Stability for a Class of Foliations Covered by a Product

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Introduction

In this paper, we study transversely orientable, codimension-1 C^1 foliations of Riemannian 3-manifolds. In particular, we examine foliations of a manifold M that are covered by the canonical foliation of $\mathfrak{R}^3$ by parallel hyperplanes, which we refer to as "covered by a product". These foliations are particularly nice in the sense that they are completely determined by the induced action of $\pi_1(M)$ on the real line, which is the leaf space of the universal cover (see [20]). This family of foliations includes fibrations as well as weak stable (or unstable) foliations associated with many Anosov flows, including geodesic flows or suspensions of Anosov diffeomorphisms. Several recent works have focused on whether the associated foliations of other Anosov flows are covered by a product (e.g., [1; 3; 5]).

For closed M, foliations that are covered by a product constitute a large subset of the taut foliations. However, they are strictly a proper subset, as shown by an example in Section 2. While the property of being taut is stable for closed manifolds in the sense that all C^1 close foliations are also taut [21], this is not at all clear for the property of being covered by a product. (We refer to the metric on the space of C^1 foliations, defined by Hirsch in [10]). In this paper, we consider the class of foliations of a closed manifold $M \neq S^2 \times S^1$ that have a transverse loop which lifts to a copy of the leaf space in the universal cover and hence are covered by a product. We find conditions that are sufficient to ensure these foliations are stable in the sense that nearby foliations are also in this class.

More precisely, we find a condition on a branched surface W constructed from a foliation of a manifold M that is sufficient to guarantee the existence of such a transverse loop τ. Under certain conditions (given in Lemma 2.3), the branched surface W can then be modified to obtain a branched surface W' with the property that, for every foliation carried by W' and covered by a product, τ is a transverse loop that is covered by a copy of the leaf space. We denote by $\hat{W}'$ the lift of W' to the universal cover of M. The covering of τ is a curve that is transverse to $\hat{W}'$, and if it intersects a set of smooth submanifolds in a particular manner (which we make explicit at the end of Section 1) then we call it a *global transversal* for $\hat{W}'$. In this case we have that, for every foliation carried by W', τ is a transverse loop that is covered by a copy of the leaf space. In short, we show the following.

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THEOREM 2.5. *Let F_p be a foliation that is covered by a product, and let τ be a transverse loop that is not freely homotopic to an integral loop of F_p. If F_p is carried by a branched surface for which the hypothesis of Lemma 2.3 is satisfied, then there exists a branched surface W' carrying F_p such that every foliation carried by W' is covered by a product if and only if the covering of τ is a global transversal for $\hat{W}'$.*

In particular, when the manifold M is closed, Theorem 2.5 provides a sufficient condition for the stability of the property "covered by a product". This follows from the fact that if a foliation F of a closed manifold is carried by a branched surface, then all sufficiently close foliations are carried by this branched surface.

The paper is divided into two parts. Section 1 contains the preliminary information necessary to understand the results presented here. Details can be found in [2] or [17]. Section 2 contains the main results.

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1. Branched Surfaces Constructed from Foliations

The idea of using branched surfaces to study foliations can be traced back to Williams [23]. In [2], Christy and Goodman describe the construction of branched surfaces from transversely oriented, codimension-1 foliations of a Riemannian 3-manifold without boundary. The branched surfaces obtained in this manner are the same as those constructed by Floyd and Oertel [4] from laminations of 3-manifolds. Therefore, they are more restricted than those used by Williams.

The construction of a branched surface from a foliation is not essential to understanding the results presented here. Hence we provide only a brief description of branched surfaces constructed in this manner. We also outline the relationship between a given branched surface and the foliation used for its construction. More details are given in [17].

In what follows, W represents a branched surface constructed from a foliation of a manifold M without boundary. By the construction, W is imbedded in M and is transversely orientable. The complement $M - W$ is the union of 3-manifolds. The boundary of each component D of $M - W$ is divided by a set of nondifferentiable points into two disjoint, open, planar surfaces with a common boundary. These surfaces in ∂D are homeomorphic, and on one—the "lower hemisphere"—the transverse orientation is directed toward the interior of D; on the other—the "upper hemisphere"—the transverse orientation is directed toward the exterior of D. The branched surface W is a connected 2-dimensional complex (which is compact if M is compact) with a set of charts Ψ, each defining an orientation-preserving local diffeomorphism onto one of the models in Figure 1.1, 1.2, 1.3, or 1.4; the transition maps are smooth and preserve transverse orientation. Hence W is a connected 2-manifold except on a dimension-1 subset B called the *branch set* (indicated by the dotted lines). The set B is a 1-manifold except at isolated points called *crossings*. The branched surface can always be constructed to ensure B is connected,

