p-Dirichlet Energy Minimizing Maps into a Complete Manifold

SHIAH-SEN WANG

1. Introduction

Suppose M is a compact m-dimensional $\mathcal{C}^2$ Riemannian submanifold of $\mathbb{R}^Q$ with (or without) boundary ∂M. Suppose N is an n-dimensional $\mathcal{C}^2$ complete connected Riemannian submanifold without boundary of some Euclidean space $\mathbb{R}^P$ such that

$$N \subset N_\tau = \{y \in \mathbb{R}^P : \operatorname{dist}(y, N) < \tau(|y|)\}.$$

Here N_τ is a τ-tubular neighborhood of N in $\mathbb{R}^P$ such that the nearest point projection map

$$\pi : N_\tau \to N$$

(i.e., $\operatorname{dist}(y, N) = |y - \pi(y)|$ for all $y \in N_\tau$) exists and is as smooth as N, and

$$\tau : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$$

is a monotonically decreasing function.

The p-Dirichlet energy functional is the L^p-norm of the gradient defined on the admissible mapping space

$$L^{1,p}(M, N) = \{v \in L^{1,p}(M, \mathbb{R}^P) : v(x) \in N \text{ for } \mathcal{L}^m \text{ a.e. } x \in M\},$$

where $\mathcal{L}^m$ is the m-dimensional Hausdorff measure induced by the metric of M and $1 < p \le m$. We say $u \in L^{1,p}(M, N)$ is p-Dirichlet energy-minimizing if

$$\int_M |\nabla u|^p \le \int_M |\nabla v|^p$$

for all $v \in L^{1,p}(M, N)$ in the same (relative) homotopy class of u.

Much work has been done regarding the existence and partial regularity of a p-Dirichlet energy-minimizing map, in particular the case when $p = 2$ (see [1] for references). For $\tau > c > 0$, White showed in [8] that there is such a minimizer among maps in the same $[p-1]$-homotopy class. Furthermore, if the image of a small ball of such a minimizer is contained in a compact subset of N, where N may not be compact, then Hardt and Lin's [4] (or Luckhaus's

---

Received September 6, 1994. Revision received November 1, 1995.
Research supported in part by NSC 83-0208-M-009-055.
Michigan Math. J. 43 (1996).

[6]) study on the partial regularity of energy minimizing maps between two Riemannian manifolds is readily applied to this case, that is, such a minimizing map is $\mathcal{C}^\alpha$ in the interior of M with the exception of a singular set whose Hausdorff dimension is at most $m-[p]-1$. Recently, Li [5] obtained the partial regularity of a Dirichlet energy minimizing harmonic map ($p=2$) with a singular set of Hausdorff dimension at most $m-2$ into a complete manifold, which is not necessarily a uniform tubular neighborhood retract; that is,

$$\tau(t) \to 0 \quad \text{as } t \to \infty,$$

contrary to the assumption of previous works [4; 6; 8; 9] that $\tau > c$ for some constant $c > 0$. Thus N may be a submanifold of $\mathbb{R}^P$ such that its curvature is infinite at infinity or its different branches become close near infinity, since $\tau(t) \to 0$ as $t \to \infty$.

First we note that, in order to show two maps are in the same homotopy class, it is natural to consider their affine homotopy. The image of this homotopy map will not necessarily stay on N, yet a retraction map can be applied to make this homotopy map stay on N. The assumption $\tau > c > 0$ is used to ensure the existence of such a retraction map in White's study [9] of existence theory of p-Dirichlet energy minimizing maps. A similar technique is also used in constructing a comparison map for this minimizing map in Luckhaus's study [6] of its partial regularity.

Second, in the partial regularity theory of [4] and [6], the assumption that the image of a small ball $B_r(a)$ of an energy minimizing map lies in a compact subset of N is used to ensure that the energy of this minimizing map will not be distributed to the infinity of N. This assumption also provides a monotonicity formula for the normalized energy of this p-Dirichlet energy minimizing map u on $B_r(a)$,

$$r^{p-m} \int_{B_r(a)} |\nabla u|^p,$$

to further reduce the upper bound of the Hausdorff dimension of its singular set to $m-[p]-1$ (by a Federer dimension-reducing argument). In [5, Sec. 2], Li constructed an example showing that, when this assumption is removed, an energy minimizing harmonic map from one surface to another exists with isolated singular point, even though the monotonicity formula still holds. Here we notice that this monotonicity formula can be obtained via Almgren's "squeeze" deformation of the domain $B_r(a)$ (see [4, Lemma 4.1] for details), which still works in Li's setting.

In this paper, we will show that White's results on the existence of an energy minimizing map among maps in the same homotopy class still hold in Li's setting. We will also show that such an energy minimizing map u is C^α on $M \sim \partial M \sim Z$. Here Z is defined as the set of points a in M for which the normalized energy on the ball $B_r(a)$ fails to approach zero as $r \to 0$, or for which the supremum of the absolute value of its averaged integral on $B_r(a)$,

$$\left|\frac{1}{\mathcal{L}^m(B_r(a))}\int_{B_r(a)} u\right|,$$

is infinity when $r \to 0$.

The main ingredient for making the generalization of these results possible without using assumptions of previous works [4; 6; 8; 9] is that the sup-norm of the affine homotopy map of two maps is bounded by the L^p norm of the difference of these two maps and the L^p norm of their gradients. The sizes of these quantities can be controlled when the image of one of the maps lies in the compact subset of N. We then show that the image of the affine homotopy still lies in N_τ, and the rest of the proofs simply involve some careful modifications of the previous works.

Now we recall some definitions and notation.

Definitions. If X is a polyhedral complex, we let X^k denote the k-dimensional skeleton of X. We say a d-dimensional polyhedral complex is a *regular polyhedral complex* if it is the union of its d-dimensional cells and if, for every connected open set $U \subset X$, the set $U \sim X^{d-2}$ is also connected.

From now on we will assume that M has a fixed smooth (or Lipschitz) triangulation.

We say that two continuous maps $f, g: M \to N$ are k-homotopic if their restrictions to M^k are homotopic. The k-homotopy type of a continuous map from M to N is the homotopy class of its restriction to the k-dimensional skeleton of M.

Notation. $B_t(x)$ and $S_t(x)$ (or simply B_t and S_t when the center is clear from the context) will denote the ball and the sphere respectively of radius $t > 0$ centered at x in M:

$$B_t(x) = \{z \in M: \operatorname{dist}_M(x, z) < t\}, \qquad S_t(x) = \{z \in M: \operatorname{dist}_M(x, z) = t\},$$

where $\operatorname{dist}_M(\cdot, \cdot)$ is the geodesic distance function in M. By $B_t^k(z)$ we mean the ball of radius $t > 0$ centered at z in $\mathbb{R}^k$, when the dimension is needed to be specified.

