Cyclicity and Approximation by Lacunary Power Series

Evgeny V. Abakumov

Introduction

Let $\ell^p = \ell^p(\mathbb{Z}_+)$, $1 \le p \le \infty$, be the classical Banach space of all p-summable complex sequences $x = \{x_0, x_1, x_2, \dots\}$. Define the operators S and T on ℓ^p by

$$S\{x_0, x_1, x_2, \dots\} = \{0, x_0, x_1, \dots\},$$
$$T\{x_0, x_1, x_2, \dots\} = \{x_1, x_2, x_3, \dots\}.$$

These operators are called, respectively, *the forward* (right) and *the backward* (left) *shifts.*

In dealing with S and T it is convenient to consider the spaces ℓ_A^p of those power series (or analytic functions in the unit disk $\mathbb{D}$) that are the discrete Fourier transforms of the elements of ℓ^p:

$$\ell_A^p = \left\{ f = \sum_{k=0}^{\infty} \hat{f}(k) z^k : \|f\|_p = \left(\sum_{k=0}^{\infty} |\hat{f}(k)|^p \right)^{1/p} < \infty \right\}, \quad 1 \le p < \infty;$$
$$\ell_A^\infty = \left\{ f = \sum_{k=0}^{\infty} \hat{f}(k) z^k : \|f\|_\infty = \sup_{k \ge 0} |\hat{f}(k)| < \infty \right\}$$

(we always consider ℓ_A^∞ to be endowed with the weak* topology). Clearly, the space ℓ_A^2 coincides with the well-known Hardy space H^2. We use the same letters S and T for the corresponding operators on ℓ_A^p:

$$Sf = zf,$$
$$Tf = \frac{f - f(0)}{z}.$$

A subspace E of ℓ_A^p is called *S-invariant* if $SE \subset E$, and *T-invariant* if $TE \subset E$ (subspace always means a closed linear subspace).

We denote by q the conjugate exponent of p (i.e., the number determined by $1/p + 1/q = 1$), and we identify the space ℓ_A^q with the dual space of ℓ_A^p (if $p \ne \infty$), with duality defined by

$$\langle f, \phi \rangle = \sum_{k=0}^{\infty} \hat{f}(k) \hat{\phi}(k), \quad f \in \ell_A^p, \ \phi \in \ell_A^q.$$

Received May 18, 1994. Revision received March 15, 1995.

The author was partially supported by the Spanish Science and Technology Interministerial Council.

Michigan Math. J. 42 (1995).

It is clear that $S: \ell_A^q \to \ell_A^q$ is the adjoint operator of $T: \ell_A^p \to \ell_A^p$, so that the S-invariant subspaces of ℓ_A^q are precisely the annihilators of the T-invariant subspaces of ℓ_A^p.

The problem of characterizing shift-invariant subspaces in the spaces ℓ_A^p was raised by Beurling [3] in 1949 and is still far from being resolved. In this situation, some less general problems such as, for instance, that of describing *S-cyclic vectors* (i.e., the elements $f \in \ell_A^p$ such that $\operatorname{span}(S^n f,\, n \geq 0) = \ell_A^p$; span$(\cdot)$ denotes the closed linear hull of $(\cdot)$) and *T-cyclic vectors* ($f \in \ell_A^p$ such that $\operatorname{span}(T^n f,\, n \geq 0) = \ell_A^p$) become of great interest. Beurling's fundamental result [3] consists of the description of S-invariant subspaces and S-cyclic vectors in the space H^2; namely, every nonzero S-invariant subspace of H^2 has the form θH^2, where θ is an inner function and a vector $f \in H^2$ is S-cyclic if and only if f is an outer function (see e.g. [6; 8] for details). In fact, the latter result was first obtained by Smirnov [10] as early as 1932.

In this paper we investigate mostly T-cyclic vectors (with a few applications to the structure of S-invariant subspaces of ℓ_A^q, $q > 2$), and we write *cyclic* (vector, function, power series) for *T-cyclic*. An additional interest in T-cyclic vectors (not related to invariant subspace problem) lies in a feeling that the cyclicity of a function $f \in \ell_A^p$ means a "good approximation ability" of f (in the sense that every element of ℓ_A^p can be approximated by linear combinations of the Taylor remainders of f).