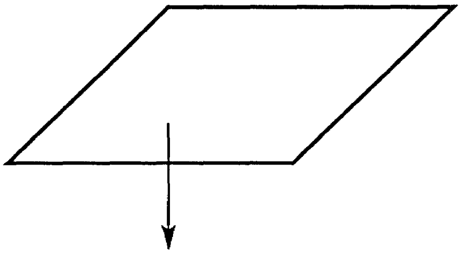

Figure 1.1

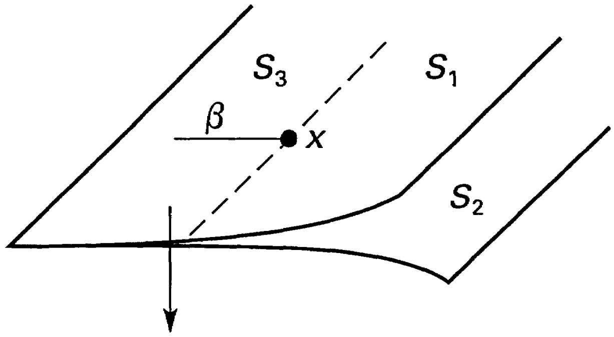

Figure 1.2

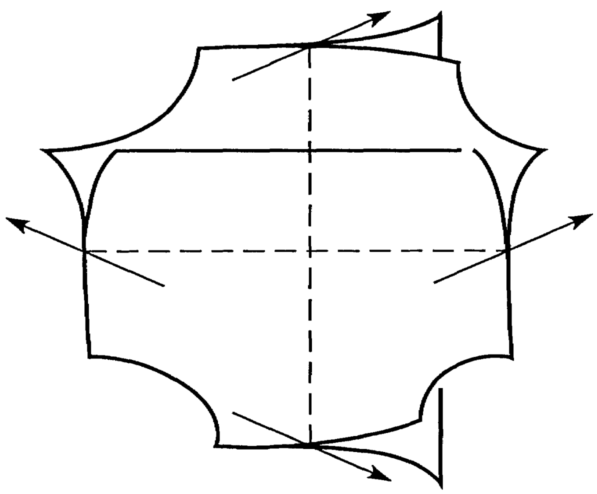

Figure 1.3

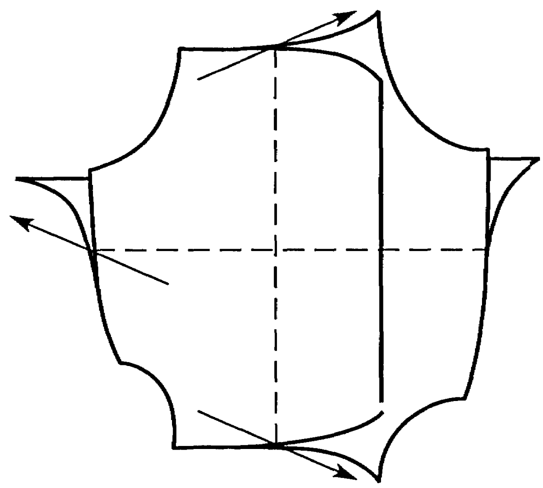

Figure 1.4

so we will assume this to be the case. A *sector* of W is a component of $W - B$ and a *branch point* of W is a point in B. At any branch point x that is not a crossing, there are locally three adjacent sectors S_1, S_2, S_3 such that $\mathrm{cl}(S_1) \cup \mathrm{cl}(S_3)$ and $\mathrm{cl}(S_2) \cup \mathrm{cl}(S_3)$ are smooth submanifolds of W (i.e., the set of charts locally defines a smooth immersion into a planar subset of $\mathfrak{R}^3$ on $\mathrm{cl}(S_1) \cup \mathrm{cl}(S_2)$ and $\mathrm{cl}(S_2) \cup \mathrm{cl}(S_3)$). We say S_1 and S_2 are *tangent* to S_3 at x. By the construction of W, there exists a flow ϕ_w on M that is transverse to W in the direction of the transverse orientation. Suppose that forward orbits (under ϕ_w) of points in S_1 flow into S_2. Then, for any curve β that begins or ends at x and contains a curve in S_3 that is also bounded by x, we say that S_2 is the *upper* sector branching from β at x and that S_1 is the *lower* sector branching from β at x (see Figure 1.2). Given a curve α containing β, any sectors branching from the beginning of β are *incoming* sectors along α, and any sectors branching from the end of β are *outgoing* sectors along α.

We will use weight systems on W as defined in [4]. Such a system assigns a nonnegative real number or *weight* to each sector so that locally, at each branch point that is not a crossing, the weight assigned to the two sectors branching from the branch point (S_1 and S_2) sum to give the weight assigned to the sector tangent to both (S_3); see Figure 1.5. In addition, there is a nonzero weight assigned to at least one sector of W.

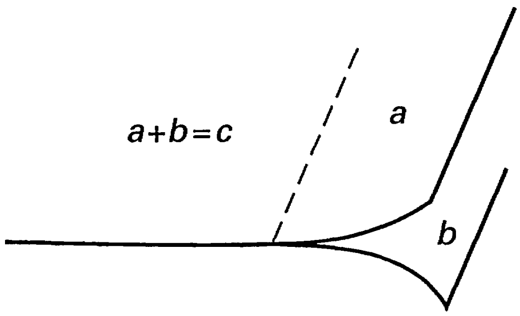

Figure 1.5

We may thicken the sectors along the transverse direction to obtain a set of disjoint, trivial, unit-interval fiber bundles over the sectors. We then piece together these thickened neighborhoods over the branch set to obtain a *neighborhood* of W, which we call $N(W)$. When we thicken W to get $N(W)$, each component of the complement shrinks to a diffeomorphic copy of itself. Hence, the complement of $N(W)$ in M is again the union of 3-manifolds. Even though $N(W)$ is not quite a fibration, we refer to the interval in $N(W)$ obtained when we thicken a point x in W as the "fiber" over x (see Figure 1.6); accordingly, we say points in this fiber "lie over" x. We note that the fiber has an orientation induced by the transverse orientation to W at x. Throughout, $\pi: N(W) \to W$ is the map for which the inverse image of any point in W is the fiber lying over it. Each $x \in \partial N(W)$ that is in the interior of some fiber of $N(W)$ is called a *furrow point* of $N(W)$; these furrow points of $N(W)$ lie over the branch points of W.

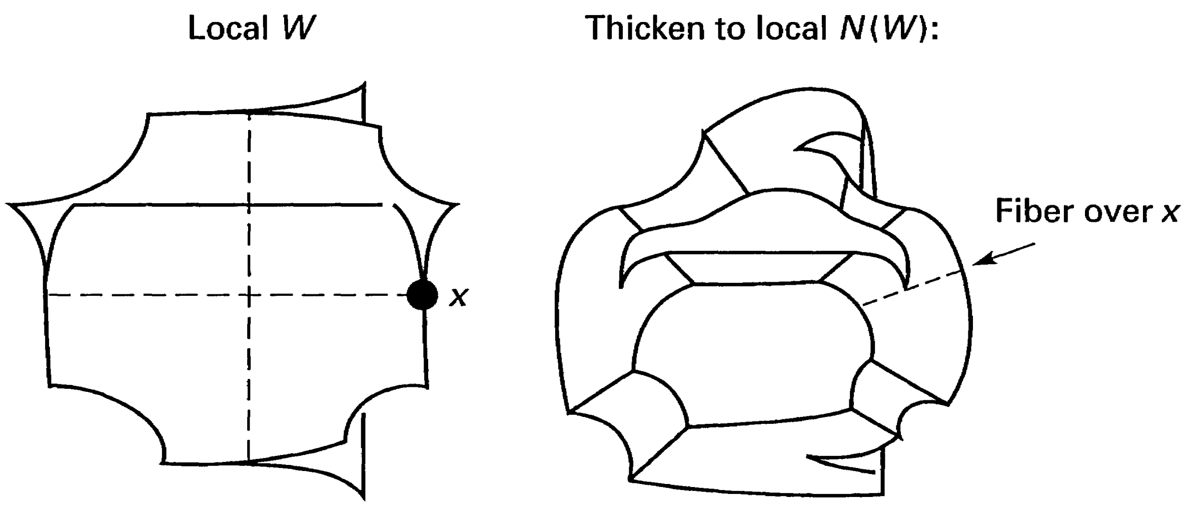

Figure 1.6

We consider only foliations of $N(W)$ by surfaces (possibly branched) that are transverse to the fibers, such that {branch points of leaves} = {furrow points of $N(W)$}. We also require that the boundary of each component of $M - N(W)$ be contained in a leaf. (Figure 1.7 shows how a foliation may appear locally.) We

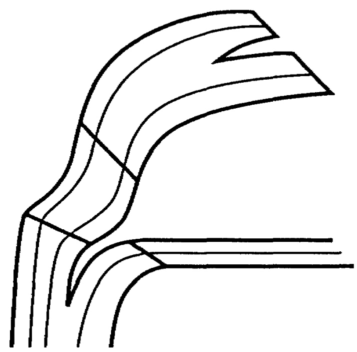

Figure 1.7

may collapse each component of $M - N(W)$ by identifying the upper and lower hemispheres of its boundary. When we do this, each foliation of $N(W)$ induces a foliation of M that we say is "carried by" W. Thus there is a natural duality between foliations of $N(W)$ and foliations of M carried by W. In particular, if W is constructed from F, then some foliation F^* of $N(W)$ yields F when we collapse the complement of $N(W)$; this foliation F^* is called the *dual foliation* to F. It is important to note that there may be many foliations carried by W that are structurally different from F, and any of these foliations can be used to construct W. However, if M is closed, then it follows from the construction that all foliations sufficiently close to F are also carried by W. (We refer to the metric on the space of C^1 foliations, defined by Hirsch in [10].)