For $v \in L^{1,p}(M, \mathbb{R}^P)$, $t > 0$, $A \subset M$, and $x \in M$, we let

$$⨍_A v = \frac{1}{\mathcal{L}^m(A)}\int_A v,$$

the averaged integral of v over A. In particular,

$$v_{x,t} = ⨍_{B_t(x)} v$$

when A is a ball, and

$$E_{x,t}(v) = ⨍_{B_t(x)} |\nabla v|^p,$$

the normalized energy of v over the ball $B_t(x)$.

The term $\mathcal{H}^k$ will denote the k-dimensional Hausdorff measure induced by the metric of the ambient space, and $[p]=n$ if $n \le p < n+1$ for some $n \in \mathbb{Z}$. Throughout, the letter C will represent various constants whose dependence on the data will be specified in the context.

This paper is organized as follows. In Section 2, we prove the existence results for the case where N is complete without boundary and with boundary. In Section 3, we prove the partial regularity of a p-Dirichlet energy minimizing map.

2. Existence Theorem of p-Dirichlet Energy Minimizing Maps

The proofs of many of the results in this section will be exactly the same as in [8] and [9], particularly those involving manipulation on the domain M only. We shall simply state such results without proof; the reader can refer to the original papers for details. The following four lemmas are the basic properties of functions in the Sobolev space with a polyhedral complex as the domain; their proofs can be found in [8] and [9].

LEMMA 2.1 (Morrey-type Inequality). *Let X be a regular d-dimensional polyhedral complex, where $d<p$ and $0<\gamma<1-d/p$. For every $\epsilon>0$, there is a positive constant $C(\epsilon)$ such that if $f: X \to \mathbb{R}$ is Lipschitz, then*

$$|f|_{0,\gamma} \le \epsilon \|\nabla f\|_p + C(\epsilon)\|f\|_p. \tag{2.1}$$

Consequently, if $f \in L^{1,p}(X,\mathbb{R})$ then f is $\mathcal{H}^d$-a.e. equal to a $C^{0,\gamma}$ function that satisfies (2.1).

LEMMA 2.2 (Local Poincaré Inequality). *Let X be a regular d-dimensional polyhedral complex with $d+1 \le p$. Let*

$$\mathcal{B}_r(x) = \{y \in X: \operatorname{dist}(y,x) \le r\},$$

where $\operatorname{dist}(\cdot,\cdot)$ *is the geodesic distance function on X. Then there is a positive constant $C=C(X,P,p)$ such that for every $x\in X$, $r>0$, and Lipschitz $f: \mathcal{B}_{3r}(x) \to \mathbb{R}^P$, there is a $v \in \mathbb{R}^P$ such that*

$$\int_{\mathcal{B}_r(x)} |f-v|^p \le r^p \int_{\mathcal{B}_{3r}(x)} |\nabla f|^p. \tag{2.2}$$

LEMMA 2.3 (Fubini-type Lemma). *Let h be a Lipschitz map from a regular d-dimensional polyhedral complex X into Ω, and let $\delta \le \operatorname{dist}(h(X),\partial\Omega)$. For each k, there exists a positive constant $C=C(\delta,k)$ such that if $F_1,\dots,F_k \in L^1$, then*

$$\int_X |F_i(h(x)+v)|\,dx < \infty \quad \textit{for } i=1,\dots,k \tag{2.3}$$

for $\mathcal{L}^Q$ a.e. $v \in B_\delta(0)$, and

$$\int_X |F_i(h(x)+v)| \le C|X| \int_\Omega |F_i| \tag{2.4}$$

on a set of $v \in B_\delta(0)$ of positive $\mathcal{L}^Q$ measure.

LEMMA 2.4. *Let X be a regular polyhedral complex and U an open subset of $\mathbb{R}^Q$. Let $h: X \to U$ be Lipschitz, let $f \in L^{1,p}(U, \mathbb{R}^P)$, and let g be a distribution derivative of f. Define $h_v: X \to U$ by*

$$h_v(x) = h(x) + v,$$

for $v \in B_\delta(0)$, where $\delta = \operatorname{dist}(h(X), \partial U)$. Then, for $\mathcal{L}^Q$ a.e. $v \in B_\delta(0)$, $f \circ h_v \in L^{1,p}$ and $g(h_v) \cdot \nabla h_v$ is a distribution derivative of $f \circ h_v$.

The proofs of the next two propositions involve some modifications—see in particular (2.7), (2.8), and (2.13)—since the retraction map $\pi: N_\tau \to N$ is used in White's original proofs.

PROPOSITION 2.5. *Let U be an open subset of $\mathbb{R}^Q$, and let $f \in L^{1,p}(U, N)$. Let X be a regular polyhedral complex of dimension $d \le [p-1]$ that is contained in the d-skeleton of a regular $(d+1)$-dimensional polyhedral complex Y. Let $h: X \to U$ be a Lipschitz map that extends to a Lipschitz map from Y to U. Then there is a homotopy class $f_\#[h]$ of continuous maps from X into N with the following properties. For $\mathcal{L}^Q$ a.e. $v \in B_\delta(0)$, where $\delta = \operatorname{dist}(h(X), \partial U)$, there is a continuous map $g^v: X \to N$ such that:*

(1) *$f \circ h_v(x) = g^v(x)$ for $\mathcal{H}^k$ a.e. $x \in X^k$ $(0 \le k \le d)$;*
(2) *g^v extends to a continuous map from Y into N; and*
(3) *$g^v \in f_\#[h]$.*

Furthermore, if $\psi \in \operatorname{Lip}(X, U)$ is homotopic to h, then $f_\#[\psi] = f_\#[h]$.

Proof. Define $\tilde{X} \subset X \times [0,1]^d$ and $\tilde{h}: \tilde{X} \to U$ by

$$\tilde{X} = \bigcup_{k=0}^{d} X^k \times \{0\}^k \times [0,1]^{d-k} \quad \text{and} \quad \tilde{h}(x,t) = h(x).$$

Then $\tilde{X}$ is a regular polyhedral complex. By Lemma 2.4, for $\mathcal{L}^Q$ a.e. $v \in B_\delta(0)$, $f \circ h_v \in L^{1,p}(\tilde{X}, N)$. By Lemma 2.1, for such a v there is a continuous map $g^v: \tilde{X} \to N$ such that

$$f \circ \tilde{h}_v(x,t) = g^v(x,t)$$

for $\mathcal{H}^d$ a.e. $(x,t) \in \tilde{X}$; that is,

$$f \circ h_v(x) = g^v(x,t) \quad \text{for } \mathcal{H}^d \text{ a.e. } (x,t) \in \tilde{X}. \tag{2.5}$$

The set $\tilde{X}_x = \{t: (x,t) \in \tilde{X}\}$ is connected for each x, so (2.5) implies that g^v is a function of x alone. Thus we may write

$$f \circ h_v(x) = g^v(x) \tag{2.6}$$

for $\mathcal{H}^d$ a.e. $(x,t) \in X^k \times \{0\}^k \times [0,1]^{d-k}$ and therefore $\mathcal{H}^k$ a.e. $x \in X^k$. This proves (1).

