $p(1) = \zeta$. For each $b \in (-\infty, \operatorname{Re} g(0))$, let t_b be the last value on the curve C such that $\operatorname{Re} g(p(t_b)) = b$. Let now $L(b)$ be the connected component of $\{s : \operatorname{Re} s = b\} \cap g(\mathbb{D})$ that contains $g(p(t_b))$. Then $g^{-1}(L(b))$ is a crosscut in $\mathbb{D}$. Say that $Q(b)$ is the component of $\mathbb{D} \setminus g^{-1}(L(b))$ for which ζ belongs to its boundary. The definition of t_b gives that $Q(b) \setminus C$ is the union of two domains $U^+(b)$ and $U^-(b)$. We have $R = \{g(p(t)) : t_b < t < 1\} \subset g(\mathbb{D})$ and therefore

$$B = \{g(p(t)) - 2\pi i : t_b \le t < 1\} \subset \mathbb{C} \setminus g(\mathbb{D}),$$

because f is univalent. If B^- denotes the Jordan curve B with the reversed orientation, then

$$J = B^- \cup [g(p(t_b)) - 2\pi i, g(p(t_b))] \cup R \cup \{\infty\}$$

is a positively oriented Jordan curve in $\mathbb{C}_\infty$. Let H^+ be its interior. Since the tangent vector to $g(C)$ at the point $g(p(t_b))$ has negative real part and

$$g(U^+(b)) \cap J = \emptyset,$$

we conclude that

$$g(U^+(b)) \subset H^+.$$

Then $g(e^{it}) \in \overline{H^+}$ for some small interval $t \in (\theta, \theta + \delta)$ with $e^{it} \in E(\zeta)$. Letting $b \to -\infty$, we infer from (3.1) and (3.2) that

$$\begin{aligned} \alpha_C(\zeta) - 2\pi &\le \alpha^+(\zeta) \le \alpha_C(\zeta), \\ \beta_C(\zeta) - 2\pi &\le \beta^+(\zeta) \le \beta_C(\zeta). \end{aligned} \tag{3.9}$$

Similarly, if we consider the Jordan arc $B^- + 4\pi i$ then we have

$$\begin{aligned} \alpha_C(\zeta) &\le \alpha^-(\zeta) \le \alpha_C(\zeta) + 2\pi, \\ \beta_C(\zeta) &\le \beta^-(\zeta) \le \beta_C(\zeta) + 2\pi. \end{aligned} \tag{3.10}$$

If we apply the inequalities (3.9) to curves that are sufficiently tangential to $\{e^{it} : t < \theta\}$ or to $\{e^{it} : t > \theta\}$, we obtain $\alpha^- - 2\pi \le \alpha^+ \le \alpha^-$ and $\beta^- - 2\pi \le \beta^+ \le \beta^-$. This and (3.10) give (3.6) and (3.7). □

It is not difficult to see that the hypothesis of continuity of f at ζ is essential in each inequality of (3.6) and (3.7). Also, the constant 2π is best possible, as will be shown by Example 1. However, if we are considering only Stolz angles, then some improvement of Proposition 1 is possible; see [Wa].

We recall that the function f is isogonal at ζ (see [Po2, p. 80]) if, for some γ,

$$\arg \frac{f(z) - w}{z - \zeta} \to \gamma \quad \text{as } z \to \zeta \text{ in every Stolz angle.}$$

It follows from this definition that, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\{z : 0 < |z - w| < \delta, \ \gamma_1 + \varepsilon < \arg(z - w) < \gamma_2 - \varepsilon\} \subset G, \tag{3.11}$$

with $\gamma_1 = \gamma + \theta + \pi/2$ and $\gamma_2 = \gamma_1 + \pi$.

PROPOSITION 2. *If f is continuous and isogonal at ζ, then $\alpha^-(\zeta) = \beta^+(\zeta) + \pi$.*

Proof. Since f is continuous at ζ, it follows from (3.11) (for $\varepsilon \to 0$) that

$$\beta^+(\zeta) \le \gamma_1, \qquad \alpha^-(\zeta) \ge \gamma_2.$$

Then $\alpha^- - \beta^+ \ge \gamma_2 - \gamma_1 = \pi$. However, by Ostrowski's theorem [Po2, p. 252], the domain does not contain any sector of vertex $f(\zeta)$ of angle larger than π. Hence $\alpha^- - \beta^+ = \pi$. □

It is interesting to point out that the inequalities $\alpha^- \le \alpha^+ + 2\pi$ and $\beta^- \le \beta^+ + 2\pi$ of Proposition 1 cannot be improved even in the case when ζ is an isogonal point, as Example 1 will show. On the other hand, if ∂G has a tangent at $f(\zeta)$ then one has $\alpha^+ = \beta^+$, $\alpha^- = \beta^-$, $\beta^- = \beta^+ + \pi$.

