Boundary Correspondence under $\mu(z)$-Homeomorphisms

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0. Introduction

We study the boundary correspondence of the upper half plane under $\mu(z)$-homeomorphisms, where $|\mu(z)|<1$ a.e. but $\|\mu\|_\infty=1$, and obtain estimates on the growth of the dilatation function and of the quasisymmetric function. Throughout this paper we denote the upper half plane by H. Let E be a compact set in H that is of σ-finite linear measure and hence of planar measure zero. Let $\mu(z)$ be a measurable function in H such that $\sup\{|\mu(z)|\,|\,z\in F\}<1$ for every compact set $F\subset H-E$. A function f is called a *self-$\mu(z)$-homeomorphism* of H if f is a homeomorphism of H onto itself and is locally $\mu(z)$-quasiconformal in $H-E$.

As the definition shows, the conditions for the $\mu(z)$-homeomorphisms are locally described. Therefore, estimates of modules by local character [4; 8] will be an efficient tool for studying $\mu(z)$-homeomorophisms. For convenience of description, we adopt the notation in [8]. Set

$$\begin{aligned}\phi(z,r,\theta)&=\frac{|1-e^{-2i\theta}\mu(z+re^{i\theta})|^2}{1-|\mu(z+re^{i\theta})|^2},\\ \phi^*(z,r)&=\frac{1}{2\pi}\int_0^{2\pi}\phi(z,r,\theta)\,d\theta.\end{aligned}\tag{0.1}$$

Let $A(z;r_1,r_2)=\{\tilde z\,|\,r_1<|\tilde z-z|<r_2\}$ be an annulus, and let f be a quasiconformal mapping in A with complex dilatation $\mu(z)$ satisfying $\|\mu(z)\|_\infty<1$. Then

$$\frac{1}{2\pi}\int_{r_1}^{r_2}\frac{dr}{r\phi^*(z,r)}\le \operatorname{mod} f(A(z;r_1,r_2)),\tag{0.2}$$

where $\operatorname{mod} f(A(z;r_1,r_2))$ is the module of the doubly connected domain $f(A(z;r_1,r_2))$. For details of the definition of the module, we refer to [1, p. 13].

By the method of Lehto [6], inequality (0.2) still holds in the case of a $\mu(z)$-homeomorphism, where the exceptional set E is of σ-finite linear measure. Furthermore, let $f(z)$ be a self-$\mu(z)$-homeomorphism of H and let

Received April 18, 1995. Revision received September 25, 1995.
Research supported by grants from the National Science Foundation of China and the Doctoral Program Foundation of Higher Education.
Michigan Math. J. 43 (1996).

$$A^+(x;r_1,r_2)=\{z\,|\,r_1<|z-x|<r_2,\,0<\arg(z-x)<\pi,\,x\in\mathbb{R}\}$$

be the upper half of $A(x;r_1,r_2)$. Set

$$\gamma_j=\{z\,||z-x|=r_j,\,0<\arg(z-x)<\pi,\,x\in\mathbb{R}\},\quad j=1,2.$$

Denote by Γ the family of curves that connect $f(\gamma_1)$ with $f(\gamma_2)$ in $f(A^+(x;r_1,r_2))$. We define the module $\operatorname{mod} f(A^+(x;r_1,r_2))$ by $\lambda(\Gamma)$, which is the extremal length of Γ. Then it is not difficult to prove that

$$\frac{1}{\pi}\int_{r_1}^{r_2}\frac{dr}{r\phi^*(x,r)}\le \operatorname{mod} f(A^+(x;r_1,r_2)),\tag{0.3}$$

where

$$\phi^*(x,r)=\frac{1}{\pi}\int_0^{\pi}\phi(x;r,\theta)\,d\theta,\quad x\in\mathbb{R}.\tag{0.4}$$

Similarly, for the dilatation function $D(z)=(1+|\mu(z)|)/(1-|\mu(z)|)$, we set

$$\begin{aligned}D^*(z,r)&=\frac{1}{2\pi}\int_0^{2\pi}D(z+re^{i\theta})\,d\theta,\\ D^*(x,r)&=\frac{1}{\pi}\int_0^{\pi}D(x+re^{i\theta})\,d\theta,\quad x\in\mathbb{R}.\end{aligned}\tag{0.5}$$

