New Cases of Almost Periodic Factorization of Triangular Matrix Functions

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1. Introduction

A function f is said to be an *almost periodic polynomial* if it can be expressed in the form

$$f(x) = \sum_{j=1}^{m} c_j e^{i\lambda_j x} \quad \text{with } c_j \in \mathbb{C} \text{ and } \lambda_j \in \mathbb{R}. \tag{1.1}$$

The set of all almost periodic polynomials forms an algebra AP_P. The closure of AP_P under the uniform norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$ gives the algebra AP of almost periodic functions. In other words, AP is the C^*-subalgebra of $L^\infty(\mathbb{R})$ generated by all functions $e_\lambda(x) = e^{i\lambda x}$, $\lambda \in \mathbb{R}$.

The *mean value* of an almost periodic function is defined as

$$\mathbf{M}(f) = \lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} f(x)\,dx,$$

and the *Fourier coefficient* $\mathbf{M}_\lambda(f) := \mathbf{M}(e_{-\lambda} f)$. (These definitions are standard; see [4] and [14].) Of course, $\mathbf{M}(f) = \mathbf{M}_0(f)$. For $f \in AP_P$ written in the form (1.1), $\mathbf{M}_{\lambda_j}(f) = c_j$.

The *Fourier spectrum* of f, denoted $\Omega(f)$, is defined as $\{\lambda \in \mathbb{R} : \mathbf{M}_\lambda(f) \neq 0\}$.

We use AP^+ (resp. AP^-) to denote the subalgebra consisting of all $f \in AP$ such that $\Omega(f) \subset [0,\infty)$ (resp. $(-\infty, 0]$). A matrix function is said to be in AP or in $AP^\pm$ if all of its entries are. We say that an $n \times n$ matrix AP function G is *AP-factorable* if it can be represented as a product

$$G(x) = G^+(x)\Lambda(x)G^-(x), \tag{1.2}$$

where $(G^+)^{\pm 1} \in AP^+$, $(G^-)^{\pm 1} \in AP^-$, and $\Lambda = \operatorname{diag}[e_{\lambda_1}, \dots, e_{\lambda_n}]$, $\lambda_j \in \mathbb{R}$. Factorization (1.2) was introduced in [10]. It was also observed there that, if G is periodic with a period T, then a simple change of variable $t = e^{ixT/2\pi}$ reduces

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(1.2) to a classical Wiener–Hopf factorization of matrix functions that are continuous on the unit circle. The latter factorization is important, in particular because of its applications to Wiener–Hopf equations (i.e., convolution type equations on the half-line); see [8], an early influential paper on the subject, and [7], a recent exposition. As it happens, a more general AP factorization arises naturally [10; 11] when convolution type equations on finite intervals are considered. Other applications of AP factorization include inverse scattering problems [1] and signal processing [15]. It is also used in extension problems for positive and contractive (matrix) AP functions [18; 17], as well as functions on a torus [2].

Some properties of the AP factorization are very similar to those of the Wiener–Hopf factorization and can be established analogously. In particular, if an AP factorization exists then the set of λ_j in (1.2) is defined uniquely; λ_j are referred to as the *partial AP indices* of G. If the partial AP indices are all equal to zero, then the multiples G^+ and G^- in (1.2) are defined up to a transformation $G^+ \mapsto G^+C$ and $G^- \mapsto C^{-1}G^-$ with a nonsingular constant matrix C, so that $\mathbf{d}(G) = \mathbf{M}(G^+)\mathbf{M}(G^-)$ is defined uniquely (see [10]). On the other hand, the existence of AP factorization and its explicit construction are much more complicated than those of the usual Wiener–Hopf factorization of continuous matrix functions. These questions are nontrivial (and still open) even for 2×2 matrices of the form

$$G_f(x) = \begin{bmatrix} e^{i\lambda x} & 0 \\ f(x) & e^{-i\lambda x} \end{bmatrix}, \tag{1.3}$$

where $\lambda > 0$ and f is an almost periodic polynomial. By the way, such matrices are of special importance because they arise in the just mentioned applications to convolution type (in particular, difference) equations in the case of one interval of length λ. We will refer to λ in (1.3) as the *diagonal exponent* of G_f. We prove AP-factorability of several new classes of matrix functions of the form (1.3), establish necessary and sufficient conditions for having zero partial AP indices, and in some cases compute $\mathbf{d}(G_f)$. This is done mainly in Sections 3, 4, and 5. Briefly, the classes of matrix functions are described in terms of the Fourier spectrum of f. For example, we prove that if $\Omega(f) \subset \{-\nu\} \cup R \cup [\lambda - \nu, \lambda)$, where $0 < \nu < \frac{1}{2}\lambda$ and R is a suitably chosen interval in $(0, \lambda - 2\nu]$, then G_f is AP-factorable (see Theorem 3.1). In Section 4 we study the case when f is a trinomial: $f(x) = c_{-1}e_{-\nu} + c_0 e_\mu + c_1 e_\delta$. We establish new cases of AP-factorability of G_f when $\mu \neq 0$. Some generalizations of the trinomial case are given in Section 5. The structure-preserving transformation introduced in [3] is the main tool in our investigation. It turns out that virtually all previously known cases of AP-factorability of matrix functions (1.3) can be verified using this transformation. The transformation is described in full detail in Section 2, where some previously known results are presented as well. The matrix generalization of the functions (1.3), where diagonal entries are changed to $e^{\pm i\lambda x}I_n$ and f is an almost periodical $n \times n$ matrix, is studied in Section 6. Here, AP-factorization is proved for several classes of such matrix functions. New phenomena appear for the matrix generalization of (1.3); for example, in contrast with (1.3), not every matrix function of the form

$$G(x) = \begin{bmatrix} e^{i\lambda x} I_n & 0 \\ F(x) & e^{-i\lambda x} I_m \end{bmatrix},$$

where $F(x)$ is an almost periodic polynomial $n \times m$ matrix with nonnegative Fourier spectrum and $m + n > 2$, admits an AP-factorization. Applications to the Fredholm properties of systems of convolution equations on a finite interval (both continuous and discrete types) are given in Section 7.

Certain properties of almost periodic matrix functions $G(x)$ and their AP-factorability are well known and can be easily established.

LEMMA 1.1. *For any two matrix functions P and Q with $P^{\pm 1} \in AP^+$ and $Q^{\pm 1} \in AP^-$, PGQ and G are simultaneously AP-factorable (PGQ is AP-factorable if and only if G is) and have the same partial AP indices, and if the partial AP indices of G are zero then $\mathbf{d}(PGQ) = \mathbf{M}(P)\mathbf{d}(G)\mathbf{M}(Q)$.*

LEMMA 1.2. *G and G^* (the conjugate transpose of G) are AP-factorable only simultaneously, the partial AP indices of G^* are the negatives of the partial AP indices of G, and if the partial AP indices of G are zero then $\mathbf{d}(G^*) = (\mathbf{d}(G))^*$.*

For the matrix function G_f of the form (1.3), we have more specific information.

LEMMA 1.3. *Let G_f be given as in* (1.3).

(i) *If G_f is AP-factorable, then its partial AP indices are of the form $\pm\alpha$, where $0 \le \alpha \le \lambda$.*

(ii) *G_f and $G_{\tilde{f}}$ are simultaneously AP-factorable with the same partial AP indices, and*

$$\mathbf{d}(G_{\tilde{f}}) = J(\mathbf{d}(G_f))^* J, \quad \text{where } J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(iii) *Define $f'(x) = \sum_{\nu \in \Omega(f) \cap (-\lambda, \lambda)} \mathbf{M}_\nu(f) e^{i\nu x}$. Then*

$$G_{f'} = \begin{bmatrix} e_\lambda & 0 \\ f' & e_{-\lambda} \end{bmatrix} \quad \text{and} \quad G_f = \begin{bmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{bmatrix}$$

are simultaneously factorable with the same partial AP indices. If the partial AP indices are zero then, in addition,

$$\mathbf{d}(G_{f'}) = \begin{bmatrix} 1 & 0 \\ -\mathbf{M}_\lambda(f) & 1 \end{bmatrix} \mathbf{d}(G) \begin{bmatrix} 1 & 0 \\ -\mathbf{M}_{-\lambda}(f) & 1 \end{bmatrix}.$$

Part (i) follows from the general fact (discussed in [10]) that the sum of partial AP indices of any AP-factorable almost periodic $n \times n$ matrix function with constant determinant will be zero, and from [13, Lemma 1.2]. Parts (ii) and (iii) can also be found in [10], though they are not formulated there as separate statements.

In view of Lemma 1.3(iii), it will be implicitly assumed throughout the rest of the paper that $\Omega(f) \subset (-\lambda, \lambda)$.

Throughout the paper we denote by $\mathbb{Z}$ and $\mathbb{Z}^+$ the set of integers and the set of nonnegative integers, respectively.

2. The Transformation

The transformation that is the principal tool of the present paper is introduced in [3] as Theorem 3.1. For lack of a better name, we will refer to the technique as the *BKST transformation* after the authors of the paper. We describe now the BKST transformation in detail. We begin with a matrix in the form (1.3) and f written as

$$f(x) = ae^{-i\nu x}\left(1 - \sum_{k=1}^{m} b_k e^{i\gamma_k x}\right), \tag{2.1}$$

where $a \neq 0$, $0 < \gamma_1, \gamma_2, \ldots, \gamma_m < \lambda + \nu$, and $\nu \in (-\lambda, \lambda)$. For convenience, it will be assumed that γ_j are arranged in the increasing order: $\gamma_1 < \cdots < \gamma_m$. We let Γ be the m-vector $(\gamma_1, \ldots, \gamma_m)$, and N will denote any m-vector $(n_1, \ldots, n_m)$ with $n_j \in \mathbb{Z}^+$; $\langle N, \Gamma\rangle$ will be the usual inner product $n_1\gamma_1 + \cdots + n_m\gamma_m$, and $|N| = \sum n_j$. Finally, for any vector N and polynomial in the form (2.1), let

$$y_N = y_N(f) = \frac{(n_1 + n_2 + \cdots + n_m)!}{n_1!\, n_2! \cdots n_m!} b_1^{n_1} b_2^{n_2} \cdots b_m^{n_m}. \tag{2.2}$$

We define $M_1^+(x) = \sum_{N:\langle N,\Gamma\rangle<\lambda+\nu} y_N a^{-1} e^{i\langle N,\Gamma\rangle x}$ and

$$M_2^+(x) = \sum_{k=1}^{m} \sum_{N:\lambda+\nu-\gamma_k\leq\langle N,\Gamma\rangle<\lambda+\nu} y_N b_k e^{i(\langle N,\Gamma\rangle+\gamma_k-\lambda-\nu)x}.$$

Then direct calculation yields that

$$\begin{aligned}
e^{i\nu x}&M_1^+ f + e^{i(\lambda+\nu)x} M_2^+ \\
&= \left(1 - \sum_{k=1}^{m} b_k e^{i\gamma_k x}\right) \sum_{N:\langle N,\Gamma\rangle<\lambda+\nu} y_N e^{i\langle N,\Gamma\rangle x} \\
&\quad + \sum_{k=1}^{m} \sum_{N:\lambda+\nu-\gamma_k\leq\langle N,\Gamma\rangle<\lambda+\nu} y_N b_k e^{i(\langle N,\Gamma\rangle+\gamma_k)x} \\
&= \sum_{N:\langle N,\Gamma\rangle<\lambda+\nu} y_N e^{i\langle N,\Gamma\rangle x} - \sum_{k=1}^{m} b_k e^{i\gamma_k x} \sum_{N:\langle N,\Gamma\rangle<\lambda+\nu} y_N e^{i\langle N,\Gamma\rangle x} \\
&\quad + \sum_{k=1}^{m} \sum_{N:\lambda+\nu-\gamma_k\leq\langle N,\Gamma\rangle<\lambda+\nu} y_N b_k e^{i(\langle N,\Gamma\rangle+\gamma_k)x} \\
&= \sum_{N:\langle N,\Gamma\rangle<\lambda+\nu} y_N e^{i\langle N,\Gamma\rangle x} - \sum_{k=1}^{m} \sum_{N:\langle N,\Gamma\rangle<\lambda+\nu-\gamma_k} y_N b_k e^{i(\langle N,\Gamma\rangle+\gamma_k)x}. \qquad (2.3)
\end{aligned}$$

Every $N \neq 0$ in the left summation of (2.3) will correspond to a vector

$$N_k = (n_1, \ldots, n_k - 1, \ldots, n_m)$$

in the kth summation on the right for each k such that $n_k \neq 0$. If we let $n_{\alpha_1} \ldots n_{\alpha_q}$ denote the nonzero terms of N and $n = |N|$, then

$$\begin{aligned}\sum_{k:n_k\neq 0} & b_k y_{N_k} e^{i(\gamma_k+\langle N_k,\Gamma\rangle)x}\\
&= e^{i\langle N,\Gamma\rangle x}\Bigg(b_{\alpha_1}\frac{(n-1)!}{(n_{\alpha_1}-1)!\,n_{\alpha_2}!\cdots n_{\alpha_q}!}b_{\alpha_1}^{n_{\alpha_1}-1}b_{\alpha_2}^{n_{\alpha_2}}\cdots b_{\alpha_q}^{n_{\alpha_q}}+\cdots\\
&\qquad\qquad + b_{\alpha_q}\frac{(n-1)!}{n_{\alpha_1}!\cdots(n_{\alpha_q}-1)!}b_{\alpha_1}^{n_{\alpha_1}}\cdots b_{\alpha_q}^{n_{\alpha_q}-1}\Bigg)\\
&= e^{i\langle N,\Gamma\rangle x}\Bigg(b_{\alpha_1}^{n_{\alpha_1}}\cdots b_{\alpha_q}^{n_{\alpha_q}}\frac{(n-1)!\,n_{\alpha_1}+(n-1)!\,n_{\alpha_2}+\cdots+(n-1)!\,n_{\alpha_q}}{n_{\alpha_1}!\,n_{\alpha_2}!\cdots n_{\alpha_q}!}\Bigg)\\
&= e^{i\langle N,\Gamma\rangle x}\Bigg(b_{\alpha_1}^{n_{\alpha_1}}\cdots b_{\alpha_q}^{n_{\alpha_q}}\frac{n!}{n_{\alpha_1}!\,n_{\alpha_2}!\cdots n_{\alpha_q}!}\Bigg) = y_N e^{i\langle N,\Gamma\rangle x}.\end{aligned}\tag{2.4}$$

Hence, every term except $N=0$ in the summation on the left in (2.3) vanishes along with every term in the double summation on the right, and we are left with

$$e^{i\nu x}M_1^+(x)f(x)+e^{i(\lambda+\nu)x}M_2^+(x)=y_0e^{i0x}=1.\tag{2.5}$$

Let now

$$M^- = M_1^+e_{\nu-\lambda} = \sum_{N:\langle N,\Gamma\rangle<\lambda+\nu} y_N a^{-1}e^{i(\langle N,\Gamma\rangle-\lambda+\nu)x}$$

if $\nu\le 0$, and

$$M^-(x) = \sum_{N:\langle N,\Gamma\rangle\le\lambda-\nu} y_N a^{-1}e^{i(\langle N,\Gamma\rangle-\lambda+\nu)x}$$

if $\nu>0$. Then

$$f_1(x)\overset{\text{def}}{=}M_1^+(x)e^{-i\lambda x}-M^-(x)e^{-i\nu x}=\sum_{N:\langle N,\Gamma\rangle-\lambda\in(-\nu,\nu)} y_N a^{-1}e^{i(\langle N,\Gamma\rangle-\lambda)x}\tag{2.6}$$

if we agree, as usual, that $(-\nu,\nu)=\emptyset$ for $\nu\le 0$ and that a sum with an empty set of indices equals zero.