EXAMPLE 1. Put $D_1 = \{ z : \operatorname{Re} z > 0,\ |z| < 2 \}$. For each natural number $n \ge 1$, consider

$$C_n = \begin{cases} \{ z : z = re^{is},\ \frac{\pi}{2} \le s < \frac{3\pi}{2},\ \frac{1}{n} < r < \frac{1}{n} + \frac{1}{3^n} \} & \text{if } n \text{ is an odd number,} \\ \{ z : z = re^{is},\ \frac{\pi}{2} < s \le \frac{3\pi}{2},\ \frac{1}{n} - \frac{1}{3^n} < r < \frac{1}{n} \} & \text{if } n \text{ is an even number.} \end{cases}$$

Now $G = D_1 \cup \bigcup_{n\ge1} C_n$ is a simply connected domain. Let f a conformal map from $\mathbb{D}$ onto G with $f(1) = 0$. Because, at point 1, we have $\beta^- = 3\pi/2$, $\alpha^- = \pi/2$, $\beta^+ = -\pi/2$, and $\alpha^+ = -3\pi/2$, it follows that $\alpha^- = \alpha^+ + 2\pi$ and $\beta^- = \beta^+ + 2\pi$. The function f is continuous at 1 and so the fact that 1 is isogonal can be proved, with a little effort, by means of Ostrowski's theorem (mentioned in the proof of Proposition 2).

COROLLARY 1. *Let f map $\mathbb{D}$ conformally onto G, and assume that G is starlike with respect to $f(0) = 0$. If $\zeta \in \mathbf{T}$ and $f(\zeta) \ne \infty$, then*

$$\Delta(\zeta) \le \pi.$$

Proof. We write $\beta_1(t) = \lim_{r\to1} \arg f(re^{it})$. Then $\beta_1(t)$ exists for all t and β_1 is an increasing function [Po, p. 66]. Therefore,

$$\beta_1 \le \alpha^+ \le \beta^+ \le \beta_1 + \pi$$

and

$$\beta_1 + \pi \le \alpha^- \le \beta^- \le \beta_1 + 2\pi.$$

Hence, by (3.3), we obtain $\Delta(\zeta) \le \pi$. □

4. Proofs of Theorem 3 and Theorem 2

Proof of Theorem 3. (a) First we recall [Po2, p. 52] that

$$\log(f(z) - a) = \log|f(0) - a| + \frac{i}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \arg(f(e^{it}) - a)\, dt$$

if f maps $\mathbb{D}$ conformally onto G and $a \notin G$. Therefore, if we apply this formula to the functions f and $g(z) = z$ with the point $a = f(\zeta)$, $\zeta \in \mathbf{T}$, and afterwards take its imaginary part, we obtain the Poisson representation

$$\arg \frac{f(z)-f(\zeta)}{z-\zeta} = \frac{1}{2\pi}\int_{E(\zeta)} \frac{1-|z|^2}{|e^{it}-z|^2} \arg \frac{f(e^{it})-f(\zeta)}{e^{it}-\zeta}\, dt, \quad z \in \mathbb{D}. \tag{4.1}$$

Now the proof is a consequence of a general fact about Poisson integrals. For the sake of completeness we will present it. Let $q(z) = \operatorname{Im} h(z)$, $z \in \mathbb{D} \cup E(\zeta)$ with h the function defined in (1.4), and assume that $\beta = \limsup_{\eta\to\zeta,\, \eta\in E(\zeta)} q(\eta)$ is finite. Given $\varepsilon > 0$, there exists $0 < \delta < 1$ such that

$$q(\eta) < \beta + \varepsilon \quad \text{for all } \ \eta \in E(\zeta), \ \ |\eta - \zeta| < \delta.$$

Now we decompose the integral in (4.1) as the sum of two integrals over the sets $\eta \in E(\zeta)$, $|\eta - \zeta| > \delta$, and $\eta \in E(\zeta)$, $|\eta - \zeta| \le \delta$, respectively. Estimates of each term give

$$q(z) \le \frac{(1-|z|^2)}{\delta^2} c + \beta + \varepsilon \quad \text{for } \ z \in \mathbb{D} \ \text{ and } \ |z-\zeta| < \frac{\delta}{2}. \tag{4.2}$$

Hence (4.2) implies that

$$\limsup_{z\to\zeta,\, z\in\mathbb{D}} q(z) \le \beta. \tag{4.3}$$

The reverse inequality in (4.3) is almost trivial because $q(\xi)$ is the radial limit of $q(r\xi)$ as $r \to 1$ in each point $\xi \in E(\zeta)$.

An analogous argument gives

$$\liminf_{z\to\zeta,\, z\in\mathbb{D}} q(z) = \liminf_{\eta\to\zeta,\, \eta\in E(\zeta)} q(\eta),$$

so (2.4) holds.

(b) If f is continuous at ζ then we obtain from (3.3) and Proposition 1 that

$$\begin{aligned} \Delta(\zeta) &\le \max[\beta_C - (\alpha_C - 2\pi),\ \beta_C - \alpha_C + \pi,\ \beta_C + 2\pi - (\alpha_C - 2\pi) - \pi, \\ &\qquad \beta_C + 2\pi - \alpha_C] \\ &= \beta_C - \alpha_C + 3\pi, \end{aligned}$$

which proves the right-hand inequality (3.4) and thus assertion (2.5). The left-hand inequality (3.4) is immediate from (2.2).

