# Geodesic Conjugacies of Two-Step Nilmanifolds

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## Introduction

Two Riemannian manifolds $(M, g)$ and $(N, h)$ are said to have $C^k$*-conjugate geodesic flows* if there is a $C^k$ diffeomorphism $F: S(M, g) \to S(N, h)$ between their unit tangent bundles that intertwines their geodesic flows. A compact Riemannian manifold is said to be $C^k$*-geodesically rigid* within a given class $\mathcal{M}$ of manifolds if any Riemannian manifold $N$ in $\mathcal{M}$ whose geodesic flow is $C^k$-conjugate to that of $M$ is isometric to $M$.

Weinstein [W] exhibited a zoll surface of nonconstant curvature whose geodesic flow is conjugate to that of the round sphere. More recently, Croke and Kleiner [CK] proved that, on any smooth manifold, there exist infinite-dimensional families of pairwise nonisometric Riemannian metrics with mutually $C^\infty$-conjugate geodesic flows. On the other hand, various manifolds are known to be geodesically rigid—for example, within the class of all Riemannian manifolds, compact surfaces of negative curvature are $C^0$-geodesically rigid ([CFF], see also [C] and [O] for $C^1$-rigidity) and hyperbolic manifolds are $C^1$-geodesically rigid [BCG].

In this article, we will consider questions of geodesic rigidity for the class of compact two-step Riemannian nilmanifolds. A *$k$-step Riemannian nilmanifold* $(\Gamma\backslash N, g)$ is a quotient of a $k$-step nilpotent Lie group $G$ by a (possibly trivial) discrete subgroup $\Gamma$ together with a Riemannian metric $g$ whose lift to $G$, also denoted $g$, is left-invariant. Note that a compact one-step Riemannian nilmanifold is just a flat torus. Thus the compact two-step Riemannian nilmanifolds may be viewed as the simplest generalization of flat tori, yet they have a much richer geometry. This paper builds on work of Eberlein [E], who studied the length spectrum and marked length spectrum of compact two-step Riemannian nilmanifolds and raised the question of whether such manifolds are geodesically rigid. For other work concerning geodesic conjugacies of two-step nilmanifolds, see [BM] and [K].

We are interested in the question of rigidity of geodesic flows in part because of its relationship to spectral geometry. Two Riemannian manifolds are said to be *isospectral* if the associated Laplace–Beltrami operators have the same eigenvalue

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spectrum. A continuous family $M_t$ $(-\varepsilon < t < \varepsilon)$ of Riemannian manifolds is said to be an *isospectral deformation* of $M_0$ if the manifolds are pairwise isospectral. Since the Laplacian can be viewed as the quantum analog of the geodesic flow, one might expect that continuous families of isospectral manifolds would have conjugate geodesic flows. Wilson and the first author [GW] gave a method for constructing isospectral compact Riemannian nilmanifolds. The isospectral manifolds in that construction are of the form $(\Gamma\backslash N, g)$ and $(\Phi(\Gamma)\backslash N, g)$, where $g$ is a fixed left-invariant metric on the nilpotent Lie group $N$ and $\Phi$ is an "almost-inner automorphism" of $G$. (See Section 1 for details.) Continuous families of non-inner almost inner automorphisms $\{\Phi_t\}$ give rise to nontrivial isospectral deformations. Next, Eberlein [E] showed that, if a pair of compact two-step nilmanifolds have $C^0$-conjugate geodesic flows, they must then be isometric to a pair $(\Gamma\backslash N, g)$ and $(\Phi(\Gamma)\backslash N, g)$, where again $\Phi$ is an almost inner automorphism. In particular, compact two-step nilmanifolds with conjugate geodesic flows must be isospectral. The converse is not true, as we shall see. Moreover, there are no examples of nonisometric nilmanifolds whose geodesic flows have been proven to be conjugate.

We will work entirely in the context of compact two-step nilmanifolds. We note at the outset, however, that any compact nilmanifolds $\Gamma\backslash N$ and $\Gamma'\backslash N'$ with conjugate geodesic flows must have the same step size; in fact, the nilpotent groups $N$ and $N'$ must be isomorphic. Indeed, the fundamental group of the unit tangent bundle $S(\Gamma\backslash N)$ is isomorphic to $\Gamma$, so geodesic conjugacy of $\Gamma\backslash N$ and $\Gamma'\backslash N'$ implies that $\Gamma$ and $\Gamma'$ are isomorphic and consequently, by Malcev rigidity, that $N$ and $N'$ are isomorphic. Thus, all geodesic rigidity results concerning compact two-step nilmanifolds will be valid within the space of all compact nilmanifolds.

We first strengthen Eberlein's result by showing that, if a pair of compact two-step nilmanifolds have $C^0$-conjugate geodesic flows, then the associated almost inner automorphism $\Phi$ must satisfy a stringent additional property. We call such a $\Phi$ an almost inner automorphism "of continuous type" (see Definition 1.6). As a corollary, we obtain Theorem 1.

THEOREM 1. *There is a large class of compact two-step Riemannian nilmanifolds each of which is $C^0$-geodesically rigid within the class of all Riemannian nilmanifolds.*

The primary class of two-step nilmanifolds that is not covered by Theorem 1 is the class of so-called "nonsingular" two-step nilmanifolds (see Definition 3.1). Consequently, in Section 3 we focus on the nonsingular two-step nilmanifolds and prove Theorem 2.

THEOREM 2. *Almost all compact nonsingular two-step Riemannian nilmanifolds are $C^2$-geodesically rigid within the class of all compact Riemannian nilmanifolds.*

We also obtain rigidity results for some special classes of two-step nilmanifolds with interesting geometric properties. Perhaps the most extensively studied class of nilmanifolds are the two-step nilmanifolds of Heisenberg type first introduced by Kaplan [Kn] (see Definition 3.2). We then prove Theorem 3.

THEOREM 3. *Compact two-step nilmanifolds of Heisenberg type are $C^0$-geodesically rigid within the class of all compact Riemannian nilmanifolds.*

This article is a companion to [GMS], in which it is shown that any pair of compact two-step nilmanifolds with *symplectically* conjugate geodesic flows are isometric. The proof of that result relies on the results established here in Section 2.

The paper is organized as follows: In Section 1, we review the geometry of two-step nilmanifolds, define the notion of almost inner automorphism of continuous type and give examples, and review Eberlein's result. Section 2 establishes a number of results concerning $C^0$-conjugacies of geodesic flows between arbitrary compact two-step nilmanifolds and culminates in Theorem 1, restated as Corollaries 2.13 and 2.14. Theorem 2 is established in Section 3 along with a rigidity result for a special class of nilmanifolds. Theorem 3 is proved in Section 4.

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## 1. Preliminaries

In this section, we give a brief introduction to the geometry of two-step nilpotent Lie groups. We recommend [E] as a reference.

A Riemannian nilmanifold is a quotient $M = \Gamma\backslash N$ of a simply connected nilpotent Lie group $N$ by a discrete subgroup $\Gamma$, together with a Riemannian metric $g$ whose lift to $N$ is left-invariant. We say the nilmanifold has step size $k$ if $N$ is $k$-step nilpotent. We will be interested only in compact nilmanifolds and in their simply connected covering nilmanifolds $(N, g)$.

1.1. NOTATION AND REMARKS. A Lie group $N$ is said to be *two-step nilpotent* if its Lie algebra $\mathcal{N}$ satisfies $[\mathcal{N}, [\mathcal{N}, \mathcal{N}]] = 0$; equivalently, $[\mathcal{N}, \mathcal{N}]$ is central in $N$. Let $N$ be a two-step nilpotent Lie group and let $\mathcal{N}$ be its Lie algebra. The Lie group exponential map $\exp\colon \mathcal{N} \to N$ is a diffeomorphism; we denote its inverse by $\log$.

The Campbell–Baker–Hausdorff theorem gives the product rule

$$\exp(x)\exp(y) = \exp(x + y + \tfrac{1}{2}[x, y])$$

for all $x, y \in \mathcal{N}$. Conjugation in $N$ is thus given by

$$\exp(x)\exp(y)(\exp(x))^{-1} = \exp(y + [x, y]).$$

Let $g$ be a left-invariant Riemannian metric on $N$. Then $g$ defines an inner product $\langle\cdot,\cdot\rangle$ on the Lie algebra $\mathcal{N}$ of $N$. Let $\mathcal{Z} = [\mathcal{N}, \mathcal{N}]$, and let $\mathcal{V}$ denote the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$ relative to $\langle\cdot,\cdot\rangle$. Note that while $\mathcal{Z}$ is contained in the center of $\mathcal{N}$, it does not necessarily coincide with the full center.

For $z$ in $\mathcal{Z}$, we can define a skew symmetric linear transformation $J(z)\colon \mathcal{V} \to \mathcal{V}$ by $J(z)x = (\mathrm{ad}(x))^* z$ for $x \in \mathcal{V}$, where $(\mathrm{ad}(x))^*$ denotes the adjoint of $\mathrm{ad}(x)$. Equivalently,

$$\langle J(z)x, y\rangle = \langle [x, y], z\rangle \quad \text{for } x, y \in \mathcal{V},\ z \in \mathcal{Z}. \tag{$*$}$$

Conversely, given inner product spaces $\mathcal{V}$ and $\mathcal{Z}$ and a nonsingular linear map $J\colon \mathcal{Z} \to \mathfrak{so}(\mathcal{V})$, we can construct a two-step nilpotent Lie algebra $\mathcal{N}$, together with an inner product, by setting $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ as an inner product space and defining the Lie bracket so that $\mathcal{Z}$ is central and $[\cdot,\cdot]\colon \mathcal{V} \times \mathcal{V} \to \mathcal{Z}$ is given by $(*)$. Since $J$ is nonsingular, we have $\mathcal{Z} = [\mathcal{N}, \mathcal{N}]$. The inner product defines a left-invariant Riemannian metric on the associated simply connected nilpotent Lie group $N$, thus giving $N$ the structure of a two-step Riemannian nilmanifold.

In what follows, for notational convenience we will assume that an inner product is given on all two-step nilpotent Lie algebras considered and thus write $\mathcal{N} = \mathcal{V} + \mathcal{Z}$ as before, although in some situations (e.g., in 1.3 and 1.6) the inner product (in particular, the choice of vector space complement $\mathcal{V}$ of $\mathcal{Z}$) is irrelevant.

1.2. EXAMPLE. The Heisenberg group of dimension $2n+1$ is the simply connected Lie group with Lie algebra $\mathcal{N} = \mathrm{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$, where $[x_i, y_i] = z$, $1 \le i \le n$, and all other brackets of basis elements are zero. Giving $\mathcal{N}$ the inner product for which the foregoing basis is orthonormal, we have $\mathcal{Z} = \mathrm{span}\{z\}$, $\mathcal{V} = \mathrm{span}\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, and $J(z)x_i = y_i$ and $J(z)y_i = -x_i$. Thus $J(z)$ can be viewed as a complex structure on the vector space $\mathcal{V}$.

1.3. UNIFORM DISCRETE SUBGROUPS OF $N$. We recall some basic facts about uniform discrete subgroups of a two-step nilpotent Lie group $N$. See [R] for details. If $\Gamma$ is a uniform discrete subgroup of $N$, then $\log(\Gamma) \cap \mathcal{Z}$ is a lattice of full rank in the derived algebra $\mathcal{Z}$ and $\pi_{\mathcal{V}}(\log\Gamma)$ is a lattice of full rank in $\mathcal{V}$, where $\pi_{\mathcal{V}}\colon \mathcal{N} \to \mathcal{V}$ is the projection with kernel $\mathcal{Z}$. For $x, y \in \log\Gamma$, we have $\exp(x)\exp(y)\exp(-x)\exp(-y) = \exp([x, y]) \in \Gamma$, so $[x, y] \in \log(\Gamma) \cap \mathcal{Z}$. In particular, if we choose a basis of $\mathcal{N}$ consisting of elements of $\log\Gamma$, then the constants of structure are rational. Letting $\mathcal{N}_Q$ denote the rational span of $\log\Gamma$, it follows that $\mathcal{N}_Q$ has the structure of a rational Lie algebra. We will say a subspace of $\mathcal{N}$ is *rational* if it has a basis consisting of elements of $\log\Gamma$. For example, $\mathrm{image}(\mathrm{ad}(x))$ is rational for all $x \in \log\Gamma$.

The concept of almost inner automorphisms of nilpotent Lie groups, defined in [GW] and generalized in [G], plays a key role in the construction of isospectral nilmanifolds. After recalling the definition in 1.4, in 1.6 we define a special class of almost inner automorphisms of two-step nilpotent Lie groups that will arise naturally in our study of geodesic conjugacies.

1.4. DEFINITION. (a) Let $\Gamma$ be a uniform discrete subgroup of a simply connected nilpotent Lie group $N$. An automorphism $\Phi$ of $N$ is said to be $\Gamma$-*almost inner* if $\Phi(\gamma)$ is conjugate to $\gamma$ for all $\gamma \in \Gamma$. The automorphism is said to be *almost inner* if $\Phi(x)$ is conjugate to $x$ for all $x \in N$.

(b) A derivation $\phi$ of the Lie algebra $\mathcal{N}$ is said to be $\Gamma$-almost inner (resp. almost inner) if $\phi(x) \in \mathrm{image}(\mathrm{ad}(x))$ for all $x \in \log \Gamma$ (resp. for all $x \in \mathcal{N}$).

1.5. REMARKS. (See [GW].) (a) The $\Gamma$-almost inner automorphisms and the almost inner automorphisms form connected Lie subgroups of $\mathrm{Aut}(N)$. In many cases, these groups properly contain the group $\mathrm{Inn}(N)$ of inner automorphisms. The spaces of $\Gamma$-almost inner (resp. almost inner) derivations of $\mathcal{N}$ are the Lie algebras of these groups of automorphisms. In particular, if $\phi$ is a ($\Gamma$-)almost inner derivation, then $\phi$ generates a one-parameter group $\Phi_t$ of ($\Gamma$-)almost inner automorphisms of $N$ satisfying $\Phi_{t*} = e^{t\phi}$. Conversely, if $\Phi$ is a ($\Gamma$-)almost inner automorphism of $N$, then $\Phi_* = e^{\phi}$ for some ($\Gamma$-)almost inner derivation of $\mathcal{N}$.