To prove (2), let $\psi: Y \to U$ be a Lipschitz map that extends h. Let $\hat{Y} = X \times [0,1] \times Y \times \{1\}$ and define $\hat{\psi}: \hat{Y} \to U$ by $\hat{\psi}(y, t) = \psi(y)$. Then, for $\mathcal{L}^Q$ a.e. $v \in B_\delta(0)$, $f \circ \hat{\psi} \in L^{1,p}(\hat{Y}, N)$. Let $\nabla f \circ \hat{\psi}_v$ be a distribution derivative. As in the proof of (1), $f \circ \hat{\psi}_v$ is essentially continuous on $X \times [0,1]$.

Let $\epsilon > 0$. For $r \in (0,1]$, define $H(y,t) \in \mathbb{R}^P$ that minimizes

$$\int_{z \in \hat{Y},\, \operatorname{dist}(z,(y,t)) < \epsilon r} |f \circ \hat{\psi}_v(z) - H(z,t)|^p \, dz,$$

where $H(z,t)$ is unique by the strict convexity of L^p-norm. Clearly, $H(z,t)$ is continuous for $r > 0$, and because $f \circ \hat{\psi}_v$ is essentially continuous on $X \times [0,1]$, we may extend H continuously to all of $\hat{Y}$ so that

$$H(x,0) = f \circ \hat{\psi}_v(x) \quad \text{for } \mathcal{H}^d \text{ a.e. } x \in X.$$

Now

$$\begin{aligned} \operatorname{dist}(H(y,r), N) &< C(\epsilon r)^{-(d+1)} \int_{z \in \hat{Y},\, \operatorname{dist}(z,(y,r)) < \epsilon r} |f(\hat{\psi}_v(z)) - H(y,r)|^p \, dz \\ &\leq C \int_{z \in \hat{Y},\, \operatorname{dist}(z,(y,r)) < 3\epsilon} |\nabla (f \circ \hat{\psi}_v)(z)|^p \, dx \end{aligned} \tag{2.7}$$

by the local Poincaré inequality.

By taking $\epsilon < \min\{\tau(|H(y,r)|) : (y,r) \in \hat{Y} \times [0,1]\}$, we can make the last term in the chain of inequalities (2.7) as small as we like. In particular, H lies in the tubular neighborhood N_τ of N. Then $\pi \circ H(\cdot, 0) = g^v(\cdot)$, which is homotopic to $\pi \circ H(\cdot, 1) \mid X$, which extends to $\pi \circ H(\cdot, 1): Y \to N$. This proves (2).

To prove (3), fix a small vector $u \in \mathbb{R}^Q$. Consider the map

$$\tilde{h}: (X \times \{0\}) \cup (X \times \{1\}) \to U,$$

by $\tilde{h}(x,t) = h(x) + tu$, and notice that $\tilde{h}$ extends to a Lipschitz map of $Y = X \times [0,1]$ into U. By (1) and (2), for $\mathcal{L}^Q$ a.e. (small) v, $f \circ \hat{h}_v$ is essentially continuous and extends to a continuous map of $X \times [0,1]$ into N. But that means $f \circ \hat{h}_v$ and $f \circ \hat{h}_{v+u}$ are essentially continuous and homotopic in N. Now let $f_\#[h]$ be the common homotopy class. Then (3) is immediate.

To prove the last statement, let

$$\begin{gathered} \hat{h}: X \times [0,1] \to U, \\ \hat{h}(x,0) = h(x), \\ \hat{h}(x,1) = \psi(x). \end{gathered}$$

Then, exactly as in the proof of (3), $f \circ h_v$ and $f \circ \psi_v$ are essentially continuous and homotopic in N for $\mathcal{L}^Q$ a.e. small v. Thus $f_\#[h] = f_\#[\psi]$. □

PROPOSITION 2.6. *Let X be a regular polyhedral complex of dimension $d = [p-1]$, let U be an open set in $\mathbb{R}^Q$, and let $h \in \operatorname{Lip}(X, U)$. For every $K < \infty$, there is an $\epsilon = \epsilon(p, K, \tau) > 0$ such that if*

$$f_1, f_2 \in L^{1,p}(U, N),$$
$$\|f_i\|_{1,p} \le K \quad (i=1,2),$$
$$\|f_1 - f_2\|_p < \epsilon,$$

then $f_{1\#}[h] = f_{2\#}[h]$.

Proof. Let $\delta = \operatorname{dist}(h(x), \partial U)$. By Lemma 2.3, there is a set of $v \in B_\delta(0)$ of positive measure such that

$$\int_X |f_i \circ h_v|^p \le (C_1 K)^p, \tag{2.8}$$

$$\int_X |f_1 \circ h_v - f_2 \circ h_v|^p < (C_1 \epsilon)^p, \tag{2.9}$$

$$\int_X |(\nabla f_i) \circ h_v|^p < (C_1 K)^p, \tag{2.10}$$

where C_1 depends on X and h and where ϵ is to be chosen later. By Lemma 2.4 and Proposition 2.5, for almost every $v \in B_\delta(0)$, $f_i \circ h_v$ is essentially continuous, has distribution derivative $\nabla f_i(h_v) \circ \nabla h_v$, and

$$f_i \circ h_v \in f_{i\#}[h], \quad (i=1,2). \tag{2.11}$$

Let $0 < \eta < 1$. By Lemma 2.1,

$$\begin{aligned} |f_1 \circ h_v - f_2 \circ h_v| &\le \eta \|\nabla(f_1 \circ h_v) - \nabla(f_2 \circ h_v)\|_p + C(\eta)\|f_1 \circ h_v - f_2 \circ h_v\|_p \\ &\le 2\eta(\operatorname{Lip} h) C_1 K + C(\eta) C_1 \epsilon \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} |f_2 \circ h_v| &\le \|\nabla(f_2 \circ h_v)\|_p + C(1)\|f_2 \circ h_v\|_p \\ &\le (\operatorname{Lip} h) C_1 K + C(1) C_1 K = A. \end{aligned} \tag{2.13}$$

Choose $\eta > 0$ so that

$$2\eta(\operatorname{Lip} h) C_1 K < \tfrac{1}{3} \min\{\tau(t) : t \in [0, A]\} = L/3,$$

and choose $\epsilon > 0$ so that

$$C(\eta) C_1 \epsilon < L/3.$$

It then follows that

$$\begin{aligned} &\operatorname{dist}(t f_1(h_v(x)) + (1-t) f_2(h_v(x)), N) \\ &\quad \le |(t f_1(h_v(x)) + (1-t) f_2(h_v(x)) - f_2(h_v(x))| \\ &\quad \le |f_1 \circ h_v - f_2 \circ h_v| < 2L/3 < L \end{aligned}$$

for $t \in [0,1]$ and $x \in X$. Thus $f_1 \circ h_v$ and $f_2 \circ h_v$ are homotopic in N_τ and therefore homotopic in N. This, together with (2.11), implies that $(f_1)_\#[h] = (f_2)_\#[h]$. □

The proofs of the following results will merely consist of some easy modifications of White's orignal proofs. We shall give only the necessary replacement lists and refer to the original paper [9] for details.