(c) Finally, we construct an example where equality on the right side of (2.5) holds for $C = [0, \zeta)$.

Example 2. Consider the parabolas $P^{\pm} = \{t \pm it^2 : 0 \le t < +\infty\}$ and let H be the domain between P^+ and P^-. Let $0 < \varepsilon_n < 2^{-n}$ and $A_n = \{w : 2^{-n} < |w| < 2^{-n} + \varepsilon_n\}$. Then

$$G = H \cup \bigcup_{n \text{ even}} [(A_n \setminus \bar{H}) \cup (P^- \cap A_n)] \cup \bigcup_{n \text{ odd}} [(A_n \setminus \bar{H}) \cup (P^+ \cap A_n)]$$

is a simply connected domain. It has a cusp at 0 and moreover narrow annular corridors whose entrances accumulate at 0 and that go almost completely around 0 in the positive (n even) and negative (n odd) directions. Let f map $\mathbb{D}$ conformally onto G such that $f(1) = 0$. By (2.4) and (3.5), our construction gives

$$\Delta(1) = \left(2\pi - \frac{3\pi}{2}\right) - \left(-2\pi - \frac{\pi}{2}\right) = 3\pi.$$

Now $\{ f(r) : 0 < r < 1 \}$ has, up to multiplicative bounds, the same distance from the two parts $\{ f(e^{it}) : 0 < t < \pi \}$ and $\{ f(e^{it}) : -\pi < t < 0 \}$ of the boundary (see e.g. [PoRo, Thm. 3.3]). If ε_n tends to zero rapidly enough, it follows that $f([0,1))$ cannot cut infinitely many segments $P^- \cap A_n$ or $P^+ \cap A_n$. So $\{ f(r) : 1-\varepsilon < r < 1 \} \subset H$ for some $\varepsilon > 0$. Thus $\arg f(r) \to 0$ as $r \to 1$. □

As in Proposition 1, the hypothesis of continuity of f at ζ is essential in (2.5). To prove that we make a little modification of Example 2; we will sketch the idea. Consider $P^{\pm}(1) = \{ t \pm it^2 : 0 \le t < 1 \}$ and let H be the domain between $P^+(1)$ and $P^-(1)$. The domain G is defined as before, but now A_n is always a narrow corridor that starts in $\{ w : 2^{-n} < |w| < 2^{-n} + \varepsilon_n \} \cap P^+(1)$ with $A_n \subset \{ z : |z| \le 2 \}$ and that goes around H at least n times in the positive direction. In this situation one actually has $\alpha^- = 0$ and $\beta^- = +\infty$, and the radial oscillation is zero.

Proof of Theorem 2. By (3.5) and by our hypothesis (see (2.3)), we have

$$\arg \frac{f(e^{it}) - f(\zeta)}{e^{it} - \zeta} \to \beta - \theta - \pi/2 \quad \text{as } e^{it} \to \zeta,\ e^{it} \in E(\zeta).$$

Now the representation formula (4.1) and the properties of the Poisson kernel give that $\lim \arg \frac{f(z)-f(\zeta)}{z-\zeta}$ exists when $z \to \zeta$ $(z \in \mathbb{D})$; hence $\Delta(\zeta) = 0$.

Conversely, if $\Delta(\zeta) = 0$ then Theorem 3 implies that there exists

$$\lim \arg \frac{f(e^{it}) - f(\zeta)}{e^{it} - \zeta} = \alpha \quad \text{as } e^{it} \to \zeta,\ e^{it} \in E(\zeta).$$

Now considering $t \to \theta^{\pm}$ and (3.5), we conclude that

$$\alpha^+ = \beta^+, \quad \alpha^- = \beta^-, \quad \alpha = \beta^+ - (\theta + \pi/2) = \beta^- - (\theta + 3\pi/2).$$

From this it follows that $\beta^- = \beta^+ + \pi$. Hence (see (2.3)), ∂G has a tangent at $f(\zeta)$. □

5. Proof of Theorem 4

We say that f is "twisting" at $\zeta \in \mathbf{T}$ [Po2, p. 141] if $\alpha_R(\zeta) = -\infty$ and $\beta_R(\zeta) = +\infty$, which implies that $\Delta(\zeta) = \infty$ (recall that $\alpha_R = \alpha_{[0,\zeta]}$ in (3.2)). The McMillan twist theorem states that f is isogonal or twisting at almost all $\zeta \in \mathbf{T}$. Hence we may assume that f is isogonal at ζ. Let Λ denote the linear measure in $\mathbb{C}$ [Po2, p. 129].