Set

$$A^+=\begin{bmatrix}-fe_\nu & e_{\lambda+\nu}\\ M_2^+ & M_1^+\end{bmatrix},\qquad A^-=\begin{bmatrix}-M^- & 1\\ 1 & 0\end{bmatrix}.$$

It is clear that $A^+\in AP^+$ and $A^-\in AP^-$; (2.5) shows that A^+ has determinant -1, as does A^-, and they are therefore invertible in AP^+ and AP^- respectively.

Matrix multiplication shows that

$$\begin{aligned}A^+G_fA^- &= \begin{bmatrix}-fe_\nu & e_{\lambda+\nu}\\ M_2^+ & M_1^+\end{bmatrix}\begin{bmatrix}e_\lambda & 0\\ f & e_{-\lambda}\end{bmatrix}\begin{bmatrix}-M^- & 1\\ 1 & 0\end{bmatrix}\\
&= \begin{bmatrix}-fe_{\lambda+\nu}+fe_{\lambda+\nu} & e_\nu\\ e_\lambda M_2^+ + fM_1^+ & M_1^+e_{-\lambda}\end{bmatrix}\begin{bmatrix}-M^- & 1\\ 1 & 0\end{bmatrix}\\
&= \begin{bmatrix}0 & e_\nu\\ e_{-\nu} & M_1^+e_{-\lambda}\end{bmatrix}\begin{bmatrix}-M^- & 1\\ 1 & 0\end{bmatrix}\\
&= \begin{bmatrix}e_\nu & 0\\ -M^-e_{-\nu}+M_1^+e_{-\lambda} & e_{-\nu}\end{bmatrix} = \begin{bmatrix}e_\nu & 0\\ f_1 & e_{-\nu}\end{bmatrix} := G_{f_1}.\end{aligned}$$

Observe also that, owing to (2.4),

$$\mathbf{M}(M_2^+) = \sum_{k=1}^{m} \sum_{N_k : \langle N_k, \Gamma\rangle = \lambda+\nu-\gamma_k} y_{N_k} b_k = \sum_{N:\langle N,\Gamma\rangle=\lambda+\nu} y_N .$$

This and Lemma 1.1 yield the following theorem.

THEOREM 2.1. *With the foregoing notation, G_f and G_{f_1} are AP-factorable only simultaneously, their partial AP indices coincide, and when the partial AP indices equal* 0 *we have*

$$\mathbf{d}(G_{f_1}) = \begin{bmatrix} -a & 0 \\ X & a^{-1} \end{bmatrix} \mathbf{d}(G_f) \begin{bmatrix} -Y & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.7}$$

where $X = \mathbf{M}(M_2^+) = \sum_{N:\langle N,\Gamma\rangle=\lambda+\nu} y_N$ and

$$Y = \mathbf{M}(M^-) = \begin{cases} \sum_{N:\langle N,\Gamma\rangle=\lambda-\nu} y_N a^{-1} & \textit{if } \nu > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

In comparison with [3], we have simplified the formula for X. Also, the case $\nu \le 0$ was not considered in [3] because it corresponds to the situation $\Omega(f) \subset [0,\infty)$ disposed of earlier in [10, Thm. 2.4]. We, however, decided to demonstrate how the BKST transformation can be applied to derive this result.

THEOREM 2.2. *Suppose $\Omega(f) \subset \mathbb{R}^+ := [0,\infty)$, and let μ be the smallest (leftmost) element of $\Omega(f)$. Then G_f is AP-factorable with partial AP indices equal to $\pm\mu$, and if $\mu = 0$ then*

$$\mathbf{d}(G) = \begin{bmatrix} 0 & -1/a \\ a & M \end{bmatrix}, \tag{2.8}$$

where $a = \mathbf{M}_0(f)$ and $M = \sum_{N:\langle N,\Gamma\rangle=\lambda} y_N(f)$.

Proof. Write f in the form (2.1) with $\nu = -\mu$. Applying the BKST transformation once yields, according to (2.6),

$$G_{f_1} = \begin{bmatrix} e_{-\mu} & 0 \\ 0 & e_\mu \end{bmatrix}. \tag{2.9}$$

Hence G_{f_1} (and therefore G_f) is AP-factorable with partial AP indices $\pm\mu$.

If $\mu = 0$ then (2.9) and (2.7) tell us that

$$\mathbf{d}(G_{f_1}) = I = \begin{bmatrix} -a & 0 \\ X & a^{-1} \end{bmatrix} \mathbf{d}(G_f) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and so

$$\mathbf{d}(G_f) = \begin{bmatrix} -1/a & 0 \\ X & a \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/a \\ a & X \end{bmatrix},$$

where $X = \sum_{N:\langle N,\Gamma\rangle=\lambda} y_N(f) := M$. Formula (2.8) follows. □

The BKST transformation can also be used to verify several other previous results on factorability (additional information is available in [16]); for the reader's convenience, we state some of them.

Suppose that one point of the Fourier spectrum of f is separated from the rest of the spectrum by a "big gap", a distance at least as great as the diagonal exponent. This type of matrix reduces in one BKST transformation to the case considered in Theorem 2.2, allowing explicit calculation of partial AP indices and $\mathbf{d}(G)$.

THEOREM 2.3. *If* $-\nu \in \Omega(f) \subset \{-\nu\} \cup [\lambda - \nu, \lambda)$ *and* $\nu > 0$, *then* G_f *is* AP-*factorable*; *if we let* μ = *the leftmost element of* $\Omega(f)\backslash\{-\nu\}$, *then the partial* AP *indices of* G_f *are* $\pm(\nu + \mu - \lambda)$. *For* $\nu + \mu - \lambda = 0$,

$$\mathbf{d}(G_f) = \begin{bmatrix} -1/p & 0 \\ \mathbf{M}_{-\nu}(f)M & -p \end{bmatrix}, \tag{2.10}$$

where

$$p = \mathbf{M}_{-\nu+\lambda}(f)/\mathbf{M}_{-\nu}(f) \quad \text{and} \quad M = \sum y_N(f)p^{-|N|},$$

with the summation over all $N = (0, n_2, \ldots, n_m)$ *such that* $\sum_{j=2}^{m} n_j\delta_j = \nu$ *for* $\{\delta_2, \ldots, \delta_m\} = \{\alpha - \mu : \alpha \in \Omega(f) \setminus \{-\nu, \mu\}\}$.

Proof. Write f as in (2.1). Applying the BKST transformation yields $\Gamma = (\gamma_1, \ldots, \gamma_m)$ with $\gamma_i \geq \lambda$. Thus, $\langle N, \Gamma\rangle \geq \lambda$ unless $\langle N, \Gamma\rangle = 0$, and so $\Omega(f_1) \subseteq \{\langle N, \Gamma\rangle - \lambda\} \cap (-\nu, \nu) \subset \mathbb{R}^+$. By Theorem 2.2, G_{f_1}, and therefore G_f, is AP-factorable. Its nonnegative partial index coincides with the smallest element $\gamma_i - \lambda$. The latter, by definition, equals $\nu + \mu - \lambda$.

Consider now the case of zero partial AP indices, that is, $\nu + \mu - \lambda = 0$. Letting $a = \mathbf{M}_{-\nu}(f)$, formula (2.6) shows that

$$\mathbf{M}_0(f_1) = -p/a.$$

By (2.8),

$$\mathbf{d}(G_{f_1}) = \begin{bmatrix} 0 & a/p \\ -p/a & M_1 \end{bmatrix}, \tag{2.11}$$

where

$$M_1 = \sum_{N' : \langle N', \Gamma'\rangle = \nu} y_{N'}(f_1).$$

On the other hand, by (2.7),

$$\mathbf{d}(G_{f_1}) = \begin{bmatrix} -a & 0 \\ X & a^{-1} \end{bmatrix} \mathbf{d}(G_f) \begin{bmatrix} -Y & 1 \\ 1 & 0 \end{bmatrix}, \tag{2.12}$$

where

$$X = \sum_{N : \langle N, \Gamma\rangle = \lambda + \nu} y_N(f); \qquad Y = \sum_{N : \langle N, \Gamma\rangle = \lambda - \nu} \frac{y_N(f)}{a}.$$

In our case, $Y = 0$. Since $\lambda \leq \gamma_j < \lambda + \nu < 2\lambda$, we have also $X = 0$. Comparing the formulas (2.11) and (2.12),

$$\mathbf{d}(G_f) = \begin{bmatrix} -1/p & 0 \\ aM_1 & -p \end{bmatrix}.$$

Since $\gamma_j \geq \lambda$ and $\lambda < \lambda + \nu < 2\lambda$, the only Fourier exponents of f_1 are the numbers $\gamma_1 - \lambda = 0 < \cdots < \gamma_m - \lambda$. Thus,

$$f_1 = \sum_{j=1}^{m} \frac{y_j(f)}{a} e_{\gamma_j - \lambda},$$

where $y_j(f) = b_j$. Consequently

$$M_1 = \sum_{N' : \langle N', \Gamma' \rangle = \nu} y_{N'},$$

where Γ' is the set $\{\gamma_2 - \lambda, \dots, \gamma_m - \lambda\}$, and for $N' = \{n'_2, \dots, n'_m\} \in (\mathbb{Z}^+)^{m-1}$ we have

$$\begin{aligned} y_{N'} &= \frac{(n'_2 + \cdots + n'_m)!}{n'_2! \cdots n'_m!} \prod_{j=2}^{m} \left(\frac{b_j}{a(-\mathbf{M}_0(f_1))} \right)^{n'_j} \\ &= \frac{(n'_2 + \cdots + n'_m)!}{n'_2! \cdots n'_m!} \prod_{j=2}^{m} (b_j)^{n'_j} \frac{1}{p^{|N'|}} = y_N(f) \frac{1}{p^{|N'|}}, \end{aligned}$$

where $N = (0, n'_2, \dots, n'_m)$. This proves formula (2.10). $\square$

This case was considered earlier in [6]. However, $\mathbf{d}(G)$ was calculated there only for a trinomial f.

Suppose the Fourier spectrum of f lies in a grid $\mathbb{M} = -\nu + h\mathbb{Z}^+$, where $-\nu$ is the leftmost point in the spectrum and $h > 0$. This situation occurs (with a suitable choice of h) if and only if the distances between all the points of $\Omega(f)$ are commensurable. According to [12, Thm. 3.1], the following result holds.

THEOREM 2.4. *If $\Omega(f) \subset \mathbb{M} = -\nu + h\mathbb{Z}^+$ then G_f is AP-factorable.*

We postpone the proof (based on a recursive use of the BKST transformation) until Theorem 2.7, where a more general result will be established. Meanwhile, observe that the BKST transformation does not give us a convenient way to explicitly calculate the partial AP indices of G_f or $\mathbf{d}(G_f)$ other than recursively. However, necessary and sufficient conditions for zero partial AP indices can be found in [12, Sec. 3.2]. Combining Theorems 3.2 and 3.3 in [12] yields the following result.

THEOREM 2.5. *Let $\Omega(f) \subset \mathbb{M} = -\nu + h\mathbb{Z}^+$. Let also τ be the smallest positive element of $\mathbb{M}$, and write f as $\sum_{j=-M}^{M} c_j e^{i(\tau + jh)x}$, where $M = \{\lambda/h\}$.* (Throughout this paper, we will let $[x]$ denote the greatest integer less than or equal to x, and $\{x\}$ the greatest integer strictly less than x; $[x] = \{x\}$ for $x \notin \mathbb{Z}$ and $[x] = \{x\} + 1 = x$ for $x \in \mathbb{Z}$.) *For any positive integer n, define the matrix*

$$T_n = (c_{i-j})_{i,j=1}^{n} = \begin{bmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-n+1} \\ c_1 & c_0 & c_{-1} & \cdots & c_{-n+2} \\ c_2 & c_1 & c_0 & \cdots & c_{-n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix};$$

let Δ_1 *be the matrix obtained from* T_{M+1} *by deleting the* $(M+1)$th *row and* $(M+1)$th *column, and* Δ_2 *the matrix obtained from* T_{M+1} *by deleting the* $(M+1)$th *row and first column.*

Then G_f *is AP-factorable, and the partial AP indices of* G_f *will be zero if and only if one of the following holds:*

(a) $\nu/h \in \mathbb{Z}$, $\lambda/h \in \mathbb{Z}$, *and* $\det T_{M+1} \neq 0$;
(b) $\nu/h \in \mathbb{Z}$, $\lambda/h \notin \mathbb{Z}$, *and* $\det T_M \det T_{M+1} \neq 0$; *or*
(c) $\nu/h \notin \mathbb{Z}$, $\lambda/h \in \mathbb{Z}$, *and* $\det \Delta_1 \det \Delta_2 \neq 0$.