THEOREM 2.7. *Let $d=[p-1]$. Then each $f \in L^{1,p}(M,N)$ has a d-homotopy type $f_\#[M^d]$. This d-homotopy type is a homotopy class of continuous maps from M^d into N such that:*

(1) *if $f_i \in L^{1,p}(M,N)$, $\|f_i - f\|_p \to 0$, and $\|\nabla f_i\|_p$ is uniformly bounded, then*

$$(f_i)_\#[M^d] = f_\#[M^d]$$

for all sufficiently large i;

(2) *if $f \in L^{1,p}(M,N)$ is continuous at each $x \in M^d$, then*

$$f_\#[M^d] = [f \mid M^d];$$

(3) $\{f_\#[M^d]: f \in L^{1,p}(M,N)\} = \{[\varphi \mid M^d]: \varphi \in C^0(M^{d+1}, N)\}$.

Proof. See Theorem 3.4 of [9] with Proposition 3.2 and Proposition 3.3 in place of our Proposition 2.4 and Proposition 2.5, respectively. □

REMARK. Notice that once this compactness result is established, the existence of a minimizer for more general functionals is readily applicable. For example,

$$\mathfrak{F}(v) = \int_M f(x, u, \nabla v),$$

with

$$c^{-1}(1 - |\xi|^p) \le f(x, y, \xi) \le c(1 + |\xi|^p)$$

for some positive constant c and all $(x, y, \xi) \in M \times N \times \mathbb{R}^{mP}$.

The next proposition shows that our assertions in Theorem 2.7 are independent of the triangulation of M.

PROPOSITION 2.8. *Let $f_1, f_2 \in L^{1,p}(M,N)$, let X be a regular polyhedral complex of dimension $\le [p-1]$, and let φ be a Lipschitz map from X to M. If $(f_1)_\#[M^{[p-1]}] = (f_2)_\#[M^{[p-1]}]$, then $(f_1)_\#[\varphi] = (f_2)_\#[\varphi]$.*

Proof. Replace Proposition 3.2 in the proof of Theorem 3.5 in [9] by our Proposition 2.5. □

We are now ready to discuss the existence results regarding the Dirichlet problems. For their proofs we will refer to Section 4 of [8] with Proposition 3.2 and Theorem 3.4 there in place of Proposition 2.6 and Theorem 2.7 of this section. Here we say that $f, g: M^k \to N$ are *homotopic relative to* ∂M if there is a homotopy $H: [0,1] \times M^k \to N$ from f to g such that $H(\cdot, x) = f(x) = g(x)$ for all $x \in \partial M \cap M^k$. The corresponding equivalence class of a continuous map f is called the *homotopy class* (rel ∂M) and is denoted by $[f(\mathrm{rel}\,\partial M)]$.

THEOREM 2.9. *Let φ be a Lipschitz map from ∂M to N. Then there exists a map $f \in L^{1,p}(M,N)$ such that $f \mid \partial M = \varphi$ if and only if φ can be extended continuously from $\partial M \cup M^{[p]}$ into N.*

THEOREM 2.10. *Suppose* $\varphi \in \operatorname{Lip}(\partial M, N)$. *Then each* $f \in L^{1,p}(M,N)$ *with* $f \mid \partial M = \varphi$ *has a* $[p-1]$*-homotopy type* $f_\#[M^{[p-1]}(\operatorname{rel}\partial M)]$. *This* $[p-1]$*-homotopy type is a homotopy class* $(\operatorname{rel}\partial M)$ *of continuous maps from* $M^{[p-1]}$ *into* N *such that:*

(1) *if* $f_i \in L^{1,p}(M,N)$ *so that* $f_i \mid \partial M = \varphi$, $\|f_i - f\|_p \to 0$, *and* $\|f_i\|_p$ *is uniformly bounded, then*

$$(f_i)_\#[M^{[p-1]}(\operatorname{rel}\partial M)] = f_\#[M^{[p-1]}(\operatorname{rel}\partial M)]$$

for all sufficiently large i;

(2) *if* $f \in L^{1,p}(M,N)$ *such that* $f \mid \partial M = \varphi$ *and is continuous at each* $x \in M^{[p-1]}$, *then*

$$f_\#[M^{[p-1]}(\operatorname{rel}\partial M)] = [(f \mid M^{[p-1]}(\operatorname{rel}\partial M)];$$

(3) *the set*

$$\{f_\#[M^{[p-1]}(\operatorname{rel}\partial M)] : f \in L^{1,p}(M,N) \text{ and } f \mid \partial M = \varphi\}$$

is equal to

$$\{[(\psi \mid M^{[p-1]}(\operatorname{rel}\partial M)] : \psi \in C^0(M^{[p-1]+1}, N) \\ \text{and } \psi(x) = \varphi(x) \text{ for } x \in M^{[p-1]} \cup \partial M\}.$$

3. Interior Partial Regularity for p-Dirichlet Energy Minimizers

Throughout this section M is assumed to be a bounded open set in $\mathbb{R}^m$, since partial regularity is a local property and the modification for the M with curvature can be easily made (as in [4, Sec. 7]).

The following lemma is a variation of Luckhaus's lemma [6, Lemma 1]. We shall prove it (following the idea of L. Simon for the case $p = 2$) in detail since in our setting it is slightly different from Luckhaus's version.

LEMMA 3.1. *Suppose for each* $\Gamma, \Upsilon > 0$ *there is an* $\eta_0 > 0$ *such that, for* $f, g \in L^{1,p}(S_1, N)$,

$$\int_{S_1} |\nabla_{\tan} f|^p + |\nabla_{\tan} g|^p < \Gamma, \qquad \int_{S_1} |f-g|^p < \eta_0,$$

and $f(S_1) \subset B^P_\Upsilon(0) \cap N$. *Then, for each* $\sigma \in (0,1)$, *there is a map*

$$h \in L^{1,p}(B_1(0) \sim B_{1-\sigma}(0), N)$$

satisfying the following properties:

$$h(x) = \begin{cases} g(x) & \text{if } x \in S_1, \\ f(x/(1-\sigma)) & \text{if } x \in S_{1-\sigma}; \end{cases}$$

$$\int_{B_1(0)\sim B_{1-\sigma}(0)} |\nabla h|^p\,dx \le C\Big(\sigma\int_{S_1}(|\nabla_{\tan} f|^p+|\nabla_{\tan} g|^p)\,dS + \sigma^{1-p}\int_{S_1}|f-g|^p\,dS\Big), \tag{3.1}$$

where $C=C(m,p,\Upsilon,N_\tau)$.