(i) First we consider the case $0 < \Delta(\zeta) < \pi$. It follows from (1.5) and (3.5) that

$$\operatorname*{osc}_{z\to\zeta,\, z\in\mathbb{D}} \arg[f(z) - f(\zeta)] \le \pi + \Delta(\zeta) < 2\pi. \tag{5.1}$$

Hence Theorem 1 shows that f is continuous at ζ with at most countably many exceptions. Thus it suffices to show that $\Lambda(L) = 0$, where

$$L = \{ \zeta \in \mathbf{T} : f \text{ is isogonal and continuous at } \zeta \text{ and } 0 < \Delta(\zeta) < \pi \}.$$

If E is an arbitrary subset of $\mathbb{C}$ and $w \in E$, let $T_E(w)$ denote the union of all rays $\{ w + te^{i\lambda} : 0 \le t < \infty \}$ such that there exist $w_n \in E$ with $w_n \to w$ and $\arg(w_n - w) \to \lambda$ as $n \to \infty$. The Kolmogoroff–Verčenko theorem (see [Sa, Chap. 9; Po2, p. 127]) states that

$$E = E_0 \cup \{ w \in E : T_E(w) \text{ is } \mathbb{C} \text{ or a half-plane or a full line} \},$$

where $\Lambda(E_0) = 0$.

Let (I_n) be the countable collection of all different intervals $\{ e^{it} : q < t < q' \}$ with q, q' rational numbers. Fix $n \ge 1$ and put

$$B_n = \bigcap_{r<1} \overline{\bigcup_{r<s<1} \{ f(se^{it}) : t \in I_n \}}. \tag{5.2}$$

Now we apply the Kolmogoroff–Verčenko theorem to B_n and obtain

$$\begin{gathered} B_n = E_n \cup \{ w \in B_n : T_{B_n}(w) \text{ is } \mathbb{C} \text{ or a half-plane or a full line} \}, \\ \Lambda(E_n) = 0. \end{gathered} \tag{5.3}$$

Let $\zeta = e^{i\theta}$ for $\zeta \in L$. The definition of L, together with (5.1) and (3.11), tells us that there exists a $\delta > 0$ such that

$$f(e^{it}) \in S^{\pm}(\zeta) \quad \text{for almost all } \theta < t < \theta + \delta \text{ or } \theta - \delta < t < \theta, \tag{5.4}$$

where $S^{\pm}(\zeta)$ are two open disjoint sectors with vertex $f(\zeta)$. Choose $n(\zeta)$ such that

$$\zeta \in I_{n(\zeta)} \subset \{ e^{it} : |t - \theta| < \delta \}.$$

By (5.2), (5.4), and the density on G of the set of radial limits, one has

$$B_{n(\zeta)} \subset S^{+}(\zeta) \cup S^{+}(\zeta).$$

Because $S^{+}(\zeta) \cap S^{+}(\zeta) = \emptyset$, we obtain that $T_{B_{n(\zeta)}}(f(\zeta))$ can neither be the full plane nor a half-plane. If $T_{B_{n(\zeta)}}(f(\zeta))$ were a full line then the continuity of f at ζ would imply that $\Delta(\zeta) = 0$, in contradiction to the definition of L. So by (5.3) one has

$$f(\zeta) \in E_{n(\zeta)}.$$

Thus we have seen that $f(L) \subset E$, where $E = \bigcup_{n=1}^{\infty} E_n$. But (5.3) implies

$$0 \le \Lambda(f(L)) \le \Lambda(E) = 0.$$

Since f is isogonal at each point $\zeta \in \mathbf{T}$, we conclude (cf. [Mc; Po2, p. 146]) that $\Lambda(L) = 0$.

(ii) Now we consider the case that $2\pi < \Delta(\zeta) < \infty$ for $\zeta \in \mathbf{T}$, where f is continuous at ζ. Since f is isogonal at ζ, we obtain from Proposition 2 that $\alpha^- = \beta^+ + \pi$ and from Proposition 1 that

$$\beta^+ - \alpha^+ = \alpha^- - \pi - \alpha^+ \le \pi, \qquad \beta^+ - \alpha^- + \pi = 0,$$

$$\beta^- - \alpha^+ - \pi \le \beta^+ - \alpha^- + 3\pi = 2\pi, \qquad \beta^- - \alpha^- = \beta^- - \beta^+ - \pi \le \pi.$$

Hence it follows from (3.3) that $\Delta(\zeta) \le 2\pi$, which is a contradiction with our hypothesis. $\square$

6. Proof of Theorem 5

Let $k = 2, 3, \ldots$ and $J_k = [3^{-k}, 3 \cdot 3^{-k}]$. We write $\lambda_k = (k-1)/(2k)$ and, for simplicity, $\ell_k = 2/3^k$. In each interval J_k we construct a Cantor set B_k as follows.

We delete the central open interval of length ℓ_k/k. In each of the two remaining closed intervals of length $\ell_k\lambda_k$, we delete the central open interval of length $\ell_k\lambda_k/k$. Then we obtain the union of four intervals of length $\ell_k\lambda_k^2$. In each of them we delete the open central interval of length $\ell_k\lambda_k^2/k$, and so on. Then let B_k be the intersection of all sets obtained this way. The Hausdorff dimension (see [Fa, p. 15]) satisfies

$$\dim B_k = \frac{\log 2}{\log(1/\lambda_k)} = \frac{\log 2}{\log 2 + \log[k/(k-1)]}. \tag{6.1}$$

The intersection of B_k with B_{k+1} is one point, and all the B_k lie on the lower boundary of $H = \{z : |\operatorname{Re} z| < 1,\ 0 < \operatorname{Im} z < 1\}$. At each interval of length $\ell_k\lambda_k^n/k$ with $n \geq 0$ deleted from J_k, we attach downwards an isosceles open triangle of height $\ell_k\lambda_k^n/2$. These triangles (for all generations n and for all k) together with H form a Jordan domain G. Let f and h be the continuous extensions to $\bar{\mathbb{D}}$ of the conformal maps from $\mathbb{D}$ onto G and H, respectively.