If $\Omega(f) \cap (-\lambda, \lambda)$ consists of two points only, then condition $\Omega(f) \subset \mathbb{M}$ is obviously satisfied. In this case, one step of the BKST transformation provides an AP factorization of G_f. This leads to the formulas for partial AP indices, which can also be extracted from the proof of Theorem 2.3 in [10] as follows.

THEOREM 2.6. *Let* $f = ae_{-\nu} + be_{\mu}$, $-\lambda < -\nu < \mu < \lambda$. *Then* G_f *is AP-factorable with partial AP indices equal to*

(a) $\pm\nu$ *if* $\nu \leq 0$ *or* $b = 0$,
(b) $\pm\mu$ *if* $\mu \leq 0$ *or* $a = 0$,
(c) $\pm\min\{\mu, \nu, \min_{k\in\mathbb{Z}}|k(\mu+\nu) - \lambda|\}$ *if* $\mu, \nu > 0$, *and* $ab \neq 0$.

The following case is introduced in [6, Thm. 3.1 and Thm. 3.6] as a generalization of both the commensurable distances situation (Theorem 2.4) and the case of one-sided f (Theorem 2.2). Define ε as the positive distance from zero to the negative portion of the union of the two grids $\mathbb{M} = -\nu + h\mathbb{Z}$ and $\mathbb{M}' = -\lambda + h\mathbb{Z}$. (Strictly, $\varepsilon = \min(\lambda - h\{\lambda/h\}, \nu - h\{\nu/h\})$.) Theorem 3.6 of [6], which also can be verified with the BKST transformation, may be stated as follows.

THEOREM 2.7. *If* $\Omega(f) \subset \mathbb{M} \cup [\lambda - \varepsilon, \lambda)$, *then* G_f *is AP-factorable.*

We will first show that BKST transformation reduces G_f to another matrix in the same class with smaller λ.

LEMMA 2.8. *If* $\Omega(f) \subset \mathbb{M} \cup [\lambda - \varepsilon, \lambda)$, *then there exists a matrix function* G' *such that*

$$G_f = \begin{bmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{bmatrix} \quad \text{and} \quad G' = \begin{bmatrix} e_{\lambda'} & 0 \\ f' & e_{-\lambda'} \end{bmatrix}$$

are simultaneously AP-factorable, $\Omega(f') \subset \mathbb{M} \cup [\lambda' - \varepsilon, \lambda')$, *and either* $f' = 0$ *or* $\lambda' \in \{\lambda + h\mathbb{Z}\} \cap [\varepsilon, \lambda - h]$.

Proof. If $f(x) = 0$ then we are done. Otherwise, we construct G' by applying the BKST transformation not more than twice to G_f. The first time, write f as $f(x) = ae^{-i\nu x}(1 - \sum b_j e^{i\gamma_j x} - \sum c_k e^{i\delta_k x})$, with $\gamma_j \in h\mathbb{Z}$ and $\delta_k \geq \lambda + \nu - \varepsilon$ and $\Gamma = (\gamma_1, \dots, \gamma_p, \delta_1, \dots, \delta_q)$. $\langle N, \Gamma\rangle$ is therefore either a multiple of h or $\geq \lambda + \nu - \varepsilon$, so $\Omega(f_1) \subset \{\langle N, \Gamma\rangle - \lambda\} \cap (-\nu, \nu) \subset \{-\lambda + h\mathbb{Z}\} \cup [\nu - \varepsilon, \nu)$. If $f_1(x) = 0$, let $G' = G_1$ and we are done. Otherwise, let $-\lambda'$ be the leftmost point in $\Omega(f_1)$. If $-\lambda' \geq 0$ then $f_1 \in AP^+$ and, by Theorem 2.2, G_1 is AP-factorable with partial AP indices $\pm\lambda'$ and so we let $G' = \operatorname{diag}[e_{\lambda'}, e_{-\lambda'}]$. Suppose $-\lambda' < 0$. Since $\varepsilon \le \nu - h\{\nu/h\} \le \nu$, the inclusion $-\lambda' \in \Omega(f_1) \subset \{-\lambda + h\mathbb{Z}\} \cup [\nu - \varepsilon, \nu)$ implies that $\lambda' \in \{\lambda + h\mathbb{Z}\}$. From here and the inequalities $0 < \lambda' < (\nu <) \lambda$ one can conclude that, in fact, $\lambda' \ge \lambda - h\{\lambda/h\} \ge \varepsilon$ and $\lambda' \le \lambda - h$. Transforming a second time, the elements of Γ_1 are either multiples of h or $\ge \lambda' + \nu - \varepsilon$, and the diagonal exponents of G_1 are $\pm\nu$, so $\langle N, \Gamma_1\rangle$ is either a multiple of h or $\ge \lambda' + \nu - \varepsilon$. Hence $\Omega(f_2) \subset \{\langle N, \Gamma_1\rangle - \nu\} \subset \{-\nu + h\mathbb{Z}\} \cup [\lambda' - \varepsilon, \lambda')$. We may now let $G' = G_2$. □

Proof of Theorem 2.7. Applying Lemma 2.8 repeatedly, we arrive at a matrix,

$$G_{f_n} = \begin{bmatrix} e_{\lambda_n} & 0 \\ f_n & e_{-\lambda_n} \end{bmatrix},$$

for which either $f_n = 0$ or $\lambda_n = \lambda - h\{\lambda/h\}$, $\Omega(f_n) \subset \{-\nu + h\mathbb{Z}\} \cup [\lambda_n - \varepsilon, \lambda_n)$. In the first case, G_{f_n} is obviously AP-factorable (with partial AP indices $\pm\lambda_n$). In the second case, consider the two subcases separately.

(i) $\lambda - h\{\lambda/h\} \le \nu - h\{\nu/h\}$. Then $\varepsilon = \lambda_n$, $\{-\nu + h\mathbb{Z}\} \cap (-\lambda_n, 0) = \emptyset$, and $f_n \in AP^+$. Hence, by Theorem 2.2, G_{f_n} is AP-factorable.

(ii) $\lambda - h\{\lambda/h\} > \nu - h\{\nu/h\}$. Then $\varepsilon = \nu - h\{\nu/h\} < \lambda_n$, and the intersection $\{-\nu + h\mathbb{Z}\} \cap (-\lambda_n, 0)$ consists of exactly one point, $-\varepsilon$, the distance of which from the rest of $\Omega(f_n)$ is at least λ_n. By Theorem 2.3, G_{f_n} is AP-factorable.

In both subcases, Lemma 2.8 implies that G_f is AP-factorable as well. □

Again, this method gives no way other than recursively to explicitly calculate the partial AP indices of G_f or $\mathbf{d}(G_f)$ (if the partial indices are zero).

3. A Generalization of the Big-Gap Result

Theorem 2.3 shows the factorability of polynomials f with one negative exponent and the rest of the exponents a distance of at least λ away. It relies on the fact that, under the BKST transformation, such f yield $f_1 \in AP^+$, a known factorable case. We can generalize this result by allowing additional points in the Fourier spectrum of f which lie within a certain closed interval but which still cause $f_1 \in AP^+$. Given $\lambda \in \mathbb{R}^+$, $\nu \in (0, \lambda)$, $s \in \mathbb{Z}^+$, and $s < \lambda/\nu - 1$, define

$$R_s(\lambda, \nu) = \begin{cases} \left[\frac{\lambda}{s+1} - \nu, \frac{\lambda-\nu}{s} - \nu\right] & \text{if } s \ge 1, \\ \emptyset & \text{if } s = 0. \end{cases} \tag{3.1}$$

THEOREM 3.1. *If $\Omega(f) \subset \{-\nu\} \cup R_s \cup [\lambda - \nu, \lambda)$ for some s, then the following statements hold.*

(1) *G_f is AP-factorable.*
(2) *If we let $a = \mathbf{M}_{-\nu}(f)$, $b = -\frac{1}{a}\mathbf{M}_{\lambda/(s+1)-\nu}(f)$, and $c = -\frac{1}{a}\mathbf{M}_{\lambda-\nu}(f)$, then G_f will have partial AP indices equal to zero if and only if $c + b^{s+1} \neq 0$.*
(3) *When the partial AP indices equal zero,*

$$\mathbf{d}(G_f) = \begin{bmatrix} 1/ad & Y/ad \\ -X/d - aM & ad - XY/d - aMY \end{bmatrix},$$

where

(a) $d = c/a + b^{s+1}/a$,

(b) $Y = a^{-s-1}(-\mathbf{M}_{(\lambda-\nu)/s-\nu}(f))^s$,

(c) $X = \sum_{N:\langle N,\Gamma\rangle=\lambda+\nu} y_N(f)$, *and*

(d) $M = \sum_{N':\langle N',\Gamma'\rangle=\nu} y_{N'}(f_1)$, *where* Γ' *is the vector of nonzero elements of* $\Omega(f_1)$.

Observe that $R_s \subset (0, \lambda-2\nu]$ and $R_i \cap R_j = \emptyset$ for $i \neq j$. Thus, Theorem 3.1 contains several independent statements in which the intermediate part of $\Omega(f)$ is allowed to lie entirely in any one of the disjoint intervals $R_1, \ldots, R_{\{\lambda/\nu\}-1}$. The case $s = 0$ corresponds to the setting of Theorem 2.3, where $\Omega(f) \subset \{-\nu\} \cup [\lambda-\nu, \lambda)$; in this case, $b = X = Y = 0$.

Proof. Write

$$f(x) = ae^{-i\nu x}\left(1 - \sum_{j=1}^{p} b_j e^{i\gamma_j x} - \sum_{k=1}^{q} c_k e^{i\delta_k x}\right),$$

with $\nu < \lambda/(s+1) \leq \gamma_1 < \gamma_2 < \cdots < \gamma_p \leq (\lambda-\nu)/s < \lambda \leq \delta_1 < \delta_2 < \cdots < \delta_q < \lambda+\nu$. Let n and m denote p- and q-vectors of nonnegative integers, and let $N = (n|m)$; let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$, $\delta = (\delta_1, \delta_2, \ldots, \delta_q)$, and $\Gamma = (\gamma|\delta)$; note that $\langle N, \Gamma\rangle = \langle n, \gamma\rangle + \langle m, \delta\rangle = \sum n_j\gamma_j + \sum m_k\delta_k$.

(1) Theorem 2.1 states that G_f will be simultaneously factorable with

$$G_{f_1} = \begin{bmatrix} e_\nu & 0 \\ f_1 & e_{-\nu} \end{bmatrix},$$

where $\Omega(f_1) \subseteq \{\langle N, \Gamma\rangle - \lambda\} \cap (-\nu, \nu)$. We will show that $\Omega(f_1) \subset \mathbb{R}^+$, proving that G_{f_1}, and thus G_f, are AP-factorable.

Since $\delta_k \geq \lambda$, if $|m| \geq 1$ then $\langle N, \Gamma\rangle - \lambda \geq \lambda - \lambda = 0$. Since $\gamma_j \leq (\lambda-\nu)/s$, if $|m| = 0$ and $|n| \leq s$ then $\langle N, \Gamma\rangle = \langle n, \gamma\rangle \leq s((\lambda-\nu)/s) = \lambda-\nu$, so $\langle N, \Gamma\rangle - \lambda \notin (-\nu, \nu)$. And since $\gamma_j \geq \lambda/(s+1)$, if $|m| = 0$ and $|n| \geq s+1$, then $\langle N, \Gamma\rangle = \langle n, \gamma\rangle \geq (s+1)(\lambda/(s+1)) = \lambda$ and so $\langle N, \Gamma\rangle - \lambda \geq 0$. Thus, $\Omega(f_1) \in \mathbb{R}_+$, so G_{f_1} is AP-factorable by Theorem 2.2 and so G_f is AP-factorable as well.

(2) Theorem 2.1 also states that G_f and G_{f_1} will have the same partial AP indices, that is, G_f will have zero partial AP indices if and only if G_{f_1} does. But $f_1(x) \in AP^+$, so G_{f_1} will have zero partial indices if and only if $\mathbf{M}_0(f_1) \neq 0$ (Theorem 2.2). Now we will consider in what cases $\langle N, \Gamma\rangle - \lambda = 0$, that is, what N will contribute to the constant term of G_{f_1}. If $|m| \geq 1$ then $\langle N, \Gamma\rangle = \lambda$ if and only if $|n| = 0$, $m = (1, 0, 0, \ldots, 0)$, and $\delta_1 = \lambda$; that is, $\mathbf{M}_{\lambda-\nu}(f) \neq 0$. The contribution to $\mathbf{M}_0(f_1)$ is

$$\frac{y_N}{a} = \frac{1!}{1!}c_1^1 a^{-1} = \frac{c_1}{a}.$$

If $|m| = 0$, then we know that $\langle N, \Gamma\rangle \leq \lambda - \nu$ if $|n| \leq s$ and that $\langle N, \Gamma\rangle \geq \lambda + \nu$ if $|n| \geq s+2$. Since $\gamma_1 \geq \lambda/(s+1)$ and $\gamma_j > \lambda/(s+1)$ for $j \neq 1$, it follows if $|n| = s+1$ that $\langle N, \Gamma\rangle = \lambda$ if and only if $n = (s+1, 0, 0, \ldots, 0)$ and $\gamma_1 = \lambda/(s+1)$; that is, $\mathbf{M}_{\lambda/(s+1)-\nu}(f) \neq 0$. In this case, the contribution to $\mathbf{M}_0(f_1)$ is

$$\frac{(s+1)!}{(s+1)!} b_1^{s+1} a^{-1} = \frac{b_1^{s+1}}{a}.$$

Thus, if we define $b = -\frac{1}{a}\mathbf{M}_{\lambda/(s+1)-\nu}(f)$ and $c = -\frac{1}{a}\mathbf{M}_{\lambda-\nu}(f)$ ($b = b_1$ or 0, $c = c_1$ or 0), then $\mathbf{M}_0(f_1) = c/a + b^{s+1}/a$, and so G_{f_1} and G_f will have partial AP indices equal to zero if and only if $c + b^{s+1} \neq 0$.