(b) Note that a $\Gamma$-almost inner derivation $\phi$ satisfies $\phi(\mathcal{N}) \subseteq [\mathcal{N}, \mathcal{N}]$ and $\phi(z) = 0$ if $Z$ is central. In particular, if $\mathcal{N}$ is two-step nilpotent, then (letting $\mathcal{Z} = [\mathcal{N}, \mathcal{N}]$ as before) we have $\phi(\mathcal{N}) \subseteq \mathcal{Z}$ and $\phi(\mathcal{Z}) = 0$, so $\phi^2 = 0$. Thus $e^{t\phi} = \mathrm{Id} + t\phi$.

1.6. DEFINITION. Let $\phi$ be an almost inner derivation of a two-step nilpotent Lie algebra $\mathcal{N}$. By Definition 1.4, there exists a map $\xi : \mathcal{N} \to \mathcal{N}$ such that $\phi(x) = [\xi(x), x]$ for all $x \in \mathcal{N}$. For $x \in \mathcal{Z}$, $\xi(x)$ is completely arbitrary; only the values of $\xi$ on $\mathcal{V}$ are of interest. Even on $\mathcal{V}$, the map $\xi$ is not uniquely defined. We will say $\phi$ is *of continuous type* if $\xi$ can be chosen to be continuous on $\mathcal{V} \setminus \{0\}$. We will also say the almost inner automorphisms $\Phi_t$ of $N$ generated by $\phi$ as in 1.5 are of continuous type. We emphasize that this definition is independent of the inner product on $\mathcal{N}$; the space $\mathcal{V}$ can be taken to be any vector space complement of $\mathcal{Z}$ in $\mathcal{N}$.

1.7. EXAMPLE. (i) (See [GM] and [GW].) Let $N$ be the six-dimensional simply connected nilpotent Lie group with Lie algebra

$$\mathcal{N} = \mathrm{span}\{X_1, X_2, Y_1, Y_2, Z_1, Z_2\}$$

satisfying

$$[X_1, Y_1] = [X_2, Y_2] = Z_1, \quad [X_1, Y_2] = Z_2,$$

and with all other brackets of basis vectors trivial.

The almost inner derivations of $\mathcal{N}$ are the linear maps that send $X_2$ and $Y_1$ to multiples of $Z_1$, send $X_1$ and $Y_2$ into $\mathrm{span}\{Z_1, Z_2\}$, and send $Z_1$ and $Z_2$ to zero. These form a six-dimensional subspace containing the inner derivations as a four-dimensional subspace. The derivations $\phi$ (resp. $\psi$) that send $X_1$ (resp. $Y_2$) to $Z_2$ and send all other basis vectors to zero span a two-dimensional space of noninner almost inner derivations. No noninner almost inner derivation of $\mathcal{N}$ is of continuous type. To see this, we examine the derivation $\psi$; the other noninner almost inner derivations behave similarly. For $U = \sum_{i=1,2}(x_i X_i + y_i Y_i + z_i Z_i) \in \mathcal{N}$ with $x_i, y_i, z_i \in \mathbf{R}$, we have $\phi(U) = [\xi(U), U]$ for all $U \in \mathcal{N}$, where $\xi$ must satisfy

$$\xi(U) \equiv X_1 - \frac{y_1}{y_2} X_2 \ \mathrm{mod}\ \mathrm{span}\{Y_1, Y_2, Z_1, Z_2\} \quad \text{if } \ y_2 \neq 0.$$

When $y_2 = 0$, we can take $\xi(U) = 0$. Thus $\phi$ is an almost inner derivation. However, $\xi$ can not be chosen continuously on $\mathcal{V} \setminus \{0\}$ where $\mathcal{V}$ is any complement of the derived algebra $\mathcal{Z} = \operatorname{span}\{Z_1, Z_2\}$. Thus $\phi$ is not of continuous type. See [GM] for further examples of this type.

(ii) If $\mathcal{N}$ is a two-step nilpotent Lie algebra with the property that $\mathcal{Z} = \operatorname{image}(\operatorname{ad}(x))$ for all nonzero $x \in \mathcal{V}$, then every linear map $\phi$ that sends $\mathcal{V}$ into $\mathcal{Z}$ and $\mathcal{Z}$ to $\{0\}$ is an almost inner derivation. One can show that these almost inner derivations are all of continuous type. (The authors would like to thank P. Eberlein for pointing out the latter fact to them.)

The Lie algebras satisfying the property in Example 1.7(ii) are said to be *nonsingular* and will be studied in Section 3. Except on nonsingular two-step nilpotent Lie algebras, examples of noninner almost inner derivations of continuous type appear to be quite rare. The only examples we have been able to construct are on Lie algebras which are almost nonsingular in that $\mathcal{Z} = \operatorname{image}(\operatorname{ad}(x))$ for almost all $x \in \mathcal{V}$, and even in this situation examples seem to be rare. Proposition 1.8 indicates one reason for this apparent scarcity of almost inner derivations of continuous type. In contrast, there are plentiful examples of noninner almost inner derivations (not of continuous type) on singular two-step nilpotent Lie algebras. The reader may find it helpful to compare Proposition 1.8 with Example 1.7(i).

1.8. PROPOSITION. *We use the notation of 1.1. Let $\mathcal{N}$ be a two-step nilpotent Lie algebra with an inner product $\langle \cdot, \cdot \rangle$, and let $\phi$ be an almost inner derivation of continuous type on $\mathcal{N}$, say $\phi(x) \equiv [\xi(x), x]$ with $\xi$ continuous on $\mathcal{V} \setminus \{0\}$. In the notation of 1.1, let $z \in \mathcal{Z}$ and $y \in \ker(J(z))$. Then $\langle \phi(x), z \rangle = \langle [\xi(y), x], z \rangle$ for all $x \in \mathcal{N}$.*

Note in particular that if $\mathcal{V}$ contains a central element $y$ (i.e., if the center of $\mathcal{N}$ properly contains the derived algebra), then Proposition 1.8 states that every almost inner derivation of continuous type on $\mathcal{N}$ is inner. In contrast, central elements in $\mathcal{V}$ are irrelevant in the construction of almost inner derivations that are not of continuous type.

*Proof*. For $y$ and $z$ as in the proposition, for $x \in \mathcal{V}$ and for $t \in \mathbf{R}$ we have

$$\langle \phi(y + tx), z \rangle = \langle [\xi(y + tx), y + tx], z \rangle$$

and thus

$$\langle \phi(y), z \rangle + t\langle \phi(x), z \rangle = \langle [\xi(y + tx), y], z \rangle + t\langle [\xi(y + tx), x], z \rangle. \tag{1.8.1}$$

The first term on the right-hand side of 1.8.1 is zero since $J(z)(y) = 0$, and the first term on the left-hand side is zero since

$$\langle \phi(y), z \rangle = \langle [\xi(y), y], z \rangle = -\langle J(z)(y), \xi(y) \rangle = 0.$$

Consequently, (1.8.1) says that

$$\langle \phi(x), z \rangle = \langle [\xi(y + tx), x], z \rangle$$

for all nonzero $t \in \mathbf{R}$ and all $x \in \mathcal{V}$. We let $t$ approach 0 and use continuity of $\xi$ to conclude the proof. $\square$

Almost inner derivations can be used to construct isospectral deformations of nilmanifolds (of arbitrary step size) as follows.

1.9. Proposition. *Let $(\Gamma\backslash N, g)$ be a compact nilmanifold and let $\Phi$ be a $\Gamma$-almost inner automorphism of N. Then $(\Phi(\Gamma)\backslash N, g)$ is isospectral to $(\Gamma\backslash N, g)$. Conversely, if N is two-step nilpotent and if $(\Gamma_t)_{t\geq 0}$ is a continuous family of discrete subgroups of N such that the family of manifolds $(\Gamma_t\backslash N, g)$ are all isospectral, then there exists a family $\{\Phi_t\}$ of $\Gamma_0$-almost inner automorphisms of N such that $\Gamma_t = \Phi_t(\Gamma_0)$ for all t.*

The first statement is proven in [GW] for almost inner automorphisms and in [G] for $\Gamma$-almost inner automorphisms. The converse is given in [OP] and [P].

1.10. Remark. (See [GW].) If $\Phi$ is an inner automorphism of $N$ (say, $\Phi$ is conjugation by $a \in N$), then $(\Phi(\Gamma)\backslash N, g)$ is isometric to $(\Gamma\backslash N, g)$. The isometry is induced from the left translation $L_a$ of $(N, g)$. However, if $\phi$ is a $\Gamma$-almost inner derivation that is not inner and $\{\Phi_t\}$ is the corresponding family of automorphisms, then the deformation $(\Phi_t(\Gamma)\backslash N, g)$ is nontrivial.

As we will discuss in Section 2, compact nilmanifolds with conjugate geodesic flows must have the same marked length spectrum. After recalling the definition of marked length spectrum, we will review Eberlein's classification of compact two-step nilmanifolds with the same marked length spectrum.

1.11. Notation and Remarks. The fundamental group of a nilmanifold $\Gamma\backslash N$ is isomorphic to $\Gamma$. Recall that free homotopy classes of loops in a manifold correspond to conjugacy classes in the fundamental group. In our situation, we will denote both the conjugacy class of an element $\gamma$ in $\Gamma$ and the associated free homotopy class of loops by $[\gamma]_\Gamma$. A unit speed closed geodesic in $\Gamma\backslash N$ of length $l$ lies in the free homotopy class $[\gamma]_\Gamma$ if and only if it lifts to a geodesic $\sigma$ in $N$ satisfying $\sigma(t+l) = \gamma\sigma(t)$ for all $t \in \mathbf{R}$.

1.12. Definition. Two Riemannian manifolds $M_1$ and $M_2$ are said to have the same *marked length spectrum* if there exists an isomorphism (called a *marking*) $\Phi\colon \pi_1(M_1) \to \pi_1(M_2)$ such that, for each $\gamma \in \pi_1(M_1)$, the collection of lengths (counting multiplicities) of closed geodesics in the free homotopy class $[\gamma]$ of $M_1$ coincides with the analogous collection in the free homotopy class $[\Phi(\gamma)]$ of $M_2$.

1.13. Proposition [E]. *Two compact two-step nilmanifolds $(\Gamma\backslash N, g)$ and $(\Gamma^*\backslash N^*, g^*)$ have the same marked length spectrum if and only if there exists a $\Gamma$-almost inner automorphism $\Phi$ of N such that $(\Gamma^*\backslash N^*, g^*)$ is isometric to $(\Phi(\Gamma)\backslash N, g)$. In this case, any marking between the length spectra of $(\Gamma\backslash N, g)$ and $(\Gamma^*\backslash N^*, g^*)$ is of the form $\Psi_*\circ\Phi|_\Gamma$, where $\Psi\colon (\Phi(\Gamma)\backslash N, g) \to (\Gamma^*\backslash N^*, g^*)$ is an isometry and $\Psi_*\colon \Phi(\Gamma) \to \Gamma^*$ is the induced map on fundamental groups.*

1.14. Remarks. The "if" statement in Proposition 1.13 is elementary and is valid for compact nilmanifolds of arbitrary step size, but the converse requires a careful study of the geodesics and is, in fact, not valid for higher-step nilmanifolds, as examples of Gornet [Gt1; Gt2] illustrate. The paper [Gt2] even gives continuous families of compact higher-step nilmanifolds which have the same marked length spectrum but which are not related by almost inner automorphisms in the sense of Proposition 1.13.

In contrast to Proposition 1.13, the second statement in Proposition 1.9 is valid only for continuous families, as opposed to pairs, of isospectral two-step nilmanifolds. Indeed, there exist many examples of pairs of isospectral two-step nilmanifolds that are not related by almost inner automorphisms, including many examples with nonisomorphic fundamental groups.

## 2. $C^0$-Geodesic Conjugacies

Throughout this section, $(N, g)$ will denote an arbitrary simply connected two-step nilpotent Lie group with left-invariant Riemannian metric; $\Gamma$ will denote a uniform discrete subgroup of $N$. We continue to use the notation $\mathcal{N} = \mathcal{Z} + \mathcal{V}$ introduced in 1.1 for the Lie algebra of $N$.

2.1. NOTATION AND REMARKS. (a) The left-invariant vector fields on $N$ induce global vector fields on $\Gamma\backslash N$. Thus the tangent bundles of both $N$ and $\Gamma\backslash N$ are completely parallelizable, and the unit tangent bundles may be identified with

$$S(N, g) = N \times S(\mathcal{N}),$$

$$S(\Gamma\backslash N, g) = \Gamma\backslash N \times S(\mathcal{N}),$$

where $S(\mathcal{N})$ is the unit sphere in $\mathcal{N}$ relative to the Riemannian inner product. We will write $S(N)$ for $S(N, g)$ if $g$ is understood.

(b) For $x \in N$, the left action $L_x : N \to N$ induces a diffeomorphism $(L_x)_*$ of $S(N)$. Under the identification in (a),

$$(L_x)_*(n, u) = (xn, u)$$

for $n \in N$ and $u \in S(\mathcal{N})$.