Proof. With Γ,Υ given and η_0 to be chosen satisfying the above properties, we show that there is a map $\tilde h\in L^{1,p}(S_1\times[0,\sigma],N)$ satisfying

$$\tilde h\,|\,S_1\times\{0\}=f,\qquad \tilde h\,|\,S_1\times\{\sigma\}=g,$$

$$\int_{S_1\times[0,\sigma]}|D\tilde h|^p\le C\Big(\sigma\int_{S_1}(|\nabla_{\tan} f|^p+|\nabla_{\tan} g|^p)\,dS+\sigma^{1-p}\int_{S_1}|f-g|^p\,dS\Big),$$

where D is the gradient operator on $S_1\times[0,\sigma]$, and

$$\begin{aligned}\operatorname{dist}(\tilde h(\omega,s),N)&\le|\tilde h(\omega,s)-f(\omega)|\\&\le C\Big(\int_{S_1}(|\nabla_{\tan} f|^p+|\nabla_{\tan} g|^p)\Big)^{1/p^2}\Big(\int_{S_1}|f-g|^p\Big)^{(p-1)/p^2}\\&\quad+C\Big(\int_{S_1}|f-g|^p\Big)^{1/p}\end{aligned}$$

for $(\omega,s)\in S_1\times[0,\sigma]$ and where $C=C(m,p)$. The assertion then follows because: (1) there is a bi-Lipschitz map Φ from $B_1(0)\sim B_{1-\sigma}(0)$ to $S_1\times[0,\sigma]$ such that $\Phi\colon S_1(0)\to S_1(0)\times\{0\}$ and $\Phi\colon S_{1-\sigma}(0)\to S_1(0)\times\{\sigma\}$ are also bi-Lipschitz; and (2) the left-hand side of the last inequality can be chosen so small that $\tilde h(S_1\times[0,\sigma])\subset N_\tau$ when η_0 is small and taking $h(x)=\pi(\tilde h(\Phi^{-1}(x)))$.

If $m=2$ then f,g have representatives in their corresponding equivalence classes, also denoted by f,g, which are absolutely continuous on S^1. Thus, on S^1 we have

$$\begin{aligned}|f-g|^p&\le\int_{S_1}|\nabla_{\tan}(f-g)||f-g|^{p-1}+\frac{1}{2\pi}\int_{S_1}|f-g|^p\\&\le\Big(\int_{S^1}|\nabla_{\tan}(f-g)|^p\Big)^{1/p}\Big(\int_{S^1}|f-g|^p\Big)^{(p-1)/p}+\frac{1}{2\pi}\Big(\int_{S^1}|f-g|^p\Big).\end{aligned}\tag{3.2}$$

For $(\theta,s)\in S^1\times[0,\sigma]$, define

$$w(\theta,s)=f(\theta)+s/\sigma(g(\theta)-f(\theta)).$$

We have

$$\begin{aligned}\operatorname{dist}(w(\theta,s),N)&\le\frac{s}{\sigma|g(\theta)-f(\theta)|}\\&\le\Big(\int_{S^1}|\nabla_{\tan}(f-g)|^p\Big)^{1/p^2}\Big(\int_{S^1}|f-g|^p\Big)^{(p-1)/p^2}\\&\quad+\Big(\frac{1}{2\pi}\Big)^{1/p}\Big(\int_{S^1}|f-g|^p\Big)^{1/p}\end{aligned}$$

and

$$|Dw|^p \le C((|\nabla_{\tan} f|^p + |\nabla_{\tan} g|^p) + \sigma^{-p}|f-g|^p)$$

with $C = C(p)$. Thus

$$\int_{S^1\times[0,\sigma]} |Dw|^p \le C\Big(\sigma\int_{S^1}(|\nabla_{\tan} f|^p + |\nabla_{\tan} g|^p) + \sigma^{1-p}\int_{S^1}|f-g|^p\Big).$$

This completes the proof in the case $m=2$; from now on we assume that $m \ge 3$.

Let Q be the cube centered at zero with edges E parallel to the axes and with vertices on $\partial B_{1/2}(0)$. Let $\tilde{Q} = 2Q$. Notice that the edges E of Q have length $1/\sqrt{m}$.

For $a\in\mathbb{R}^m$, $|a| < 1/(4\sqrt{m})$, define $E_a = E + a$ and $F_a^{(l)} = F^{(l)} + a$, $l = 1, 2, \ldots, m-1$, where $F^{(l)}$ is the l-dimensional face of Q. Observe that, for any nonnegative $\mathcal{L}^m$-measurable function g, we have

$$\int_{|a|<1/(4\sqrt{m})}\int_{F_a^{(l)}} g \le C\int_\Omega g,$$

where $\Omega = B_1(0) \sim B_{1/(2\sqrt{n})}(0)$ and C is an arbitrarily large number (say, 100) by Fubini's theorem. Hence

$$\int_{F_a^{(l)}} g \le C\theta^{-1}\int_\Omega g \quad \text{for all } l = 1, 2, \ldots, m-1$$

and $|a| < 1/(4\sqrt{m})$ with the possible exception of a set of $\mathcal{L}^m$-measure θ.

Now we extend f, g homogeneously of degree 0 so that f, g are defined on $B_1(0) \sim \{0\}$; that is, $f(r\omega) \equiv f(\omega)$ and $g(r\omega) \equiv g(\omega)$ for $r \in (0,1]$ and $\omega \in S_1(0)$. We see that

$$\int_\Omega |\nabla f|^p + |\nabla g|^p \le C\int_{S^{m-1}} |\nabla_{\tan} f|^p + |\nabla_{\tan} g|^p, \tag{3.3}$$

$$\int_\Omega |f-g|^p \le C\int_{S^{m-1}} |f-g|^p. \tag{3.4}$$

Now we select the representatives of f, g so that $f(\tilde{Q})$, $g(\tilde{Q}) \subset N$ and for $\mathcal{L}^m$ almost all $|a| < 1/(4\sqrt{m})$ we have f, g absolutely continuous on E_a. By one-dimensional calculus,

$$\sup_{E_a}|f-g| \le C\bigg(\Big(\int_{E_a}|\nabla(f-g)|^p\Big)^{1/p^2}\Big(\int_{E_a}|f-g|^p\Big)^{(p-1)/p^2} + \Big(\int_{E_a}|f-g|^p\Big)^{1/p}\bigg), \tag{3.5}$$

where $C = C(m)$. Let $\tilde{S}$ be the set of positive $\mathcal{L}^m$-measure in the set $\{a\in\mathbb{R}^m : |a| < 1/(4\sqrt{m})\}$ such that

$$\begin{aligned}\int_{F_a^{(l)}} |\nabla f|^p + |\nabla g|^p &\le C\int_\Omega |\nabla f|^p + |\nabla g|^p,\\ \int_{F_a^{(l)}} |f-g|^p &\le \int_\Omega |f-g|^p,\end{aligned} \tag{3.6}$$

and (3) holds, for all possible faces $F_a^{(l)}$ and edges, for all $a \in \tilde{S}$. Then, for all $a \in \tilde{S}$ and any given $\sigma > 0$, we can define an $\mathbb{R}^P$-valued function w on $Q_a \times [0, \sigma]$ by the following inductive procedure. We first define w on $Q_a \times \{0\}$ and $Q_a \times \{\sigma\}$ by

$$w \,|\, Q_a \times \{0\} = f \,|\, Q_a \times \{0\}, \qquad w \,|\, Q_a \times \{\sigma\} = g \,|\, Q_a \times \{\sigma\}. \tag{3.7}$$

Now we extend w to each $F^{(1)} \times [0, \sigma]$ of Q_a by defining

$$w(x, s) = f(x) + s/\sigma(g(x) - f(x)).$$

By (3.5) and (3.6) we have

$$\operatorname{dist}(w, N) \le |w - f| \le \max_{\text{1-faces of } Q_a} \max_{F^{(1)}} |f - g| < R, \tag{3.8}$$

where $R = C((\int_\Omega |f-g|^p)^{(p-1)/p^2} (\int_\Omega |\nabla f|^p + |\nabla g|^p)^{1/p^2} + (\int_\Omega |f-g|^p)^{1/p})$.