Noting that $\phi\le D$ and $\phi^*\le D^*$ and using (0.3), we deduce

$$\frac{1}{\pi}\int_{r_1}^{r_2}\frac{dr}{rD^*(x,r)}\le \operatorname{mod} f(A^+(x,r_1,r_2)),\quad x\in\mathbb{R}.\tag{0.6}$$

For a quasiconformal mapping of H onto itself, Beurling and Ahlfors [2] have studied its boundary correspondence and obtained fundamental results. It will be of interest to generalize their results to the case of a $\mu(z)$-homeomorphism. In Section 1, for a self-$\mu(z)$-homeomorphism f of H, we give sufficient conditions so that f can be extended to a homeomorphism of $\bar{H}$ onto itself. Denoting the boundary function by $h(x)=\lim_{z\to x}f(z)$ and following [2], we introduce the quasisymmetric function

$$\rho(x,t)=\frac{h(x+t)-h(x)}{h(x)-h(x-t)},\quad t>0.\tag{0.7}$$

In this case, $\rho(x,t)$ may tend to $+\infty$ when $t\to+0$. In Section 2 we derive an estimation of $\rho(x,t)$ when the mean dilatation function $D^*(x,r)$ is under certain control. In Section 3, we study the Beurling–Ahlfors extension f of a self-homeomorphism of $\mathbb{R}$ with unbounded $\rho(x,t)$, and obtain an estimation of the dilatation function $D(x+iy)$ of f when y is small.

1. Boundary Extension of $\mu(z)$-Homeomorphisms

An important difference between a quasiconformal mapping and a $\mu(z)$-homeomorphism is that the former can be homeomorphically extended to the boundary, whereas that is not yet clear for the latter. Under some appropriate conditions, we have the following result.

THEOREM 1. *Let $f(z)$ be a self-$\mu(z)$-homeomorphism of H. If, for every $x \in \mathbb{R}$,*

$$\int_{r_1}^{r_2} \frac{dr}{r(1+\phi^*(x,r))}$$

is positive and tends to $+\infty$ as $r_1 \to 0$ or as $r_2 \to +\infty$, then $f(z)$ can be extended to be a homeomorphism of $\bar{H}$ onto itself.

Proof. First we prove that for all $x \in \mathbb{R}$, $f(z)$ converges when z tends to x in H.

Otherwise, there exist two sequences $\{z_n^{(j)}\}$, $j=1,2$, such that $f(z_n^{(j)}) \to u_j$ as $z_n^{(j)}$ tends to $x \in \mathbb{R}$ in H where u_1 and u_2 are different. We may as well suppose that $-\infty < u_1 < u_2 < +\infty$, or we can make it so by a linear transformation. Let $A^+ = A^+(x; r_1, r_2)$ and

$$U^+(x,r) = \{z \,|\, |z-x| < r, 0 < \arg(z-x) < \pi\}, \quad x \in \mathbb{R}.$$

Let $\tilde{\Gamma}$ be the family of curves that connect $f(\gamma_2)$ with the real line segment $[u_1, u_2]$ in the simply connected domain $f(U^+(x, r_2))$. When r_2 is fixed, the extremal length $\lambda(\tilde{\Gamma})$ is a positive constant C independent of r_1.

Since $f(A^+) \subset f(U^+(x, r_2))$, it is obvious that for every curve $\tilde{\gamma} \in \tilde{\Gamma}$ there exists a curve $\gamma \in \Gamma$ such that $\gamma \subset \tilde{\gamma}$. Hence $\Gamma < \tilde{\Gamma}$, which implies [1, p. 11]

$$\lambda(\Gamma) < \lambda(\tilde{\Gamma}).$$

It then follows from (0.3) that

$$\frac{1}{\pi}\int_{r_1}^{r_2} \frac{dr}{r\phi^*(x,r)} \le \operatorname{mod} f(A^+) = \lambda(\Gamma) < \lambda(\tilde{\Gamma}) = C < +\infty \tag{1.2}$$

holds for any $r_1 > 0$. According to the conditions of the theorem, the left-hand side of (1.2) tends to $+\infty$ as $r_1 \to 0$, which is a contradiction. Therefore, $f(z)$ converges as $z \to x$.