Formally, a *curve* in M is a continuous map from a connected subset of $\mathfrak{R}$ into M. However, we will consider a curve to be the image of such a map where the map parameterizes the curve. The *beginning* and *end* of a curve refer to its negative and positive boundary, respectively. A *loop* is a curve that begins and ends at the same point. An *integral* curve of a foliation is a curve that is contained in a leaf. A *transversal to* W will be a curve in M that intersects W, everywhere transversely, and whose orientation is consistent with the positive (or negative) orientation of W at each point of this intersection. We only consider transversals to W that intersect the components of $M - W$ in a union of disjoint arcs extending from the lower hemisphere to the upper (respectively, a union of disjoint arcs extending from the upper hemisphere to the lower).

Given a foliation F carried by W, let F^* be the corresponding dual foliation of $N(W)$. A *transversal to* F^* will be a curve that intersects F^*, everywhere transversely, and whose orientation is consistent with the positive (or negative) transverse orientation of F^*. We consider only transversals to F^* that intersect the components of $M - N(W)$ in a union of disjoint arcs extending from the lower hemisphere to the upper (respectively, a union of disjoint arcs extending from the upper hemisphere to the lower). If we have a curve γ^* transverse to F^*, we may collapse each component D of $M - N(W)$ so that points in the same arc of $\gamma^* \cap D$ are identified. This yields a curve transverse to F. In particular, for each loop transverse to F^* there exists a dual loop transverse to F. Conversely, for any curve γ transverse to F there is clearly a curve transverse to F^*, unique up to homotopy in $\mathrm{cl}(M - N(W))$, that yields γ when we collapse the complement of $N(W)$ in M. Hence, up to homotopy in $\mathrm{cl}(M - N(W))$, there is a natural duality (to which we will often refer) between curves transverse to F and curves transverse to F^*. It is also worth noting that each curve transverse to W extends (under π^{-1}) to a curve transverse to F^* and so gives rise to a curve transverse to F.

At each branching of a leaf L of F^* there is a component of $\partial N(W)$ contained in L. We may restrict L so that it contains only one hemisphere of this component. After restricting L at each of its branchings, we have a surface in $N(W)$ that yields the same leaf as L when we collapse $M - N(W)$. In the definitions that follow, whenever we consider an image under π of a branched leaf L (or some curve in L), we assume that L is restricted in this manner. (This technical remark is fundamental and will be implicitly used throughout this article. It will disallow backtracking across the branch set in various situations.)

DEFINITION. A curve γ on W is an *F-curve* if it is the image under π of an integral curve of F^*. A surface S in W is called an *F-surface* if it is the image under π of a leaf of F^*. We say a curve (surface) is a *W-curve* (*-surface*) if it is an F-curve (-surface) for some foliation F carried by W.

Note that we restrict the domain when applying π to a branched leaf, and so W-curves cannot locally switch from an upper sector to a lower sector (or vice versa) at a branching of W. For example, a W-curve cannot cross the branch set and then backtrack, going from a lower sector to an upper sector.

Let S be an F-surface and let Γ be a transversal to W that intersects S. We say the intersection of S by Γ is *proper with respect to F* if any leaf of F^* lying over S intersects the fibers over the set $\Gamma \cap S$ only once.

DEFINITION. Let F be a foliation carried by W, and let Γ be a transversal to W. The curve Γ is a *global transversal for F* if Γ intersects every F-surface exactly once and this intersection is proper with respect to F; Γ is a *global transversal for W* if it is a global transversal for every foliation carried by W.

For example, Figure 1.8 is a noncompact branched 1-manifold W imbedded in $\mathfrak{R}^2$. Given any foliation F carried by W, the indicated transversal Γ intersects every

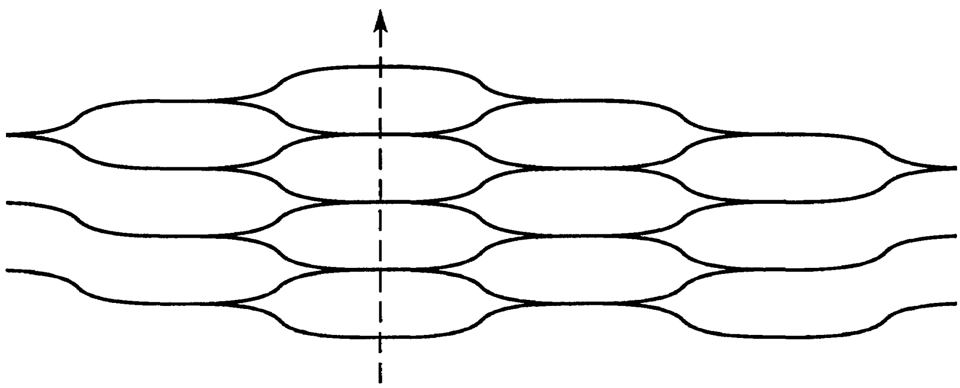

Figure 1.8

F-surface exactly once. Further, each of these intersections is proper with respect F. Hence Γ is a global transversal for W.

Let p be a continuous surjective map from a branched surface $\hat{W}$ with charts $\hat{\Psi}$ onto a branched surface W with charts Ψ. An open set U of W is *evenly covered* if the inverse image $p^{-1}(U)$ can be written as a union of disjoint open sets V_α in $\hat{W}$ such that, for each α, the restriction of p to V_α is a diffeomorphism onto U that maps neighborhoods of the types shown in Figures 1.1–1.4 onto neighborhoods of the same type. If every point in W has an evenly covered neighborhood and if each element of the composition $\Psi * p * \hat{\Psi}^{-1}$ is smooth and orientation-preserving, then p is called a *covering map*. In this case, we say "$\hat{W}$ covers W" or "$\hat{W}$ is the lift of W".