Douglas, Shapiro, and Shields gave in [4] a description of T-cyclic vectors in the Hardy space H^2 in terms of so-called pseudocontinuation. Namely, they proved that a function $f \in H^2$ is noncyclic if and only if there exists a Nevanlinna type meromorphic function $\tilde{f}$ in $\hat{\mathbb{C}} \setminus \operatorname{clos} \mathbb{D}$ such that the boundary values of f and $\tilde{f}$ coincide almost everywhere on $\{|z| = 1\}$. This result has several interesting corollaries concerning the structure of cyclic vectors in H^2, but leaves some questions unresolved. For instance, it is still difficult to apply the Douglas–Shapiro–Shields theorem when working with the Fourier (Taylor) coefficients of f.

We work mainly with Fourier coefficients and consider relations between the *spectrum* (frequency spectrum) of a function $f \in \ell_A^p$, that is, the set

$$\sigma(f) = \{k \geq 0 : \hat{f}(k) \neq 0\}$$

and the approximation ability of f in the sense just mentioned. The first result in this direction, that every Hadamard lacunary function is cyclic in the space H^2, was obtained in [4]. (By definition, an element $f \in \ell_A^p$ is called a *Hadamard lacunary function* (or *power series*) if its spectrum $\sigma(f)$ is a lacunary (in the Hadamard sense) sequence, that is, an increasing sequence of nonnegative integers $\{n_k\}_{k=1}^{\infty}$ such that $\inf_k (n_{k+1}/n_k) > 1$. We also say that f is a (Hadamard) lacunary function of order at least λ (or, briefly, of order λ) if $n_{k+1}/n_k > \lambda > 1$ for all $k \geq 1$.) This fact confirms the general principle that "lacunary series are noncontinuable" (recall in this connection the well-known Fabry gap theorem on the impossibility of an analytic continuation of a function with a sparse spectrum across any point of the boundary of its circle of convergence; the reader is referred to [5] for a detailed discussion and relations with the uncertainty principle in harmonic analysis). It is curious that, as far as the author knows, there is no direct proof (i.e., not using the shift operator, cyclic vectors, and so on) of the fact that a Hardy space lacunary series does not admit a pseudocontinuation. We refer also to the paper [7] by Nikolski where the cyclicity of Hadamard lacunary series is proved for a class of weighted spaces $\ell_A^2(w_n)$ and for a family of Fréchet spaces, in particular, for the space of all entire functions.

The present paper is organized as follows. Section 1 includes several auxiliary lemmas. In Section 2 we give some generalizations of the Douglas-Shapiro-Shields theorem on the cyclicity of Hadamard lacunary series in H^2. Namely, we prove that, for cyclicity, it is sufficient for $\sigma(f)$ to be a finite union of Hadamard lacunary sequences. Also we show that cyclicity is conserved under overconvergent perturbations of lacunary series (more precisely, the sum of a Hadamard lacunary function from H^2 and a function analytic in a neighborhood of the closed unit disk is either cyclic or rational (and hence noncyclic)). At the end of Section 2 we present a scale of conditions on the sparseness of $\sigma(f)$ and on a specific decreasing of Taylor coefficients of f to imply cyclicity.

The main result of the paper is that cyclicity of lacunary series is not a universal fact but depends heavily on metric properties of the space and on a specific decreasing of Taylor coefficients of the function. In Section 3 we give necessary and sufficient conditions for a Hadamard lacunary function of order 2 to be cyclic in the spaces ℓ_A^p, $1 < p < \infty$ (Theorem 3.1) and in the space ℓ_A^1 (Theorem 3.2). The main idea allowing us to analyze approximation ability is that the spectra of the left shifts $T^k f$, $k \geq 0$, are "almost disjoint" for a lacunary (of order 2) function f; the precise meaning is revealed by Lemma 1.5. It happens that for $2 < p \leq \infty$ all Hadamard lacunary functions are cyclic in the space ℓ_A^p, but this is not the case for $1 \leq p < 2$ (Theorem 3.3). One can say that the classical case of the Hardy space H^2 is a splitting point on the scale ℓ_A^p, $1 \leq p \leq \infty$, with respect to approximation ability of lacunary series. We complete Section 3 with several examples and by considering the cyclicity problem in the space c_A of those analytic functions whose Taylor coefficients tend to 0.