Also notice that, by (3.6) and direct computation, we have

$$\int_{F^1 \times [0, \sigma]} |\nabla w|^p \le C\left(\sigma \int_\Omega (|\nabla f|^p + |\nabla g|^p) + \sigma^{1-p} \int_\Omega |f - g|^p\right). \tag{3.9}$$

Assume $l \ge 2$ and that w is defined (with L^p-gradient) on all $F^{(l-1)} \times [0, \sigma]$, so that

$$\int_{F^{(l-1)} \times [0, \sigma]} |\nabla w|^p \le C\left(\sigma\left(\int_\Omega |\nabla f|^p + |\nabla g|^p\right) + \sigma^{1-p}\left(\int_\Omega |f-g|^p\right)\right) \tag{3.10}$$

for all such $F^{(l-1)}$. Notice that $\partial(F^{(l)} \times [0, \sigma])$ is the union of some $F^{(l-1)} \times [0, \sigma]$ together with $F^{(l)} \times \{0\}$ and $F^{(l)} \times \{\sigma\}$. Thus w is already well-defined on $\partial(F^{(l)} \times [0, \sigma])$, and hence we can define w on $F^{(l)} \times [0, \sigma]$, with L^p-gradient, by using a homogeneous degree-0 extension of $w \,|\, \partial(F^{(l)} \times [0, \sigma])$ into $F^{(l)} \times [0, \sigma]$ with center at the point $(q, \sigma/2)$, where q is the midpoint of $F^{(l)}$. Then by (3.10) we have

$$\begin{aligned}\int_{F^{(l)} \times [0, \sigma]} |\nabla w|^p &\le C\left(\sigma \int_{F^{(l)}} |\nabla f|^p + |\nabla g|^p\right) + C \int_{F^{(l-1)} \times [0, \sigma]} |\nabla w|^p \\ &\le C\sigma\left(\int_\Omega |\nabla f|^p + |\nabla g|^p\right) + C \sum_{\text{all } F^{(l-1)}} \int_{F^{(l-1)} \times [0, \sigma]} |\nabla w|^p, \end{aligned} \tag{3.11}$$

where we have used (3.6). Furthermore, we notice that the homogeneous degree-0 extension preserves the bound (3.8), and by induction based on (3.10) and (3.11) we conclude that w can be extended to all of $\partial Q_a \times [0, \sigma]$ such that: (3.8) holds; w has L^p-gradient on all of $F^{(m-1)} \times [0, \sigma]$; and

$$\int_{F^{(m-1)} \times [0, \sigma]} |\nabla w|^p \le C\sigma \int_\Omega |\nabla f|^p + |\nabla g|^p + C\sigma^{1-p} \int_\Omega |f - g|^p.$$

Thus

$$\int_{\partial Q \times [0, \sigma]} |\nabla w|^p \le C\left[\sigma \int_\Omega |\nabla f|^p + |\nabla g|^p + \sigma^{1-p} \int_\Omega |f - g|^p\right]. \tag{3.12}$$

Now let Ψ be the radial map from zero taking S^{m-1} onto ∂Q_a. We can then define $\tilde{h}$ on $S^{m-1} \times [0, \sigma]$ by

$$\tilde{h}(\omega, s) = w(\Psi(\omega), s).$$

This, together with (3.1) and (3.2), completes our proof. □

The following proposition shows that if the averaged integral of a p-Dirichlet energy minimizing map is bounded and the normalized energy is small on a small ball of M, then the normalized energy improves on a smaller ball of M. The method of proof is a blowing-up argument due to Luckhaus [6].

PROPOSITION 3.2. *For any positive* Γ, *there exist positive numbers* $\epsilon \in (0,1)$ *and* $\theta \in (0, 1/2)$ *such that, if u is a p-Dirichlet energy minimizing map with the properties*

$$|u_{x_0,\rho}| \leq \Gamma, \tag{3.13}$$

$$E_{x_0,\rho}(u) < \epsilon^p, \tag{3.14}$$

then

$$E_{x_0,\theta\rho}(u) < \tfrac{1}{2} E_{x_0,\rho}(u). \tag{3.15}$$

Proof. If the assertion were false then there would exist sequences of balls $\{B_{\rho_k}(x_k)\}$ and of positive real numbers $\{\epsilon_k\}$ such that

$$|u_{x_k,\rho_k}| < \Gamma, \tag{3.16}$$

$$E_{x_k,\rho_k}(u) = \epsilon_k^p \to 0 \quad \text{as } k \to \infty, \tag{3.17}$$

while

$$E_{x_k,\theta\rho_k}(u) > \tfrac{1}{2}\epsilon_k^p. \tag{3.18}$$

Notice that

$$\operatorname{dist}(u_{x_k,\rho_k}, N)^p \leq |u - u_{x_k,\rho_k}|^p,$$

and that integrating over the ball $B_{\rho_k}(x_k)$ yields

$$\begin{aligned}\operatorname{dist}(u_{x_k,\rho_k}, N)^p &\leq \fint_{B_{\rho_k}(x_k)} |u - u_{x_k,\rho_k}| \\ &\leq C\rho_k^{p-m} \int_{B_{\rho_k}(x_k)} |\nabla u|^p = C E_{x_k,\rho_k}(u) = C\epsilon_k^p,\end{aligned}$$

where C is the Poincaré inequality constant. By passing to a subsequence and rearranging the indices, we define $v_k \colon B_1(0) \to \mathbb{R}^p$ by

$$v_k(x) = \frac{u(x_k + \rho_k x) - y_k}{\epsilon_k} \quad \text{for } x \in B_1(0) \text{ and } k \in \mathbb{N},$$

where $y_k = \pi(u_{x_k,\rho_k})$. Then, by computation,

$$\int_{B_1(0)} |v_k|^p \leq 2^{p-1}\epsilon_k^{-p}\left(\int_{B_1(0)} |u(x_k + \rho_k x) - u_{x_k,\rho_k}|^p + \int_{B_1(0)} |u_{x_k,\rho_k} - y_k|^p\right) \leq C$$

by the Poincaré inequality and the previous inequality, and

$$\int_{B_1(0)} |\nabla v_k|^p = 1 \quad \text{for all } k \in \mathbb{N}$$

by (3.17) and the definition of v_k. Thus, by Rellich's compactness theorem and further passing to subsequences (without changing notations), we may assume that

$$\begin{aligned}
&y_k \to y_0 \in N, \\
&v_k \to v \in L^{1,p}(B_1(0), \mathrm{Tan}(N, y_0)), \quad \mathcal{L}^m \text{ pointwise a.e., and in the } L^p\text{-norm} \\
&\nabla v_k \to \nabla v, \qquad\qquad \text{weakly in } L^p(B_1(0), \mathbb{R}^P).
\end{aligned}$$