If $\hat{W}$ is a branched surface covering W, then every foliation carried by W is covered by a foliation carried by $\hat{W}$. Conversely, if W carries a foliation F and $\hat{F}$ covers F, then W lifts to a branched surface $\hat{W}$ carrying $\hat{F}$ [18]. For the remainder of this paper, $\hat{W}$ will denote the lift of W to the universal cover of the manifold M.

2. Main Results

If the manifold M is closed, then any foliation of M that is covered by a product is taut. That is, every leaf has a transverse loop passing through it; a standard modification of these transversals produces a single transverse loop meeting every leaf [8]. For suppose a foliation F of a closed manifold is covered by a product, yet there is some leaf L that is not met by a transverse loop. Then L is a compact surface of genus 1 [7]. Since F is covered by a trivial product of hyperplanes in $\mathfrak{R}^3$, each curve in M can be homotoped to be transverse to F or tangent to F. In particular, if $x_0 \in L$ and γ is a loop based at x_0, then γ is homotopic to a loop in L. Since each leaf of $\hat{L}$ is a plane, L is incompressible and hence $\pi_1(L, x_0) = \pi_1(M, x_0)$. This implies that M fibers over a simply connected 1-manifold with fiber L [9, Thm. 10.6], contradicting the assumption that M is closed.

In this section, we consider the class of taut foliations of a closed manifold that are covered by a product, and find conditions on a transverse loop which ensure that all nearby foliations are also in this class. Since taut foliations of a closed manifold are stable, it is first worth noting that not all taut foliations are covered by a product. That is, a transverse loop meeting each leaf is not necessarily covered by a copy of the leaf space. This is illustrated by the following example, constructed by S. Goodman.

Begin with a foliation of $\Sigma_2 \times S^1$ by copies of Σ_2 (surface of genus 2). Insert two tori, transverse (for now) to the foliation, as shown in Figure 2.1. Now sweep the leaves meeting these tori asymptotically around the tori, in the S^1-fiber direction, so that the tori are now leaves. However, be careful to sweep them in opposite directions on either side of the toral leaves so that (in cross section) we have the result shown in Figure 2.2. There exits a transverse loop through both toral leaves, so the foliation is taut. However, there are non-Hausdorff points in the leaf space of the universal cover—namely, those points representing the equivalence classes of leaves that cover the toral leaves.

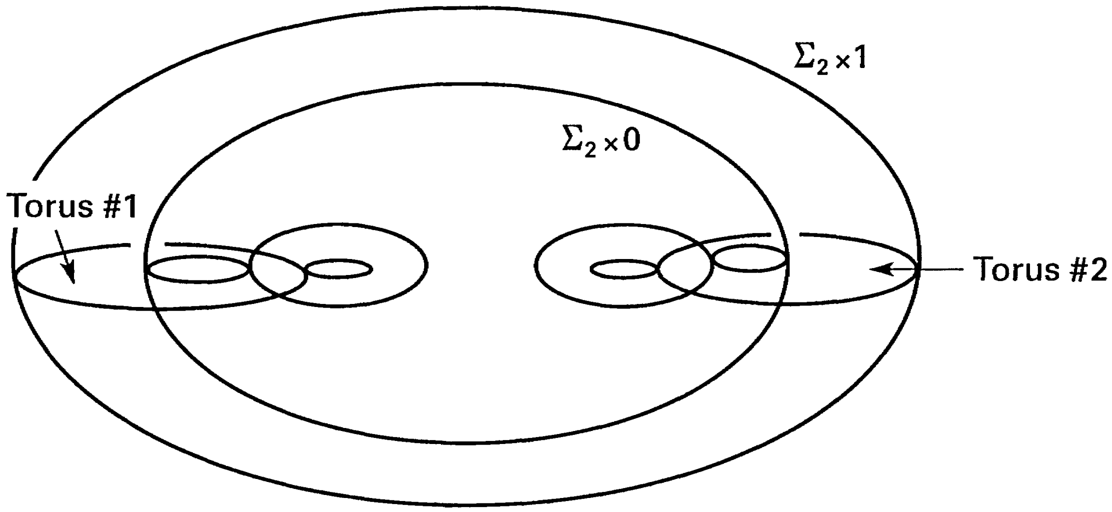

Figure 2.1

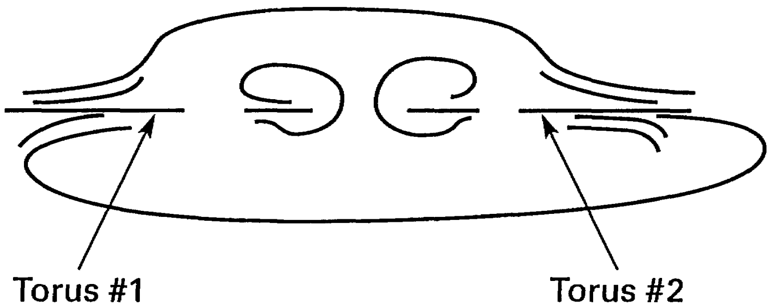

Figure 2.2

In what follows, M is a smooth Riemannian 3-manifold without boundary (which is not necessarily compact, unless this is specified) and F_p is a codimension-1 C^1 foliation of M that is covered by a product. So the universal cover of M, which we denote by $\hat{M}$, is $\mathfrak{R}^3$ [13]. For any $b_0 \in M$, the elements of $\pi_1(M, b_0)$ can be used to define deck transformations of $\hat{M}$ [11]. Moreover, each of these deck transformations maps the covering product foliation onto itself (i.e., is a topological equivalency map) and so induces a homeomorphism of any imbedded copy R of the leaf space of this product foliation. In this manner, each $\tau \in \pi_1(M, b_0)$ induces a homeomorphism of R. A result in [20] suggests the following lemma.

LEMMA 2.1. *Let F_p be a foliation that is covered by a product. Given a loop τ transverse to F_p, if τ is not freely homotopic to an integral loop of F_p then τ is covered by a copy of the leaf space for the product.*

Proof. We assume that the loop τ is not null-homotopic and show the contrapositive. Let Q be a curve in $\mathfrak{R}^3$ that is transverse to the product foliation covering F_p and meets each leaf exactly once (i.e., Q is an imbedded copy of the leaf space). Suppose τ is based at a point b_0 and is not covered by a copy of the leaf space.