(In fact, G_{f_1} and therefore also G_f will have partial AP indices that are equal to $\pm\min((s+1)\gamma_1 - \lambda, \delta_1 - \lambda)$ unless $(s+1)\gamma_1 = \delta_1$ and $c_1 = -b_1^{s+1}$, and, if this is the case, then G_f will have partial AP indices equal to $\pm\min(s\gamma_1 + \gamma_2 - \lambda, \delta_2 - \lambda)$ unless $s\gamma_1 + \gamma_2 = \delta_2$ and $c_2 = -(s+1)b_1^s b_2$.)

(3) According to Theorem 2.1,

$$\mathbf{d}(G_{f_1}) = \begin{bmatrix} -a & 0 \\ X & a^{-1} \end{bmatrix} \mathbf{d}(G_f) \begin{bmatrix} -Y & 1 \\ 1 & 0 \end{bmatrix},$$

where

$$X = \sum_{N:<N,\Gamma>=\lambda+\nu} y_N(f) \quad \text{and} \quad Y = \sum_{N:<N,\Gamma>=\lambda-\nu} y_N(f) a^{-1}.$$

Note that every $N = (n|m)$ that contributes to X will have $m = 0$, because $\delta_j \in [\lambda, \lambda + \nu)$ and so $\lambda + \nu - \delta_j \in (0, \nu] \neq \langle N_j, \Gamma\rangle$ for any N. As for Y, we note that if $|m| \neq 0$ then $\langle N, \Gamma\rangle > \lambda > \lambda - \nu$. If $|n| \geq s+1$ then $\langle N, \Gamma\rangle \geq \lambda > \lambda - \nu$; if $m = 0$ and $|n| \leq s$ then $\langle N, \Gamma\rangle \leq \lambda - \nu$, with equality holding only when $n = (0, 0, \dots, 0, s)$ and $\gamma_p = (\lambda - \nu)/s$. In this case, $y_n = (s!/s!)b_p^s$ and $b_p = -\frac{1}{a}\mathbf{M}_{(\lambda-\nu)/s-\nu}(f)$, so $Y = (-\mathbf{M}_{(\lambda-\nu)/s-\nu}(f))^s a^{-s-1}$.

We know from Theorem 2.2 that

$$\mathbf{d}(G_{f_1}) = \begin{bmatrix} 0 & -d^{-1} \\ d & M' \end{bmatrix},$$

where $d = \mathbf{M}_0(f_1) = c/a + b^{s+1}/a$ and $M' = \sum_{N':\langle N',\Gamma'\rangle=\nu} y_{N'}(f_1)$ (note that what is referred to in Theorem 2.2 as λ is ν here, and μ there is here the leftmost element of f_1, which is 0; Γ' represents the vector of nonzero elements of $\Omega(f_1)$).

Matrix inversion and multiplication yields

$$\begin{aligned} \mathbf{d}(G_f) &= \begin{bmatrix} -1/a & 0 \\ X & a \end{bmatrix} \begin{bmatrix} 0 & -1/d \\ d & M' \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & Y \end{bmatrix} \\ &= \begin{bmatrix} 1/ad & Y/ad \\ -X/d - aM & ad - XY/d - aMY \end{bmatrix}. \end{aligned}$$ □

4. Trinomials

In this section we consider almost periodic matrices of the form

$$G = \begin{bmatrix} e_\lambda & 0 \\ c_{-1}e_{-\nu} + c_0 e_\mu + c_1 e_\delta & e_{-\lambda} \end{bmatrix}$$

with $-\nu < \mu < \delta$ and with c_{-1}, c_0, and c_1 complex numbers. Thus, $G = G_f$, where $f(x) = c_{-1}e_{-\nu} + c_0 e_\mu + c_1 e_\delta$. Such G are not always factorable (see Theorem 4.1), nor does there exist a universal test for factorability of trinomials. There are, however, many special cases in which factorization is possible.

If $-\nu \geq 0$ or $\delta \leq 0$ then $f(x) \in AP^{\pm}$ and so G is explicitly factorable by Theorem 2.2, so we can assume without any loss of generality that $\nu, \delta > 0$. We will also assume that $\mu \geq 0$, since if $\mu < 0$ then we can instead use the matrix $G_{\bar{f}} = JG^*J$ with $\bar{f} = c_1 e^{-i\delta x} + c_0 e^{i|\mu|x} + c_{-1}e^{i\nu x}$. If $\beta := (\nu+\mu)/(\delta-\mu)$ is rational then the distances between points are commensurable and Theorem 2.4 applies, so we will assume β is irrational. Finally, we will assume throughout this section that $c_{-1}c_0c_1 \neq 0$; if this is not true, then $f(x)$ is a binomial and Theorem 2.6 applies.

Let us first consider the case where $\nu + \delta \geq \lambda$, which is referred to as the case of small λ. The $\mu = 0$ case is covered completely in [13, Thm. 5.1, Cor. 5.2, and Thm. 6.1] (see also [3, Sec. 2]). These results can be summarized as follows.

THEOREM 4.1. *If $\nu + \delta = \lambda$ and $\mu = 0$, then*:

(a) *G is AP-factorable if and only if $|c_1^{\beta} c_{-1}| \neq |c_0^{1+\beta}|$ ($\beta = \nu/\delta$ irrational)*;
(b) *G has zero partial AP indices if it is AP-factorable*; *and*
(c) $$\mathbf{d}(G) = \begin{cases} \begin{bmatrix} 0 & c_0^{-1} \\ -c_0 & 0 \end{bmatrix} & \text{if } |c_1^{\beta}c_{-1}| < |c_0|^{\beta+1}, \\ \begin{bmatrix} -c_1^{-1}c_{-1} & 0 \\ -2c_0 & -c_1c_{-1}^{-1} \end{bmatrix} & \text{if } |c_1^{\beta}c_{-1}| > |c_0|^{\beta+1}. \end{cases}$$

THEOREM 4.2. *If $\nu + \delta > \lambda$ and $\mu = 0$, then G is AP-factorable. Its partial AP indices are zero if and only if there exists a positive integer l such that either*

(1) $\delta + \nu = \beta_{l-1} + \sum_{s=1}^{l} n_s\beta_{s-1} > \lambda > \beta_{l-2} + \sum_{s=1}^{l-1} n_s\beta_{s-1}$ *and* $c_{(-1)^l,l} \neq 1$, *or*
(2) $\delta + \nu > \beta_{l-1} + \sum_{s=1}^{l} n_s\beta_{s-1} \geq \lambda > \beta_{l-2} + \sum_{s=1}^{l-1} n_s\beta_{s-1}$,

where

$$n_1 + \cfrac{1}{n_2 + \frac{1}{n_3+\cdots}} = \text{the unique continued fraction for } \beta = \frac{\nu}{\delta} \ (n_j \in \mathbb{Z}^+)$$

$$\beta_{-1} := \nu, \quad \beta_0 := \delta, \quad \beta_k := \beta_{k-2} - n_k\beta_{k-1},$$

$$c_{1,0} := c_0^{-1}c_1, \quad c_{-1,0} := c_0^{-1}c_{-1},$$

and

$$c_{-1,k} := \begin{cases} c_{1,k-1}^{n_k} c_{-1,k-1}, \\ c_{-1,k-1}, \end{cases} \quad c_{1,k} := \begin{cases} c_{1,k-1}, \\ c_{1,k-1}c_{-1,k-1}^{n_k}, \end{cases} \quad \text{for } \begin{cases} k \text{ odd}, \\ k \text{ even}. \end{cases}$$

The BKST transformation allows us to understand the small-λ case when $\mu \neq 0$, which we present as Theorems 4.3 and 4.4.

THEOREM 4.3. *If $\nu + \delta = \lambda$ and $\mu > 0$, then G is AP-factorable. The partial AP indices of G will be zero if and only if either*

(a) *$w(\nu+\mu) \geq \lambda - \mu$, where $w := 1 + [(\lambda-\nu)/(\nu+\mu)]$ (which is always true when $\mu \geq \nu$) or*
(b) *one of the conditions for zero AP indices in Theorem 4.2 holds for $\beta_{-1} = w(\nu+\mu) - \lambda$, $\beta_0 = (w+1)(\nu+\mu) - \lambda$,*

$$\beta = \frac{w(\nu+\mu)-\lambda}{(w+1)(\nu+\mu)-\lambda} = n_1 + \frac{1}{n_2 + \frac{1}{n_3+\cdots}},$$

$$c_{1,0} = -\frac{(-c_0)^{w+1}}{c_1 c_{-1}^{w}}, \quad c_{-1,0} = -\frac{(-c_0)^{w}}{c_1 c_{-1}^{w-1}},$$

and $c_{\pm1,k}$ and β_k defined as in Theorem 4.2.

Proof. Applying the BKST transformation once will transform $G(x)$ to an already understood case. We have $\Gamma = (\nu+\mu, \lambda)$, so $\Omega(f_1) \subset \{n_1(\nu+\mu)+n_2\lambda-\lambda\} \cap (-\nu, \nu)$. If $n_2 \geq 2$ then $\langle N, \Gamma\rangle - \lambda \geq \lambda > \nu \notin \Omega(f_1)$. If $n_2 = 1$ then $n_1 = 0$ gives $0 \in \Omega(f_1)$, but $n_2 = 1$ and $n_1 \geq 1$ gives $\langle N, \Gamma\rangle - \lambda \geq \nu+\mu > \nu \notin \Omega(f_1)$. If $n_2 = 0$, define $w = 1 + [(\lambda-\nu)/(\nu+\mu)]$; that is, make w the smallest integer such that $w(\nu+\mu) > \lambda-\nu$; clearly, $(w+2)(\nu+\mu) > w(\nu+\mu)+2\nu > \lambda+\nu$, so at worst, f_1 is a trinomial with $\Omega(f_1) = \{w(\nu+\mu)-\lambda, 0, (w+1)(\nu+\mu)-\lambda\}$ and the diagonal terms of G_{f_1} are $e^{\pm i\nu x}$. (Degenerate cases are when $(w+1)(\nu+\mu)$ or $w(\nu+\mu) \geq \lambda+\nu$, in which case f_1 is binomial or monomial, both of which are AP-factorable, so we need only consider the above case.) However, since $\nu+\mu > \nu$, G_{f_1} meets the conditions of Theorem 4.2 and so G_{f_1}, and thus G, must be AP-factorable.

As for the partial AP indices, if $w(\nu+\mu) \geq \lambda-\mu$ then $(w+1)(\nu+\mu)-\lambda \geq \nu \notin \Omega(f_1)$, and so f_1 is binomial with Fourier spectrum $\{w(\nu+\mu)-\lambda, 0\}$ and therefore has zero partial AP indices. (If $w(\nu+\mu)-\lambda > \nu$, then f_1 is monomial with Fourier spectrum $\{0\}$ and therefore zero partial AP indices.) Otherwise, we construct

$$\begin{aligned} f_1 &= \frac{w!}{w!}\frac{1}{c_{-1}}\left(-\frac{c_0}{c_{-1}}\right)^{w} e^{i(w(\nu+\mu)-\lambda)x} \\ &\quad + \frac{1!}{1!}\frac{1}{c_{-1}}\frac{-c_1}{c_{-1}}e^{i0x} + \frac{(w+1)!}{(w+1)!}\frac{1}{c_{-1}}\left(-\frac{c_0}{c_{-1}}\right)^{w+1} e^{i((w+1)(\nu+\mu)-\lambda)x} \\ &= (-c_0)^{w} c_{-1}^{-w-1} e_{w(\nu+\mu)-\lambda} - c_1 c_{-1}^{-2} + (-c_0)^{w+1} c_{-1}^{-w-2} e_{(w+1)(\nu+\mu)-\lambda} \end{aligned}$$

and apply Theorem 4.2. □

THEOREM 4.4. *If $\nu+\delta > \lambda$ and $\mu > 0$, then G is AP-factorable and its partial AP indices are equal to*

$$\begin{cases} \pm\min_{j\in\mathbb{Z}}|j(\nu_k+\mu_k)-\lambda_k| & \textit{if } -\nu_k < 0 \quad \textit{and} \\ \pm\min(-\nu_k, \mu_k) & \textit{if } -\nu_k \geq 0 \end{cases}$$

where

$$\lambda_0 := \lambda, \quad \nu_0 := \nu, \quad \mu_0 := \mu, \quad \delta_0 := \delta,$$

$$w_n := 1 + \left[\frac{\lambda_n - \nu_n}{\nu_n + \mu_n}\right],$$

$$\lambda_{n+1} := \nu_n,$$

$$\nu_{n+1} := -w_n(\nu_n+\mu_n) + \lambda_n,$$

$$\mu_{n+1} := \nu_n + \delta_n - \lambda_n = \mu_{n-1},$$
$$\delta_{n+1} := (w_n + 1)(\nu_n + \mu_n) - \lambda_n,$$

and

$$k := \textit{the smallest natural number } j \textit{ such that } \delta_j \geq \lambda_j.$$

Specifically, the partial AP indices of G are zero if and only if either $\nu_k = 0$, *or* $-\nu_k < 0$ *and* $\lambda_k/(\nu_k + \mu_k) \in \mathbb{Z}$.

Proof. We will show inductively that for $n \leq k$, after n iterations of the BKST transformation, G transforms to

$$G_n = \begin{bmatrix} e_{\lambda_n} & 0 \\ c_{-1}^{(n)} e_{-\nu_n} + c_0^{(n)} e_{\mu_n} + c_1^{(n)} e_{\delta_n} & e_{-\lambda_n} \end{bmatrix}$$

with $0 < \mu_n < \lambda_n$, $\nu_n + \delta_n > \lambda_n$, $0 < \delta_n < \lambda_n$ for $n < k$, and $\delta_k \geq \lambda_k$ for some $k < \infty$, so $f_k(x)$ is binomial and therefore G_k is factorable with partial AP indices computable by Theorem 2.6.