Thus, by (3.17) and (3.18) we see that

$$E_{0,1}(v_k) = 1, \tag{3.19}$$

while

$$E_{0,\theta}(v_k) \geq \tfrac{1}{2}. \tag{3.20}$$

By the lower semicontinuity of L^p-norm with respect to the weak convergence topology in L^p spaces, we have

$$\lim_{k\to\infty} \int_{B_\rho(0)} |\nabla v_k|^p \geq \int_{B_\rho(0)} |\nabla v|^p \quad \text{for } \rho \in [0, 1]. \tag{3.21}$$

Claim 1:

$$\lim_{k\to\infty} \int_{B_r(0)} |\nabla v_k|^p = \int_{B_r(0)} |\nabla v|^p.$$

Claim 2:

$$\int_{B_r(0)} |\nabla \tilde{v}|^p \geq \int_{B_r(0)} |\nabla v|^p$$

for $r \in (0, 1/2]$ and $\tilde{v} \in L^{1,p}(B_1(0), \mathrm{Tan}(N, y_0))$ with $\tilde{v} = v$ on $B_1(0) \sim B_{1/2}(0)$.

If the claims hold then by Claim 2 v is a p-Dirichlet energy minimizing map among maps in $L^{1,p}(B_1(0), \mathrm{Tan}(N, y_0))$, with the same trace as v on $S_{1/2}(0)$. Thus, by [7, Prop. 5.1], v is $C^{1,\alpha}$ in $B_{1/2}(0)$, so we would have

$$E_{0,\tau}(v) < C\tau^p \quad \text{where } C = C(m, p) \text{ and } \tau < 1/3,$$

which contradicts Claim 1 and (3.21) if we further choose τ so that $C\tau^p < 1/2$. The assertion of the proposition follows.

Proof of the Claims. It suffices, by (3.21), to show that for all such $\tilde{v}$ we have

$$\int_{B_t(0)} |\nabla \tilde{v}|^p \geq \lim_{k\to\infty} \int_{B_t(0)} |\nabla v_k|^p, \quad t \in [1/2, 1]. \tag{3.22}$$

Fix $\rho_0 \in [r, 1]$, by Fubini's theorem, so that

$$\int_{S_{\rho_0}} |\nabla v_k|^p + |\nabla v|^p < C \int_{B_1} |\nabla v_k|^p + |\nabla v|^p$$

and

$$\int_{S_{\rho_0}} |v_k - v|^p < C \int_{B_1} |v_k - v|^p.$$

Let $u_k = \epsilon_k v_k + y_k$ and $g_k(\omega) = u_k(\rho_0 \omega)$. Then we have

$$\int_S |\nabla g_k|^p \leq \rho_0^{p+1-m} \epsilon_k^p \int_{S_{\rho_0}} |\nabla v_k|^p = C\epsilon_k^p \int_{S_{\rho_0}} |\nabla v_k|^p. \tag{3.23}$$

Let $\tilde{v}_k = \tilde{v} R_k / \max(|\tilde{v}|, R_k)$ and $\tilde{\tilde{v}}_k = \pi(y_k + \epsilon_k \tilde{v}_k)$, where R_k is so chosen that $R_k \to \infty$,

$$\|(\pi - \mathrm{Id}) \,|\, B_{\epsilon_k R_k}(y_k) \cap y_k + \mathrm{Tan}(N, y_k)\|_\infty = o(\epsilon_k),$$

and

$$\|\nabla(\pi - \mathrm{Id}) \,|\, B_{\epsilon_k R_k}(y_k) \cap y_k + \mathrm{Tan}(N, y_k)\|_\infty = o(\epsilon_k)/\epsilon_k.$$

Thus we see that

$$\tilde{\tilde{v}}_k \to \tilde{v} \quad \text{and} \quad \nabla \tilde{\tilde{v}}_k \to \nabla \tilde{v}$$

in L^p norm and pointwise $\mathcal{L}^m$ a.e. $x \in B_1$.

Let $f_k(\omega) = \tilde{\tilde{v}}_k(\rho_0 \omega)$. We have

$$\begin{aligned}
\int_{S_1} |\nabla f_k|^p &\leq \rho_0^{p+1-m} \epsilon_k^p \int_{S_{\rho_0}} |\nabla \pi(y_k + \epsilon_k \tilde{v}_k) \cdot \nabla \tilde{v}_k|^p \\
&\leq 2^{p-1} \rho_0^{p+1-m} \epsilon_k^p \left(\int_{S_{\rho_0}} |\nabla(\pi - \mathrm{Id})(y_k + \epsilon_k \tilde{v}_k) \cdot \nabla \tilde{v}_k|^p + |\nabla \tilde{v}_k|^p \right) \\
&\leq C\epsilon_k^p \int_{S_{\rho_0}} |\nabla \tilde{v}|^p + o(\epsilon_k^p).
\end{aligned} \tag{3.24}$$

Compute

$$\begin{aligned}
\int_{S_1} |f_k - g_k|^p &= \rho_0^{p+1-m} \int_{S_{\rho_0}} |u_k - \tilde{\tilde{v}}_k|^p \\
&\leq 2^{p-1} \rho_0^{p+1-m} \epsilon_k^p \left(\int_{S_{\rho_0}} |v_k - \tilde{v}_k|^p + |(\pi - \mathrm{Id})(y_k + \epsilon_k \tilde{\tilde{v}}_k)|^p \right) \\
&\leq C\epsilon_k^p \int_{S_{\rho_0}} |v_k - \tilde{v}|^p + o(\epsilon_k^p).
\end{aligned} \tag{3.25}$$

Now we may apply Proposition 3.1 to f_k, g_k, ϵ_k in place of f, g, σ and so define

$$w_k(x) = \begin{cases} u_k(x) & \text{if } |x| \geq \rho_0, \\ \tilde{\tilde{v}}_k(x/(1-\epsilon_k)) & \text{if } |x| \leq (1-\epsilon_k)\rho_0. \end{cases}$$

Thus we have the following chain of inequalities:

$$\begin{aligned}
\lim_{k \to \infty} \int_{B_{\rho_0}} |\nabla v_k|^p &= \lim_{k \to \infty} \epsilon_k^{-p} \int_{B_{\rho_0}} |\nabla u_k|^p \leq \lim_{k \to \infty} \epsilon_k^{-p} \int_{B_{\rho_0}} |\nabla w_k|^p \\
&= \lim_{k \to \infty} \left(\epsilon_k^{-p} \int_{B_{\rho_0(1-\epsilon_k)}} |\nabla \tilde{\tilde{v}}_k|^p + \epsilon_k^{-p} \int_{B_{\rho_0} \sim B_{\rho_0(1-\epsilon_k)}} |\nabla w_k|^p \right) \\
&= \int_{B_{\rho_0}} |\nabla \tilde{v}|^p.
\end{aligned}$$