The results of Section 3 yield some information about the structure of the set of cyclic vectors in the spaces ℓ_A^p, $1 \leq p < 2$. In particular, in these spaces there exist two noncyclic vectors whose sum is cyclic (which is impossible in the space ℓ_A^2 in view of the main theorem in [4]). As a dual fact to these considerations, we obtain in Section 4 the following corollary concerning the forward shift operator S: In every space ℓ_A^q, $2 < q \leq \infty$, there exists a family $\{E_\lambda\}_{\lambda \in \mathbb{C}}$ of nonzero S-invariant subspaces such that $E_\lambda \cap E_\mu = \{\mathbb{O}\}$ whenever $\lambda \neq \mu$ (Theorem 4.2). Note that in the spaces ℓ_A^2 and ℓ_A^1 the intersection of two nonzero S-invariant subspaces is always nonzero (for ℓ_A^2 this follows from Beurling's theorem, and for ℓ_A^1 this is obvious since ℓ_A^1 is a convolution algebra without zero divisors).

The technique developed in this paper can be applied to similar problems for a larger class of weighted sequence spaces, which will be the subject of the paper [1].

1. Preparatory Lemmas

In this section we state several lemmas used throughout the paper. Some of them are known.

LEMMA 1.1. *Let* $\{c_k\}_{k=1}^{\infty}$ *be a sequence of positive numbers such that* $\sum_{k=1}^{\infty} c_k < \infty$. *Put* $R_k = c_k / \sum_{j>k} c_j$, $k \geq 1$. *Then the series* $\sum_{k=1}^{\infty} R_k$ *diverges.*

The proof is well-known; see [9, Chap. 3, Exer. 12(a)].

REMARK 1. One can easily prove that, conversely, for every divergent series $\sum_{k=1}^{\infty} R_k$ with positive terms there exists a convergent series $\sum_{k=1}^{\infty} c_k$ with positive terms such that $c_k/(\sum_{j>k} c_j) = R_k$, $k \geq 1$, and such a series is unique up to multiplying by a positive constant.

REMARK 2. We refer to $\{R_k\}_{k=1}^{\infty}$ as the *remainder sequence* for $\{c_k\}_{k=1}^{\infty}$. By the remainder sequence for a function $f \in \ell_A^p$ we mean the one for the sequence of the pth powers of the moduli of its nonzero Taylor coefficients.

LEMMA 1.2. *Let* $\{n_k\}_{k=1}^{\infty}$ *be a Hadamard lacunary sequence. Then there exists a number* M *such that no integer* K *has more than* M *representations of the form* $K = n_j - n_k$, $j, k \geq 1$.

The proof of this lemma is contained, for instance, in [4].

In the following three lemmas we study the intersection structure of the spectra of left shifts of a Hadamard lacunary series of order 2. We will think of the backward shift T as defined on the set of all formal power series.

LEMMA 1.3. *Let* f *be a Hadamard lacunary power series of order* 2, *and let* $d_1 \neq d_2$ *be nonnegative integers. Then the set* $\sigma(T^{d_1}f) \cap \sigma(T^{d_2}f)$ *contains at most one element.*

Proof. Let $d_1 > d_2$. Suppose that there exist two integers $r < s$ such that $\{r, s\} \subset \sigma(T^{d_1}f) \cap \sigma(T^{d_2}f)$. Then $s+d_1, r+d_1, s+d_2, r+d_2 \in \sigma(f)$. Since f is lacunary of order 2, we obtain a contradiction:

$$(s+d_1)+(r+d_2) > (s+d_1) = \tfrac{1}{2}(s+d_1)+\tfrac{1}{2}(s+d_1) > (s+d_2)+(r+d_1). \qquad \square$$

LEMMA 1.4. *Let* f *be a Hadamard lacunary power series of order* 2, *and let* $d_1, d_2, \ldots, d_s$ *be distinct nonnegative integers such that*

$$\sigma(T^{d_i}f) \cap \sigma(T^{d_{i+1}}f) \neq \emptyset, \quad 1 \leq i \leq s,$$

where d_{s+1} *is defined to be* d_1. *Then* $\bigcap_{i=1}^{s} \sigma(T^{d_i}f) \neq \emptyset$.