We define $B = \bigcup_k B_k$ and $A = f^{-1}(B)$. The function $g = f^{-1} \circ h$ maps $\bar{\mathbb{D}}$ into itself and such that $g(h^{-1}(B)) = A \subset \mathbf{T}$. Applying a theorem of Hamilton [Ha; Po2, p. 235] to g, we derive that $\dim g(h^{-1}(B)) \geq \dim h^{-1}(B)$. Hence

$$\dim A \geq \dim h^{-1}(B) = \dim B = \sup_k \dim B_k = 1. \tag{6.2}$$

The first equality in (6.2) is true because h is analytic on $h^{-1}(B)$, and the last one follows from (6.1).

Consider now $A_1 \subset A$, the set of points whose images by f are not extreme points of some of the deleted intervals involved in the construction of the sets B_k. Let $\zeta \in A_1$. Then $p = f(\zeta) \in B_k$ for some k. For each generation, p lies in one of the remaining intervals I of length $\ell_k\lambda_k^n$. To the right of I there is an adjacent deleted interval I' of length $\ell_k\lambda_k^m/k$ with $m < n$. The triangle attached to I' has height $3^{-k}\lambda_k^m$. Hence, there are points $w \in \partial G$ near and to the right of p with $\arg(w - p) \leq -\pi/4$. On the other hand, $\arg(w-p) \geq -\arctan k$ for all $w \in \partial G$ near and to the right of p, so

$$-\arctan k \leq \alpha^+(\zeta) \leq -\pi/4.$$

It easy to see that $\beta^+(\zeta) = 0$. Similarly, we have $\alpha^-(\zeta) = \pi$ and $5\pi/4 \leq \beta^-(\zeta) \leq \pi + \arctan k$. Hence we obtain from (3.3) that

$$\pi/2 \leq \Delta(\zeta) \leq 2\arctan k < \pi \quad \text{for all } \zeta \in A_1,$$

and (2.6) follows from (6.2). □

7. Topological Results Related to Theorem 1

We assume throughout this section that f maps $\mathbb{D}$ conformally onto G. We shall only consider $\zeta \in \mathbf{T}$ for which the angular limit $f(\zeta) \neq \infty$ exists.

We consider, for each $\delta > 0$, the sets $U_\delta(\zeta) = \{ z \in \mathbb{D} : |z - \zeta| < \delta \}$ and the one-sided neighborhoods

$$U_\delta^+(\zeta) = \{ z \in U_\delta(\zeta) : \arg z > \arg \zeta \},$$

$$U_\delta^-(\zeta) = \{ z \in U_\delta(\zeta) : \arg z < \arg \zeta \}.$$

The one-sided cluster sets at ζ are defined by

$$C^\pm(f, \zeta) = \bigcap_{\delta > 0} \overline{f(U_\delta^\pm(\zeta))} \subset \partial G \tag{7.1}$$

and the total cluster set by $C(f, \zeta) = C^+(f, \zeta) \cup C^-(f, \zeta)$. The prime end $\hat{f}(\zeta)$ of G is called symmetric if $C^+(f, \zeta) = C^-(f, \zeta)$. The Collingwood symmetry theorem (see [Po2, p. 38]) states that there are at most countably many nonsymmetric prime ends. The symmetry of prime ends has some curious consequences.

THEOREM 7. *Let f map $\mathbb{D}$ conformally onto G, let $\zeta, \zeta' \in \mathbf{T}$ $(\zeta \neq \zeta')$, and let E be a continuum. Assume that:*

(i) *the angular limits $f(\zeta)$, $f(\zeta')$ exist and $f(\zeta) \in E$, $f(\zeta') \in E$;*
(ii) *there exists a neighborhood V of ζ such that $f(V) \cap E = \emptyset$; and*
(iii) *the prime end $\hat{f}(\zeta)$ is symmetric.*

Then $C(f, \zeta) \subset E \cap \partial G$.

Proof. We may assume that $V = \{ z \in \mathbb{D} : |z - \zeta| < \rho \}$ and $\zeta' \notin \bar{V}$. Let $z_0 = (1 - \rho)\zeta$ and $\Gamma_0 = [z_0, \zeta)$. Now take a circular arc Γ from z_0 to ζ' such that $\Gamma \subset \mathbb{D} \setminus V$. By (i) and (ii) we have

$$f(z_0) \notin E, \qquad f(\zeta') \in E.$$

Hence there exists a first point $z_1 \in \Gamma$ where $f(\Gamma)$ meets E (it is possible that $z_1 = \zeta'$). We consider the open arc Γ_1 between z_0 and z_1. Then the Jordan arc $C = C_0 \cup C_1$, where $C_0 = f(\Gamma_0)$ and $C_1 = f(\Gamma_1)$, satisfies $C \cap E = \emptyset$ and $\bar{C} \cap E = \{f(\zeta), f(z_1)\}$. The components of $\mathbb{C} \setminus E$ are simply connected domains. By (ii) there exists a component H of $\mathbb{C} \setminus E$ such that $f(V) \subset H$. The Jordan arc C lies in H, so C is a crosscut of H and thus $H \setminus C$ has exactly two components H^+ and H^-. If we write