(c) Now suppose that $(\Gamma^*\backslash N^*, g^*)$ is another compact two-step nilmanifold and that

$$F : S(\Gamma\backslash N, g) \to S(\Gamma^*\backslash N^*, g^*)$$

is a homeomorphism intertwining the geodesic flows. By (a) we can write

$$F : \Gamma\backslash N \times S(\mathcal{N}) \to \Gamma^*\backslash N^* \times S(\mathcal{N}^*).$$

Consider the universal coverings $\pi : N \times S(\mathcal{N}) \to \Gamma\backslash N \times S(\mathcal{N})$ and $\pi^* : N^* \times S(\mathcal{N}^*) \to \Gamma^*\backslash N^* \times S(\mathcal{N}^*)$. Choose an arbitrary lift $\tilde{F} : N \times S(\mathcal{N}) \to N^* \times S(\mathcal{N}^*)$ of $F$, that is, a map satisfying $\pi^* \circ \tilde{F} = F \circ \pi$. Note that the group of deck transformations of $\pi$ consists of the left translations $dL_\gamma : (n, U) \mapsto (\gamma n, U)$ with $\gamma \in \Gamma$ and similarly for $\pi^*$. Thus for every $\gamma \in \Gamma$ there is a unique $\gamma^* \in \Gamma^*$ such that $\tilde{F} \circ dL_\gamma = dL_{\gamma^*} \circ \tilde{F}$, and $\gamma \mapsto \gamma^*$ is an isomorphism that we denote by $F_* : \Gamma \to \Gamma^*$. Note that $\tilde{F}$ intertwines the geodesic flows of $(N, g)$ and $(N^*, g^*)$

since $F$ does so on the quotients. This, together with $\tilde{F} \circ dL_\gamma = dL_{F_*(\gamma)} \circ \tilde{F}$, implies that the isomorphism $F_*$ induces a marking of the length spectra of $(\Gamma\backslash N, g)$ and $(\Gamma^*\backslash N^*, g^*)$ (see Definition 1.12). Since the fundamental group of $S(\Gamma\backslash N)$ is isomorphic to $\Gamma$, $F$ induces an isomorphism

$$F_*\colon \Gamma \to \Gamma^*.$$

Proposition 1.13 and the discussion in 2.1(c) give us the following lemma.

2.2. Lemma. *Suppose that $(\Gamma\backslash N, g)$ and $(\Gamma^*\backslash N^*, g^*)$ are compact two-step nilmanifolds and that $F\colon S(\Gamma\backslash N) \to S(\Gamma^*\backslash N^*)$ is a homeomorphism intertwining their geodesic flows. Then there exists a $\Gamma$-almost inner automorphism $\Phi$ of $N$ (see Definition 1.4) such that $(\Gamma^*\backslash N^*, g^*)$ is isometric to $(\Phi(\Gamma)\backslash N, g)$. Moreover the isomorphism $F_*\colon \Gamma \to \Gamma^*$ is given by $F_* = \Psi_* \circ \Phi|_\Gamma$, where $\Psi\colon (\Phi(\Gamma)\backslash N, g) \to (\Gamma^*\backslash N^*, g^*)$ is an isometry and $\Psi_*\colon \Phi(\Gamma) \to \Gamma^*$ is the induced map on fundamental groups.*

2.3. Notation and Remarks. (a) In view of Lemma 2.2, we will assume throughout the rest of the paper that we are given a compact two-step nilmanifold $(\Gamma\backslash N, g)$, a $\Gamma$-almost inner automorphism $\Phi$ of $N$, and a geodesic conjugacy $F\colon S(\Gamma\backslash N, g) \to S(\Phi(\Gamma)\backslash N, g)$ such that $F_* = \Phi|_\Gamma$. Our ultimate goal is to show that $\Phi$ is an inner automorphism, so that $(\Phi(\Gamma)\backslash N, g)$ is isometric to $(\Gamma\backslash N, g)$ (see Remark 1.10). At various times, additional hypotheses will be imposed on $(\Gamma\backslash N, g)$ and various regularity conditions will be imposed on $F$. In the current section, we consider arbitrary compact two-step nilmanifolds and prove (Theorem 2.12) that the $\Gamma$-almost inner automorphism $\Phi$ must be an almost inner automorphism of continuous type (see Definition 1.6).

(b) Let

$$\tilde{F}\colon S(N) \to S(N)$$

be a lift of $F$. In the notation of 2.1(b), we have

$$\tilde{F} \circ dL_\gamma = dL_{\Phi(\gamma)} \circ \tilde{F}$$

for all $\gamma \in \Gamma$. Also, denoting by $G^t$ the geodesic flow of $(N, g)$, we have

$$\tilde{F} \circ G^t = G^t \circ \tilde{F}.$$

(c) (See Remark 1.5.) There exists a $\Gamma$-almost inner derivation of $\mathcal{N}$, which we denote by $\phi$, such that the differential $\Phi_*\colon \mathcal{N} \to \mathcal{N}$ is given by $\Phi_* = \mathrm{Id} + \phi$. We have $\phi(\mathcal{Z}) = 0$ and $\phi(\mathcal{V}) \subseteq \mathcal{Z}$. Note that $\Phi(x) = \exp((\mathrm{Id} + \phi)(\log(x)))$ for $x \in N$.

2.4. The Riemannian versus the Lie Group Exponential Maps. Given an arbitrary left-invariant Riemannian metric on a Lie group $G$, the integral curves of the left-invariant vector fields are in general not geodesics. For $x$, $y$, and $z$ left-invariant vector fields, the covariant derivative is given by

$$\langle \nabla_x y, z\rangle = \tfrac{1}{2}\{\langle [x, y], z\rangle + \langle [z, x], y\rangle + \langle [z, y], x\rangle\}.$$

In particular,

$$\langle \nabla_x x, z\rangle = \langle [z, x], x\rangle.$$

Thus, for $a \in G$, the integral curves $a\exp(tx)$ of $x$ are geodesics if and only if $x \perp \operatorname{image}(\operatorname{ad}(x))$.

In the case of a two-step nilpotent Lie group $N$, this condition holds for all $x \in \mathcal{V}$ and for all $x \in \mathcal{Z}$ and, more generally, for any $x$ of the form $v + z$ with $v \in \mathcal{V}$ and $z \in \mathcal{Z}$ and with $z \perp [v, \mathcal{N}]$. If, moreover, $\gamma = \exp(x) \in \log(\Gamma)$, where $\Gamma$ is a uniform discrete subgroup of $N$, then the geodesic $\exp(t(x/|x|))$ and suitable left translates of this geodesic (those translates $a\exp(t(x/|x|))$ for which $a\gamma a^{-1} \in [\gamma]_\Gamma$) descend to closed geodesics in $\Gamma\backslash N$ in the free homotopy class $[\gamma]_\Gamma$.

2.5. LEMMA [E]. (a) *Let* $(\Gamma\backslash N, g)$ *be a compact Riemannian two-step nilmanifold. Suppose* $\gamma \in \Gamma$ *and* $\gamma = \exp(v + z)$ *with* $0 \neq v \in \mathcal{V}$ *and* $z \in \mathcal{Z}$*. Write* $z = z_0 + z_1$ *with* $z_0 \perp \operatorname{image}(\operatorname{ad}(v))$ *and* $z_1 \in \operatorname{image}(\operatorname{ad}(v))$*. Then all the longest geodesics in the free homotopy class* $[\gamma]_\Gamma$ *are projections to* $\Gamma\backslash N$ *of geodesics of the form* $n\exp(tu)$ *where* $u = (v + z_0)/|v + z_0|$.

(b) *Let* $v + z$ *be a unit vector with* $v \in \mathcal{V}$ *and* $z \in \mathcal{Z}$ *and with* $z \perp [v, \mathcal{N}]$*. Let* $x \in \mathcal{N}$ *and let* $n = \exp(x) \in N$*. Then the geodesic* $n\exp t(v + z)$ *descends to a* (*not necessarily prime*) *closed geodesic in* $\Gamma\backslash N$ *of length* $l$ *if and only if*

$$\exp l(v + [x, v] + z) \in \Gamma.$$

The second statement follows easily from the multiplication formula in 1.1. The first statement sounds surprising at first glance; one might expect these geodesics to be the shortest ones. In case $z = 0$, these geodesics are actually the only ones in their respective free homotopy classes, and they minimize length not only in their free homotopy classes but also in their homology classes. On the other hand, when $v = 0$ so that $\gamma$ is central, there are many shorter geodesics in the free homotopy class $[\gamma]_\Gamma$.

2.6. LEMMA [E]. *As in 1.3, let* $\pi_{\mathcal{V}} : \mathcal{N} \to \mathcal{V}$ *be the projection with kernel* $\mathcal{Z}$*. Given* $v \in \pi_{\mathcal{V}}(\log\Gamma)$ *and* $\varepsilon > 0$, *there exists a positive integer* $k$ *and an element* $z_\varepsilon \in \mathcal{Z}$ *with* $|z_\varepsilon| < \varepsilon$ *such that* $kv + z_\varepsilon \in \log\Gamma$.

2.7. PROPOSITION. *In the notation of 1.1 and 2.3, if*

$$F : (S(\Gamma\backslash N), g) \to (S(\Phi(\Gamma)\backslash N), g)$$

*is a* $C^0$*-geodesic conjugacy, then*:

(a) $F : \Gamma\backslash N \times \{z\} \to \Phi(\Gamma)\backslash N \times \{z\}$ *for all* $z \in \mathcal{Z}$.
(b) $F : \Gamma\backslash N \times \{v\} \to \Phi(\Gamma)\backslash N \times \{v\}$ *for all* $v \in \mathcal{V}$.

*Proof*. We use the notation of 1.1, 1.11 and 2.3.

(a) Let

$$\mathcal{Z}_u = \{z \in \mathcal{Z},\ |z| = 1\}$$

and

$$\mathcal{Z}_\Gamma = \{\, z \in \mathcal{Z}_u \mid rz \in \log(\Gamma) \text{ for some } r > 0 \,\}.$$

Since $\Gamma$ is a uniform lattice, $\mathcal{Z}_\Gamma$ is dense in $\mathcal{Z}_u$ (see 1.3). Thus we need only prove (a) for $z \in \mathcal{Z}_\Gamma$.

The almost inner automorphism $\Phi$ restricts to the identity on the center of $N$ and, in particular, on the central subgroup $\Gamma \cap [N, N]$. Thus, by 2.3(a), for $\gamma \in \Gamma \cap [N, N] = \Gamma \cap \exp(\mathcal{Z})$, the geodesic conjugacy maps closed orbits of the geodesic flow in the free homotopy class $[\gamma]_\Gamma$ of $S(\Gamma\backslash N)$ to orbits in the free homotopy class $[\gamma]_{\Phi(\Gamma)}$ in $S(\Phi(\Gamma)\backslash N)$.

Let $z_0 \in \mathcal{Z}_\Gamma$ and let $r_0 \neq 0$ satisfy $\gamma := \exp(r_0 z_0) \in \Gamma$. By Lemma 2.5, the longest geodesics in the free homotopy class $[\gamma]_\Gamma$ are precisely the projections to $\Gamma\backslash N$ of $\exp(tz_0)$ and all its left translations. Thus the submanifold $\Gamma\backslash N \times \{z_0\}$ of $S(\Gamma\backslash N)$ is foliated by all the longest periodic orbits of the geodesic flow in the free homotopy class $[\gamma]_\Gamma$ (viewed now as a free homotopy class of curves in $S(\Gamma\backslash N)$). Similarly, $\Phi(\Gamma)\backslash N \times \{z_0\}$ is foliated by the longest periodic orbits of the flow in the class $[\gamma]_{\Phi(\Gamma)}$ of $S(\Phi(\Gamma)\backslash N)$. Hence the geodesic conjugacy $F$ must map $\Gamma\backslash N \times \{z_0\}$ onto $\Phi(\Gamma)\backslash N \times \{z_0\}$. This proves (a).

(b) The proof of (b) is similar but more complicated. Let

$$\mathcal{V}_u = \{\, v \in \mathcal{V} \mid |v| = 1 \,\}$$

and

$$\mathcal{V}_\Gamma = \{\, v \in \mathcal{V}_u \mid rv \in \pi_{\mathcal{V}}(\log \Gamma) \text{ for some } r > 0 \,\},$$

where $\pi_{\mathcal{V}} \colon \mathcal{N} \longrightarrow \mathcal{V}$ is the orthogonal projection. Since $\Gamma$ is a uniform lattice, $\mathcal{V}_\Gamma$ is dense in $\mathcal{V}_u$ (see 1.3) and we need only prove (b) for all $v \in \mathcal{V}_\Gamma$.

For a fixed $v \in \mathcal{V}_\Gamma$, there is an $r > 0$ such that $rv \in \pi_{\mathcal{V}}(\log \Gamma)$. According to Lemma 2.6, for $\varepsilon > 0$ there is a positive integer $k_\varepsilon$ and an element $z_\varepsilon \in z$ with $|z_\varepsilon| < \varepsilon$ such that

$$k_\varepsilon rv + z_\varepsilon \in \log(\Gamma). \tag{2.7.1}$$

We temporarily fix $\varepsilon$ and drop the subscripts in (2.7.1). Write $z = z_0 + z_1$ with $z_1 \in \text{image}(\text{ad}(v))$ and $z_0 \perp \text{image}(\text{ad}(v))$. Set $v_0 = krv$ and let $u = (v_0 + z_0)/|v_0 + z_0|$. Let

$$N_0 = \{\, n = \exp(x) \in N : \exp l(v_0 + [x, v_0] + z_0) \in \Gamma \text{ for some } l \in \mathbf{R}^+ \,\}. \tag{2.7.2}$$

For $n = \exp(x) \in N_0$, Lemma 2.5 says that the geodesic $n \exp(tu)$ descends to a closed geodesic in $\Gamma\backslash\mathcal{N}$ of length $l|v_0 + z_0|$. This closed geodesic is of maximal length in the free homotopy class $[\alpha]_\Gamma$ where $\alpha = \exp(lv_0 + l[x, v_0] + lz_0)$. Thus $F$ sends this geodesic to a closed geodesic in $\Phi(\Gamma)\backslash N$ of maximal length in the free homotopy class $[\Phi(\alpha)]_{\Phi(\Gamma)}$.