Proof. We denote by l_i, $1 \le i \le s$, the unique (by Lemma 1.3) element of $\sigma(T^{d_i}f) \cap \sigma(T^{d_{i+1}}f)$. Set $l = \max_{1 \le i \le s} l_i$. We assume, without loss of generality, that $d_1 = \min_{i:\, l \in \sigma(T^{d_i}f)} d_i$ and that $l \in \sigma(T^{d_2}f)$. It is sufficient to prove that $l \in \sigma(T^{d_i}f)$ implies $l \in \sigma(T^{d_{i+1}}f)$ for $i = 2, 3, \ldots, s$.

For this, suppose that $l \in \sigma(T^{d_i}f)$. Then $\{l\} = \sigma(T^{d_1}f) \cap \sigma(T^{d_i}f)$ and hence $\{l + d_1\} = \sigma(f) \cap \sigma(T^{d_i - d_1}f)$. Since $d_i - d_1 > 0$, it is easy to deduce from the fact that f is lacunary of order 2 that $l + d_1 = \min \sigma(T^{d_i - d_1}f)$ and so $l = \min \sigma(T^{d_i}f)$. But $l \ge l_i$, so that $\sigma(T^{d_i}f) \cap \sigma(T^{d_{i+1}}f) = \{l\}$ and $l \in \sigma(T^{d_{i+1}}f)$, as required. □

LEMMA 1.5. *Let f be a Hadamard lacunary power series of order 2, and let $D_1, D_2, \ldots, D_r$ be mutually disjoint finite subsets of $\mathbb{Z}_+$. For every d, $1 \le d \le r$, define*

$$U_d = \bigcup_{k \in D_1 \cup D_2 \cup \cdots \cup D_d} \sigma(T^k f).$$

Suppose that $\bigcap_{k \in D_1} \sigma(T^k f) \ne \emptyset$, and that for any d, $2 \le d \le r$, and for any $k \in D_d$,

$$\sigma(T^k f) \cap U_{d-1} \ne \emptyset. \tag{$*$}$$

Then all the sets

$$\sigma(T^k f) \cap U_{r-1}, \quad k \in D_r,$$

are singletons, and the sets

$$\sigma(T^k f) \setminus U_{r-1}, \quad k \in D_r,$$

are mutually disjoint.

Proof. By the hypothesis, $\sigma(T^k f) \cap U_{r-1} \ne \emptyset$ for any $k \in D_r$. Fix $k_0 \in D_r$. For every $l \in \sigma(T^{k_0}f) \cap U_{r-1}$ one can construct, using ($*$), the sequences of integers $\{k_i\}_{i=1}^t$ and $r > r_1 > r_2 > \cdots > r_t = 1$ so that $l \in \sigma(T^{k_1}f)$, $k_i \in D_{r_i}$, and

$$\sigma(T^{k_{i-1}}f) \cap \sigma(T^{k_i}f) \ne \emptyset \quad \text{for } 1 \le i \le t.$$