$$V^+ = \{ z \in U_\rho(\zeta) : \arg z > \arg \zeta \},$$

$$V^- = \{ z \in U_\rho(\zeta) : \arg z < \arg \zeta \},$$

then we may assume that

$$f(V^\pm) \cap H^\pm \neq \emptyset.$$

Since f is univalent, we have

$$C_0 \cap f(V^\pm) = \emptyset, \quad C_1 \cap f(V^\pm) = \emptyset, \quad E \cap f(V^\pm) = \emptyset$$

by (ii). Therefore,

$$(E \cup C) \cap f(V^\pm) = \emptyset$$

and so $f(V^\pm) \subset H^\pm$, which implies

$$C^{\pm}(f,\zeta) \subset \overline{f(V^{\pm})} \cap \partial G \subset \overline{H^{\pm}} \cap \partial G.$$

Using (iii) and the fact that $C \cap \partial G = \emptyset$, we obtain

$$\begin{aligned} C(f,\zeta) &= C^{+}(f,\zeta) \cap C^{-}(f,\zeta) \subset (\overline{H^{+}} \cap \overline{H^{-}}) \cap \partial G \subset (C \cup E) \cap \partial G \\ &= (C \cap \partial G) \cup (E \cap \partial G) = E \cap \partial G. \end{aligned}$$ □

COROLLARY 2. *Suppose that $\hat{f}(\zeta)$ is symmetric and that f is not continuous at ζ. Let H be a domain and A a Jordan arc such that $A \setminus \{w\} \subset H$ and that A begins at $w = f(\zeta)$. If $f(U_\delta(\zeta)) \subset \mathbb{C} \setminus H$ for some $\delta > 0$ then*

$$f(\mathbb{D}) = G \subset \mathbb{C} \setminus H.$$

Proof. Suppose that $f(U_\delta(\zeta)) \subset \mathbb{C} \setminus H$ but there exists $z \in \mathbb{D}$ with $f(z) \in H$. We may assume that A ends at $f(z)$. Let w_1 be the last point where A meets $\mathbb{C} \setminus f(\mathbb{D})$. By [Po2, p. 29] we know that $w_1 = f(\zeta')$ for some $\zeta' \neq \zeta$; it is possible that $w_1 = w$. We consider the subarc A_1 of A between w and w_1. Now $E = A_1$ satisfies all hypotheses of Theorem 7, so $C(f,\zeta) \subset E \cap \partial G \cap f(U_\delta(\zeta)) = \{w\}$ and f is continuous [Po2, p. 35] at ζ, which contradicts our assumption. □

Let H be a connected component of $\mathbb{C} \setminus G$. The angular limit $w = f(\zeta)$ is called a *transition point* of G with respect to H if there exists a Jordan arc A that begins at w and such that $A \setminus \{w\} \subset H$. It is easy to see that is equivalent to saying there exists a continuum K (i.e., a compact connected set with more than one point) with $K \subset H \cup \{w\}$.

A point $b \in \partial G$ is *accessible* from G if there exists a Jordan arc L that lies in G except for the endpoint b. We shall now study the relationship between the concepts of transition point, continuity point, and symmetric prime end.

PROPOSITION 3. *Let f map $\mathbb{D}$ conformally onto G, and let $w = f(\zeta)$ be a transition point of G with respect to H. Let*

$$\tilde{H} = \{ b \in \partial G : \text{ there exists a continuum } F \subset \mathbb{C} \setminus G,\ \partial H \not\subset F, \text{ and } b, w \in F \}.$$

Assume that f is not continuous at ζ and that $\hat{f}(\zeta)$ is symmetric. Then

$$\partial H \cup \tilde{H} \subset C(f,\zeta). \tag{7.2}$$

Proof. First we will show that $\partial H \subset C(f,\zeta)$. Assume this inclusion is not true; then there exists $b \in \partial H$ with $b \notin C(f,\zeta)$. Choose a disc V of center b such that $V \cap C(f,\zeta) = \emptyset$. Since $b \in \partial H \subset \partial G$, we can select two points

$$c \in G \cap V \quad \text{and} \quad c' \in H \cap V.$$

Let $A \subset H$ be the arc of transition and let $a \neq w$ be the other endpoint of A. Consider an arc $L \subset H$ joining a with c'. Now take c'' as the first point of $[c, c'] \in V$ that meets G^c. The set $E = A \cup L \cup [c'', c']$ allows us to apply Theorem 7, and we conclude that

$$C(f,\zeta) \subset E \cap V^c \cap \partial G \subset \{w\},$$

in contradiction to the hypothesis that f is not continuous at ζ. Thus $\partial H \subset C(f,\zeta)$.