This completes the proof of the claims and so the proposition. □

PROPOSITION 3.3. *For any positive* Γ, *there are positive numbers* $\epsilon \in (0,1)$ *and* $\theta \in (0, 1/2)$, *where* θ *is as in Proposition 3.3, such that if* u *is a* p*-Dirichlet energy minimizing map and if* $|u_{b,2R}| < \Gamma/2$ *and* $R^{p-m}\int_{B_{b,2R}}|\nabla u|^p < \epsilon^p$, *then*

$$E_{a,r}(u) < \theta^{p-m-\alpha}\epsilon(r/R)^\alpha \quad \textit{for all } a \in B_R(b) \textit{ and } 0 \le r \le R,$$

where $B_{2R}(b)$ *is in the interior of* M *and* α *is given by the condition* $\theta^\alpha = 1/2$. *Hence* u *is Hölder continuous on* $B_R(b)$.

Proof. Suppose that $|u_{b,2R}| < \Gamma/2$ and $R^{p-m}\int_{B_{b,2R}}|\nabla u|^p < \epsilon^p$, with ϵ to be chosen later. Let $\Gamma_1 = 2\Gamma$, and let ϵ_1 be its corresponding positive constant as in Proposition 3.3. Since $|u_{x,t}|$ and $E_{x,t}(u)$ are both continuous functions of x when $u \in L^{1,p}(M, \mathbb{R}^P)$ is given, we may choose a neighborhood U_b of b such that

$$|u_{a,R}| < \Gamma \quad \text{and} \quad E_{a,R}(u) < \epsilon_1^p$$

for all $a \in U_b$.

Notice that, for $0 < s < t \le R$, we have

$$|u_{x,s} - u_{x,t}|^p \le 2^{p-1}(|u - u_{x,s}|^p + |u - u_{x,t}|^p).$$

Integrating both sides over $B_s(x)$, we have

$$\begin{aligned} |u_{x,s} - u_{x,t}|^p &\le 2^{p-1}\left(\fint_{B_s(x)} |u - u_{x,s}|^p + \left(\frac{t}{s}\right)^m \fint_{B_t(x)} |u - u_{x,t}|^p\right) \\ &\le C2^{p-1}\left(E_{x,s}(u) + \left(\frac{t}{s}\right)^m E_{x,t}(u)\right), \end{aligned}$$

by the Poincaré inequality. Suppose $E_{x,\theta^k R}(u) \le \frac{1}{2}\mathrm{E}_{x,\theta^{k-1}R}$ for all $k = 1, 2, \dots, l$ with $l \ge 1$. Thus, with $\theta^k R$ and θ^{k-1} in place of s and t in the above inequality, we obtain

$$|u_{x,\theta^k R} - u_{x,\theta^{k-1}R}| \le C(p)(1/2 + \theta^{-m})^{1/p} E_{x,\theta^{k-1}R}(u).$$

Computation yields

$$\begin{aligned} |u_{x,\theta^{l+1}R}| &\le |u_{x,R}| + \sum_{k=1}^{l+1} |u_{x,\theta^{k+1}R} - u_{x,\theta^k R}| \\ &\le \Gamma + \sum_{k=1}^{l+1} C(p)^{1/p}(1/2 + \theta^{-m})^{1/p}(1/2)^{1/(kp)}\epsilon_1 \\ &\le \Gamma + C(m,p)\epsilon_1 \frac{1}{1 - (1/2)^{1/p}} \le 2\Gamma = \Gamma_1, \end{aligned}$$

as long as we choose ϵ_1 so small that the last inequality holds.

For each $r \in (0, R)$, choose $j \in \mathbb{N}$ so that $\theta^{j+1}R < r \le \theta^j R$. Then we see that

$$\begin{aligned} E_{x,r}(u) &< (\theta^{j+1}R)^{p-m}(\theta^j R)^{m-p} E_{x,\theta^j R}(u) \le \theta^{p-m} E_{x,\theta^j R}(u) \\ &\le \theta^{p-m}(1/2)^j \epsilon^p = \theta^{p-m}\theta^{\alpha k}\epsilon^p \le \theta^{p-m-\alpha}\epsilon^p (r/R)^\alpha. \end{aligned}$$

Hence the last assertion follows from Morrey's growth estimate lemma. $\square$

REMARK. Notice that once the interior C^α regularity is established here, the proof for the Hölder continuity of the gradient is exactly the same as in [4, Sect. 3] with the assumption that the image of a small ball of a p-Dirichlet energy minimizing map lies in a compact subset of N, since C^α regularity ensures this condition already.

THEOREM 3.5. *Any p-Dirichlet energy minimizer $u \in L^{1,p}(M,N)$ is locally Hölder continuous on $M \sim \partial M \sim Z$, where*

$$Z = \{a \in M \sim \partial M\colon \sup_R |u_{a,R}| = \infty\} \cup \{a \in M \sim \partial M\colon \limsup_{r\to 0} E_{a,r}(u) > 0\}.$$

Moreover, the Hausdorff dimension of Z is at most $m-p$.

Proof. The set $\{a \in M\colon \limsup_{r\to 0} E_{a,r}(u) > 0\}$ has $(m-p)$-dimensional Hausdorff measure zero by an elementary covering argument (see [4, Cor. 2.7]). The set $\{a \in M\colon \sup_R |u_{x,R}| = \infty\}$ has $(m-p+\epsilon)$-dimensional Hausdorff measure zero, for all $\epsilon > 0$, by [3, Chap. IV, Thm. 2.1]. □

References

[1] J. Eells and L. Lemaire, *Another report on harmonic maps,* Bull. London Math. Soc. 20 (1988), 385–524.

[2] L. C. Evans and R. F. Gariepy, *Blowup, compactness and partial regularity in the calculus of variations,* Indiana Univ. Math. J. 36 (1987), 361–371.

[3] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems,* Princeton Univ. Press, Princeton, NJ, 1983.

[4] R. Hardt and F. H. Lin, *Mappings minimizing L^p-norm of the gradient,* Comm. Pure Appl. Math. 40 (1987), 555–588.

[5] M. Li, *Partial regularity of energy minimizing harmonic map into complete manifold,* preprint, 1994.

[6] S. Luckhaus, *Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold,* Indiana Univ. Math. J. 37 (1988), 349–367.

[7] P. Tolksdorff, *Everywhere-regularity for some quasilinear systems with a lack of ellipticity,* Ann. Mat. Pura Appl. (4) 134 (1983), 241–266.

[8] B. White, *Infima of energy functionals in homotopy classes of mappings,* J. Differential Geom. 23 (1986), 127–142.

[9] ———, *Homotopy classes in Sobolev spaces and the existence of energy minimizing maps,* Acta Math. 160 (1988), 1–17.

Department of Applied Mathematics
National Chiao Tung University
Hsinchu
Taiwan

swang@math.nctu.edu.tw