Now suppose that $\{l, l'\} \subset \sigma(T^{k_0}f) \cap U_{r-1}$. Construct the sequences $\{k_i\}_{i=1}^t$ for l and $\{k_i'\}_{i=1}^{t'}$ for l' as just described. Note that $k_t, k_{t'}' \in D_1$, and so

$$\sigma(T^{k_t}f) \cap \sigma(T^{k_{t'}'}f) \ne \emptyset$$

by the hypothesis. If the numbers $k_0, k_1', k_2', \ldots, k_{t'}', k_t, k_{t-1}, \ldots, k_2, k_1$ are pairwise distinct, then they satisfy the conditions on $\{d_i\}_{i=1}^r$ in Lemma 1.4; otherwise take the minimal index i for which $k_i' = k_j$ for some j, $1 \le j \le t$, and apply Lemma 1.4 to $k_0, k_1', k_2', \ldots, k_i', k_{j-1}, \ldots, k_2, k_1$. In both cases we obtain that the set $\sigma(T^{k_0}f) \cap \sigma(T^{k_1'}f) \cap \cdots \cap \sigma(T^{k_1}f)$ contains a number, say l_0. Using Lemma 1.3 we conclude that

$$\{l\} = \sigma(T^{k_0}f) \cap \sigma(T^{k_1}f) = \{l_0\} = \sigma(T^{k_0}f) \cap \sigma(T^{k_1'}f) = \{l'\}.$$

The first assertion of the lemma is proved.

To establish the second one, note that, by Lemma 1.3, for any distinct $k, k' \in D_r$, the set $\sigma(T^k f) \cap \sigma(T^{k'} f)$ is either empty or contains a single element l. In the latter case we can construct two sequences $k_1, k_2, \ldots, k_t$ and $k'_1, k'_2, \ldots, k'_{t'}$ as before and use Lemma 1.4 to show that

$$\sigma(T^k f) \cap \sigma(T^{k_1} f) \cap \cdots \cap \sigma(T^{k'_1} f) \cap \sigma(T^{k'} f) = \{l\}.$$

Thus, $l \in U_{r-1}$ and hence $[\sigma(T^k f) \backslash U_{r-1}] \cap [\sigma(T^{k'} f) \backslash U_{r-1}] = \emptyset$, as required. □

LEMMA 1.6. *Let $f = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ be a T-cycle vector in the space ℓ_A^p, $1 \le p < \infty$. Then $T^m f$ is also cyclic for any $m \ge 1$.*

Proof. It suffices to show that Tf is cyclic. Suppose the contrary. Then $\operatorname{span}(T^k f,\ k \ge 1)$ is a T-invariant subspace of ℓ_A^p of codimension 1. Therefore, its annihilator is a one-dimensional S-invariant subspace of $(\ell_A^p)^* = \ell_A^q$. But this is impossible. □

LEMMA 1.7. *Let $1 \le p < \infty$, and let Ω be a T-invariant subset of ℓ_A^p (i.e., if $f \in \Omega$ then $Tf \in \Omega$). Suppose that the inclusion $1 \in \operatorname{span}(T^k f,\ k \ge 0)$ holds for any $f \in \Omega$. Then all elements of Ω are cyclic vectors in ℓ_A^p.*

Proof. Let

$$f = \sum_{j=0}^{\infty} \hat{f}(j) z^j \in \Omega.$$

By the hypothesis, $1 \in \operatorname{span}(T^k f,\ k \ge 0)$. It suffices to prove that z^m lies in $\operatorname{span}(T^k f,\ k \ge 0)$ for all $m > 0$. Using an induction argument, suppose that $1, z, z^2, \ldots, z^{m-1}$ belong to $\operatorname{span}(T^k f,\ k \ge 0)$. Denote by P the projection $\sum_{j=0}^{\infty} \hat{f}(j) z^j \mapsto \sum_{j=m}^{\infty} \hat{f}(j) z^j$ in the space ℓ_A^p (clearly $Pf = S^m(T^m f)$). By the hypothesis, $T^m f \in \Omega$ and so $1 \in \operatorname{span}(T^k(T^m f),\ k \ge 0)$. Therefore,

$$z^m \in \operatorname{span}(S^m(T^m(T^k f)),\ k \ge 0) = P(\operatorname{span}(T^k f,\ k \ge 0)).$$

This means that there exists a polynomial $g = \sum_{j=0}^{m-1} \hat{g}(j) z^j$ such that $g + z^m \in \operatorname{span}(T^k f,\ k \ge 0)$. Since $g \in \operatorname{span}(T^k f,\ k \ge 0)$ by the induction hypothesis, we conclude that $z^m \in \operatorname{span}(T^k f,\ k \ge 0)$. The lemma follows. □

REMARK. When we apply Lemma 1.7, it is usually clear what Ω is considered to be (and the verification of T-invariance are trivial). Note that, in particular, the classes of Hadamard lacunary series and of lacunary series of order 2 are T-invariant.