From the assumption that

$$\int_{r_1}^{r_2} \frac{dr}{r(1+\phi^*(x,r))}$$

tends to $+\infty$ as $r_2 \to +\infty$, we can similarly prove that $f(z)$ converges as $z \to \infty$. Now we may define the boundary function by $h(x) = \lim_{z \to x} f(z)$. At most by a linear transformation, h can be normalized by $h(+\infty) = +\infty$, $h(-\infty) = -\infty$. Furthermore, it is not difficult to see that $h(x)$ is continuous.

Next we shall prove that h is injective on $\mathbb{R}$.

Otherwise, there exists $u \in \mathbb{R}$ and $-\infty < x_1 < x_2 < +\infty$ such that $h(x_1) = h(x_2) = u$, which implies $h(x_3) = u$ for any $x_3 \in (x_1, x_2)$ as is known from topology. Consider the upper half annulus $A^+ = A^+((x_1+x_2)/2; (x_2-x_1)/4, (x_2-x_1)/2)$. By the definition of module, we have

$$\operatorname{mod} f\left(A^{+}\left(\frac{x_1+x_2}{2}; \frac{x_2-x_1}{4}, \frac{x_2-x_1}{2}\right)\right) = 0. \tag{1.3}$$

But it follows from (0.3) and assumption (1.1) that

$$\operatorname{mod} f\left(A^{+}\left(\frac{x_1+x_2}{2}; \frac{x_2-x_1}{4}, \frac{x_2-x_1}{2}\right)\right) > 0.$$

The contradiction shows that h must be injective on $\mathbb{R}$.

From the preceding discussion of $f(z)$ and the well-known consequences of topology, it is easy to see that f can be extended to a homeomorphism of $\bar{H}$ onto itself. □

Theorem 1 provides a sufficient condition under which any self-$\mu(z)$-homeomorphism of H can be extended to a self-homeomorphism of $\bar{H}$.

2. Estimates for the Growth of Quasisymmetric Functions

Let $f(z)$ be a self-$\mu(z)$-homeomorphism of H where $\mu(z)$ fulfills condition (1.1). According to Theorem 1, $f(z)$ can be extended to a self-homeomorphism of $\bar{H}$. Then the boundary function h,

$$h(x) = \lim_{z \to x} f(z), \quad x \in \mathbb{R},$$

is a homeomorphism of $\mathbb{R}$ onto itself with ∞ fixed. In the present section, assume that $D^*(x,r)$ is controlled by $M \log(1/r)$ when r is small. Then we shall obtain an estimate of $\rho(x,t)$ for small t.

Let T_p be a Teichmüller domain whose complements are $[-1,0]$ and $[p,+\infty)$. Set

$$\operatorname{mod} T_p = \frac{1}{2\pi} \log \Psi(p).$$

Obviously, Ψ is a strictly increasing function of p.

LEMMA 1. *Let f be a self-$\mu(z)$-homeomorphism of H, and let $D^*(x,r)$ be the mean dilatation function. If $f(z)$ can be extended to a self-homeomorphism ot $\bar{H}$, then*

$$\Psi^{-1}\left(\exp\left(\int_{t/2}^{3t/2} \frac{dr}{rD^*(x-t/2,r)}\right)\right) \le \rho(x,t)$$
$$\le \left(\Psi^{-1}\left(\exp\left(\int_{t/2}^{3t/2} \frac{dr}{rD^*(x+t/2,r)}\right)\right)\right)^{-1},$$

where $\rho(x,t)$ is the quasisymmetric function.