Let p be the universal covering map, and let e_0 be a point in the preimage of b_0. Consider the cover $\hat{\tau}$ of τ through e_0. Since $\hat{\tau}$ is not a copy of the leaf space, the intersection of the imbedded leaf space Q with the set of leaves met by $\hat{\tau}$ is either bounded above or bounded below (where the ordering of points in Q is determined by the orientation of Q). Without loss of generality, we assume this set is bounded above. For each positive integer n, lift τ^n to an arc beginning at e_0 and ending at some point $e_n \in \hat{\tau}$. Let $\underline{e_n} \in Q$ be the intersection of Q with the leaf through e_n, for every n. Since the sequence $\{\underline{e_n}\}$ is increasing and bounded above in Q, it converges to some point $x \in Q$. Consider the deck transformation h_τ defined by τ for which $h_\tau(e_0) = e_1$; h_τ induces a homeomorphism of the imbedded leaf space Q for which x is a fixed point. Hence there is an integral curve β of the product from x to $h_\tau(x)$. In addition, there exists a curve α in M that lifts to curves α_0 and α_1 from e_0 to x and from e_1 to $h_\tau(x)$, respectively. Since $\hat{M}$ is simply connected, τ is fixed-point homotopic to $p(\alpha_1^{-1}) * p(\beta) * p(\alpha_0) = \alpha^{-1} * p(\beta) * \alpha$. That is, τ is freely homotopic to the integral loop $p(\beta)$ of F_p. We may conclude that if τ is not freely homotopic to an integral loop of F_p, then τ is covered by a copy of the leaf space. □

If W carries a foliation F, then an F-*loop* is a loop in W that is the projection of an integral loop in the dual foliation F^*. A loop is freely homotopic to an F-loop if and only if it is freely homotopic to an integral loop of F^*. This is equivalent to being freely homotopic to an integral loop of F. Recall that if W carries a foliation F, then each transversal to W extends to give a transversal to F^* and hence gives rise to a dual transversal to F. If a loop τ transverse to W is not freely homotopic to an F-loop, then the transversal to F^* obtained by extending τ along fibers of $N(W)$ is not freely homotopic to an F-loop. In this case, τ gives rise to a transversal to F that is not freely homotopic to an integral loop of F. If, in addition, F is covered by a product, then (by Lemma 2.1) τ gives rise to a loop that is covered by a copy of the leaf space. In the following theorem, we use this fact to give a condition on W that guarantees any foliation carried by W and covered by a product has a transverse loop that is covered by a copy of the leaf space. Recall that a W-curve is the projection of an integral curve in some foliation of $N(W)$. Similarly, a W-loop is a loop that is the projection of an integral loop of some foliation of $N(W)$.

THEOREM 2.2. *If there exists a loop τ transverse to a branched surface W and not freely homotopic to a W-loop, then for each foliation that is covered by a trivial product of hyperplanes and carried by W, τ gives rise to a transverse loop that is covered by a copy of the leaf space.*

For example, Figure 2.3 shows a branched 1-manifold W imbedded in a planar model of the torus. Its contains two W-loops, each represented in the diagram by a horizontal line segment. Every foliation carried by W is covered by a trivial product of hyperplanes in $\mathfrak{R}^2$. As noted in Section 1, we may lift W to a branched 1-manifold $\hat{W}$ in $\mathfrak{R}^2$ carrying these product foliations. The dotted line segment

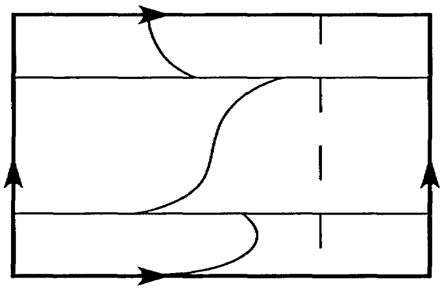

Figure 2.3

depicts a transverse loop τ that is not freely homotopic to a W-loop. It is covered by an imbedding $\hat{\tau}$ of $\mathfrak{R}$ that is transverse to $\hat{W}$. It is easy to verify that every smooth curve in $\hat{W}$ meets $\hat{\tau}$ exactly once, so clearly $\hat{\tau}$ gives rise to a copy of the leaf space for any foliation carried by $\hat{W}$. That is, for each foliation carried by W, τ gives rise to a transverse loop that is covered by a copy of the leaf space.

Given a branched surface W carrying a foliation F, any W-curve γ that is not an F-curve contains an F-curve γ' that has two incoming sectors branching from its beginning and two outgoing sectors branching from its end (see Figure 2.4). In addition, γ' has the property that a "splitting in F" along γ' deletes γ [17]. That is, we may extend a component D of $M - N(W)$ by further splitting the leaf of F^* that contains ∂D; by choosing the extension so that $N(W)$ splits along an integral curve over γ', we destroy any curve in $N(W)$ over γ. The effect of this splitting of $N(W)$ is that W is split along a strip containing γ'. (Figure 2.4 shows the extension and the corresponding splitting of W.) Details of these splittings can be found in [17].

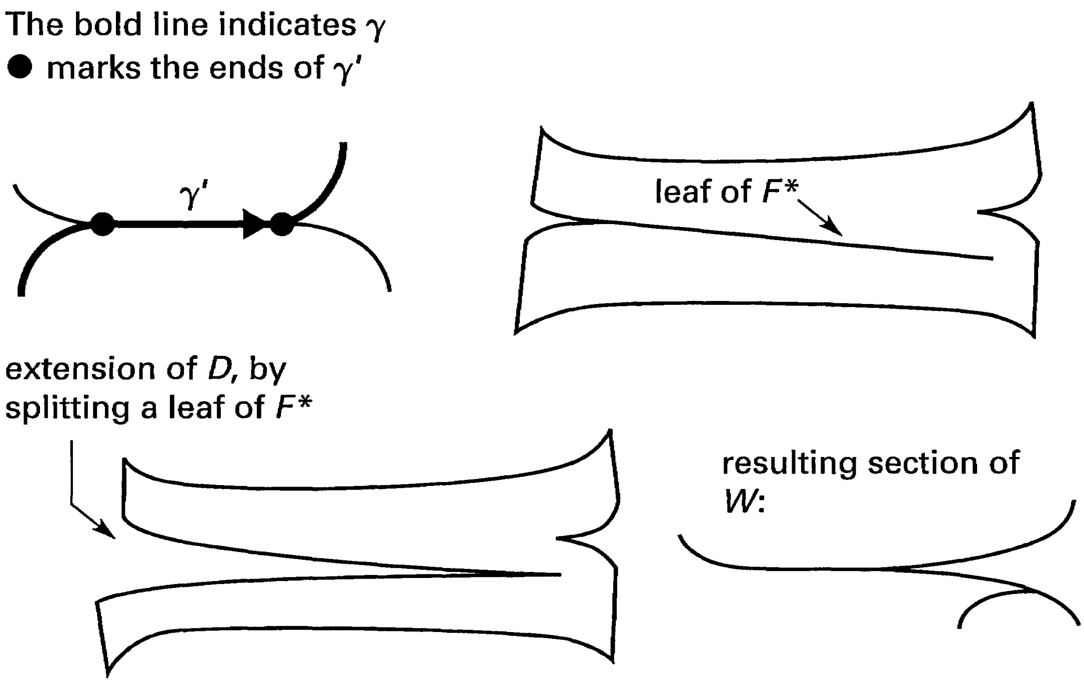

Figure 2.4

For example, the left diagram in Figure 2.5 shows a branched 1-manifold W imbedded in a planar model of the torus. The dotted line segment indicates an imbedded loop τ. We consider a foliation F carried by W and dual to the foliation F^* shown in the middle diagram. Clearly τ is not freely homotopic to an F-loop, so τ is not freely homotopic to an integral loop of F. The branched 1-manifold W' shown in the right diagram also carries F; it is obtained when we delete the W-loop that is freely homotopic to τ by a splitting in F. In fact, there are no W'-loops that are freely homotopic to τ. Thus for every foliation that is covered by a product and carried by W', τ gives rise to a transverse loop that is covered by a copy of the leaf space (Theorem 2.2).