Assume now that there exists a $b \in \tilde{H}$ and that $b \notin C(f, \zeta)$. Consider the continuum $E = F \cup [b, c'']$, where F is as in the definition of $\tilde{H}$ and where c'' is chosen as before. It follows from Theorem 7 that

$$C(f, \zeta) \subset (F \cap C(f, \zeta)) \cup ([c'', b] \cap C(f, \zeta)) \subset F.$$

But this contradicts the facts that $\partial H \subset C(f, \zeta)$, $b \in \tilde{H}$, and $\partial H \not\subseteq F$, so $\tilde{H} \subset C(f, \zeta)$ and therefore (7.2) is proved. $\square$

We have the following consequence.

COROLLARY 3. *Except for at most countable many points $\zeta \in \mathbf{T}$, if $f(\zeta)$ is a transition point then f is continuous at ζ.*

Proof. By the Collingwood symmetry theorem and the fact that there are only countably many components H of $\mathbb{C} \setminus G$, it is enough to show that in each component H there is at most a transition point $w = f(\zeta)$, with $\hat{f}(\zeta)$ symmetric and f not continuous at ζ, where for this point ζ the set $f^{-1}(\zeta)$ is a singleton. Assume that $w = f(\zeta)$ and $w' = f(\zeta')$, $w \neq w'$, are two transition points with respect to H. Then let B be an arc joining the endpoints of the corresponding arcs A, A' and put $E = A \cup B \cup A'$. We can apply Theorem 7 to conclude that f must be continuous at ζ and ζ'. If $w = w'$ and $\zeta \neq \zeta'$, we can take $E = \{w\}$ to infer that f would be continuous at ζ; hence $\zeta = \zeta'$. $\square$

Proof of Theorem 1. By the Collingwood symmetry theorem, we may assume that $\hat{f}(\zeta)$ is symmetric. It follows from (2.1) that there is a sector $H(\zeta)$ of vertex $f(\zeta)$ with $f(U_\delta(\zeta)) \subset \mathbb{C} \setminus H(\zeta)$ for some $\delta = \delta(\zeta) > 0$. If f is continuous at ζ then we are finished. Otherwise, $G \subset \mathbb{C} \setminus H(\zeta)$ by Corollary 2. We conclude that $f(\zeta)$ is a transition point (with A the midline of $H(\zeta)$), and Corollary 3 implies that f is continuous at ζ with at most countably many exceptions.

In order to show that the constant 2π is best possible we present an example in which there are uncountable many points where the oscillation (2.1) equals 2π and where the function is not continuous. Let K be the usual Cantor set and let Q be the square $\{x + iy : 0 < x < 1, 0 < y < 2\}$. Now take any conformal map f from $\mathbb{D}$ onto the simply connected domain $Q \setminus (K + i[0, 1])$. For each point $w = a + i$ $(a \in K)$ there exists a point $\zeta \in \mathbf{T}$ such that $f(\zeta) = w$ [Po2, p. 29]; let us denote by E the set of such points. Now we have finished because the set E is uncountable and in each point $\zeta \in E$ one has

$$\liminf_{z \to \zeta} \operatorname{Im} f(z) = 0 \quad \text{and} \quad \operatorname*{osc}_{z \to \zeta,\, z \in \mathbb{D}} \arg[f(z) - f(\zeta)] = 2\pi. \qquad \square$$

8. Additional Topological Results

In the proof of Corollary 3, we have seen what occurs if there are two transition points with respect to the same component H. It is interesting to study what happens if we relax this hypothesis and assume that one of those points is accessible only from G. Before stating the next result, we will need the following definition:

A continuum E is *indecomposable* if it cannot be written as the union of two proper subcontinua (see [Na, pp. 7–14] for more information).

THEOREM 8. *Let f map $\mathbb{D}$ conformally onto G, and let $\zeta \in \mathbf{T}$. Assume that*:

(a) *the radial limit $f(\zeta) = w$ exists and is a transition point with respect to the component H of $\mathbb{C} \setminus \bar{G}$*;
(b) *the prime end $\hat{f}(\zeta)$ is symmetric*;
(c) *f is not continuous at ζ; and*
(d) *there exists a point $w' \in \partial H$ $(w' \neq w)$ that is accessible from G.*

Then $C(f, \zeta) = \partial H$ and ∂H is indecomposable.

Proof. We already know that

$$\partial H \subset C(f, \zeta). \tag{8.1}$$

Assume that ∂H is decomposable; then there exist continua A, B such that

$$\partial H = A \cup B \quad (A \neq \partial H,\ B \neq \partial H, \text{ and } w' \in A). \tag{8.2}$$

First we will prove the following.

Claim: If $b \in \partial H$ is accessible from H then $b \in B$.

In order to prove the claim, assume that $b \notin B$; thus $b \in A$ by (8.2). There exists a curve $C \subset H \cup \{w, b\}$ joining w and b. Then $E = A \cup C \subset \mathbb{C} \setminus G$ is a continuum that contains two accessible points w and w', so Theorem 7 entails

$$C(f, \zeta) \subset E \cap \partial G = (A \cap \partial G) \cup (C \cap \partial G) = A \cup \{w\}. \tag{8.3}$$

Recall that $w \in C(f, \zeta)$. Therefore, if $w \notin A$ then (8.3) implies $C(f, \zeta) = \{w\}$, which is impossible by (c). If $w \in A$ then (8.3) and (8.2) imply that

$$C(f, \zeta) \subset A \cup \{w\} = A \subsetneq \partial H,$$

which contradicts (8.1). Therefore $b \in B$ and the claim is proved.