By 2.3(c) and Definition 1.4, $\Phi(\alpha) = \exp(lv_0 + l[x + \xi, v_0] + lz_0)$ for some $\xi \in \mathcal{V}$. Again by Lemma 2.5, a maximal length geodesic in $[\Phi(\alpha)]_{\Phi(\Gamma)}$ must have the form $n^* \exp(tu)$ for some $n^* \in N$, with $u$ as before. Thus, $\tilde{F}(n, u) = (n^*, u)$ and we have

$$\tilde{F} \colon N_0 \times \{u\} \to N \times \{u\}. \tag{2.7.3}$$

We next show that $N_0$ is dense in $N$. First, $N_0$ is nonempty: by the definition of $z_1$, there exists $y \in \mathcal{V}$ such that $[y, v_0] = z_1$ and 2.7.1 implies that

$$\exp(v_0 + [y, v_0] + z_0) \in \Gamma. \tag{2.7.4}$$

Thus $\exp(y) \in N_0$.

Let $\mathcal{Z}_Q$ be the rational span of $\mathcal{Z} \cap \log(\Gamma)$; that is, $\mathcal{Z}_Q = \mathcal{Z} \cap \mathcal{N}_Q$ in the notation of 1.3. Let

$$A(v_0) = \{ x \in \mathcal{V} : [x, v_0] \in \mathcal{Z}_{\mathbf{Q}} \}. \tag{2.7.5}$$

We claim that

$$\exp(x + y) \in N_0 \quad \text{for all } \ x \in A(v_0), \tag{2.7.6}$$

where $y$ is defined as in the previous paragraph. To verify 2.7.6, observe that for $x \in A(v_0)$ there exists a $q \in \mathbf{Z}$ such that

$$q[x, v_0] \in \log(\Gamma) \cap \mathcal{Z}. \tag{2.7.7}$$

We then have

$$\exp q(v_0 + [x + y, v_0] + z_0) = \exp q(v_0 + [y, v_0] + z_0) \exp q[x, v_0] \tag{2.7.8}$$

by 1.1, since $[x, v_0] \in \mathcal{Z}$. Equations 2.7.4, 2.7.7, and 2.7.8 imply that the left-hand side of 2.7.8 belongs to $\Gamma$ and hence that $\exp(x + y) \in N_0$, verifying 2.7.6.

Now, by 1.3, $\mathcal{Z}_{\mathbf{Q}} \cap \text{image}(\text{ad}(v_0))$ is dense in $\text{image}(\text{ad}(v_0))$. Since $\text{ad}(v_0)$ is linear and since $A(v_0)$, as defined in 2.7.5, is the inverse image of $\mathcal{Z}_{\mathbf{Q}} \cap \text{image}(\text{ad}(v_0))$ under $\text{ad}(v_0)$, it follows that $A(v_0)$ is dense in $\mathcal{V}$. By 2.7.6 and the fact that $n \exp(\mathcal{Z}) \subset N_0$ whenever $n \in N_0$ (as can be seen from 2.7.2), we conclude that $N_0$ is dense in $N$.

Equation 2.7.3 thus implies that $\tilde{F} \colon N \times \{u\} \to N \times \{u\}$. Recall that $u$, as defined just after equation 2.7.1, depends on $\varepsilon$. As $\varepsilon$ goes to 0, $u$ converges to the unit vector $v$. Consequently, we have $\tilde{F} \colon N \times \{v\} \to N \times \{v\}$ whenever $v \in \mathcal{V}_\Gamma$. As noted previously, this completes the proof of Proposition 2.7. □

2.8. NOTATION AND REMARKS. By Proposition 2.7 and the fact that $\tilde{F}$ commutes with the geodesic flow $G^t$ of $S(N)$, we see that $\tilde{F}$ carries orbits $(n \exp(tv), v)$ of $G^t$ to orbits of the same form when $v \in \mathcal{V}$; that is, it simply left-translates such orbits. Hence there exists an element $f(n, v) \in N$ such that $\tilde{F}(n \exp(tv), v) \equiv (f(n, v) n \exp(tv), v)$. A similar statement holds with $v$ replaced by $z \in \mathcal{Z}$. For later use (Section 3), we extend the definition of $f$ as follows. Define $f \colon S(N) \to N$ and $g \colon S(N) \to \mathcal{N}$ by the condition

$$\tilde{F}(n, u) = (f(n, u)n, g(n, u)).$$

In particular, for $u \in \mathcal{V}$ or $u \in \mathcal{Z}$, the previous statement implies that

$$f(n \exp tu, u) = f(n, u) \quad \text{and} \quad g(n, u) = u.$$

Write

$$\log(f(n, u)) = A(n, u) + B(n, u)$$

with $A(n, u) \in \mathcal{Z}$ and $B(n, u) \in \mathcal{V}$.

2.9. PROPOSITION. *In the notation of 2.3 and 2.8, for $(n, u) \in S(N)$ and $\gamma \in \Gamma$ we have*:

(a) $g(n, u) = u$ *if* $u \in \mathcal{V}$ *or* $u \in \mathcal{Z}$;
(b) $f(G^t(n, u)) = f(n, u)$ *if* $u \in \mathcal{V}$ *or* $u \in \mathcal{Z}$ *(in particular,* $A(G^t(n, u)) = A(n, u)$ *and* $B(G^t(n, u)) = B(n, u)$ *in this case*);
(c) $g(\gamma n, u) = g(n, u)$;
(d) $f(\gamma n, u) = \Phi(\gamma) f(n, u)\gamma^{-1}$ *(in particular,* $B(\gamma n, u) = B(n, u)$ *and* $A(\gamma n, u) = A(n, u) + \phi(\log\gamma) - [B(n, u), \log\gamma]$*).*

*Proof*. (a) and (b) are a restatement of Proposition 2.7 and the comments in 2.8. (c) and the first equation in (d) are a restatement of the equation $\tilde{F} \circ dL_\gamma = dL_{\Phi(\gamma)} \circ \tilde{F}$, noted in 2.3; see also 2.1(b). To obtain the last two equations in (d), note that by 2.3(c) and the fact that $\phi(\mathcal{N}) \subset \mathcal{Z}$ (see 1.5(b)) we have

$$\Phi(\gamma) = \exp(\log\gamma + \phi(\log\gamma)) = \gamma \exp(\phi(\log\gamma))$$

with $\exp(\phi(\log\gamma))$ central in $N$, so

$$\Phi(\gamma) f(n, u)\gamma^{-1} = \gamma f(n, u)\gamma^{-1} \exp(\phi(\log\gamma)).$$

By 1.1 and the first statement in (d), we thus have that

$$\log(f(\gamma n, u)) = \log(f(n, u)) + [\log(\gamma), \log(f(n, u))] + \phi(\log(\gamma)).$$

The last two equations in (d) now follow from the Definition 2.8 of $A$ and $B$ and the fact that $[\mathcal{N}, \mathcal{N}] = \mathcal{Z}$. □

2.10. NOTATION AND REMARKS. The derived group $[N, N]$ of $N$ is a simply connected central subgroup with Lie algebra $[\mathcal{N}, \mathcal{N}] = \mathcal{Z}$, and $\exp\colon \mathcal{Z} \to [N, N]$ is a vector space isomorphism. By 1.3, $(\Gamma \cap [N, N])\backslash[N, N]$ is a torus; we denote it by $T$. Note that $T$ acts isometrically on $\Gamma\backslash N$ by left translations.

By Proposition 2.9, $B(\gamma n, u) = B(n, u)$ for all $\gamma \in \Gamma$, and $A(\gamma n, u) = A(n, u)$ for $\gamma \in [N, N] \cap \Gamma$. Thus we can define the following averages over $T$:

$$\bar{B}(n, u) = \int_T B(z \cdot n, u)\, dz,$$

$$\bar{A}(n, u) = \int_T A(z \cdot n, u)\, dz$$

for $(n, u) \in S(N)$. Here $dz$ is the Haar measure on $T$ of total volume one.

2.11. PROPOSITION. *In the notation of 1.1, 2.3, 2.8, and 2.10,*

$$\phi(v) = [\bar{B}(n, v), v] \quad \text{and} \quad \bar{B}(n, v) = \bar{B}(e, v)$$

*for all* $(n, v) \in S(N)$ *with* $v \in \mathcal{V}$.

*Proof*. Let $\pi_{\mathcal{V}}\colon \mathcal{N} \to \mathcal{V}$ be the orthogonal projection. $N/[N, N]$ is a simply connected abelian Lie group. Letting $\pi\colon N \to N/[N, N]$ be the projection,

$\pi(\Gamma)\backslash\pi(N)$ is a torus $\bar{T}$. The Lie algebra of $N/[N,N]$ is $\mathcal{N}/\mathcal{Z}$. Under the identification of $\mathcal{N}/\mathcal{Z}$ with $\mathcal{V}$, the exponential map $\mathcal{V}\to N/[N,N]$ carries $\pi_{\mathcal{V}}(\log\Gamma)$ isometrically to $\pi(\Gamma)$. Thus the torus $\bar{T}$ is isomorphic to $\pi_{\mathcal{V}}(\log\Gamma)\backslash\mathcal{V}$.

By 2.8 and 2.10,

$$\begin{aligned}\bar{A}(n\exp(tv),v)&=\bar{A}(n,v),\\ \bar{B}(n\exp(tv),v)&=\bar{B}(n,v)\end{aligned}\tag{2.11.1}$$

for all $t\in\mathbf{R}$ and $v\in\mathcal{V}$.

We first prove that $\bar{B}(n,v)=\bar{B}(e,v)$ for all $(n,v)\in S(N)$ with $v\in\mathcal{V}$. Since $\bar{B}(yn,v)=\bar{B}(n,v)$ for $y\in[N,N]$ by Definition 2.10, and since $\bar{B}(\gamma n,v)=\bar{B}(n,v)$ for $\gamma\in\Gamma$ by Proposition 2.9(d), the map $\bar{B}(\cdot,v)$ may be viewed as a map on the torus $\bar{T}$. In particular, if $v\in\mathcal{V}$ is a unit vector such that the projection of $v$ to the torus $(\pi_{\mathcal{V}}(\log\Gamma))\backslash\mathcal{V}\cong\bar{T}$ is a generator of the torus—that is, if the projection of $\{tv\mid t\in\mathbf{R}\}$ is dense in the torus—then $\bar{B}(n,v)=\bar{B}(e,v)$ for all $n$ by (2.11.1). By Kronecker's theorem, the generators form a dense set in the torus. Hence $\bar{B}(n,v)\equiv\bar{B}(e,v)$ for all unit vectors $v\in\mathcal{V}$ and all $n\in N$. (Remark: We can't expect to have $\bar{A}(n,v)=\bar{A}(e,v)$, since $\bar{A}(\gamma n,v)\neq\bar{A}(n,v)$ for general $\gamma\in\Gamma$.)

We now turn to the expression for $\phi(v)$. By Proposition 2.9(d), for all $n\in N$ and $u\in\mathcal{V}$ with $|u|=1$ and $\gamma\in\Gamma$, we have

$$\phi(\log\gamma)=A(\gamma n,u)-A(n,u)+[B(n,u),\log\gamma];$$

therefore,

$$\phi(\log\gamma)=\bar{A}(\gamma n,u)-\bar{A}(n,u)+[\bar{B}(n,u),\log\gamma].\tag{2.11.2}$$

Recalling that $\phi(\mathcal{Z})=0$, (2.11.2) with $n=e$ implies for $\eta\in\pi_{\mathcal{V}}(\log(\Gamma))$ that

$$\phi(\eta)=\bar{A}(\exp(\eta),u)-\bar{A}(e,u)+[\bar{B}(e,u),\eta].\tag{2.11.3}$$

It suffices to prove $\phi(v)=[\bar{B}(n,v),v]$ for those unit vectors $v$ of the form $v=r\eta$ with $\eta\in\pi_{\mathcal{V}}(\log(\Gamma))$ and $r\in\mathbf{R}$, since such vectors $v$ form a dense subset of the unit sphere in $\mathcal{V}$. For $v$ of this form, we have

$$\phi(v)=r\phi(\eta)=r(\bar{A}(\exp(\eta),u)-\bar{A}(e,u))+[\bar{B}(e,u),v].\tag{2.11.4}$$

Setting $u=v$ in equation (2.11.4), recalling that $\eta$ is a real multiple of $v$, and applying (2.11.1), we obtain $\phi(v)=[\bar{B}(e,v),v]$ as desired. □

2.12. THEOREM. *Suppose* $(\Gamma\backslash N,g)$ *and* $(\Gamma^*\backslash N^*,g^*)$ *are compact two-step nilmanifolds whose geodesic flows are* $C^0$*-conjugate. Then there exists an almost-inner automorphism* $\Phi$ *of* $N$ *of continuous type* (*see Definition 1.6*) *such that* $(\Gamma^*\backslash N^*,g^*)$ *is isometric to* $(\Phi(\Gamma)\backslash N,g)$.

The theorem follows immediately from 2.3 and 2.11 since $\bar{B}$ is continuous. Theorem 2.12 and Remark 1.10 yield the following corollary.

2.13. COROLLARY. *Let* $N$ *be a two-step nilpotent Lie group that does not admit any noninner almost inner automorphisms of continuous type. Then any associated compact nilmanifold* $(\Gamma\backslash N,g)$ *is* $C^0$*-geodesically rigid within the class of all compact nilmanifolds.*

The conclusion that $(\Gamma\backslash N, g)$ is $C^0$-geodesically rigid within the class of all compact nilmanifolds, and not just in the class of two-step nilmanifolds, follows from the remarks in the introduction.

2.14. COROLLARY. *There exist compact two-step nilmanifolds* $(\Gamma\backslash N, g)$ *that satisfy the following conditions.*

(a) *Any compact nilmanifold whose geodesic flow is* $C^0$*-conjugate to* $(\Gamma\backslash N, g)$ *must be isometric to* $(\Gamma\backslash N, g)$.