Proof. For any $x \in \mathbb{R}$ and positive number t, consider the half annulus $A^+(x+t/2; t/2, 3t/2)$. To simplify notation, we write $\rho = \rho(x,t)$. By the basic properties of extremal length [1, pp. 11 & 16], we have

$$\operatorname{mod} f\left(A^+\left(x+\frac{t}{2};\frac{t}{2},\frac{3t}{2}\right)\right) \le 2 \operatorname{mod} T_{1/\rho} = \frac{1}{\pi}\log\Psi\left(\frac{1}{\rho}\right).$$

From (0.6) it follows that

$$\frac{1}{\pi}\int_{t/2}^{3t/2}\frac{dr}{rD^*(x+t/2,r)} \le \operatorname{mod} f\left(A^+\left(x+\frac{t}{2};\frac{t}{2},\frac{3t}{2}\right)\right).$$

Combining the above two inequalities, we have

$$\frac{1}{\pi}\int_{t/2}^{3t/2}\frac{dr}{rD^*(x+t/2,r)} \le \frac{1}{\pi}\log\Psi\left(\frac{1}{\rho}\right). \tag{2.1}$$

After rearrangement, we obtain

$$\rho(x,t) \le \left(\Psi^{-1}\left(\exp\left(\int_{t/2}^{3t/2}\frac{dr}{rD^*(x+t/2,r)}\right)\right)\right)^{-1}.$$

With a similar discussion on another half annulus $A^+(x-t/2;t/2,3t/2)$ as above, with ρ still denoting $\rho(x,t)$, we have

$$\operatorname{mod} f\left(A^+\left(x-\frac{t}{2};\frac{t}{2},\frac{3t}{2}\right)\right) \le 2 \operatorname{mod} T_\rho = \frac{1}{\pi}\log\Psi(\rho)$$

and

$$\frac{1}{\pi}\int_{t/2}^{3t/2}\frac{dr}{rD^*(x-t/2,r)} \le \frac{1}{\pi}\log\Psi(\rho). \tag{2.2}$$

After rearrangement, we obtain

$$\rho(x,t) \ge \Psi^{-1}\left(\exp\left(\int_{t/2}^{3t/2}\frac{dr}{rD^*(x-t/2,r)}\right)\right). \qquad \square$$

THEOREM 2. *Let f be a self-$\mu(z)$-homeomorphism of H, and let $D^*(x,r)$ be the mean dilatation function. If there exists $0<\delta<1$ such that, for any $0<r<\delta$,*

$$D^*(x,r) \le M\log(1/r) \tag{2.3}$$

where M is a constant independent of x and r, then

$$(t/2)^A < \rho(x,t) < (t/2)^{-A}$$

holds for $0<t<2\delta/3$ with constant $A=\sqrt{2}M\pi^2/\log 3$, where $\rho(x,t)$ is the quasisymmetric function.

Proof. From the assumption $D^*(x,r)\le M\log(1/r)$, $0<r<\delta$, it follows that when $t<2\delta/3$,

$$\begin{aligned}\int_{t/2}^{3t/2}\frac{dr}{rD^*(x\pm t/2,r)} &\ge \int_{t/2}^{3t/2}\frac{dr}{Mr\log(1/r)}\\ &= \frac{1}{M}\left|\log\frac{|\log 3t/2|}{|\log t/2|}\right|\\ &= \frac{1}{M}\left|\log\left(1+\frac{\log 3}{\log t-\log 2}\right)\right|.\end{aligned}$$

Since $0 < t < 2\delta/3 < 2/3$, it is obvious that

$$\left|\log\left(1+\frac{\log 3}{\log t-\log 2}\right)\right| \geq \frac{\log 3}{\log 2-\log t}.$$

Hence we have

$$\int_{t/2}^{3t/2} \frac{dr}{rD^*(x\pm t/2, r)} > \frac{\log 3}{M(\log 2-\log t)} \tag{2.4}$$

for $0 < t < 2\delta/3$.