We know that $G(x)$ meets the criteria $\nu + \delta > \lambda$ and $\mu > 0$; we will show inductively that f_{n+1} is trinomial with these properties if f_n is. Transforming f_n by the BKST method, we have $\Gamma_n = (\nu_n + \mu_n, \nu_n + \delta_n)$, so $\Omega(f_{n+1}) \subset \{n_1(\nu_n + \mu_n) + n_2(\nu_n + \delta_n) - \lambda_n\}$. For $n_2 \geq 2$, $\langle N, \Gamma\rangle - \lambda_n > \lambda_n > \nu_n \notin \Omega(f_{n+1})$; likewise, for $N = (1, 1)$, $\langle N, \Gamma\rangle - \lambda_n > \nu_n + \lambda_n - \lambda_n = \nu_n \notin \Omega(f_{n+1})$. Let $w_n = 1 + [(\lambda_n - \nu_n)/(\nu_n + \mu_n)]$ be the smallest integer such that $w_n(\nu_n + \mu_n) - \lambda_n > -\nu_n$. If $n_1 \geq w_n + 2$, then $\langle N, \Gamma\rangle - \lambda_n > \lambda_n - \nu_n + 2(\nu_n + \mu_n) - \lambda_n > \nu_n \notin \Omega(f_{n+1})$. So the only vectors N that could contribute to $\Omega(f_{n+1})$ are $N = (w_n, 0)$, $(0, 1)$, and $(w_n + 1, 0)$, so at worst $\Omega(f_{n+1}) = \{w_n(\nu_n + \mu_n) - \lambda_n, \nu_n + \delta_n - \lambda_n, (w_n + 1)(\nu_n + \mu_n) - \lambda_n\} = \{-\nu_{n+1}, \mu_{n+1}, \delta_{n+1}\}$. Here, $\mu_{n+1} = \nu_n + \delta_n - \lambda_n > 0$ by assumption, and $\nu_{n+1} + \delta_{n+1} = -w_n(\nu_n + \mu_n) + \lambda_n + (w_n + 1)(\nu_n + \mu_n) - \lambda_n = \nu_n + \mu_n > \nu_n = \lambda_{n+1}$. Further, if $n + 1 < k$ then $-\nu_{n+1} < 0$ because otherwise $\delta_{n+1} = -\nu_{n+1} + \nu_n + \mu_n > \nu_n = \lambda_{n+1}$. Also, $\mu_{n+1} < \nu_n$ because otherwise $\nu_n + \delta_n - \lambda_n \geq \nu_n$, that is, $\delta_n \geq \lambda_n$, and if $\delta_{n+1} \geq \lambda_{n+1}$ then we are merely in the $n = k$ case, and so we are done.

It is worth noting that $\mu_{n+2} = \nu_{n+1} + \delta_{n+1} - \lambda_{n+1} = -w_n(\nu_n + \mu_n) + \lambda_n + (w_n + 1)(\nu_n + \mu_n) - \lambda_n - \nu_n = \mu_n$.

Now we must show that k is finite. We let y denote the smallest number j such that $\nu_j + \mu_j \geq \lambda_j$ and claim $k \leq \min(y + 1, 2[\lambda_0/\mu_0]) < \infty$. If y is finite, then $w_y = 1$ because $\nu_y + \mu_y - \lambda_y \geq 0 > -\nu_y$; then $\delta_{y+1} = (w_y + 1)(\nu_y + \mu_y) - \lambda_y = 2(\nu_y + \mu_y) - \lambda_y \geq \nu_y + \mu_y > \nu_y = \lambda_{y+1}$ and so $k \leq y + 1$. If y is infinite, then at every step we know that $\nu_j < \lambda_j - \mu_j$, that is, $\lambda_{j+1} < \lambda_j - \mu_j$; therefore, $\lambda_{j+2} \leq \lambda_j - \mu_j - \mu_{j+1}$, and since either μ_j or $\mu_{j+1} = \mu_0$, it follows that $\lambda_{j+2} \leq \lambda_j - \mu_0$. Therefore, if we let $z = 2[\lambda_0/\mu_0]$ then we have $\lambda_z \leq \lambda_0 - [\lambda_0/\mu_0]\mu_0 \leq \mu_0 = \mu_z$, and since (as we showed before) $\mu_j \geq \lambda_j$ implies $\delta_{j-1} \geq \lambda_{j-1}$, we know that $k < z < \infty$.

Finally, after the kth transformation, $\delta_k \geq \lambda_k$. We know $0 < \mu_k < \lambda_k$ since $\nu_{k-1} + \delta_{k-1} > \lambda_{k-1}$ and $\delta_{k-1} < \lambda_{k-1}$. So if $-\nu_k \geq 0$ then we are in the one-sided case, which is factorable with partial AP indices of $\pm\min(-\nu_k, \mu_k)$ by

Theorem 2.2. (If $-\nu_k \geq \lambda_k$, then f_k is monomial and so G_k is factorable with partial AP indices $\pm\mu_k$, which is the same as $\pm\min(-\nu_k, \mu_k)$ since $\mu_k < -\nu_k$.) If $-\nu_k < 0$ then we are in the conventional binomial case, and G_k is factorable with partial AP indices $\pm\min_{j\in\mathbb{Z}}|j(\nu_k + \mu_k) - \lambda_k|$. So G_k, and therefore G, is AP-factorable with partial AP indices as stated previously. $\square$

Thus, we have a complete understanding of the small-λ case, at least in terms of AP-factorability and partial AP indices; combining Theorems 4.1, 4.2, 4.3, and 4.4 yields our next result as follows.

COROLLARY 4.5. *Let* $f(x) = c_{-1}e^{-i\nu x} + c_0 e^{i\mu x} + c_1 e^{i\delta x}$ *with* $\nu, \delta > 0$, $\mu \geq 0$, *and* $\nu + \delta \geq \lambda$. *Then* G_f *is AP-factorable unless all of the following hold*:

(a) $\nu + \delta = \lambda$,
(b) $\mu = 0$,
(c) $\beta = \nu/\delta$ *is irrational, and*
(d) $|c_1^{\beta} c_{-1}| = |c_0^{1+\beta}| \neq 0$.

We now use the foregoing results to generalize to other classes of trinomials f that can be shown to be AP-factorable. The following result contains Theorem 4.4 except for the explicit calculation of partial AP indices; however, Theorem 4.4 is used in the proof, so it needed to be stated and proven separately.

THEOREM 4.6. *If* $\mu > 0$ *and* $\nu+\mu+\delta \geq \lambda$, *then G is AP-factorable. Moreover, the partial AP indices are zero if and only if one of the following holds*:

(a) $\lambda/(\nu + \mu) \in \mathbb{Z}$;
(b) $w(\nu + \mu) \geq \lambda - \mu$ (*always true if* $\mu > \nu$) *and* $\nu + \delta = \lambda$, *or* $\nu/(\nu + \delta - w(\nu + \mu)) \in \mathbb{Z}$ *and* $(\nu + \delta - \lambda)(w(\nu + \mu) - \lambda) < 0$;
(c) $w(\nu + \mu) = \delta$ *or* $w(\nu + \mu) = \delta + \nu - \mu < \lambda - \mu$;
(d) $\delta < w(\nu + \mu) < \min\{\delta + \nu - \mu, \lambda - \mu\}$ *and the matrix*

$$\begin{bmatrix} e_\nu & 0 \\ (-c_0)^w c_{-1}^{-w-1} e_{w(\nu+\mu)-\lambda} - c_1 c_{-1}^{-2} e_{\nu+\delta-\lambda} + (-c_0)^{w+1} c_{-1}^{-w-2} e_{(w+1)(\nu+\mu)-\lambda} & e_{-\nu} \end{bmatrix}$$

satisfies the conditions for zero partial AP indices of Theorem 4.2 if $\nu + \delta = \lambda$ *and of Theorem 4.4 if* $\nu + \delta \neq \lambda$.

Proof. We have $\Gamma = (\nu + \mu, \nu + \delta)$, so $\Omega(f_1) \subseteq \{n_1(\nu + \mu) + n_2(\nu + \delta) - \lambda\} \cap (-\nu, \nu)$. If $n_2 \geq 2$ then $\langle N, \Gamma\rangle - \lambda \geq 2\nu + 2\delta - \lambda > 2\nu + \delta + \mu - \lambda \geq \nu$. If $n_2 = 1$ and $n_1 \geq 1$, we have $\langle N, \Gamma\rangle - \lambda \geq 2\nu + \mu + \delta - \lambda \geq \nu$. Define $w = 1 + [(\lambda - \nu)/(\nu + \mu)]$; that is, w is the smallest integer such that $w(\nu + \mu) > \lambda - \nu$. By definition, $(w - 1)(\nu + \mu) - \lambda \notin (-\nu, \nu)$, and since $\nu + \mu > \nu$ we see that $(w + 2)(\nu + \mu) - \lambda > w(\nu + \mu) - \lambda + 2\nu > \nu$. Hence the only vectors such that $\langle N, \Gamma\rangle - \lambda \in (-\nu, \nu)$ are $(w, 0)$, $(w + 1, 0)$, and $(0, 1)$, so at worst we have $\Omega(f_1) = \{\nu + \delta - \lambda, w(\nu + \mu) - \lambda, (w + 1)(\nu + \mu) - \lambda\}$, with the diagonal exponents of G_{f_1} equal to $\pm\nu$. If $\nu + \delta < w(\nu + \mu)$ or $\nu + \delta > (w + 1)(\nu + \mu)$, we are in the big-gap case, which is always AP-factorable by Theorem 2.3; if not, we are in the small-λ trinomial case (Theorems 4.2 and 4.4), which are always AP-factorable. Thus, G_{f_1}, and therefore also G, are AP-factorable.

As for the partial AP indices, if $w(\nu+\mu) \geq \lambda - \mu$ then $(w+1)(\nu+\mu) - \lambda \geq \nu$ and f_1 is binomial with Fourier spectrum $\{w(\nu+\mu) - \lambda,\ \nu+\delta-\lambda\}$. Owing to Theorem 2.6, the partial AP indices will be zero if either of the numbers in $\Omega(f_1)$ is zero or they are of opposite signs and their difference divides ν. (If λ is a multiple of $\nu+\mu$ then $w(\nu+\mu) = \lambda > \lambda - \mu$, so this condition is sufficient on its own.) Otherwise, f_1 is trinomial. If $\nu+\delta-\lambda \geq w(\nu+\mu) - \lambda + \nu$ (i.e., if $\delta \geq w(\nu+\mu)$) then we are in the situation of Theorem 2.3 (one point in $\Omega(f)$ separated from the rest by a distance of λ, or in this case ν), in which case the partial AP indices are zero if and only if equality holds, and likewise if $(w+1)(\nu+\mu) - \lambda \geq \nu + \delta - \lambda + \nu$ (i.e., if $w(\nu+\mu) \geq \delta + \nu - \mu$). If none of these hold then we are in a nondegenerate trinomial case with a distance of $\nu + \mu > \nu$ between highest and lowest exponents, and so we construct G_{f_1} and apply Theorem 4.2 if the middle exponent is zero and Theorem 4.4 otherwise. □

If μ and δ are sufficiently close then G will transform to a case of commensurable distances between exponents, and we will be able to prove AP-factorability and give necessary and sufficient conditions for zero partial AP indices based on Theorem 2.5.

THEOREM 4.7. *Let $k = [(\lambda+\nu)/(\nu+\mu)]$. If $0 < \mu < \delta \leq (\lambda-\nu)/(k-1) - \nu$, then G is AP-factorable. If we let $g = \nu + \mu$ and $h = \nu + \delta$, the partial AP indices of G will be zero if and only if*:

(a) $(kg-\lambda)/(h-g) \in \mathbb{Z}$, $\nu/(h-g) \in \mathbb{Z}$, *and* $\det\, T_{M+1} \neq 0$;
(b) $(kg-\lambda)/(h-g) \in \mathbb{Z}$, $\nu/(h-g) \notin \mathbb{Z}$, *and* $\det\, T_M \det\, T_{M+1} \neq 0$; *or*
(c) $(kg-\lambda)/(h-g) \notin \mathbb{Z}$, $\nu/(h-g) \in \mathbb{Z}$, *and* $\det\, \Delta_1 \det\, \Delta_2 \neq 0$,

where

$$p = \left\lceil \frac{kg-\lambda}{h-g} \right\rceil, \quad c_j = \binom{k}{j-p} \frac{b_1^{k-j+p} b_2^{j-p}}{a}, \quad M = \left\{ \frac{\nu}{h-g} \right\},$$

$T_n = (c_{i-j})_{i,j=1}^n$, $\Delta_1 = T_{M+1}$ *without its* $(M+1)$th *row and column, and* $\Delta_2 = T_{M+1}$ *without its* $(M+1)$th *row and first column*.

Proof. First, note that $k > 0$ because otherwise $(\lambda+\nu)/(\nu+\mu) < 1$ and so $\mu > \lambda$; if $k = 1$, then $\lambda + \nu < 2(\nu+\mu)$, that is, $\nu + 2\mu > \lambda$. Since $\delta > \mu$ by definition, we have $\nu + \mu + \delta > \nu + 2\mu > \lambda$, and so Theorem 4.6 holds and G is AP-factorable.

Otherwise, we will show that if $\langle N, \Gamma\rangle - \lambda \in (-\nu, \nu)$ then $|N| = k$, and that this causes $\Omega(f_1)$ to lie within a grid $\xi + h\mathbb{Z}$, a sufficient condition for AP-factorability by Theorem 2.4.

Write f in the form $f(x) = ae^{-i\nu x}(1 - b_1 e^{igx} - b_2 e^{ihx})$, $g < h \leq (\lambda-\nu)/(k-1)$. Since $\Gamma = (g, h)$ with $h > g$, if $|N| \leq k-1$ then $\langle N, \Gamma\rangle - \lambda \leq (k-1)h - \lambda \leq (k-1)((\lambda-\nu)/(k-1)) - \lambda = \lambda - \nu - \lambda = -\nu$ and so $\langle N, \Gamma\rangle \notin \Omega(f_1)$. If $|N| \geq k+1$ then $\langle N, \Gamma\rangle \geq (k+1)g$. By definition, $k+1 > (\lambda+\nu)/g$, so $\langle N, \Gamma\rangle > ((\lambda+\nu)/g)g = \lambda + \nu$ and hence $\langle N, \Gamma\rangle - \lambda \notin (-\nu, \nu)$.