From the claim we see that the set of points accessible from H lies in B. Since this is a dense subset of ∂H, it follows that

$$B = \bar{B} = \partial H,$$

which contradicts (8.2). Therefore ∂H is indecomposable.

Because ∂H is a continuum that contains $f(\zeta)$ and another point $w' \in \partial H$, accessible from G and with $\partial H \subset C(f, \zeta)$, the minimality property of cluster sets (see Proposition 2.22 of [Po2, p. 38]) implies that $\partial H = C(f, \zeta)$. □

It is a natural question to ask whether there exists a conformal map under the hypotheses of the previous theorem. The answer is *yes,* as the following result shows.

PROPOSITION 4. *There exists a continuum E with the following properties*:

(a) *$\mathbb{C} \setminus E = G \cup H$, where G and H are disjoint domains and G is simply connected*;
(b) $\partial G = \partial H = E$;

(c) *E is indecomposable*;
(d) *there is a point $\zeta \in \mathbf{T}$ where the conformal map from $\mathbb{D}$ onto G has a radial limit $f(\zeta) = w$ and*

$$C^{+}(f, \zeta) = C^{-}(f, \zeta) = E;$$

(e) *w is a transition point of f with respect to H.*

Proof. To fix ideas, let us take $a_k = (-1)^{k+1}\pi/2^{k+1}$ and put $z_k = e^{ia_k}$. Also, we consider a sequence of points (w_k) such that, for all $k \geq 1$,

$$|w_k| > 1, \quad \operatorname{Im} w_{2k+1} > \operatorname{Im} w_{2k+3} > 0, \quad \operatorname{Im} w_{2k} < \operatorname{Im} w_{2k+2} < 0$$

with $\lim_{k\to\infty} w_k = 1$.

Write $G_0 = \mathbb{D}$ and $H_0 = \{z : |z| > 1\}$. To start the construction, select $\varepsilon_1 > 0$ such that z_3 does not belong to the arc in $\mathbf{T}$ between $z_1 e^{i\varepsilon_1}$ and $z_1 e^{-i\varepsilon_1}$. Then consider an open strip B_1 with $\overline{B_1} \subset H_0$ of width $2\varepsilon_1$ joining the arc $(z_1 e^{i\varepsilon_1}, z_1 e^{-i\varepsilon_1})$ in $\mathbf{T}$ with the point w_2 and satisfying, moreover,

$$\overline{B_1} \cap [1, \infty) = \emptyset \quad \text{and} \quad d(x, \partial B_1) < 1 \quad \text{for all points } x \in \mathbf{T}.$$

Now we put $G_1 = G_0 \cup B_1$ and $H_1 = \mathbb{C} \setminus \overline{G_1}$. Assume that we have already constructed the domains G_n and H_n. Consider an open strip B_{n+1} with $\overline{B_{n+1}} \subset H_n$ of width $2\varepsilon_{n+1}$, for suitably chosen $\varepsilon_{n+1} < \varepsilon_n$, joining the arc $I_{n+1} = (z_{n+1} e^{i\varepsilon_{n+1}}, z_{n+1} e^{-i\varepsilon_{n+1}})$ with the first point w_k $(k \geq n+2)$ such that $w_k \notin \bigcup_{s\leq n} B_s$ and with the additional restrictions

$$\begin{gathered}\overline{B_{n+1}} \cap [1, \infty) = \emptyset, \quad z_{n+3} \notin I_{n+1} \quad \text{and} \\ d(x, \partial B_{n+1}) < \varepsilon_n \quad \text{for } x \in \partial G_n.\end{gathered} \tag{8.4}$$

We must show that this construction is always possible. One way to see this is to consider a continuous conformal map from $\mathbb{D}$ onto H_n, mapping 0 to ∞, and then apply its uniform continuity.

As before, we denote $G_{n+1} = G_n \cup B_{n+1}$ and $H_n = \mathbb{C} \setminus \overline{G_{n+1}}$. Then the domains are $G = \bigcup_{n=1}^{\infty} G_n$ and $H = \mathbb{C} \setminus \bar{G}$. A conformal map from $\mathbb{D}$ onto G with $f(1) = 1$ gives our example. The point 1 is a transition point, since $[0, 1) \subset G$ and $(1, \infty) \subset H$. The prime end $\hat{f}(1)$ is symmetric and $C(f, 1) = E$ because members of the family of crosscuts $(z_n e^{i\varepsilon_n}, z_n e^{-i\varepsilon_n})$ are alternatively above and below the real axis and, by (8.4), each point of E is an accumulation point in ∂B_n; hence (d) holds. The continuum $E = \partial H = \partial G$ is indecomposable. A direct proof is possible, but it is enough to apply Theorem 8 because E has a dense set of accessible points from the component H. □

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