(b) $(\Gamma\backslash N, g)$ *is isospectrally deformable. That is, there exists a continuous family* $\{M_t\}$ *of two-step compact nilmanifolds with* $M_0 = (\Gamma\backslash N, g)$ *such that* $M_t$ *is isospectral but not isometric to* $M_0$ *for all* $t$.

*Proof*. In Example 1.7, we exhibited a nilmanifold $(\Gamma\backslash N, g)$ such that (i) every almost inner derivation of $\mathcal{N}$ of continuous type is inner and (ii) $\mathcal{N}$ admits a $\Gamma$-almost inner derivation $\phi$ that is noninner. By Corollary 2.13, condition (i) implies (a). By 1.9 and 1.10, condition (ii) implies (b). □

## 3. Special Classes of Nilmanifolds

3.1. DEFINITION. In the notation of 1.1, a simply connected two-step Riemannian nilmanifold $(N, g)$ is said to be *nonsingular* if $J(z)$ is a nonsingular linear transformation for all $z \in \mathcal{Z}$, $z \neq 0$. Equivalently, $\mathrm{image}(\mathrm{ad}(x)) = \mathcal{Z}$ for all $x \in \mathcal{V}$. (Note that this definition is independent of the choice of $g$; i.e., it is a property only of the Lie algebra structure.) We will also say that the associated Lie algebra and any associated compact nilmanifold $\Gamma\backslash N$ are nonsingular in this case.

The rigidity results of Corollary 2.13 are not applicable to nonsingular nilmanifolds. Indeed, by Example 1.7(ii), the space of almost inner derivations of a nonsingular two-step nilpotent Lie algebra has dimension $\dim(\mathcal{V})\dim(\mathcal{Z})$, whereas the space of inner derivations has dimension only $\dim(\mathcal{V})$. Thus, except in the case in which $\dim(\mathcal{Z}) = 1$, the Lie algebra admits many noninner almost inner derivations. Moreover, as noted in Example 1.7(ii), all these almost inner derivations are of continuous type. Thus Proposition 1.13 says that every nonsingular compact two-step nilmanifold can be continuously deformed through an $m$-parameter family of pairwise nonisometric nilmanifolds all having the same marked length spectrum, where $m = \dim(\mathcal{V})(\dim(\mathcal{Z}) - 1)$, and the results of Section 2 do not rule out the possibility that these nilmanifolds have $C^0$-conjugate geodesic flows.

Nonetheless, we shall see (in Theorem 3.10) that generic compact nonsingular two-step nilmanifolds (defined in Definition 3.2(a)) are at least $C^2$-geodesically rigid within the space of all compact nilmanifolds. We will also establish rigidity results for some special classes of nonsingular compact two-step nilmanifolds (those in Definition 3.2(c,d)).

3.2. DEFINITION. Let $(N, g)$ be a simply connected two-step Riemannian nilmanifold. We use the notation of 1.1.

(a) $(N, g)$ is said to be *irrational* if, for $z$ in a dense set of $\mathcal{Z}$, the set $\{1, \theta_1(z), \ldots, \theta_{p(z)}(z)\}$ is linearly independent over the field of rational numbers, where

$\{\pm\sqrt{-1}\theta_i(z),\ i = 1, 2, \ldots, p(z)\}$ are the distinct eigenvalues of $J(z)$. (We choose $\theta_i(z) \geq 0$ and are allowing the eigenvalues to have arbitrary multiplicities.)

(b) $(N, g)$ is said to be *in resonance* if for all $0 \neq z \in \mathcal{Z}$, every ratio of nonzero eigenvalues of $J(z)$ is rational; equivalently, for each $z \in \mathcal{Z}$, there exists a constant $t$ such that $e^{tJ(z)} = \mathrm{Id}$, where $t$ may depend on $z$.

(c) $(N, g)$ is said to be *strongly in resonance* if, for each $z \in \mathcal{Z}$, there exists a constant $t$ such that $e^{tJ(z)} = -\mathrm{Id}$, where $t$ may depend on $z$. Equivalently, every ratio of nonzero eigenvalues is of the form $p/q$ with $p$ and $q$ odd integers.

(d) $(N, g)$ is said to be of *Heisenberg* type if $(J(z))^2 = -|z|^2 \mathrm{Id}_{\mathcal{V}}$ for all $z \in \mathcal{Z}$.

If $(N, g)$ has any of the properties defined in (a)–(d), we will also say that any associated compact nilmanifold $(\Gamma\backslash N, g)$ has the respective property.

3.3. REMARKS. The notion of Lie group of Heisenberg type is due to Kaplan [Kn]. These groups have a surprisingly rich and varied geometry and have been studied by many authors (e.g. [E; Rm]). The noncompact rank-one symmetric spaces are solvable extensions of certain of the Lie groups of Heisenberg type. More generally, Damek and Ricci [DR] showed that similarly defined solvable extensions of all the Lie groups of Heisenberg type are harmonic manifolds. The horospheres in these harmonic manifolds are isometric to the Lie groups of Heisenberg type.

Eberlein [E] defined the term "in resonance". The condition of resonance of $(N, g)$ is closely connected with the condition that the set of vectors tangent to closed geodesics in any associated compact nilmanifold $\Gamma\backslash N$ be dense in $S(\Gamma\backslash N)$ (see [E; LP; M]). The nilmanifolds of Heisenberg type are in resonance—in fact, strongly in resonance. Mast [M] gave the first examples of nilmanifolds in resonance which are not of Heisenberg type. We give an additional example as follows.

3.4. EXAMPLE. Let $\mathcal{V}$ and $\mathcal{Z}$ be inner product spaces with orthonormal bases $\{X_1, \ldots, X_8\}$ and $\{Z_1, Z_2\}$, respectively, and define $J\colon \mathcal{Z} \to \mathfrak{so}(\mathcal{V})$ so that, for $Z = z_1Z_1 + z_2Z_2$, $J(Z)$ has the following matrix representation with respect to the foregoing basis of $\mathcal{V}$:

$$\begin{pmatrix} 0 & 0 & -\lambda_1 z_1 & -\lambda_1 z_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_1 z_2 & -\lambda_1 z_1 & 0 & 0 & 0 & 0 \\ \lambda_1 z_1 & -\lambda_1 z_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 z_2 & \lambda_1 z_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 z_1 & -\lambda_2 z_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 z_2 & -\lambda_2 z_1 \\ 0 & 0 & 0 & 0 & \lambda_2 z_1 & -\lambda_2 z_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 z_2 & \lambda_2 z_1 & 0 & 0 \end{pmatrix}.$$

The distinct eigenvalues of $J(Z)$ are $\{\pm\sqrt{-1}\lambda_1|Z|, \pm\sqrt{-1}\lambda_2|Z|\}$.

As in 1.1, the data $(\mathcal{V}, \mathcal{Z}, J)$ defines a simply connected two-step nilpotent Lie group $N$ with a left-invariant Riemannian metric $g$. If, say, $\lambda_2 = 2\lambda_1$, then $(N, g)$ is in resonance but not strongly in resonance. If, say, $\lambda_2 = 3\lambda_1$, then $(N, g)$ is strongly in resonance but is not of Heisenberg type.

We next show that generic nonsingular compact two-step nilmanifolds are irrational in the sense of Definition 3.2. First recall the following facts concerning the behavior of eigenvalues under perturbations of linear maps (see [Kt] for details). As $T$ varies over a subspace $W$ of linear transformations of $\mathbf{R}^m$, the number of distinct eigenvalues of $T$ is a constant $r$ except on a nowhere dense set $U$ of measure zero in $W$ where eigenvalues coalesce, resulting in a smaller number of distinct eigenvalues. (Here $r$ may be less than $m$.) Moreover, the eigenvalues constitute branches of analytic functions with singularities only on $U$. More precisely, we may define $r$ continuous functions, $\alpha_1, \ldots, \alpha_r$, on $W$ which are locally analytic off $U$ and such that, for all $T \in W$, $\alpha_1(T), \ldots, \alpha_r(T)$ are the eigenvalues of $T$ with $\alpha_i(T) < \alpha_{i+1}(T)$, except possibly on $U$ where equality may occur.

3.5. NOTATION. Let $(N, g)$ be a nonsingular simply connected two-step Riemannian nilmanifold. We let $2p$ be the maximum number of distinct eigenvalues of $J(z)$ as $z$ varies over $\mathcal{Z}$, and we define continuous functions $\alpha_1, \ldots, \alpha_p$ on $\mathcal{Z}$ so that, except on a nowhere dense set $S$ of measure zero where some of these functions may coincide, $\{\pm\sqrt{-1}\alpha_i(z),\ i = 1, 2, \ldots, p\}$ are the distinct eigenvalues of $J(z)$. Moreover, these functions are continuous and consist of branches of analytic functions with singularities only on $S$. Note that we allow $2p < m = \dim(\mathcal{V})$.

3.6. LEMMA. *Let $(N, g)$ be a nonsingular simply connected two-step nilmanifold. In the notation of 3.5, either $(N, g)$ is irrational or else there exist rationals $a_1, \ldots, a_p$ such that*

$$a_1\alpha_1 + \cdots + a_p\alpha_p \equiv 0$$

*on $\mathcal{Z}$.*

*Proof.* Let

$$\mathcal{Z}_I = \{ z \in \mathcal{Z} : \{1, \alpha_1(z), \ldots, \alpha_p(z)\} \text{ is rationally dependent} \}.$$

If $a_1, \ldots, a_{p+1}$ are rational numbers, then the map

$$z \to a_1\alpha_1(z) + \cdots + a_p\alpha_p(z) + a_{p+1}$$

is continuous and locally analytic on $\mathcal{Z} \setminus S$, so its zero set is either nowhere dense or else is all of $\mathcal{Z}$. In case $a_{p+1}$ is nonzero, the latter possibility cannot occur since $\alpha_i(sz) = s\alpha_i(z)$ for $s \in \mathbf{R}^+$. Consequently either there exist rationals $a_1, \ldots, a_p$ such that

$$a_1\alpha_1 + \cdots + a_p\alpha_p \equiv 0$$

or else $\mathcal{Z}_I$ is a countable union of nowhere dense sets. In the latter case, $\mathcal{Z}_I$ is of the first category, and its complement is dense in $\mathcal{Z}$; that is, $(N, g)$ is irrational. □

3.7. REMARK. Viewing $\mathfrak{so}(m)$ as a vector space with the Lebesgue measure, the set $\mathcal{M}$ of matrices $A$ in $\mathfrak{so}(m)$ for which the distinct eigenvalues of $A^2$ are rationally independent is a dense set of full measure. Consider the set of all nonsingular simply connected two-step nilmanifolds $(N, g)$ for which $\mathcal{Z}$ is $k$-dimensional and $\mathcal{V}$ is

$m$-dimensional. By Lemma 3.6, the irrational ones are associated with $k$-planes in $\mathfrak{so}(m)$ that are not entirely contained in the complement of $\mathcal{M}$. Relative to the Riemannian volume form on the Grassmanian of $k$-planes in the vector space $\mathfrak{so}(m)$, it is not difficult to see that the set of such $k$-planes is a dense set of full measure in the Grassmannian. This shows that generic *simply connected* nonsingular two-step Riemannian nilmanifolds are irrational. By Definition 3.2, a compact Riemannian nilmanifold is nonsingular if and only if its universal covering is nonsingular. However, only certain simply connected nilpotent Lie groups (namely, those for which the associated Lie algebra has a basis with rational structure constants) admit uniform discrete subgroups. Thus we cannot immediately conclude that generic *compact* nonsingular two-step Riemannian nilmanifolds are nonsingular. To reach this conclusion, we will show that for *any* given simply connected nonsingular two-step nilpotent Lie group $N$, generic left-invariant metrics on $N$ are irrational.

3.8. NOTATION AND REMARKS. Let $N$ be a simply connected nonsingular two-step nilpotent Lie group. A choice of left-invariant Riemannian metric $g$ on $N$ (equivalently, of an inner product on $\mathcal{N}$) depends on (i) a choice of inner product on the derived algebra $\mathcal{Z}$, (ii) a choice of a vector space complement $\mathcal{V}$ of $\mathcal{Z}$ in $\mathcal{N}$ (to be the orthogonal complement of $\mathcal{Z}$), and (iii) a choice of inner product on $\mathcal{N}/\mathcal{Z}$ (which then defines an inner product on $\mathcal{V}$ via the canonical identification). We claim that the third choice alone determines whether $(N, g)$ is irrational. The irrelevance of the choice (ii) of $\mathcal{V}$ is an elementary consequence of Definitions 1.1 and 3.2(a) and the fact that $\mathcal{Z}$ is central. Fix a choice of inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{N}/\mathcal{Z}$ and $\langle\langle\cdot,\cdot\rangle\rangle$ on $\mathcal{Z}$. These choices give rise to a linear map $J\colon \mathcal{Z} \to \mathfrak{so}(\mathcal{N}/\mathcal{Z}, \langle\cdot,\cdot\rangle)$ as in 1.1. (Here we are identifying $\mathcal{N}/\mathcal{Z}$ with a vector space complement $\mathcal{V}$ of $\mathcal{Z}$ in $\mathcal{N}$.) Let $\mathcal{S}(\mathcal{N}/\mathcal{Z})$ (resp. $\mathcal{S}(\mathcal{Z})$) denote the set of positive-definite symmetric linear transformations of $(\mathcal{N}/\mathcal{Z}, \langle\cdot,\cdot\rangle)$ (resp., of $(\mathcal{Z}, \langle\langle\cdot,\cdot\rangle\rangle)$). Given an arbitrary inner product $g$ on $\mathcal{N}$, there exists a unique $B \in \mathcal{S}(\mathcal{Z})$ such that $g(z, w) = \langle\langle z, Bw\rangle\rangle$ for all $z, w \in \mathcal{Z}$ as well as a unique $A \in \mathcal{S}(\mathcal{N}/\mathcal{Z})$ such that the inner product induced on $\mathcal{N}/\mathcal{Z}$ by $g$ is given by $\langle\cdot, A\cdot\rangle$. The linear map $J'$ associated with $g$ as in 1.1 (after identifying $\mathcal{Z}^\perp$ with $\mathcal{N}/\mathcal{Z}$) is given by

$$J'(z) = A^{-1}J(Bz)$$

since

$$\langle AJ'(z)x, y\rangle = g(J'(z)x, y) = \langle\langle [x, y], Bz\rangle\rangle = \langle J(Bz)x, y\rangle$$

for all $x, y \in \mathcal{N}/\mathcal{Z}$ and $z \in \mathcal{Z}$. Since $B$ is an isomorphism, the question of irrationality of $(N, g)$ depends only on $A$—that is, on the choice (iii) of inner product on $\mathcal{N}/\mathcal{Z}$, and the claim is proved.