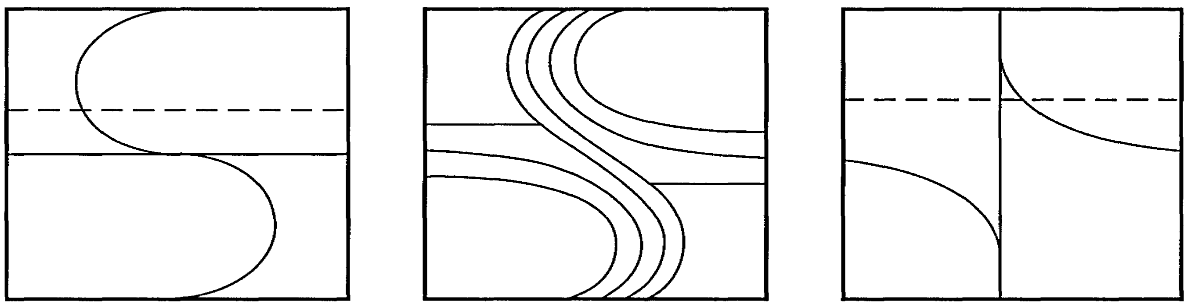

Figure 2.5

A branched surface W can be constructed from a given foliation F of a 3-manifold so that each component of $M - W$ is a 3-ball. It follows that free homotopy maps in M between curves in W can be chosen so that the range is contained in W. We may also ensure that each sector of W is homeomorphic to a disk. In what follows, we assume these properties.

Since W has only disk sectors, any curve γ' in W is smoothly homotopic to a curve γ in the branch set B, and we may choose γ so that γ and γ' are piecewise homotopic in the closure of sectors in W (Figure 2.6). Clearly, if we split W along some curve then, for the modified W, γ' is a W-curve if and only if γ is a W-curve.

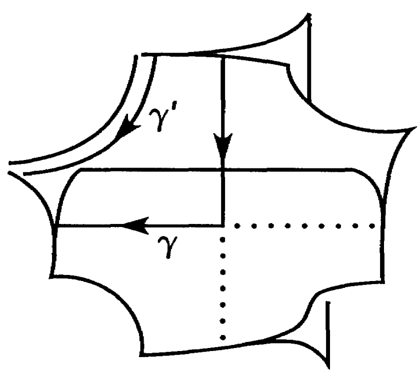

Figure 2.6

There is a more general relationship between two curves in W under which the existence of one of the curves, γ', in a modification of the branched surface is dependent on the existence of the other curve, γ. This relationship will be important since many curves are related to the same curve γ in this manner, so deleting γ deletes each of these curves. The relationship is defined as follows.

DEFINITION. Let γ and γ' be curves in W and suppose that there is a continuous family, parameterized by $t \in I$, of curves $f_t\colon [0, 1] \to W$ such that:

(i) $f_0 = \gamma'$;
(ii) $f_1 = \gamma$;
(iii) for any fixed $x_0 \in [0, 1]$, the curve $f_t(x_0)$ does not locally switch between incoming sectors at any branch point along this curve; and
(iv) if for some fixed $x_0 \in [0, 1]$ there exists $t_0 \in I$ such that the curve $f_t(x_0)$ has an outgoing sector at $f_{t_0}(x_0)$, then for every $x \in I$ the curve $f_t(x)$ has an outgoing sector at $f_{t_0}(x)$.

We say the curve γ' is *subordinate* to the curve γ.

If a curve γ' is subordinate to γ, then there is a particular type of smooth homotopy in W from γ' into γ that can be parameterized to meet requirement (iv). This requirement restricts the type of range allowed for the smooth homotopy from γ' into γ to ensure that every curve in γ with W branching from both ends is smoothly homotopic to a curve in γ' with the same property. This is necessary to avoid the case shown in Figure 2.7, where splitting along a curve to delete γ does not eliminate γ'. In this figure, γ is subordinate to γ', but the converse does not hold because no smooth homotopy from γ' into γ satisfies condition (iv).

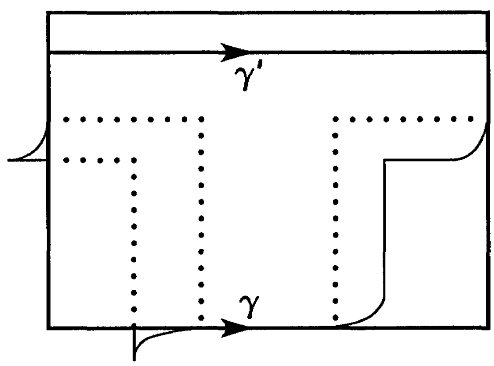

Figure 2.7

The branch set B is a graph where each crossing is a vertex. Hence B contains a set of arcs between adjacent crossings such that every loop in B that does not backtrack (i.e., does not contain a curve of the form $\alpha\alpha^{-1}$) can be represented by a word in these arcs. For each such loop γ in B, there are a finite number of cyclic permutations—that is, loops represented by a cyclic permutation of the word representing γ. If γ and γ' are loops in B that do not backtrack and γ' is subordinate to the curve γ, then we say that γ' and its cyclic permutations are subordinate to γ and to each of its cyclic permutations.

For Lemma 2.3 and the subsequent theorems, we delete all W-loops that are freely homotopic to a transverse loop to W. For this, we need only consider those W-loops in B that do not backtrack: For any W-curve that backtracks, the corresponding integral curve over the backtracking portion begins and ends at the same point; hence, if the W-curve in W obtained when we leave out this backtracking portion is deleted by a splitting in the foliation, the original curve is deleted as well. In what follows we will therefore assume that W-loops in B do not backtrack and that each is represented by a finite word as above.