If $p \in (0,1)$, we claim that

$$\frac{1}{\pi}\log\Psi(p) < \frac{\sqrt{2}\pi}{|\log p|}. \tag{2.5}$$

In fact, it follows from [1, p. 40] that

$$\frac{1}{2\pi}\log\Psi(p) = \frac{a}{2b},$$

where

$$a = \int_0^p \frac{dx}{\sqrt{(x+1)x(p-x)}}, \qquad b = \int_p^{+\infty} \frac{dx}{\sqrt{(x+1)x(x-p)}}.$$

It is not difficult to obtain the following estimates:

$$a \leq \int_0^p \frac{dx}{\sqrt{(x(p-x)}} = \int_0^1 \frac{du}{\sqrt{u(1-u)}} = \pi,$$

$$b > \int_p^1 \frac{dx}{\sqrt{(x-1)x(x-p)}} > \frac{1}{\sqrt{2}}\int_p^1 \frac{dx}{x} = \frac{1}{\sqrt{2}}|\log p|.$$

Thus, (2.5) follows immediately.

Obviously, assumption (2.3) fulfills condition (1.1), so $f(z)$ can be extended to a self-homeomorphism of $\bar{H}$ by Theorem 1. We still write $\rho = \rho(x,t)$ as in Lemma 1.

If $\rho(x,t) > 1$, (2.5) implies that $(1/\pi)\log\Psi(1/\rho) < \sqrt{2}\pi/|\log 1/\rho|$. Combining this inequality with (2.1) and (2.4), we have

$$\frac{1}{\pi}\frac{\log 3}{M(\log 2-\log t)} < \frac{\sqrt{2}\pi}{|\log 1/\rho|}.$$

Hence we obtain

$$\rho(x,t) < (t/2)^{-A}, \tag{2.6}$$

with constant $A = \sqrt{2}M\pi^2/\log 3$.

If $\rho(x,t) < 1$, (2.5) implies that $(1/\pi)\log\Psi(\rho) < \sqrt{2}\pi/|\log\rho|$. Combining this inequality with (2.2) and (2.4), we have

$$\frac{1}{\pi}\frac{\log 3}{M(\log 2-\log t)} < \frac{\sqrt{2}\pi}{|\log\rho|}.$$

Hence, we obtain

$$\rho(x,t) > (t/2)^{A}. \tag{2.7}$$

By (2.6) and (2.7), we finally conclude that

$$(t/2)^A < \rho(x,t) < (t/2)^{-A}$$

holds for $0 < t < 2\delta/3$, with constant $A = \sqrt{2}M\pi^2/\log 3$. □

REMARK. If $f(z)$ in this theorem is K-q.c., then $D^*(x,r) \le K$. Under this assumption, (2.1) and (2.5) yield

$$\frac{1}{\pi}\frac{\log 3}{K} \le \frac{\sqrt{2}\pi}{|\log 1/\rho|};$$

hence

$$\rho \le \exp\left(\frac{\sqrt{2}}{\log 3}\pi^2 K\right).$$

This inequality remains consistent on the order of $\log\rho$ with the best possible estimate $\rho < (1/16)e^{\pi K}$ [1, p. 65].

3. Estimate for Growth of the Dilatation Function

Let h be a homeomorphism of $\mathbb{R}$ onto itself with ∞ fixed. Let $\phi(z)$ be the Beurling–Ahlfors extension of h:

$$\phi(z) = u(x,y) + iv(x,y), \tag{3.1}$$

where

$$\begin{aligned} u(x,y) &= \frac{1}{2y}\int_{x-y}^{x+y} h(t)\,dt, \\ v(x,y) &= \frac{1}{2y}\left(\int_{x}^{x+y} h(t)\,dt - \int_{x-y}^{x} h(t)\,dt\right). \end{aligned} \tag{3.2}$$

Since $\phi(z) \in C^1$, it is easy to see that $\phi(z)$ is a homeomorphism of H onto itself whose dilatation function $D(z)$ is finite at every point in H. Hence $\phi(z)$ is the so-called self-$\mu(z)$-homeomorphism of H. In this section, we shall obtain estimates on the growth of $D(x+iy)$ from that of $\rho(x,t)$.