The set $\mathcal{I}$ of all inner products on $\mathcal{N}$ (i.e., of all left-invariant metrics on $N$) may be identified with the collection of all $n \times n$ positive-definite symmetric matrices and hence with an open subset of $\mathbf{R}^{n(n+1)/2}$, where $n = \dim(\mathcal{N})$. The Lebesgue measure on $\mathbf{R}^{n(n+1)/2}$ thus defines a measure on $\mathcal{I}$. Similarly, $\mathcal{S}(\mathcal{N}/\mathcal{Z})$ may be viewed as an open subset of $\mathbf{R}^{m(m+1)/2}$, where $m = \dim(\mathcal{V})$, with the Lebesgue measure.

3.9. PROPOSITION. *Let $N$ be a simply connected nonsingular two-step nilpotent Lie group, and let $\mathcal{I}$ denote the set of all left-invariant metrics on $N$ as in 3.8. Then the set of $g$ in $\mathcal{I}$ for which $(N, g)$ is irrational is a dense subset of $\mathcal{I}$ of full measure.*

*Proof*. We first prove that the irrational metrics are dense. Let $g \in \mathcal{I}$. Choose any nonzero element $z$ of $\mathcal{Z}$. There exists an orthonormal basis $\{X_1, \ldots, X_l, Y_1, \ldots, Y_l\}$ of $\mathcal{V}_g$, the orthogonal complement of $\mathcal{Z}$ in $\mathcal{N}$, such that $J(z)X_i = \sqrt{-1}\theta_i Y_i$ and $J(z)Y_i = -\sqrt{-1}\theta_i X_i$ for some $\theta_i \in \mathbf{R}$. By perturbing the norms of the $X_i$ and otherwise leaving the metric unchanged, we perturb the $\theta_i$. We can find arbitrarily small perturbations that result in values for $\theta_i(z)$ for which $\{\theta_1, \ldots, \theta_l\}$ is rationally independent. The resulting left-invariant metrics on $N$ are irrational by Lemma 3.6.

For the second statement of the proposition it suffices to show that, under the mapping from $\mathcal{I}$ to $\mathcal{S}(\mathcal{N}/\mathcal{Z})$ described in 3.8, the image of the set of irrational metrics has full measure in $\mathcal{S}(\mathcal{N}/\mathcal{Z})$. Let $J$ be as in 3.8. Fix a nonzero element $z$ of $\mathcal{Z}$ and consider the eigenvalues of $AJ(z)$ as $A$ varies over $\mathcal{S}(\mathcal{N}/\mathcal{Z})$. By the proof of the first statement of the proposition, there exists a choice of $A$ for which the eigenvalues of $AJ(z)$ are all distinct and rationally independent. As $A$ varies, we can write the eigenvalues of $AJ(z)$ as $\pm\sqrt{-1}\alpha_1(A), \ldots, \pm\sqrt{-1}\alpha_l(A)$, where the $\alpha_i(A)$ are continuous and consist of branches of analytic functions with singularities only on a set of measure zero where different eigenvalues coalesce. Here $l = \frac{1}{2}\dim(\mathcal{V})$. By Lemma 3.6, a sufficient condition for the left-invariant metrics associated with $A$ to be irrational is that the $\alpha_i(A)$ be rationally independent. Let $a_1, \ldots, a_m$ be rationals. The function $a_1\alpha_1(A) + \cdots + a_m\alpha_m(A)$ on $\mathcal{S}(\mathcal{N}/\mathcal{Z})$ is locally analytic. Since we know it is not identically zero, the zero set must be a nowhere dense set of measure zero in $\mathcal{S}(\mathcal{N}/\mathcal{Z})$. Since there are only countably many choices of the $a_i$, it follows that the set of $A$ for which the $\alpha_i(A)$ are rationally independent has full measure in $\mathcal{S}(\mathcal{N}/\mathcal{Z})$. This completes the proof. □

The primary goal of this section is to prove the following theorem.

3.10. THEOREM. *Irrational compact two-step nilmanifolds are $C^2$-geodesically rigid within the class of all compact nilmanifolds.*

This theorem, together with Proposition 3.9, yields Theorem 2 in the introduction.

Our proof of Theorem 3.10 will also yield the following result.

3.11. THEOREM. *Compact two-step nilmanifolds that are strongly in resonance are $C^2$-geodesically rigid within the class of all compact nilmanifolds.*

The results in Section 2 used the behavior of only those geodesics that were orbits of one-parameter groups of isometries. In our study of nonsingular two-step nilmanifolds, we will need to use all the geodesics.

3.12. LEMMA [E]. *Let $(N, g)$ be a two-step nilpotent Lie group with a left-invariant metric. For $u \in \mathcal{N}$, let $\sigma(\cdot, u)$ denote the geodesic in $N$ with $\sigma(0, u) = e$*

*and $\sigma'(0,u) = u$, where $\sigma'(\cdot,u)$ denotes the time derivative. Write $u = x + z$ with $x \in \mathcal{V}$ and $z \in \mathcal{Z}$ and write $\sigma(t,u) = \exp(x(t) + z(t))$ with $x(t) \in \mathcal{V}$ and $z(t) \in \mathcal{Z}$ (so $x'(0) = x$ and $z'(0) = z$). Write $J^{-1}(z)$ for $(J(z))^{-1}$. Then*:

(a) *under the usual identification of the tangent space at any point of $N$ with the Lie algebra $\mathcal{N}$, we have $\sigma'(t,u) = e^{tJ(z)}x + z$ for all $t \in \mathbf{R}$*;

(b) *$x(t) = tx_1 + (e^{tJ(z)} - id)J^{-1}(z)x_2$ for $t \in \mathbf{R}$, where $x_1 \in \mathrm{Ker}(J(z))$, $x_2 \in (\mathrm{Ker}(J(z)))^{\perp}$, and $x_1 + x_2 = x$.*

Observe that, in the nonsingular case, we have $x_1 = 0$ and $x_2 = x$ in (b).

Eberlein also computed the (somewhat long) formula for $z(t)$. We will only need that formula in the special case that $(N, g)$ is of Heisenberg type, in which case the formula simplifies considerably (see 4.1).

3.13. NOTATION. We continue to use the notation of 2.3 and 2.8. In addition, we write

$$g(n,u) = H(n,u) + I(n,u)$$

with $H(n,u) \in \mathcal{Z}$ and $I(n,u) \in \mathcal{V}$.

3.14. PROPOSITION. *Let $(\Gamma\backslash N, g)$ be an arbitrary compact two-step nilmanifold. In the notation of 2.3, 2.8, 3.12, and 3.13, if $F\colon (S(\Gamma\backslash N), g) \to (S(\Phi(\Gamma)\backslash N), g)$ is a $C^0$-geodesic conjugacy, then*:

(a) $I(G^t(n,u)) = e^{tJ(H(n,u))}I(n,u)$;

(b) $H(G^t(n,u)) = H(n,u)$;

(c) $B(G^t(n,u)) = B(n,u) + x(t, g(n,u)) - x(t,u)$.

*Proof*. Statements (a) and (b) follow from Lemma 3.12(a) and the fact that $\tilde{F}$ commutes with the geodesic flow $G^t$ of $(N, g)$. For (c), note that, since the left translations on $N$ are isometries, we have that

$$G^t(n,u) = (n\sigma(t,u), \sigma'(t,u))$$

in the notation of 3.12 and, in the notation of 2.8 and 3.13, that

$$\begin{aligned}
&(f(G^t(n,u))n\sigma(t,u), g(G^t(n,u))\\
&\quad = \tilde{F}(G^t(n,u))\\
&\quad = G^t(\tilde{F}(n,u))\\
&\quad = G^t(f(n,u)n, g(n,u))\\
&\quad = (f(n,u)n\sigma(t, g(n,u)), H(n,u) + e^{tJ(H(n,u))}I(n,u)).
\end{aligned}$$

In particular,

$$f(G^t(n,u)) = f(n,u)n\sigma(t, g(n,u))(\sigma(t,u))^{-1}n^{-1}.$$

The expression (c) for $B(G^t(n,u))$ now follows from the fact that $\log(nyn^{-1}) \equiv \log(y) \bmod \mathcal{Z}$ for any $y \in N$ (see 1.1). $\square$

As in 2.3, we assume we are given a compact two-step nilmanifold $(\Gamma\backslash N, g)$, a $\Gamma$-almost inner automorphism $\Phi$ of $N$, and a geodesic conjugacy $F: S(\Gamma\backslash N, g) \to S(\Phi(\Gamma)\backslash N, g)$ such that the induced map on fundamental groups is given by $F_* = \Phi|_\Gamma$. We want to show that $\Phi$ is an inner automorphism. Observe that if $\Phi$ is indeed inner—say, $\Phi$ is conjugation by $a \in N$—then, since the metric $g$ is left-invariant, the left translation $L_a: S(N) \to S(N)$ given by $L_a(n, u) = (an, u)$ commutes with the geodesic flow of $(N, g)$ and induces a map from $S(\Gamma\backslash N)$ to $S(\Phi(\Gamma)\backslash N)$ intertwining the geodesic flows. Of course, composing this geodesic conjucacy with any flow translation $G^{t_0}$ gives another geodesic conjugacy. For such conjugacies, the map $H$ defined in 3.13 satisfies $H(n, v + z) = z$ for all $v \in \mathcal{V}$ and $z \in \mathcal{Z}$ (see Lemma 3.12(a)). Proposition 3.14 then implies that $e^{tJ(z)}I(n, v + z) = I(n\sigma(t, v + z), e^{tJ(z)}v + z)$, where $t \to \sigma(t, v + z)$ is the geodesic in $N$ with initial tangent vector $(e, v + z)$. In Section 4 we will see that these conditions on $H$ and $I$ hold for arbitrary geodesic conjugacies when $\Gamma\backslash N$ is of Heisenberg type; this fact will be the key lemma in obtaining geodesic rigidity for such nilmanifolds. The following technical lemma gives a weak version of these conditions for arbitrary $C^2$-conjugacies when $\Gamma\backslash N$ is only assumed to be nonsingular. Observe that we already know from Proposition 2.7 that $H(n, z) = z$.

3.15. LEMMA. *We use the notation of 2.3, 2.8, and 3.13. Assume that* $(\Gamma\backslash N, g)$ *is a nonsingular compact two-step nilmanifold and that*

$$F: S(\Gamma\backslash N, g) \to S(\Phi(\Gamma)\backslash N, g)$$

*is a* $C^2$*-geodesic conjugacy. Then, for* $n \in N$ *and for unit vectors* $v \in \mathcal{V}$ *and* $z \in \mathcal{Z}$ *we have*:

(a) $$\lim_{s\to 0} \frac{H(n, \cos(s)v + \sin(s)z)}{s} = \frac{d}{ds}\bigg|_{s=0} H(n, \cos(s)v + \sin(s)z) = z;$$

(b) $$\lim_{s\to 0} \frac{d}{ds}\left(\frac{H(n, \cos(s)v + \sin(s)z)}{s}\right) = 0;$$

(c) $$e^{tJ(z)} \frac{d}{ds}\bigg|_{s=0} I(n, \cos(s)v + \sin(s)z)$$
$$= \lim_{s\to 0} \frac{d}{d\lambda}\bigg|_{\lambda=0} I\left(n\sigma\left(\frac{t}{s}, \cos(s)v + \sin(s)z\right), e^{tJ(z)} \cos(\lambda)v + \sin(\lambda)z\right).$$

*where* $\sigma(t, v + z)$ *is the geodesic in* $N$ *with* $\sigma(0) = e$ *and* $\sigma'(0) = v + z$.

The proof of Lemma 3.15 will be given at the end of this section.

Our approach to proving geodesic rigidity for the generic and the special classes of nonsingular compact two-step nilmanifolds is suggested by the following proposition, which is valid for arbitrary two-step nilmanifolds.