Recall that F_p represents a foliation that is covered by a product. For the proof of Lemma 2.3, we suppose F_p is carried by W and there is a transverse loop τ that is not freely homotopic to an integral loop of F_p. We consider all W-loops that are freely homotopic to τ, so we regard fixed-point homotopic loops as distinct rather than equating them. Our approach is to begin with W-loops in the branch set B that are freely homotopic to τ and are represented by a word in reduced form (i.e., reduced W-loops). We assume there exists a finite subset Ω of these loops to which all others are subordinate. We then use Ω to define a finite number of splittings in F_p which eliminates all W-loops that are freely homotopic to τ.

LEMMA 2.3. *Let F_p be a foliation which is covered by a product and carried by a branched surface W, and let τ be a transverse loop that is not freely homotopic to an integral loop of F_p. Suppose there is a finite set Ω of loops in B such that every reduced W-loop in B that is freely homotopic to τ is subordinate to a loop Ω. Then there exists a branched surface W' carrying F_p such that τ is not freely homotopic to any W'-loop.*

Proof. Suppose such a finite set Ω of loops in B exists. Choosing a smaller set if necessary, we may assume that every loop in Ω is freely homotopic to τ. Since each $\beta \in \Omega$ is not an F_p-loop, there exists an $N \in Z$ such that β^N is not a F_p-curve. For suppose not, and let I be the fiber of $N(W)$ over the beginning of β. For each $n \in Z$, consider the nonempty subset S_n of I consisting of points that lie at the beginning of an integral curve (of F_p^*) over β^n. It is straightforward to verify that each S_n is a closed interval in I. Given integers n_1 and n_2, if $n_2 \geq n_1 \geq 0$ then $S_{n_1} \supseteq S_{n_2}$ by definition. Likewise, if $n_1 \leq n_2 \leq 0$ then $S_{n_2} \supseteq S_{n_1}$. In either case, $S_{n_1} \cap S_{n_2} \neq \emptyset$. If $n_2 > 0 > n_1$ then, since $S_{n_2-n_1}$ is nonempty, there is an integral curve over $\beta^{n_2-n_1} = \beta^{n_2}\beta^{-n_1}$. This curve begins with an arc over $\beta^{-n_1} = (\beta^{n_1})^{-1}$ which ends at a point $x \in S_{n_2}$. Clearly x is also in S_{n_1}, so $S_{n_1} \cap S_{n_2} \neq \emptyset$. It follows that the intersection $\bigcap S_n$ over all n is nonempty. Order the points in I according to the orientation of I and let x be the lowest point in $\bigcap S_n$. Then x must be contained in an integral loop over β, contradicting the assumption β that is not an F_p-loop. Hence there exists an $N \in Z$ such that β^N is not a F_p-curve.

We may use a splitting in F_p to delete β^N. This destroys every integral curve over β^N and so deletes the W-loop β. Clearly the argument above can be applied to any cyclic permutation β^* of β, so each of these W-loops can be deleted by a splitting in F_p. Since there are finitely many cyclic permutations of β, deleting all of them requires finitely many splittings in F_p.

Repeating this for each element of Ω, we obtain a branched surface W'. Suppose α is a reduced W-loop in B that is freely homotopic to τ. Then α is subordinate to a loop $\beta \in \Omega$. Thus, for some permutations α^* and β^* of α and β, respectively, there exists a smooth homotopy (satisfying the four requirements to ensure that α^* is subordinate to β^*) from α^* onto β^*; this ensures that every curve in β^* with W branching from both ends is smoothly homotopic (under the reverse of this homotopy) to a curve in α^* with the same property. It follows that such a homotopy exists from $(\alpha^*)^n$ onto $(\beta^*)^n$ for every $n \in Z$. As a result, when we delete the loop $(\beta^*)^N$ for some $N \in Z$, we also delete $(\alpha^*)^N$. Consequently, α^* is not a W'-loop, so neither is α. Further, any word obtained by simply adding identities to the word representing α represents a loop that is not a W'-loop, since holonomy maps are trivial over each loop represented by an identity. It follows that no W-loop freely homotopic to τ is a W'-loop. □

We note that—for the most obvious examples when the manifold M is closed—there are only a finite number of W-loops in the branch set B that are freely homotopic to the given loop τ, and in these cases the hypothesis of Lemma 2.3 is trivially satisfied. For example, any branched surface carrying a fibration over S^1 by a compact surface can be modified to give a branched surface W', where no W'-loop is freely homotopic to a given S^1-fiber. Equivalently, the S^1-fiber is not freely homotopic to an integral loop of any foliation carried by W'. However, if instead B contains an infinite number of loops that are freely homotopic to τ, then except under very particular conditions (given in [18]), the hypothesis of Lemma 2.3 is necessary as well as sufficient to guarantee the existence of the modification W'. For closed M, however, we know of no examples where this is the case.

As before, suppose that F_p is covered by a product and that τ is a transverse loop to F_p which is not freely homotopic to an integral loop of F_p. Any branched surface W carrying F_p can be modified by a finite number of extensions so that τ (up to homotopy in M) is transverse to W. Thus we have the following.

THEOREM 2.4. *Suppose F_p is a foliation of a closed manifold M that is covered by a product, and let τ be a transverse loop that is not freely homotopic to an integral loop of F_p. If F_p is carried by a branched surface for which the hypothesis of Lemma 2.3 is satisfied, then for all foliations that are sufficiently close to F_p (using the C^1 metric in* [10]) *and covered by a product, τ gives rise to a transverse loop that is covered by a copy of the leaf space.*

It was shown in [18] that a curve transverse to $\hat{W}$ gives rise to a copy of the leaf space for a foliation carried by $\hat{W}$ if and only if it is a global transversal for that foliation. Since τ is not freely homotopic to an integral loop of F_p, τ is covered by a global transversal $\hat{\tau}$ for the product foliation covering F_p. Under the technical conditions of Lemma 2.3, a branched surface carrying F_p can be modified to obtain a branched surface W' which is also transverse to τ such that τ is not freely homotopic to any W'-loop. Thus we have the following.

THEOREM 2.5. *Let F_p be a foliation of a manifold that is covered by a product, and let τ be a transverse loop that is not freely homotopic to an integral loop of F_p. If F_p is carried by a branched surface for which the hypothesis of Lemma 2.3 is satisfied, then there exists a branched surface W' carrying F_p such that every foliation carried by W' is covered by a product if and only if the covering of τ is a global transversal for $\hat{W}'$.*

COROLLARY 2.6. *Let F_p be a foliation of a closed manifold that is covered by a product, and let τ be a transverse loop that is not freely homotopic to an integral loop of F_p. If F_p is carried by a branched surface for which the hypothesis of Lemma 2.3 is satisfied, and if the covering of τ is a global transversal for $\hat{W}'$, then all foliations sufficiently close to F_p are covered by a product.*

DEFINITION. Given a foliation, two distinct leaves are said to be in the same *Novikov component* if there exists a loop transverse to the foliation that meets both leaves. A Novikov component is *proper* if it is not the entire manifold.