Suppose that the quasisymmetric function $\rho(x,t) = (h(x+t)-h(x))/(h(x)-h(x-t))$ satisfies the following condition:

$$\rho(t)^{-1} \le \rho(x,t) \le \rho(t) \tag{3.3}$$

holds for all $x \in \mathbb{R}$ and $0 < t < \delta$, where $\rho(t)$ is a decreasing function in $(0,\delta)$. Because it is permitted that $\rho(t)$ tend to $+\infty$ in any order as $t \to +0$, condition (3.3) for $h(t)$ is a generalization of the ordinary ρ-condition of quasisymmetric functions. Then we have the following theorem.

THEOREM 3. *Let h be a homeomorphism of $\mathbb{R}$ onto itself with $h(\pm\infty) = \pm\infty$. If the quasisymmetric function $\rho(x,t)$ of h satisfies condition* (3.3), *then the dilatation of the Beurling–Ahlfors extension $\phi(z)$ of h at point $z_0 = x_0 + iy_0$ has the following estimation:*

$$D(x_0 + iy_0) \le 4\rho(y_0) + c \tag{3.4}$$

for all $x_0 \in \mathbb{R}$ and $0 < y_0 < \delta$, where the constant c can be chosen to be 4.25.

Before we begin our proof of the theorem, we make the following remarks. First, let $h^*(t)=h(y_0t+x_0)$ and denote the Beurling–Ahlfors extension of h^* by $\phi^*(z)$. Then $\phi^*(z)=\phi(y_0z+x_0)$, where $\phi(z)$ is the Beurling–Ahlfors extension of h. Because $D_{f^*}(i)=D_f(x_0+iy_0)$ and

$$\rho^*(x,t)=\frac{h^*(x+t)-h^*(x)}{h^*(x)-h^*(x-t)}=\rho(y_0x+x_0,y_0t),$$

we know that it suffices to estimate the dilatation of $\phi(z)$ at only one point $z_0=i$, with the quasisymmetric function satisfying

$$\rho(y_0t)^{-1}\le\rho(x,t)\le\rho(y_0t) \tag{3.5}$$

for all $x\in\mathbb{R}$ and $0<t<\delta/y_0$, where $\rho(y_0t)$ is decreasing in $t\in(0,\delta/y_0)$.

Second, we can assume that h fulfills the normalized condition $h(0)=0$, $h(1)=1$, $h(\infty)=\infty$. Otherwise, we can choose $h^*(t)=(h(t)-h(0))/h(1)$ instead of $h(t)$, which changes neither the condition (3.5) nor the dilatation of the Beurling–Ahlfors extension at $z_0=i$.

Let h be a normalized self-homeomorphism of $\mathbb{R}$ whose quasisymmetric function $\rho(x,t)$ satisfies condition (3.5), and let $\phi(z)$ be the Beurling–Ahlfors extension of h. Denote the local dilatation of $\phi(z)$ at $z_0=i$ by D. It follows from [2] that

$$D+\frac{1}{D}=\frac{1}{\xi+\eta}\left[\beta(1+\eta^2)+\frac{1}{\beta}(1+\xi^2)\right], \tag{3.6}$$

where $\beta=-h(-1)$, $\xi=1-\int_0^1h(t)\,dt$, and $\eta=1+(1/\beta)\int_{-1}^0h(t)\,dt$. The following estimates for β,ξ,η are obvious:

$$\rho(y_0)^{-1}\le\beta\le\rho(y_0), \tag{3.7}$$

$$0<\xi,\eta<1. \tag{3.8}$$

For the proof of Theorem 3, we need the following.