Thus, if $\langle N, \Gamma\rangle - \lambda \in \Omega(f_1)$ then $|N| = k$. That is, $\Omega(f_1) \in \{(k-j)g + jh - \lambda\} = \{(kg-\lambda) + j(h-g)\} = \xi + (h-g)\mathbb{Z}$ and so, by Theorem 2.4, G_{f_1} and therefore G is AP-factorable.

As for the partial AP indices, since $|N| = k$ for $\langle N, \Gamma\rangle - \lambda$ to appear in $\Omega(f_1)$,

$$f_1(x) = \sum_{j=j_0}^{j_1} \frac{k!}{(k-j)!\, j!} \frac{b_1^{k-j} b_2^{j}}{a} e^{i((kg-\lambda)+j(h-g))x} \tag{4.1}$$

From Theorem 2.5 we know that, for G_{f_1} to have zero partial AP indices, either $(kg-\lambda)/(h-g)$ or $\nu/(h-g)$ must be an integer. Define as before $p = [(kg-\lambda)/(h-g)]$, so that $\tau := kg - \lambda - p(h-g) < h - g$. Define

$$c_j = \binom{k}{j-p} \frac{b_1^{k-j+p} b_2^{j-p}}{a}.$$

Then (4.1) simplifies to

$$f_1 = \sum_{j=-M}^{M} c_j e^{i(\tau + j(h-g))x}$$

with M defined as before, so then Theorem 2.5 applies. □

COROLLARY 4.8. *If $\mu \geq \nu$ and $\lambda < 2\nu + 3\mu$, then G is AP-factorable.*

Proof. If $\lambda < 2\nu + 3\mu$ then $\lambda + \nu < 3(\nu + \mu)$, so $k = [(\lambda+\nu)/(\nu+\mu)] \leq 2$. If $\mu \geq \nu$ then $(\lambda-\nu)/(\lambda-\mu) \geq 1$, so $k - 1 \leq 1 \leq (\lambda-\nu)/(\lambda-\mu)$ and so either $k - 1 = 0$ or $\lambda - \mu \leq (\lambda-\nu)/(k-1)$. If $k-1 = 0$ then, as noted previously, $\lambda + \nu < 2\nu + 2\mu$ and so $\nu + \mu + \delta > \nu + 2\mu > \lambda$ and Theorem 4.6 holds. Otherwise, for any value of δ we must have either $\delta + \nu \geq \lambda - \mu$ or $\delta + \nu \leq (\lambda-\nu)/(k-1)$; we know from Theorems 4.6 and 4.7 that, in either of these cases, $G(x)$ is AP-factorable. □

Note that Corollary 4.8 represents the only nontrivial instance of $k - 1 \leq (\lambda-\nu)/(\lambda-\mu)$, because if $k = 1$ then Theorem 4.6 holds as already noted, and if $(\lambda-\nu)/(\lambda-\mu) \geq 2$ then either $\lambda - \nu \geq 2(\lambda - \mu)$ or $2\mu \geq \lambda + \nu$. Since $\delta > \mu$, it follows that $\mu + \delta + \nu > 2\mu + \nu \geq \lambda + 2\nu > \lambda$ and so Theorem 4.6 holds.

The following is a subcase of Theorem 4.6, and is included only because the partial AP indices have been calculated explicitly.

THEOREM 4.9. *If $\mu \geq (\lambda-\nu)/2$, then G is AP-factorable and the partial AP indices are*

(a) $\pm\nu$ *if* $\delta \leq \lambda - 2\nu$,
(b) $\pm(\nu + \delta - \lambda)$ *if* $\nu + \delta - \lambda \leq 0$ *or* $\mu \leq \lambda - 2\nu < \delta$,
(c) $\pm(\nu + \mu - \lambda)$ *if* $\nu + \mu \geq \lambda$, *or*
(d) $\pm\min_{n\in\mathbb{Z}} |n(\delta - \mu) - \nu|$ *otherwise.*

Proof. Both elements of Γ are not less than $(\lambda-\nu)/2 + \nu = (\lambda+\nu)/2$, so if $|N| \geq 2$ then $\langle N, \Gamma\rangle \geq \lambda + \nu$ and hence $\langle N, \Gamma\rangle - \lambda \notin (-\nu, \nu)$. Thus, only $N =$

$(1, 0)$ and $(0, 1)$ contribute to $\Omega(f_1)$, so $f_1(x)$ is at worst binomial; thus G_{f_1}, and therefore G, is AP-factorable. As for the partial AP indices, if $\delta \le \lambda - 2\nu$ then $\nu + \delta - \lambda \le -\nu$; since $\mu < \delta$, the same is true for μ, and $f_1(x) = 0$. This means that $G_{f_1}(x) = \operatorname{diag}[e^{i\nu x}, e^{-i\nu x}]$. If only $\mu \le \lambda - 2\nu$, then f_1 is a monomial with exponent $\nu + \delta - \lambda$, so the partial AP indices are $\pm(\nu + \delta - \lambda)$. If $\nu + \delta - \lambda \le 0$, then $f_1(x) \in AP^-$ with highest exponent $\nu + \delta - \lambda$, so the partial AP indices are again $\pm(\nu + \delta - \lambda)$. If $\nu + \mu - \lambda \ge 0$, we have $f_1 \in AP^+$ with leftmost exponent $\nu + \mu - \lambda$. If none of these degenerate cases hold, $f_1(x)$ is a two-sided binomial and the diagonal terms of G_{f_1} are $\pm\nu$, so the partial AP indices are $\pm\min_{n\in\mathbb{Z}}|n(\delta - \mu) - \nu)|$. □

5. Generalized Trinomials

Many of the results we obtained for trinomials can be generalized by considering polynomials f that have—instead of three points in the Fourier spectrum—a Fourier spectrum lying on a sort of "double-grid" generated by three points. If we choose our first three points $-\nu, \mu, \delta$ then we can consider an almost periodic polynomial f with $\Omega(f) \subset \{-\nu + (\nu + \mu)\mathbb{Z} + (\nu + \delta)\mathbb{Z}\}$ because (as will be shown), under the BKST transformation, this new matrix will behave the same as a trinomial.

Unfortunately, the case where $\Omega(f) \subset \{-\nu + g\mathbb{Z} + h\mathbb{Z}\}$ has not yet been solved in full generality. (If it were then this would give a full understanding of trinomials as a special case.) However, certain restrictions can be placed on g and h to make the matrix behave well under the BKST transformation.

Double-grids are considered in [3, Sec. 4], but the ones considered there are unions of two shifted grids with the same size step, and the size of the step is equal to the absolute value of the leftmost exponent; [3] also requires that λ lie in one of the two grids whose union contains $\Omega(f)$. We, on the other hand, propose four different restrictions that would suffice to make G_f AP-factorable, but we do not offer necessary conditions or more general results. There is some small overlap between the cases considered here and in [3] (the case covered in Theorem 5.2, for instance, meets the criteria in [3]), but in general we are using different conditions.

THEOREM 5.1. *If*

$$G_f(x) = \begin{bmatrix} e^{i\lambda x} & 0 \\ f(x) & e^{-i\lambda x} \end{bmatrix}$$

and $\Omega(f) \subset \mathbb{M} = \{-\nu + g\mathbb{Z}^+ + h\mathbb{Z}^+\}$ *with* $\nu, h, g > 0$, *and either*

(a) $h > g \ge \nu$ *and* $h > \lambda$,

(b) $h > g > \nu$ *and* $g + h \ge \lambda + \nu$,

(c) $h > g > \nu$ *and* $h \le (\lambda - \nu)/([(\lambda + \nu)/g] - 1)$, *or*

(d) $h > g > 2\nu$ *and* $3g > \lambda + \nu$,

then G_f *is* AP*-factorable.*

Note that, except for the calculation of partial AP indices, this covers Theorems 4.2–4.7 and Corollary 4.8 as special cases where $\mu = -\nu + g$, $\delta = -\nu + h$, and

$\mathbf{M}_{-\nu+ig+jh}(f) = 0$ for $i + j > 1$. Also note that, in the first two cases, the "double grid" $\mathbb{M}$ is nothing more than a single grid $-\nu + g\mathbb{Z}$ with the single point $-\nu + h$ added. Also, if h/g is rational then we can write $h/g = p/q$ with $p, q \in \mathbb{Z}$ and let $\xi = g/q$; then $\Omega(f) \subset \{-\nu + g\mathbb{Z}^+ + h\mathbb{Z}^+\} \subset \{-\nu + \xi\mathbb{Z}\}$ and so Theorem 2.4 is applicable. Therefore, we are only interested in h/g irrational.

Proof. Applying the BKST transformation, the terms of Γ are of the form $\alpha_1^{(i)} g + \alpha_2^{(i)} h$ ($\alpha_j^{(i)} \in \mathbb{Z}^+$), so $\Omega(f_1) \subset \{\langle N, \Gamma\rangle - \lambda\} = \{(n_1\alpha_1^{(1)} + \cdots + n_p\alpha_1^{(p)})g + (n_1\alpha_2^{(1)} + \cdots + n_p\alpha_2^{(p)})h - \lambda\} = \{n_1' g + n_2' h - \lambda\}$.

In the first case, if $n_2' \geq 2$ or $n_2' = 1$ and $n_1' \geq 1$, then $\langle N, \Gamma\rangle - \lambda \geq g + h - \lambda > g \geq \nu$. Define as usual $w = 1 + [(\lambda - \nu)/g]$. Then if $n_2' = 0$ and $n_1' < w$, $\langle N, \Gamma\rangle \leq [(\lambda - \nu)/g]g \leq ((\lambda - \nu)/g)g = \lambda - \nu$, and if $n_2' = 0$ and $n_1' \geq w + 2$, $\langle N, \Gamma\rangle \geq wg + 2g \geq \lambda - \nu + 2g \geq \lambda + \nu$; so $\Omega(f_1) \subset \{wg - \lambda, (w + 1)g - \lambda, h - \lambda\}$. Since $g \geq \nu$ and $h - \lambda > 0$, it follows that G_f is AP-factorable according to Theorem 4.4 or 4.3.

In the second case, we again have $2h > h + g \geq \lambda + \nu$, so again $\Omega(f_1) \subset \{wg - \lambda, (w + 1)g - \lambda, h - \lambda\}$. Since $g > \nu$, the function G_f is AP-factorable by Theorem 4.2 if $h = \lambda$ and by Theorem 4.4 otherwise.

In the third case, as in the proof of Theorem 4.7, if we let $k = [(\lambda + \nu)/g]$, then if $n_1' + n_2' < k$, $\langle N, \Gamma\rangle \leq (k - 1)h \leq (k - 1)((\lambda - \nu)/(k - 1)) = \lambda - \nu$, and if $n_1' + n_2' > k$, $\langle N, \Gamma\rangle \geq (k + 1)g > ((\lambda + \nu)/g)g = \lambda + \nu$. Therefore, $\Omega(f_1) \subset \{n_1' g + n_2' h - \lambda : n_1' + n_2' = k\} = \{(kg - \lambda) + n_2'(h - g)\} = \xi + (h - g)\mathbb{Z}$, which we know from Theorem 2.4 is a sufficient condition for G_{f_1}, and therefore G_f, to be AP-factorable.

In the fourth case, $g > 2\nu$ implies $\lambda + \nu - g \leq \lambda - \nu$, or $(\lambda - \nu)/(\lambda + \nu - g) \geq 1$. Now $3g > \lambda + \nu$ implies that $[(\lambda + \nu)/g] \leq 2$, so $[(\lambda + \nu)/g] - 1 \leq 1 \leq (\lambda - \nu)/(\lambda + \nu - g)$. Hence either $[(\lambda + \nu)/g] = 1$, in which case $g + h > 2g > \lambda + \nu$ and the second condition holds, or else $\lambda + \nu - g \leq (\lambda - \nu)/([(\lambda + \nu)/g] - 1)$ and thus, for any h, either $h \geq \lambda + \nu - g$ and the second condition holds or $h \leq (\lambda - \nu)/([(\lambda + \nu)/g] - 1)$ and the third condition holds. □

THEOREM 5.2. *Consider the case where $g = \nu$ and $h = \lambda$, that is, $\Omega(f) \subset \{-\nu + \nu\mathbb{Z}^+ + \lambda\mathbb{Z}^+\}$. Assume λ/ν is irrational. Write f as*

$$f(x) = ae^{-i\nu x}\left(1 - b_0 e^{i\lambda x} - \sum_{k=1}^{m} b_k e^{ik\nu x}\right)$$

with $a, b_0 \neq 0$. As usual, let $w = 1 + [(\lambda - \nu)/\nu] = [\lambda/\nu]$. Let

$$c_{-1} = \sum_{N:\langle N,\Gamma\rangle = w\nu} y_N a^{-1}, \quad c_0 = b_0 a^{-1},$$

$$c_1 = \sum_{N:\langle N,\Gamma\rangle = (w+1)\nu} y_N a^{-1}, \quad \textit{and} \quad \beta = \frac{\lambda - w\nu}{-\lambda + (w + 1)\nu}.$$

Then G_f has partial AP indices equal to zero if it is AP-factorable, and is AP-factorable if and only if $|c_1^{\beta} c_{-1}| \neq |c_0^{1+\beta}|$.

Note that if $b_0 = 0$ then distances between exponents are commensurable, G_f is factorable, and Theorem 2.5 gives necessary and sufficient conditions for zero partial AP indices. Furthermore, note that β is rational if and only if $1 + \beta = \nu/(-\lambda + (w + 1)\nu)$ is rational if and only if ν/λ is rational, which is false by assumption.