3.16. PROPOSITION. *Let* $(\Gamma\backslash N, g)$ *be a compact two-step nilmanifold and let* $\phi$ *be an almost inner derivation of the associated Lie algebra* $\mathcal{N}$*. Then there exists* *a map $\xi : \mathcal{V} \to \mathcal{V}$ such that $\phi(x) = [\xi(x),\ x]$ for all $x \in \mathcal{V}$. If $\xi$ can be chosen so that $\xi(e^{J(z)}x) = \xi(x)$ for all $x \in \mathcal{V}$ and $z \in \mathcal{Z}$, then $\phi$ is an inner derivation.*

*Proof.* Pick $v_1 \in \mathcal{V}$ and denote by $V_1$ the subspace of $\mathcal{V}$ spanned by all vectors

$$\{ e^{J(z_1)}e^{J(z_2)} \cdots e^{J(z_k)}v_1 \mid z_1, z_2, \ldots, z_k \in \mathcal{Z},\ k = 0, 1, 2, \ldots \}.$$

Then, pick $v_2 \in \mathcal{V}$ such that $v_2 \perp V_1$. Denote by $V_2$ the subspace of $\mathcal{V}$ spanned by

$$\{ e^{J(z_1)}e^{J(z_2)} \cdots e^{J(z_k)}v_2 \mid z_1, z_2, \ldots, z_k \in \mathcal{Z},\ k = 0, 1, 2, \ldots \}.$$

We claim that

$$V_1 \perp V_2 \quad \text{and} \quad [V_1, V_2] = 0. \tag{3.16.1}$$

In fact, let $x = e^{J(z_1)} \cdots e^{J(z_k)}v_1 \in V_1$ and $y = e^{J(\tilde{z}_1)} \cdots e^{J(\tilde{z}_l)}v_2 \in V_2$. Then

$$\begin{aligned}\langle x, y\rangle &= \langle e^{J(z_1)} \cdots e^{J(z_k)}v_1, e^{J(\tilde{z}_1)} \cdots e^{J(\tilde{z}_l)}v_2\rangle \\ &= \langle e^{J(-\tilde{z}_l)} \cdots e^{J(-\tilde{z}_1)}e^{J(z_1)} \cdots e^{J(z_k)}v_1, v_2\rangle \\ &= 0,\end{aligned}$$

since $v_2 \perp V_1$. Therefore, $V_1 \perp V_2$. Next, for $x \in V_1$, $y \in V_2$, and $z \in \mathcal{Z}$, we have $\langle e^{tJ(z)}x, y\rangle = 0$ since $e^{tJ(z)}x \in V_1$. Taking the derivative with respect to $t$ at $t = 0$ in this identity, we get $\langle J(z)x, y\rangle = 0$, that is, $\langle z, [x, y]\rangle = 0$. Since $z \in \mathcal{Z}$ is arbitrary, we conclude that $[x, y] = 0$. Therefore, we have confirmed the claim (3.16.1).

Repeating this process, we can find a finite sequence of vectors $v_1, v_2, \ldots, v_p \in \mathcal{V}$ such that

$$\mathcal{V} = V_1 \oplus V_2 \oplus \cdots \oplus V_p,$$

where $V_i$ is the subspace spanned by $\{ e^{J(z_1)}e^{J(z_2)} \cdots e^{J(z_k)}v_i \mid z_1, z_2, \ldots, z_k \in \mathcal{Z}$, $k = 0, 1, 2, \ldots \}$. Moreover, $V_i \perp V_j$ and $[V_i,\ V_j] = 0$ for $i \neq j$.

Since $\phi$ is linear and since $\xi(e^{J(z)}v) = \xi(v)$ for all $z \in \mathcal{Z}$ and $v \in \mathcal{V}$ by hypothesis, we see that $\phi(x) = [\xi(v_i), x]$ for all $x \in \mathcal{V}_i$. Thus, letting $\xi_i$ be the orthogonal projection of $\xi(v_i)$ to $V_i$, (3.16.1) implies that

$$\phi(x) = [\xi_i, x] \quad \text{for all } \ x \in \mathcal{V}_i. \tag{3.16.2}$$

Set $\xi = \xi_1 + \xi_2 + \cdots + \xi_p$. For $x \in \mathcal{V}$, write $x = x_1 + x_2 + \cdots + x_p$ with $x_i \in V_i$. By (3.16.1) and (3.16.2),

$$\begin{aligned}\phi(x) &= \Sigma_i \phi(x_i) \\ &= \Sigma_i [\xi_i,\ x_i] \\ &= \Sigma_i [\xi,\ x_i] \\ &= [\xi,\ x].\end{aligned}$$

This proves that $\phi$ is an inner derivation of $\mathcal{N}$. □

*Proof of Theorems 3.10 and 3.11.* As remarked in the introduction, it is enough to prove rigidity within the class of all compact two-step nilmanifolds. Let $(\Gamma\backslash N, g)$

be a compact two-step nilmanifold that is either nonsingular or strongly in resonance, and let $(\Gamma^*\backslash N^*, g^*)$ be a compact two-step nilmanifold whose geodesic flow is $C^2$-conjugate to that of $(\Gamma\backslash N, g)$. By Theorem 2.12, we may assume that $(\Gamma^*\backslash N^*, g^*) = (\Phi(\Gamma)\backslash N, g)$ for some almost inner automorphism $\Phi$ of $N$. We only need to show that the corresponding almost inner derivation $\phi$ is inner. We use the notation of 2.3, 2.8, 2.10, and 3.13.

According to Proposition 2.11,

$$\phi(v) = \left[\bar{B}\left(e, \frac{v}{|v|}\right), v\right] \quad \text{for } 0 \neq v \in \mathcal{V}.$$

On the other hand, since $\phi$ is linear,

$$\begin{aligned}\phi(v) &= -\phi(-v)\\ &= -\left[\bar{B}\left(e, -\frac{v}{|v|}\right), -v\right]\\ &= \left[\bar{B}\left(e, -\frac{v}{|v|}\right), v\right].\end{aligned}$$

Therefore, writing

$$\tilde{B}(v) = \frac{1}{2}\left(\bar{B}\left(e, \frac{v}{|v|}\right) + \bar{B}\left(e, -\frac{v}{|v|}\right)\right), \tag{3.10.1}$$

we have

$$\phi(v) = [\tilde{B}(v), v] \quad \text{for } 0 \neq v \in \mathcal{V}.$$

According to Proposition 3.16, we need only show that

$$\tilde{B}(e^{J(z)}v) = \tilde{B}(v)$$

for all $z \in \mathcal{Z}$ and $0 \neq v \in \mathcal{V}$.

Note that, under the hypotheses of either theorem, $N$ is nonsingular. By Lemma 3.12 and Proposition 3.14, for $(n, v + z) \in S(N)$ we have

$$\begin{aligned}&B(G^{t/s}(n, \cos(s)v + \sin(s)z) - B(n, \cos(s)v + \sin(s)z)\\ &= \left(\exp\left[tJ\left(\frac{H(n, \cos(s)v + \sin(s)z)}{s}\right)\right] - \mathrm{Id}\right)\\ &\quad \times J^{-1}(H(n, \cos(s)v + \sin(s)z))\, I(n, \cos(s)v + \sin(s)z)\\ &\quad -\left(\exp\left[tJ\left(\frac{\sin(s)}{s}z\right)\right] - \mathrm{Id}\right) J^{-1}(\sin(s)z)\cos(s)v\\ &= \left(\exp\left[tJ\left(\frac{H(n, \cos(s)v + \sin(s)z)}{s}\right)\right] - \mathrm{Id}\right)\\ &\quad \times J^{-1}\left(\frac{H(n, \cos(s)v + \sin(s)z}{s}\right)\frac{I(n, \cos(s)v + \sin(s)z)}{s}\\ &\quad -\left(\exp\left[tJ\left(\frac{\sin(s)}{s}z\right)\right] - \mathrm{Id}\right) J^{-1}\left(\frac{\sin(s)}{s}z\right)\frac{\cos(s)}{s}v.\end{aligned}$$

The last equality follows from the fact that $J^{-1}(z/s) = sJ^{-1}(z)$. (Recall we are writing $J^{-1}(z)$ for $(J(z))^{-1}$.)

Define the average $\bar{I}$ of $I$ over the torus $T = (\Gamma \cap [N,N])\backslash[N,N]$ in the same way that $\bar{A}$ and $\bar{B}$ were defined in 2.10. On the left-hand side of the identity above, first take the average over $T$ as in 2.10 and then let $s \to 0$, using the second assertion of 2.11 to see that the limit makes sense. On the right-hand side, first let $s \to 0$ and then average over $T$. (The bounded convergence theorem assures that this procedure is equivalent to averaging first over $T$ and then taking the limit.) Using Lemma 3.15(a) and (b) and Proposition 2.7, we obtain

$$\bar{B}(n, e^{tJ(z)}v) - \bar{B}(n,v) = (e^{tJ(z)} - \mathrm{Id})\, J^{-1}(z) \frac{d}{ds}\bigg|_{s=0} \bar{I}(n, \cos(s)v + \sin(s)z). \tag{3.10.2}$$

By Proposition 2.11, $\bar{B}(n,v)$ is independent of $n$; thus, so is

$$\frac{d}{ds}\bigg|_{s=0} \bar{I}(n, \cos(s)v + \sin(s)z).$$

Therefore, by Lemma 3.15(c),

$$e^{tJ(z)} \frac{d}{ds}\bigg|_{s=0} \bar{I}(n, \cos(s)v + \sin(s)z) = \frac{d}{ds}\bigg|_{s=0} \bar{I}(n, e^{tJ(z)}\cos(s)v + \sin(s)z). \tag{3.10.3}$$

If $N$ is strongly in resonance, we can pick a $t$ such that $e^{tJ(z)} = -\mathrm{Id}$. If $N$ is irrational, there is a sequence $\{t_i\}$ such that $\lim_{i\to\infty} e^{t_i J(z)} = -\mathrm{Id}$. In either case, (3.10.3) implies that

$$-\frac{d}{ds}\bigg|_{s=0} \bar{I}(n, \cos(s)v + \sin(s)z) = \frac{d}{ds}\bigg|_{s=0} \bar{I}(n, -\cos(s)v + \sin(s)z).$$

Substituting this identity into (3.10.2) with $t = 1$ and recalling (3.10.1), we have $\tilde{B}(e^{J(z)}v) - \tilde{B}(v) = 0$. This completes the proofs of Theorems 3.10 and 3.11. □

*Proof of Lemma 3.15.* (a) The first equality in (a) follows from Proposition 2.7. In particular, the limit exists. According to Proposition 3.14(a),

$$\begin{aligned} &I(n\sigma(t, \cos(s)v + \sin(s)z, e^{tJ(\sin(s)z)}\cos(s)v + \sin(s)z) \\ &\qquad = e^{tJ(H(n,\cos(s)v+\sin(s)z))} I(n, \cos(s)v + \sin(s)z). \end{aligned}$$

Replacing $t$ by $t/s$ in this equality yields

$$\begin{aligned} &I\left(n\sigma\left(\frac{t}{s}, \cos(s)v + \sin(s)z\right), \exp\left[tJ\left(\frac{\sin(s)z}{s}\right)\right]\cos(s)v + \sin(s)z\right) \\ &\qquad = \exp\left[tJ\left(\frac{H(n,\cos(s)v+\sin(s)z)}{s}\right)\right] I(n, \cos(s)v + \sin(s)z). \end{aligned} \tag{3.15.1}$$

Since $\Gamma\backslash N$ is compact we see that, for an arbitrary sequence $s_n \to 0$, there exists a subsequence $s_{n_i}$ such that $n\sigma(t/s_{n_i}, \cos(s_{n_i})v + \sin(s_{n_i})z) \to n^* \bmod(\Gamma)$ for some element $n^*$ in $N$ ($n^*$ may depend on the subsequence). Therefore, since $I(\gamma n, v+z) = I(n, v+z)$ for $\gamma \in \Gamma$ (see Proposition 2.9(c)), we have

$$\lim_{i\to\infty} I\left(n\sigma\left(\frac{t}{s_{n_i}}, \cos(s_{n_i})v + \sin(s_{n_i})z\right),\right.$$
$$\left.\exp\left[tJ\left(\frac{\sin(s_{n_i})z}{s_{n_i}}\right)\right]\cos(s_{n_i})v + \sin(s_{n_i})z\right)$$
$$= I(n^*, e^{tJ(z)}v) = e^{tJ(z)}v,$$

where the last equality follows from Proposition 2.7. Therefore,

$$\lim_{s\to 0} I\left(n\sigma\left(\frac{t}{s}, \cos(s)v + \sin(s)z\right), \exp\left[tJ\left(\frac{\sin(s)z}{s}\right)\right]\cos(s)v + \sin(s)z\right)$$
$$= e^{tJ(z)}v. \quad (3.15.2)$$

Using first Proposition 2.7 and then (3.15.1) and (3.15.2), we see that

$$\lim_{s\to 0} \exp\left[tJ\left(\frac{H(n, \cos(s)v + \sin(s)z)}{s}\right)\right]v$$
$$= \lim_{s\to 0} \exp\left[tJ\left(\frac{H(n, \cos(s)v + \sin(s)z)}{s}\right)\right]I(n, \cos(s)v + \sin(s)z) = e^{tJ(z)}v,$$

so

$$\exp\left[tJ\left(\lim_{s\to 0} \frac{H(n, \cos(s)v + \sin(s)z)}{s}\right)\right]v = e^{tJ(z)}v$$

for all $t$. Therefore,

$$J\left(\lim_{s\to 0} \frac{H(n, \cos(s)v + \sin(s)z)}{s}\right)v = J(z)v;$$

that is,

$$J\left(\lim_{s\to 0} \frac{H(n, \cos(s)v + \sin(s)z)}{s} - z\right)v = 0.$$

Since $N$ is nonsingular, $\lim_{s\to 0} H(n, \cos(s)v + \sin(s)z)/s = z$.

(b) First, we show that $\lim_{s\to 0} \frac{d}{ds}(H(n, \cos(s)v + \sin(s)z)/s)$ exists.

By Proposition 2.7, $H(n, v) = 0$ and by (a),

$$\lim_{s\to 0} \frac{H(n, \cos(s)v + \sin(s)z)}{s} = z.$$

Temporarily writing $h(s) = H(n, \cos(s)v + \sin(s)z)$, we have

$$\lim_{s\to 0} \frac{d}{ds}\left(\frac{H(n, \cos(s)v + \sin(s)z)}{s}\right) = \lim_{s\to 0} \frac{sh'(s) - h(s)}{s^2}$$
$$= \frac{1}{2}h''(0)$$
$$= \frac{1}{2}\frac{d^2}{ds^2}\bigg|_{s=0} H(n, \cos(s)v + \sin(s)z),$$

where the third line follows from L'Hôpital's rule.