LEMMA 2. *Let h be a normalized self-homeomorphism of $\mathbb{R}$, and let $\rho(t)$, ξ,η,β be defined by* (3.5) *and* (3.6). *Then*

$$2\xi+\frac{\beta}{1+\rho(y_0)}\eta\ge\frac{\beta}{1+\rho(y_0)}. \tag{3.9}$$

Proof. Let $t\in(0,\frac{1}{2})$. It follows from

$$\frac{h(1)-h(t)}{h(t)-h(2t-1)}\ge\rho((1-t)y_0)^{-1}$$

that

$$h(t)\le\frac{\rho((1-t)y_0)+h(2t-1)}{1+\rho((1-t)y_0)}.$$

Integrating both sides of the above inequality with respect to $t\in(0,\frac{1}{2})$ and noting the decreasing property of $\rho(t)$, we have

$$\begin{aligned}\int_0^{1/2} h(t)\,dt &\le \int_0^{1/2} \frac{\rho((1-t)y_0)}{1+\rho((1-t)y_0)}\,dt + \int_0^{1/2} \frac{h(2t-1)}{1+\rho((1-t)y_0)}\,dt \\ &\le \frac{1}{2}+\frac{1}{2}\frac{1}{1+\rho(y_0)}\int_{-1}^{0} h(t)\,dt.\end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned}&2\xi+\frac{\beta}{1+\rho(y_0)}\eta \\ &= 2-2\left(\int_0^{1/2} h(t)\,dt+\int_{1/2}^{1} h(t)\,dt\right)+\frac{\beta}{1+\rho(y_0)}\left(1+\frac{1}{\beta}\int_{-1}^{0} h(t)\,dt\right) \\ &\ge 2-\left(1+\frac{1}{1+\rho(y_0)}\int_{-1}^{0} h(t)\,dt+1\right)+\frac{\beta}{1+\rho(y_0)}\left(1+\frac{1}{\beta}\int_{-1}^{0} h(t)\,dt\right) \\ &= \frac{\beta}{1+\rho(y_0)},\end{aligned}$$

which completes the proof of Lemma 2. □

Proof of Theorem 3. We only discuss the case $1 \le \beta \le \rho(y_0)$, because the case $\rho(y_0)^{-1} \le \beta < 1$ can be handled by setting $h^*(t) = h(-t)/-\beta$, which is also a normalized self-homeomorphism of $\mathbb{R}$ and whose quasisymmetric function also satisfies condition (3.5). Let $\phi^*(z)$ be the Beurling-Ahlfors extension of $h^*(t) = h(-t)/-\beta$; then $\phi^*(z) = \overline{\phi(-\bar{z})}/-\beta$. It is obvious that $D_{\phi^*}(i) = D_\phi(i)$. So, without loss of generality, we assume that

$$1 \le \beta \le \rho(y_0). \tag{3.11}$$

Denote the right side of (3.6) by $H(\xi,\eta)$, that is,

$$H(\xi,\eta) = \frac{1}{\xi+\eta}\left[\beta(1+\eta^2)+\frac{1}{\beta}(1+\xi^2)\right]. \tag{3.12}$$

By (3.8) and (3.9), we know that the point (ξ,η) lies in the domain bounded by lines $\xi = 1$, $\eta = 0$, $\eta = 1$, and $2\xi+\beta/(1+\rho(y_0))\eta = \beta/(1+\rho(y_0))$. It was pointed out in [2] that $H(\xi,\eta)$ is convex, so $H(\xi,\eta)$ reaches its maximum value at one of the four vertexes: $(0,1)$, $(1,1)$, $(1,0)$, and $(\beta/2(1+\rho(y_0)),0)$. Comparing the values of $H(\xi,\eta)$ at these four points, we have

$$\begin{aligned}D < D+\frac{1}{D} &\le H\left(\frac{\beta}{2(1+\rho(y_0))},0\right) \\ &= 2(1+\rho(y_0))\left(1+\frac{1}{\beta^2}\right)+\frac{1}{2(1+\rho(y_0))} < 4\rho(y_0)+c,\end{aligned}$$

where $c = 4.25$. The proof is completed. □

REMARK. Under the condition that $\rho^{-1} \le \rho(x,t) \le \rho$, where ρ is a constant, Lehtinen [5] proved that the dilatation $D(z)$ of the Beurling-Ahlfors extension of h has the estimation $D(z) \le 2\rho$. Hence the constants 4 and c in (3.4) may be not the best possible. We have reason to believe that when inequalities (3.8) and (3.9) are improved, better estimation can be expected.

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