Proof. Under the BKST transformation, $G_f(x)$ transforms to $G_{f_1}(x)$ where $\Omega(f_1) \subset \{w\nu - \lambda, 0, (w + 1)\nu - \lambda\}$ unless $\lambda/\nu \in \mathbb{Z}$, which we know to be untrue. By definition, $c_{-1} = \mathbf{M}_{w\nu-\lambda}(f_1)$, $c_0 = \mathbf{M}_0(f_1)$, and $c_1 = \mathbf{M}_{(w+1)\nu-\lambda}(f_1)$; by applying Theorem 4.1 we find that G_{f_1} and therefore G_f is AP-factorable with zero partial AP indices if $|c_1^{\beta} c_{-1}| \neq |c_0^{1+\beta}|$. (The last case of Theorem 4.1, where $c_1 c_{-1} = c_0 = 0$, violates our assumption that $b_0 \neq 0$ since $c_0 = b_0 a^{-1}$.) $\square$

6. The Matrix Case

The BKST transformation technique can be also applied (under certain restrictions) to matrix functions of the form

$$G_F = \begin{bmatrix} e_\lambda I_n & 0 \\ F & e_{-\lambda} I_n \end{bmatrix}, \tag{6.1}$$

where $\lambda > 0$ and F is an $n \times n$ matrix whose entries are almost periodic polynomials. Defining the Fourier coefficients $\mathbf{M}_\alpha(F) = \mathbf{M}(e_{-\alpha} F)$ entrywise, we let $\Omega(F) = \{\alpha \in \mathbb{R} : \mathbf{M}_\alpha(F) \neq 0\}$. As in Section 2, denote by $-\nu$ the smallest point of $\Omega(F) \cap (-\lambda, \lambda)$, and by a the corresponding Fourier coefficient $\mathbf{M}_{-\nu}(F)$. We have now to impose the additional condition that the $n \times n$ matrix a is invertible; of course, in the scalar case ($n = 1$) this condition was satisfied automatically. Then we can write, analogously to (1.3),

$$F = a e_{-\nu} \left(I - \sum_{k=1}^{m} b_k e_{\gamma_k} \right), \tag{6.2}$$

where $0 < \gamma_1 < \cdots < \gamma_m < \lambda + \nu$ and where $b_1, \ldots, b_m$ are nonzero $n \times n$ matrices. For any $N = (n_1, \ldots, n_m)$ with $n_j \in \mathbb{Z}^+$, define

$$y_N(F) = \sum b_{j_1} b_{j_2} \cdots b_{j_w}, \tag{6.3}$$

where $w = n_1 + \cdots + n_m$ and the sum in (6.3) is taken over all ordered w-tuples $(j_1, j_2, \ldots, j_w)$ of integers exactly n_k of which are equal to k for $k = 1, \ldots, m$. Using (6.3) in place of (2.2), a calculation similar to the one given in Section 2 yields the equality

$$e_\nu M_1^+ F + e_{\lambda+\nu} M_2^+ = I.$$

Thus, the analog of Theorem 2.1 holds.

Of course, for commuting matrices $b_1, \ldots, b_m$ formula (6.3) can be written in the same form (2.2) as in the scalar ($n = 1$) case. The applicability of the BKST transformation in this setting was observed in [3, Sec. 7].

The result of Theorem 2.2 (for F with nonnegative Fourier spectrum) remains valid for G_F provided an invertibility condition is satisfied.

PROPOSITION 6.1. *Suppose $\Omega(F) \subset [0, \lambda)$ and assume that $\mathbf{M}_\mu(F)$ is invertible, where μ (≥ 0) is the smallest element of $\Omega(F)$. Then G_F is AP-factorable with partial AP indices equal to $\pm\mu$ (n pairs). If $\mu = 0$, then*

$$\mathbf{d}(G_F) = \begin{bmatrix} 0 & -a^{-1} \\ a & L \end{bmatrix}, \tag{6.4}$$

where $a = \mathbf{M}(F)$ and

$$L = a \sum_{N:\langle N,\Gamma\rangle=\lambda} y_N(F)a^{-1}.$$

Observe that L can also be written as $\sum_{N:\langle N,\Gamma\rangle=\lambda} y_N(F^{(1)})$ with $F^{(1)}(x) = aF(x)a^{-1} = ae_{-\nu}(I - \sum b_k^{(1)} e_{\gamma_k})$, $b_k^{(1)} = ab_k a^{-1}$. Coefficients $b_k^{(1)}$ appear naturally if, instead of (6.2), a representation

$$F = \left(I - \sum_{k=1}^m b_k^{(1)} e_{\gamma_k}\right) ae_{-\nu}$$

is used. The corresponding form of (6.4), along with other statements of Proposition 6.1, was established in [13]. We give here a different proof based on the BKST transformation.

Proof. Setting $-\nu = \mu$ in (6.2) and applying the BKST transformation, we obtain

$$G_{F_1} = \begin{bmatrix} e_{-\mu} I & 0 \\ 0 & e_\mu I \end{bmatrix}. \tag{6.5}$$

Hence, G_{F_1} (and therefore G_F) is AP-factorable with partial AP indices μ (n times) and $-\mu$ (n times).

If $\mu = 0$, (6.5) and (2.7) tell us that

$$\mathbf{d}(G_{F_1}) = I_{2n} = \begin{bmatrix} -a & 0 \\ X & a^{-1} \end{bmatrix} \mathbf{d}(G_F) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where $X = \sum_{N:\langle N,\Gamma\rangle=\lambda} y_N$. Thus,

$$\mathbf{d}(G_F) = \begin{bmatrix} -a^{-1} & 0 \\ aXa^{-1} & a \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} 0 & -a^{-1} \\ a & aXa^{-1} \end{bmatrix},$$

as required. □

It turns out that the invertibility hypothesis is essential in Proposition 6.1, in view of the following result.

THEOREM 6.2. *Let m, n be positive integers, at least one of them larger than 1, and let $\mu \in [0, \lambda)$ and $\delta \in (0, (\lambda - \mu)/2)$ be such that $(\lambda - \mu)/\delta$ is irrational. Then there exists an AP polynomial $m \times n$ matrix F such that*

$$\Omega(F) = \left\{\mu, \mu + \delta, \frac{\lambda + \mu}{2}, \delta + \frac{\lambda + \mu}{2}\right\} \subset [0, \lambda)$$

and the $(m + n) \times (m + n)$ matrix function

$$G = \begin{bmatrix} e_\lambda I_n & 0 \\ F & e_{-\lambda} I_m \end{bmatrix}$$

is not AP-factorable.

For the proof of Theorem 6.2 we need a lemma.

LEMMA 6.3 [19, Lemma 2.1]. *Let G be a block diagonal AP matrix*

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix},$$

and let one of its diagonal blocks G_1, G_2 be AP factorable. Then G itself is AP factorable only simultaneously with its other diagonal block.

Proof of Theorem 6.2. It is easy to see that, in view of Lemma 6.3, we need only consider two cases: (1) $m = 1$, $n = 2$; and (2) $m = 2$, $n = 1$.

Consider case (1). Let

$$F = [0\ 1]e_\mu + [h\ 0]e_{(\lambda+\mu)/2},$$

where h is an AP-polynomial with

$$\Omega(h) = \left\{\delta - \frac{\lambda - \mu}{2}, 0, \delta\right\}$$

for which the matrix function $\begin{bmatrix} e_{(\lambda-\mu)/2} & 0 \\ h & e_{-(\lambda-\mu)/2} \end{bmatrix}$ is not AP-factorable. The existence of such h is guaranteed by Theorem 4.1. Then

$$G = \begin{bmatrix} e_\lambda & 0 & 0 \\ 0 & e_\lambda & 0 \\ he_{(\lambda+\mu)/2} & e_\mu & e_{-\lambda} \end{bmatrix}. \tag{6.6}$$

We perform now the following elementary operations: subtract from the second row the third row multiplied by $e_{\lambda-\mu}$; subtract from the third column the second column multiplied by $e_{-\lambda-\mu}$; add the first row multiplied by $he_{(\lambda-\mu)/2}$ to the second row; interchange the second and third rows. Call the resulting matrix $\tilde{G}$:

$$\tilde{G} = \begin{bmatrix} e_\lambda & 0 & 0 \\ he_{(\lambda+\mu)/2} & e_\mu & 0 \\ 0 & 0 & -e_{-\mu} \end{bmatrix}.$$

By Lemma 1.1, the matrices G and $\tilde{G}$ are simultaneously AP-factorable (observe here that $\Omega(he_{(\lambda-\mu)/2}) = (\lambda - \mu)/2 + \Omega(h) \subset \mathbb{R}^+$). On the other hand, in view of the choice of h, the matrix

$$\begin{bmatrix} e_\lambda & 0 \\ he_{(\lambda+\mu)/2} & e_\mu \end{bmatrix} = e_{(\lambda+\mu)/2}\begin{bmatrix} e_{(\lambda-\mu)/2} & 0 \\ h & e_{-(\lambda-\mu)/2} \end{bmatrix}$$

is not AP-factorable. Hence (by Lemma 6.3) $\tilde{G}$, and therefore G, is not AP-factorable.

Consider now case (2). Define G by (6.6). Then, as we have seen already, G is not AP-factorable. Therefore,

$$(G^{-1})^T = \begin{bmatrix} e_{-\lambda} & 0 & -he_{(\lambda+\mu)/2} \\ 0 & e_{-\lambda} & -e_\mu \\ 0 & 0 & e_\lambda \end{bmatrix}$$

is also not AP-factorable. It remains to observe that

$$\begin{bmatrix} e_\lambda & 0 & 0 \\ -he_{(\lambda+\mu)/2} & e_{-\lambda} & 0 \\ -e_\mu & 0 & e_{-\lambda} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} (G^{-1})^T \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{6.7}$$

Thus the left-hand side of (6.7) satisfies the requirements of Theorem 6.2. □

Next, we state the "big-gap" result (a matrix generalization of Theorem 2.3). It will be convenient to treat the case of zero partial AP indices separately.

THEOREM 6.4. *Assume that* $\Omega(F) \subset \{-\nu\} \cup [\lambda - \nu, \lambda)$, *where* $\nu \in (0, \lambda)$. *Assume further that the matrices* $\mathbf{M}_{-\nu}(F)$ *and* $\mathbf{M}_\mu(F)$ *are invertible, where* μ *is the leftmost element of* $\Omega(F) \setminus \{-\nu\}$. *Then* G_F *is* AP*-factorable and the partial* AP *indices of* G_F *are* $\pm(\nu + \mu - \lambda)$ (n *pairs*).

Proof. Applying the BKST transformation, $\Gamma = (\gamma_1, \dots, \gamma_m)$ with $\gamma_i \geq \lambda$; therefore, $\langle N, \Gamma \rangle \geq \lambda$ unless $\langle N, \Gamma \rangle = 0$. Hence $\Omega(F_1) \subseteq \{\langle N, \Gamma \rangle - \lambda\} \cap (-\nu, \nu) \subset \mathbb{R}^+$. By Proposition 6.1, G_{F_1}, and therefore G_F, is AP-factorable, with nonnegative partial AP indices equal to the smallest element $\gamma_i - \lambda$, which by definition equals $\nu + \mu - \lambda$. □

THEOREM 6.5. *Assume that* $\Omega(F) \subset \{-\nu\} \cup [\mu, \lambda)$, *where* $\nu \in (0, \lambda)$ *and* $\mu = \lambda - \nu$. *Then* G_F *is* AP*-factorable with zero partial* AP *indices if and only if the matrices* $\mathbf{M}_{-\nu}(F)$ *and* $\mathbf{M}_\mu(F)$ *are invertible. In this case,*

$$\mathbf{d}(G_F) = \begin{bmatrix} -q & 0 \\ L\mathbf{M}_{-\nu}(F) & -p \end{bmatrix}, \tag{6.8}$$

where $p = \mathbf{M}_\mu(F)(\mathbf{M}_{-\nu}(F))^{-1}$, $q = (\mathbf{M}_\mu(F))^{-1}\mathbf{M}_{-\nu}(F)$, *and*

$$L = \sum \left(-\mathbf{M}_{\gamma_{j_1}-\nu}(F)\mathbf{M}_\mu(F)^{-1}\right) \cdots \left(-\mathbf{M}_{\gamma_{j_w}-\nu}(F)\mathbf{M}_\mu(F)^{-1}\right). \tag{6.9}$$

The summation in (6.9) *is over all* $N = (0, n_2, \dots, n_m) \in \left(\mathbb{Z}^+\right)^m$ *such that* $\sum_{j=2}^m n_j \delta_j = \nu$, *where*

$$\{\delta_2, \dots, \delta_m\} = \{\alpha - \mu : \alpha \in \Omega(F) \setminus \{-\nu, \mu\}\},$$

and, for every such N, *over all* w*-tuples of indices* $\{j_1, \dots, j_w\}$, *exactly* n_k *of which are equal to* k, $k = 2, \dots, m$. *Here* $-\nu < \gamma_1 - \nu = \mu < \cdots < \gamma_m - \nu$ *are the numbers in* $\Omega(F)$.

A lemma is needed for the proof of Theorem 6.5.

LEMMA 6.6. *Let* F *be an* AP *polynomial* $n \times n$ *matrix such that* $\Omega(F) \cap (-\lambda, 0]$ *consists of at most one point. If* G_F *is* AP*-factorable with zero partial* AP *indices, then the set* $\Omega(F) \cap (-\lambda, 0]$ *is indeed nonempty, and the corresponding Fourier coefficient is nonsingular.*

Proof. Arguing by contradiction, we may assume that

$$F = \begin{bmatrix} 0 & 0 \\ 0 & I_m \end{bmatrix} e_{-\nu} + F_1, \tag{6.10}$$

where $m < n$, $\nu \in (0, \lambda)$ and $\Omega(F_1) \subset (0, \infty)$. Consider the homogeneous Riemann boundary problem

$$\phi^+ + G_F \phi^- = 0, \tag{6.11}$$

where ϕ^+ and ϕ^- are unknown vectors with components in AP^+ and AP^-, respectively. It follows from (6.10) that the problem (6.11) has an infinite dimensional set of solutions. Indeed, denote by ε the smallest point in $\Omega(F_1)$; then for every $g \in AP$ with $\Omega(g) \subset (-\varepsilon, 0)$ we have the solution

$$\phi^- = [g, 0, \ldots, 0]^T, \qquad \phi^+ = -G_F[g, 0, \ldots, 0]^T = -G_{F_1}[g, 0, \ldots, 0]^T.$$

On the other hand, the AP-factorability of G_F with zero partial AP indices would imply that (6.11) has only constant solutions. □

A similar idea was used in [3] in the case of F with pairwise commuting coefficients.