Now, taking the derivative with respect to $s$ in the identity (3.15.1) and then letting $s \to 0$, we obtain (again using part (a) and the fact that $I(n, v) = v$)

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} & I\left(n\sigma\left(\frac{t}{s}, \cos(s)v+\sin(s)z\right), e^{tJ(\tilde{s}z)}\cos(s)v+\sin(s)z\right) \\
&= te^{tJ(z)}J\left(\frac{d}{ds}\Big|_{s=0}\frac{H(n,\cos(s)v+\sin(s)z)}{s}\right)v \\
&\quad + e^{tJ(z)}\frac{d}{ds}\Big|_{s=0} I(n,\cos(s)v+\sin(s)z).
\end{aligned}$$

(*Note:* For ease of notation, here and in the sequel we use $\tilde{s}$ to denote $\frac{\sin(s)}{s}$.) Equivalently,

$$\begin{aligned}
J&\left(\frac{d}{ds}\Big|_{s=0}\frac{H(n,\cos(s)v+\sin(s)z)}{s}\right)v \\
&= -\frac{1}{t}\frac{d}{ds}\Big|_{s=0} I(n,\cos(s)v+\sin(s)z) \\
&\quad + \frac{1}{t}e^{-tJ(z)}\frac{d}{ds}\Big|_{s=0} I\left(n\sigma\left(\frac{t}{s}, \cos(s)v+\sin(s)z\right),\right. \\
&\qquad\qquad \left. e^{tJ(\tilde{s}z)}\cos(s)v+\sin(s)z\right). \qquad (3.15.3)
\end{aligned}$$

Extend the domain of the function $I$ from $S(N,g) = N\times S(\mathcal{N})$ to $T(N)-0 = N\times(\mathcal{N}-0)$ by requiring that $I$ be homogeneous of degree one in the second factor. Now, by Proposition 2.7 and (3.15.2),

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0} & I\left(n\sigma\left(\frac{t}{s}, \cos(s)v+\sin(s)z\right), e^{tJ(\tilde{s}z)}\cos(s)v+\sin(s)z\right) \\
&= \lim_{s\to 0}\frac{1}{s}\left[I\left(n\sigma\left(\frac{t}{s}, \cos(s)v+\sin(s)z\right), e^{tJ(\tilde{s}z)}\cos(s)v+\sin(s)z\right)\right. \\
&\qquad\qquad \left. - e^{tJ(z)}v\right] \\
&= \lim_{s\to 0}\frac{1}{s}\left[I\left(n\sigma\left(\frac{t}{s}, \cos(s)v+\sin(s)z\right), e^{tJ(\tilde{s}z)}\cos(s)v+\sin(s)z\right)\right. \\
&\qquad\qquad \left. - e^{tJ(\tilde{s}z)}\cos(s)v\right] \\
&= \lim_{s\to 0}\frac{1}{s}\left[I\left(n\sigma\left(\frac{t}{s}, \cos(s)v+\sin(s)z\right), e^{tJ(\tilde{s}z)}\cos(s)v+\sin(s)z\right)\right. \\
&\qquad\qquad \left. - I\left(n\sigma\left(\frac{t}{s}, \cos(s)v+\sin(s)z\right), e^{tJ(\tilde{s}z)}\cos(s)v\right)\right] \\
&= \lim_{s\to 0}\frac{d}{d\lambda}\Big|_{\lambda=\bar{s}} I\left(n\sigma\left(\frac{t}{s}, \cos(s)v+\sin(s)z\right), e^{tJ(\tilde{s}z)}\cos(s)v+\sin(\lambda)z\right),
\end{aligned}$$
$$(3.15.4)$$

where $0\le \bar{s}\le s$.

Using this equality and the facts that $I(\gamma n, v+z) = I(n, v+z)$ and $F$ is $C^2$, we see that

$$\frac{d}{ds}\bigg|_{s=0} I\left(n\sigma\left(\frac{t}{s}, \cos(s)v + \sin(s)z\right), e^{tJ(\tilde{s}z)}\cos(s)v + \sin(s)z\right)$$

is bounded for all $t \in \mathbf{R}$. This, together with the fact that the $e^{tJ(z)}$ are unitary operators, shows that the right-hand side of (3.15.3) goes to zero when $t \to \infty$. Therefore, the identity (3.15.3) becomes

$$J\left(\frac{d}{ds}\bigg|_{s=0} \frac{H(n, \cos(s)v + \sin(s)z)}{s}\right)v = 0.$$

Since $N$ is nonsingular, we get $\frac{d}{ds}\big|_{s=0}[H(n, \cos(s)v + \sin(s)z)/s] = 0$.

(c) To derive (c), first put $J\left(\frac{d}{ds}\big|_{s=0}[H(n, \cos(s)v + \sin(s)z)/s]\right)v = 0$ back into the identity (3.15.3). Then use (3.15.4) and the fact that $\frac{d}{ds}\big|_{s=0}\cos(s) = 0$. □

## 4. Nilmanifolds of Heisenberg Type

Because the compact nilmanifolds of Heisenberg type (as defined in 3.2) are strongly in resonance, Theorem 3.11 shows that they are $C^2$-rigid within the class of all nilmanifolds. We will now strengthen this result to $C^0$-rigidity.

4.1. Lemma [E]. *Suppose $(N, g)$ is of Heisenberg type. Let $\sigma(t, u)$ denote the geodesic in $N$ with $\sigma(0, u) = e$ and $\sigma'(0, u) = u$. Writing $u = x + z$ with $x \in \mathcal{V}$ and $z \in \mathcal{Z}$ and writing $\sigma(t, u) = \exp(x(t) + z(t))$ with $x(t) \in \mathcal{V}$ and $z(t) \in \mathcal{Z}$ as in Lemma 3.12, we have*

$$z(t) = t\left(1 + \frac{|x|^2}{2|z|^2}\right)z - \frac{\sin(t|z|)|x|^2}{2|z|^3}z.$$

4.2. Lemma. *In the notation of 2.8 and 3.13, if $(N, g)$ is of Heisenberg type then*

$$H(n, v+z) = z$$

*for all $(n, v+z) \in S(N)$, where $v \in \mathcal{V}$ and $z \in \mathcal{Z}$.*

*Proof*. For $\gamma \in \Gamma \cap [N, N]$, let

$$\begin{aligned} C(\gamma) = \{\, (n, v+z) \in S(N) : v \neq 0 \text{ and} \\ G^{t_0}(n, v+z) = dL_\gamma(n, v+z) \text{ for some } t_0 > 0\,\} \end{aligned} \qquad (4.2.1)$$

and let

$$C = \bigcup_{\gamma \in \Gamma \cap [N,N]} C(\gamma).$$

Note that $C(\gamma)$ is the set of lifts of noncentral vectors in $S(\Gamma\backslash N)$ tangent to closed geodesics in the free homotopy class $[\gamma]$. By Lemma 2.7, $z$ must be nonzero whenever $(n, v+z) \in C(\gamma)$. We will show that $H(n, v+z) = z$ for all $(n, v+z) \in C$ and that $C$ is dense in $S(N)$, thus proving the theorem.

By Lemmas 3.12 and 4.1 and (4.2.1), $(n, v+z) \in C(\gamma)$ if and only if

$$\begin{aligned}(\gamma n, v+z) &= G^{t_0}(n, v+z)\\ &= \left(n \exp\left[(e^{t_0 J(z)} - \mathrm{Id})\, J^{-1}(z) v + t_0\left(1 + \frac{|v|^2}{2|z|^2}\right) z - \frac{\sin(t_0|z|)|v|^2}{2|z|^3} z\right],\right.\\ &\qquad\qquad \left. e^{t_0 J(z)} v + z\right) \qquad (4.2.2)\end{aligned}$$

for some $t_0 > 0$. In particular, when $(n, v+z) \in C(\gamma)$ we must have $e^{t_0 J(z)} v = v$. Since $v \neq 0$, we see from Definition 3.2(d) of nilmanifolds of Heisenberg type that $t_0$ must satisfy

$$t_0 = \frac{2\pi k}{|z|} \qquad (4.2.3)$$

for some positive integer $k$ and that $e^{t_0 J(z)} = \mathrm{Id}$. Equation (4.2.2) and the fact that $|v|^2 + |z|^2 = 1$ then imply that

$$\gamma = \exp\left[t_0\left(1 + \frac{|v|^2}{2|z|^2}\right) z\right] = \exp\left[t_0\left(\frac{|z|^2 + 1}{2|z|^2}\right) z\right]. \qquad (4.2.4)$$

Conversely, an element $(n, v+z) \in S(N)$ with $v$ and $z$ nonzero lies in $C(\gamma)$ if there exists a positive integer $k$ such that (4.2.4) holds with $t_0$ defined by (4.2.3). Thus,

$$\begin{aligned}C = \Bigg\{ &(n, v+z) \in S(N) : v \neq 0 \text{ and}\\ &\frac{2\pi k}{|z|}\left(\frac{|z|^2+1}{2|z|^2}\right) z \in \log(\Gamma \cap [N, N]) \text{ for some positive integer } k \Bigg\}.\end{aligned} \qquad (4.2.5)$$

An elementary argument using (4.2.5) and the fact, mentioned in 1.3, that the rational span $\mathcal{Z}_Q$ of $\log(\Gamma \cap [N, N])$ is dense in $\mathcal{Z}$ shows that $C$ is dense in $S(N)$.

Define $\tilde{F}\colon S(N) \to S(N)$ and $\Phi$ as in 2.3(a). Write $\tilde{F}(n, v+z) = (n_1, v_1+z_1)$. By Proposition 2.7, we have $v_1 \neq 0$ whenever $v \neq 0$. Also from 2.3 and the fact that $\Phi(\gamma) = \gamma$ when $\gamma \in \Gamma \cap [N, N]$, we see that $\tilde{F}$ commutes both with $dL_\gamma$ and with $G^t$ for all $t \in \mathbf{R}$. Consequently, by (4.2.1), $\tilde{F}$ leaves $C(\gamma)$ invariant. Moreover, for $(n, v+z) \in C(\gamma)$, both $(n, v+z)$ and $(n_1, v_1+z_1)$ satisfy the condition of (4.2.1) for the same choice of $t_0$. Thus (4.2.4) implies that

$$\left(\frac{|z|^2+1}{2|z|^2}\right) z = \left(\frac{|z_1|^2+1}{2|z_1|^2}\right) z_1.$$

Consequently,

$$H(n, v+z)\ (= z_1) = \lambda(z)z \quad \text{with either } \lambda(z) = 1 \text{ or } \lambda(z) = \frac{1}{|z|^2}. \tag{4.2.6}$$

In summary, we have shown that (4.2.6) holds whenever $(n, v+z) \in C$. Because $C$ is dense in $S(N)$, to complete the proof of the lemma we need only show that the function $\lambda$ in (4.2.6) is identically one. By continuity of $H$, $\lambda$ is independent of $n$ and, in view of (4.2.5), also of $v$. Since $C$ is dense in $S(N)$, the continuity of $H$, in particular when $z = 0$, yields $\lambda \equiv 1$. □

4.3. THEOREM. *Compact nilmanifolds of Heisenberg type are $C^0$-geodesically rigid within the class of all compact nilmanifolds.*

*Proof*. We use the notation of the first two paragraphs of the proof of Theorems 3.10 and 3.11. As in that proof, we need only show that $\tilde{B}(e^{J(z)}v) = \tilde{B}(v)$ for all $z \in \mathcal{Z}$ and all nonzero $v \in \mathcal{V}$.

By Proposition 3.14, Lemma 3.12, and Lemma 4.2, for $(n, v+z) \in S(N)$ we have

$$B(G^t(n, v+z)) = B(n, v+z) + (e^{tJ(z)} - \mathrm{Id})J^{-1}(z)(I(n, v+z) - v) \tag{4.3.1}$$

and

$$\begin{aligned} &B(G^t(n, -v+z)) \\ &\qquad = B(n, -v+z) + (e^{tJ(z)} - \mathrm{Id})J^{-1}(z)(I(n, -v+z) + v). \end{aligned} \tag{4.3.2}$$

By Proposition 3.14 and Lemma 4.2,

$$I(G^t(n, v+z)) = e^{tJ(z)}I(n, v+z).$$

Taking $t = \pi/|z|$ and using Lemma 3.12(a) and the fact that $(\Gamma\backslash N, g)$ is of Heisenberg type, we obtain

$$I\left(n\sigma\left(\frac{\pi}{|z|}, v+z\right), -v+z\right) = -I(n, v+z), \tag{4.3.3}$$

where $\sigma(t, v+z)$ is the geodesic in $N$ with $\sigma(0) = e$ and $\sigma'(0) = v+z$.

By (4.3.1)–(4.3.3), we have

$$\begin{aligned} &B(G^t(n, v+z)) + B\left(G^t\left(n\sigma\left(\frac{\pi}{|z|}, v+z\right), -v+z\right)\right) \\ &\qquad = B(n, v+z) + B\left(n\sigma\left(\frac{\pi}{|z|}, v+z\right), -v+z\right). \end{aligned}$$

Hence, in the notation of 2.10,

$$\begin{aligned} &\bar{B}(G^t(n, v+z)) + \bar{B}\left(G^t\left(n\sigma\left(\frac{\pi}{|z|}, v+z\right), -v+z\right)\right) \\ &\qquad = \bar{B}(n, v+z) + \bar{B}\left(n\sigma\left(\frac{\pi}{|z|}, v+z\right), -v+z\right). \end{aligned}$$

After replacing $v$ by $\cos(s)v$, $z$ by $\sin(s)z$, and $t$ by $t/\sin(s)$ in this equality, recalling that $\bar{B}(n, v)$ is independent of $n$ (Proposition 2.11), and applying Lemma 3.12(a), by letting $s \to 0$ we obtain

$$\bar{B}(e, e^{tJ(z)}v) + \bar{B}(e, -e^{tJ(z)}v) = \bar{B}(e, v) + \bar{B}(e, -v).$$

That is, $\tilde{B}(e^{J(z)}v) = \tilde{B}(v)$, thus completing the proof of Theorem 4.3. □

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