As noted earlier, in order for a foliation of a closed manifold M to be covered by a product, it is necessary that it be taut; that is, M must consist of a single Novikov component. Hence, in Propositions 2.7 and 2.8, we use weight systems on a branched surface carrying a foliation to give conditions that guarantee the foliation is taut.

PROPOSITION 2.7. *Suppose W is a compact branched surface carrying a foliation F of a closed manifold M. If there is no weight system for W, then there exists a loop transverse to F that meets every leaf; that is, F is taut.*

Proof. Given the foliation F, if there is no transverse loop to F that meets every leaf then every Novikov component is proper. The boundary of each proper Novikov component is the union of compact leaves [12], and we may construct W (up to homeomorphism) to ensure the corresponding set of compact leaves in F^* do not meet $\partial N(W)$ [17]. Each is then a compact surface in the interior of $N(W)$ that is transverse to the fibers, and hence induces a weight system on W when we let the weight assigned to a sector be the number of times that leaf intersects a fiber over that sector. Thus, if F is not taut then W has a weight system. □

It was shown in [18] that a branched surface carries a trivial fibration over S^1 by a compact surface if and only if it can be assigned a strictly positive weight system. Therefore, the converse of Proposition 2.7 does not hold.

We now let W be a branched surface carrying a foliation of a closed manifold, and consider all possible imbeddings of compact surfaces in the interior of $N(W)$ where the image is transverse to the fibers. We may partition the resulting set of imbedded surfaces into a finite number of equivalence classes by letting two compact surfaces S and S' be equivalent if $\pi(S) = \pi(S')$. That is, two compact surfaces are equivalent if they lie over the same sectors of W. Form a set $\mathbb{S}$ by taking one representative of each equivalence class.

PROPOSITION 2.8. *Let F be a foliation of a closed manifold that is carried by a compact branched surface W, and let F^* be the dual foliation of $N(W)$. If F has a transverse loop such that the dual transversal to F^* meets every surface in $\mathbb{S}$, then F is taut.*

Proof. If $\mathbb{S}$ is empty, then F^* has no compact leaves. In this case, M consists of a single Novikov component (proof of Proposition 2.7), so there exists a transverse loop that meets every leaf; that is, F is taut. Hence we assume that $\mathbb{S}$ is nonempty.

Let δ be a loop transverse to F such that the dual transversal δ^* to F^* meets every surface in $\mathbb{S}$. Without loss of generality, we assume that δ^* and the fibers of $N(W)$ are oriented consistently with the transverse orientation of F^*. If F is not taut, then the leaves met by δ are contained in the same proper Novikov component. In this case, there exists a compact leaf C in F^* (corresponding to a leaf in the boundary of this component) that is not met by δ^*. For some $C' \in \mathbb{S}$, $\pi(C) = \pi(C')$ and so C and C' intersect the same fibers of $N(W)$. Assume $\delta^*(t)$ intersects C' at $t = t_0$. Take a neighborhood $N(C')$ consisting of all fibers of $N(W)$ intersecting C'. Now C is in $N(C')$. Consider the fiber containing $\delta^*(t_0) \cap C'$. If C intersects this fiber at a point above (below) $\delta^*(t_0) \cap C'$, then $\delta^*(t)$ is contained in $N(C')$ for every $t > t_0$ ($t < t_0$) because it is transverse to F^* and bounded above (below) by C. But this contradicts the fact that δ^* is a loop. $\square$

The lower diagram in Figure 2.8 shows a branched 1-manifold W imbedded in a planar model of the torus. It was constructed from the foliation F shown in the upper diagram. This foliation F is covered by a product in $\mathfrak{R}^2$ and has a single compact leaf, which sits on the horizontal boundaries identified in the planar model. Consider a loop transverse to W that is given by a vertical line segment from the

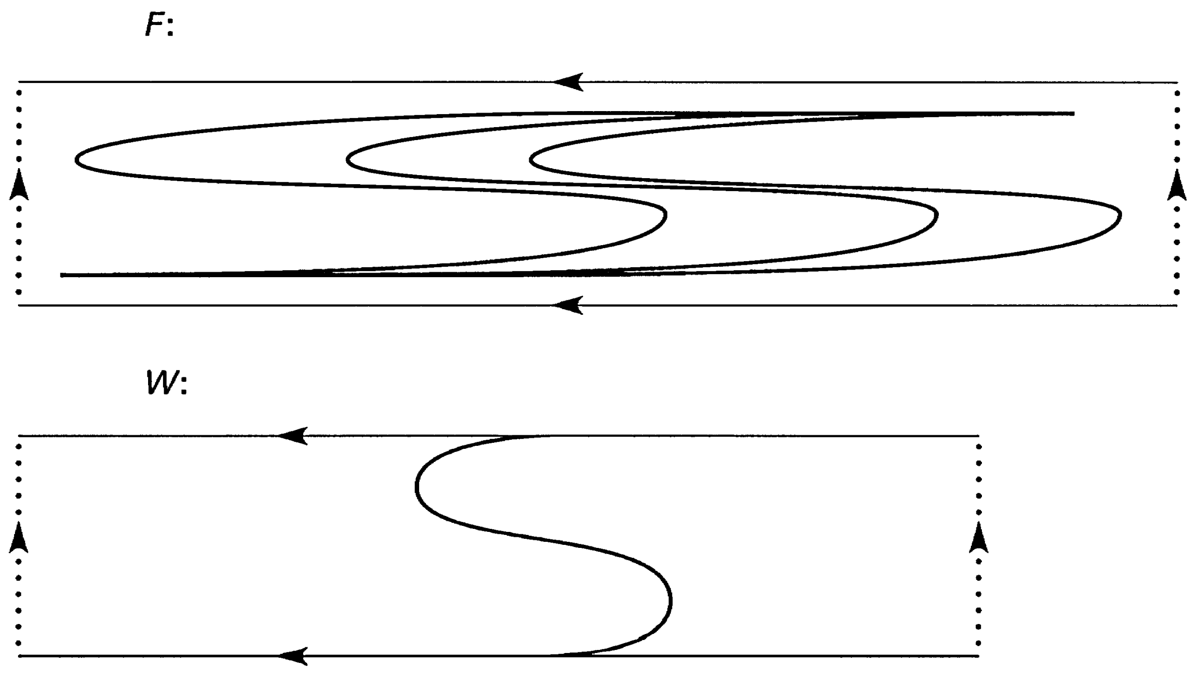

Figure 2.8

top to the bottom of the planar model containing W. This transverse loop induces a loop δ that is transverse to F and satisfies the hypothesis of Proposition 2.8. In fact, δ meets every leaf of F. Moreover, δ is not freely homotopic to an F-loop, so it gives rise to a loop that is covered by a copy of the leaf space for the product foliation.

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