Proof of Theorem 6.5. If $\mathbf{M}_{-\nu}(F)$ and $\mathbf{M}_\mu(F)$ are invertible, then G_F is AP-factorable with zero partial AP indices, by Theorem 6.4. If G_F is AP-factorable with zero partial AP indices, then (by Lemma 6.6) $\mathbf{M}_{-\nu}(F)$ is invertible and, applying the BKST transformation, we see that $\mathbf{M}_\mu(F)$ is invertible as well (cf. the proof of Theorem 6.4).

It remains to prove the formula (6.8). We argue analogously to the proof of Theorem 2.3. The formula for $F_1(x)$ shows that

$$\mathbf{M}_0(F_1) = -a^{-1}\mathbf{M}_\mu(F)a^{-1},$$

where $a = \mathbf{M}_{-\nu}(F)$. By formula (6.4) we have

$$\mathbf{d}(G_{F_1}) = \begin{bmatrix} 0 & -\mathbf{M}_0(F_1)^{-1} \\ \mathbf{M}_0(F_1) & L \end{bmatrix},$$

where

$$L = \mathbf{M}_0(F_1) \cdot \sum_{N' : \langle N', \Gamma' \rangle = \nu} y_{N'}(F_1) \cdot \mathbf{M}_0(F_1)^{-1}.$$

On the other hand, by the matrix analog of (2.7),

$$\mathbf{d}(G_{F_1}) = \begin{bmatrix} -a & 0 \\ 0 & a^{-1} \end{bmatrix} \mathbf{d}(G_F) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

Thus,

$$\mathbf{d}(G_F) = \begin{bmatrix} -\mathbf{M}_\mu(F)^{-1}a & 0 \\ aL & -\mathbf{M}_\mu(F)a^{-1} \end{bmatrix}.$$

The matrix L is computed analogously to the proof of Theorem 2.3. We have

$$F_1 = \sum_{j=1}^{m} y_j(F)a^{-1}e_{\gamma_j - \lambda},$$

where

$$y_j(F) = -(\mathbf{M}_{-\nu}(F))^{-1}\mathbf{M}_{\gamma_j-\nu}(F).$$

Straightforward algebra now yields the formula (6.8). □

We conclude this section with a matrix generalization of Theorem 3.1.

THEOREM 6.7. *Let* $0 < \nu < \lambda$, *let* $s \in \mathbb{Z}^+$ *be such that* $s < \lambda/\nu - 1$, *and define the interval* $R_s(\lambda, \nu)$ *by* (3.1). *Let* $F(x)$ *be an almost periodic* $n \times n$ *matrix polynomial such that* $\Omega(F) \subset \{-\nu\} \cup R_s(\lambda, \nu) \cup [\lambda - \nu, \lambda)$, *and assume that the matrix* $\mathbf{M}_{-\nu}(F)$ *is invertible. Let* $\Gamma = \{\alpha + \nu : \alpha \in \Omega(F) \setminus \{-\nu\}\}$ *and let* $Q = \min\Big\{ \{\langle N, \Gamma\rangle - \lambda : N \in (\mathbb{Z}^+)^m\} \cap (-\nu, \nu) \Big\}$.

(i) *If the matrix*

$$\sum_{N:\langle N,\Gamma\rangle = Q+\lambda} y_N(F) \tag{6.12}$$

is invertible, then G_F *is* AP*-factorable.*

(ii) G_F *is* AP*-factorable with zero partial* AP *indices if and only if* $Q = 0$ *and the matrix* (6.12) *is invertible.*

The proof of (i) is analogous to that of Theorem 3.1, using Proposition 6.1. To prove part (ii), Theorem 6.5 is used.

Under the hypothesis of Theorem 6.7, if G_F is AP-factorable with zero partial AP indices, a formula for $\mathbf{d}(G)$ could be given using the matrix BKST transformation and Proposition 6.1; however, the formula is too cumbersome to state and is therefore omitted.

7. Applications: Convolution Equations on a Finite Interval

Following [11], consider the convolution type equation

$$(k * u)(t) = f(t), \quad t \in E, \tag{7.1}$$

on the finite interval $E = (0, \lambda)$. We suppose that the Fourier transform $K = \mathcal{F}k$ of the $n \times n$ kernel k has AP_W-asymptotics at $\pm\infty$. The latter condition means that there exist matrix functions $K_\pm \in AP$ with absolutely convergent Fourier series $\sum_{\mu\in\Omega(K_\pm)} \mathbf{M}_\mu(K_\pm)e_\mu$ and such that

$$\lim_{x\to\pm\infty} (K(x) - K_\pm(x)) = 0. \tag{7.2}$$

Equation (7.1) will be treated in a *Bessel potentials* setting:

$$f \in \mathcal{H}^n_{\sigma,p}(E), \qquad u \in \overset{0}{\mathcal{H}}{}^n_{\sigma,p}(E), \tag{7.3}$$

where $p \in (1, \infty)$, $\mathcal{H}_{\sigma,p}$ is the Bessel potentials space $\mathcal{F}^{-1}(1+x^2)^{-\sigma/2}\mathcal{F}L_p(\mathbb{R})$ on the real line, $\mathcal{H}_{\sigma,p}(E)$ stands for its restriction on E, $\overset{0}{\mathcal{H}}_{\sigma,p}(E) = \{\phi \in \mathcal{H}_{\sigma,p}(E) : \operatorname{supp}\phi \in \bar{E}\}$, and $\sigma \in \mathbb{R}$. A particular case of $\sigma = 0$ corresponds to the more traditional setting $f, u \in L^n_p(E)$.

Recall that a linear operator A acting from a Banach space X into a Banach space Y is *Fredholm* if its range $\operatorname{Im} A$ is closed and the defect numbers $\alpha(A) = \dim \operatorname{Ker} A$ and $\beta(A) = \operatorname{codim} \operatorname{Im} A$ are finite; the difference $\alpha(A) - \beta(A)$ is called the *index* of A and is denoted by $\operatorname{ind} A$. The equation $Ax = y$ ($y \in Y$ given, $x \in X$ unknown) is, by definition, Fredholm if A is. Its defect numbers and its index are defined as the defect numbers and the index of A, respectively.

The next theorem follows from [11, Thm. 2.6].

THEOREM 7.1. *Equation* (7.1) *is Fredholm if and only if*

(1) *the matrix functions*

$$\widetilde{K_\pm} = \begin{bmatrix} e_\lambda I_n & 0 \\ K_\pm & e_{-\lambda} I_n \end{bmatrix}$$

are AP-factorable with zero partial AP indices, and

(2) *for all the eigenvalues $\xi_1, \dots, \xi_{2n}$ of $\mathbf{d}(\widetilde{K_-})^{-1}\mathbf{d}(\widetilde{K_+})$, the numbers*

$$\theta_j = \sigma - \frac{1}{2\pi} \arg \xi_j - \frac{1}{p}$$

are not integers.

Under conditions (1) *and* (2), *the index of* (7.1) *equals*

$$\sum_{j=1}^{2n} (\theta_j - [\theta_j] - 1 + 1/p - \sigma).$$

Combined with the results of Sections 3–6, this theorem yields concrete Fredholm criteria for equations (7.1) in terms of the asymptotic behavior of the Fourier transforms of their kernels. For example, Theorem 6.5 implies the following.

THEOREM 7.2. *Let the kernel k of equation* (7.1) *be such that $K_\pm$ in* (7.2) *are AP polynomials satisfying the big-gap condition*:

$$\Omega(K_+) \subset \{-\nu_+\} \cup [\mu_+, \lambda), \qquad \Omega(K_-) \subset \{-\nu_-\} \cup [\mu_-, \lambda), \tag{7.4}$$

where $\nu_\pm \in (0, \lambda)$ and $\mu_\pm = \lambda - \nu_\pm$. Then equation (7.1) *is Fredholm if and only if the following two conditions hold*:

(1) *the matrices $\mathbf{M}_{-\nu_-}(K_-)$, $\mathbf{M}_{-\nu_+}(K_+)$, $\mathbf{M}_{\mu_-}(K_-)$, $\mathbf{M}_{\mu_+}(K_+)$ are invertible*; *and*

(2) *for all the eigenvalues ξ_j of*

$$A = \mathbf{M}_{-\nu_-}(K_-)^{-1}\mathbf{M}_{\mu_-}(K_-)\mathbf{M}_{\mu_+}(K_+)^{-1}\mathbf{M}_{-\nu_+}(K_+)$$

and η_j of

$$B = \mathbf{M}_{-\nu_-}(K_-)\mathbf{M}_{\mu_-}(K_-)^{-1}\mathbf{M}_{\mu_+}(K_+)\mathbf{M}_{-\nu_+}(K_+)^{-1},$$

the numbers

$$\theta_j = \sigma - \frac{1}{2\pi} \arg \xi_j - \frac{1}{p}, \quad \omega_j = \sigma - \frac{1}{2\pi} \arg \eta_j - \frac{1}{p} \quad (j = 1, \dots, n)$$

are not integers.

Under these conditions, the index of (7.1) *is*

$$\sum_{j=1}^{n}\left(\theta_j - [\theta_j] - 1 + \frac{1}{p} - \sigma\right) + \sum_{j=1}^{n}\left(\omega_j - [\omega_j] - 1 + \frac{1}{p} - \sigma\right).$$

Proof. Condition (1) is necessary and sufficient for matrix functions $\widetilde{K_\pm}$ to be AP-factorable with zero partial AP indices. If this condition is satisfied, formula (6.8) yields

$$\mathbf{d}(\widetilde{K_-})^{-1}\mathbf{d}(\widetilde{K_+}) = \begin{bmatrix} A & 0 \\ * & B \end{bmatrix}.$$

Since eigenvalues of block triangular matrices and of their diagonal blocks coincide, condition (2) and the index formula follow immediately from those of Theorem 6.5. □

It is interesting to observe that Fredholm properties of the equation (7.1) in the setting (7.4) depend only on the two leftmost Fourier coefficients of $K_\pm$.

An important subcase of (7.2) occurs if K itself lies in AP_W:

$$K = \sum_j c_j e_{\lambda_j}, \quad \text{where} \quad \sum_j \|c_j\| < \infty.$$

Equation (7.1) can then be rewritten as

$$\sum_j c_j u(t - \lambda_j) = f(t), \quad t \in (0, \lambda). \tag{7.5}$$

The corresponding version of Theorem 7.1 reads as follows.

THEOREM 7.3. *Equation* (7.5) *is Fredholm in the Bessel potentials setting if and only if the matrix function*

$$\tilde{K} = \begin{bmatrix} e_\lambda I & 0 \\ \sum_j c_j e_{\lambda_j} & e_{-\lambda} I \end{bmatrix}$$

is AP-factorable with zero partial AP indices and $\sigma - 1/p \notin \mathbb{Z}$. If these conditions are satisfied, then the defect numbers of (7.5) *equal* $n \max\{0,\ -1 - [\sigma - 1/p]\}$ *and* $n \max\{0,\ 1 + [\sigma - 1/p]\}$.

It follows from Theorem 7.3 that equation (7.5) has a unique solution for every right-hand side if and only if $\tilde{K}$ is AP-factorable with zero partial AP indices and $\sigma \in (1/p - 1, 1/p)$. In the L_p-setting (i.e., for $\sigma = 0$) this result is stated in [9].

Proof. For $K \in AP_W$, assumption (7.2) is obviously satisfied with $K_+ = K_- = K$. Condition (1) of Theorem 7.1 is therefore equivalent to AP-factorability of $\tilde{K}$ with zero partial AP indices. In its turn, $\mathbf{d}(\widetilde{K_-})^{-1}\mathbf{d}(\widetilde{K_+}) = I_{2n}$, so that all θ_j in condition (2) are equal to $s - 1/p$. From here follows the Fredholm criterion.

To prove the formulas for defect numbers, use Theorem 2.1 of [11], according to which (7.5) is equivalent to the Wiener–Hopf operator W_S with the symbol

$$S(x) = \left(\frac{x-i}{x+i}\right)^{-\sigma} \tilde{K}(x),$$

considered on $L_p^n(\mathbb{R}^+)$. Because of the factorability of $\tilde{K}$ with zero partial AP indices, W_S has the same defect numbers as the direct sum of n copies of the operator

$$W_{((x-i)/(x+i))^{-\sigma}} : L_p(\mathbb{R}^+) \to L_p(\mathbb{R}^+).$$

It remains to apply the well-known result on one-side invertibility and the index formula for Wiener-Hopf operators with piecewise continuous symbols [5]. □

We conclude with a concrete version of Theorem 7.3 that is valid by virtue of Theorem 6.7.

THEOREM 7.4. *Let all the shifts μ_j in the difference equation*

$$u(t+\nu) - \sum_{j=1}^{J} b_j u(t-\mu_j) = f(t), \quad 0 < t < \lambda, \tag{7.6}$$

lie in $R_s(\lambda, \nu) \cup [\lambda - \nu, \lambda)$ for a certain integer $s < \lambda/\nu - 1$ and $R_s(\lambda, \nu)$ given by (3.1). *Then* (7.6) *is Fredholm* (*resp. invertible*) *if and only if $\sigma - 1/p \notin \mathbb{Z}$* (*resp.* $\sigma \in (-1 + 1/p,\ 1/p)$), $\min\big\{ \{\langle N, \Gamma\rangle - \lambda : N \in (\mathbb{Z}^+)^m\} \cap (-\nu, \nu) \big\} = 0$, *and the matrix $\sum_{N:\langle N,\Gamma\rangle=\lambda} y_N(F)$ with $y_N(F)$ defined by* (6.3) *is invertible.*

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