2. Something More about H^2

In this section we consider the case of the Hardy space H^2.

Our first theorem shows that the assertion of the Douglas–Shapiro–Shields theorem on the cyclicity of Hadamard lacunary functions [4] is still true under weaker assumptions on the spectrum.

THEOREM 2.1. *Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of nonnegative integers that can be represented as a finite union of Hadamard lacunary sequences. Let $f(z)=\sum_{k=1}^{\infty} a_k z^{n_k} \in H^2$ with $a_k \neq 0$ for $k \geq 1$. Then f is cyclic in H^2.*

Proof. Let $\{n_k\}_{k=1}^{\infty}$ be a union of r lacunary sequences of orders at least $\lambda_0 > 1$. Put $\lambda = \lambda_0^{1/r}$. One easily verifies that the sequence $n_1 < n_2 < \cdots$ can be rewritten in the form $n_{1,1} < n_{1,2} < \cdots < n_{1,\phi(1)} < n_{2,1} < n_{2,2} < \cdots < n_{2,\phi(2)} < \cdots$, so that $1 \leq \phi(k) \leq r$, $k \geq 1$, and $n_{k+1,1} > \lambda n_{k,\phi(k)}$, $k \geq 1$.

One can choose an integer M such that no positive integer K has more than M representations of the form $K = n_{l,s} - n_{i,j}$, where $l \neq i$, $1 \leq s \leq \phi(l)$, and $1 \leq j \leq \phi(i)$. (The proof is slight modification of that of Lemma 1.2.)

Fix $\epsilon > 0$. We will construct a function $g \in \operatorname{Lin}(T^k f, k \geq 0)$ such that

$$\sum_{k=1}^{\infty} |\hat{g}(k)|^2 < \epsilon^2 |\hat{g}(0)|^2$$

($\operatorname{Lin}(\cdot)$ denotes the linear hull of $(\cdot)$).

Choose δ, $0 < \delta < 1$, such that

$$2r\delta < \epsilon^2/2. \tag{$*$}$$

In what follows we will write f in the form $f = \sum_{\substack{i \geq 1 \\ 1 \leq j \leq \phi(i)}} a_{i,j} z^{n_{i,j}}$, and denote by $\{R_{i,j}\}$ the remainder sequence for f:

$$R_{i,j} = \frac{|a_{i,j}|^2}{\sum_{l \geq 1, 1 \leq s \leq \phi(l):\, n_{l,s} > n_{i,j}} |a_{l,s}|^2}, \quad i \geq 1,\ 1 \leq j \leq \phi(i).$$

Define for every $i \geq 1$ a set of integers A_i by

$$A_i = \{j : 1 \leq j \leq \phi(i) \text{ and } \delta |a_{i,j}|^2 \geq |a_{i,s}|^2 \text{ for all } s, j < s \leq \phi(i)\}.$$

The next step is to prove that

$$\sum_{\substack{i \geq 1 \\ j \in A_i}} R_{i,j} = \infty.$$

Indeed, by Lemma 1.1, $\sum_{\substack{i \geq 1 \\ 1 \leq j \leq \phi(i)}} R_{i,j} = \infty$. So, it suffices to show that for every $i \geq 1$

$$\sum_{1 \leq j \leq \phi(i)} R_{i,j} \leq (\delta^{1-r} + \delta^{2-r} + \cdots + 1) \sum_{j \in A_i} R_{i,j}.$$

Let $j_1 < j_2 < \cdots < j_m = \phi(i)$ be *all* elements of A_i. Keeping in mind that $\delta < 1$, it is easy to see that if all the numbers $j, j+1, \ldots, j+t-1$ are not in A_i but $j+t$ is in A_i, then

$$\delta |a_{i,j+t-1}|^2 \leq |a_{i,j+t}|^2,\ \delta^2 |a_{i,j+t-2}|^2 \leq |a_{i,j+t}|^2,\ \ldots,\ \delta^t |a_{i,j}|^2 \leq |a_{i,j+t}|^2,$$

and hence $\delta R_{i,j+t-1} \leq R_{i,j+t}$, $\delta^2 R_{i,j+t-2} \leq R_{i,j+t}$, $\ldots$, $\delta^t R_{i,j} \leq R_{i,j+t}$. Thus,

$$\begin{aligned} \sum_{1 \leq j \leq \phi(i)} R_{i,j} &\leq (\delta^{1-j_1} + \delta^{2-j_1} + \cdots + 1) R_{i,j_1} + \cdots + (\delta^{1-j_p} + \delta^{2-j_p} + \cdots + 1) R_{i,j_m} \\ &\leq (\delta^{1-r} + \delta^{2-r} + \cdots + 1) \sum_{j \in A_i} R_{i,j}, \end{aligned}$$

as desired.

Hence one can choose N so large that

$$\frac{2M}{\sum_{\substack{1\le i\le N\\ j\in A_i}} R_{i,j}} < \frac{\epsilon^2}{2}. \tag{**}$$

Define the function g by

$$g = \sum_{\substack{1\le i\le N\\ j\in A_i}} \frac{R_{i,j}}{a_{i,j}}(T^{n_{i,j}}f).$$

One sees that $g = \hat{g}(0) + g_1 + g_2$, where

$$g_1 = \sum_{\substack{1\le i\le N\\ j\in A_i}} \left(\frac{R_{i,j}}{a_{i,j}} \sum_{j<s\le\phi(i)} a_{i,s} z^{n_{i,s}-n_{i,j}}\right),$$

$$g_2 = \sum_{\substack{1\le i\le N\\ j\in A_i}} \left(\frac{R_{i,j}}{a_{i,j}} \sum_{\substack{l:\,l>i\\ s:\,1\le s\le\phi(l)}} a_{l,s} z^{n_{l,s}-n_{i,j}}\right).$$

Observe that $\hat{g}(0) = \sum_{1\le i\le N,\, j\in A_i} R_{i,j}$, and estimate the norms $\|g_1\|_2$ and $\|g_2\|_2$. By the definition of A_i,

$$\|g_1\|_2 \le \sum_{\substack{1\le i\le N\\ j\in A_i}} \left(R_{i,j}\left(\sum_{j<s\le\phi(i)} \frac{|a_{i,s}|^2}{|a_{i,j}|^2}\right)^{1/2}\right) \le (r\delta)^{1/2} \sum_{\substack{1\le i\le N\\ j\in A_i}} R_{i,j}.$$

Further,

$$\begin{aligned}
\|g_2\|_2^2 &= \sum_{k=1}^{\infty} \left| \sum_{\substack{1\le i\le N\\ j\in A_i}} \sum_{\substack{l:\,l>i\\ s:\,1\le s\le\phi(l),\\ n_{l,s}-n_{i,j}=k}} R_{i,j}\frac{a_{l,s}}{a_{i,j}} \right|^2 \\
&\le \sum_{k=1}^{\infty} \left(M \sum_{\substack{1\le i\le N\\ j\in A_i}} \sum_{\substack{l:\,l>i\\ s:\,1\le s\le\phi(l),\\ n_{l,s}-n_{i,j}=k}} R_{i,j}^2 \frac{|a_{l,s}|^2}{|a_{i,j}|^2}\right) \quad \text{(by the choice of } M\text{)} \\
&\le M \sum_{\substack{1\le i\le N\\ j\in A_i}} \sum_{\substack{l,s:\\ n_{l,s}>n_{i,j}}} R_{i,j}^2 \frac{|a_{l,s}|^2}{|a_{i,j}|^2} = M \sum_{\substack{1\le i\le N\\ j\in A_i}} R_{i,j}.
\end{aligned}$$

Using these estimates together with (*) and (**), we obtain

$$\sum_{k=1}^{\infty} |\hat{g}(k)|^2 \le 2\|g_1\|_2^2 + 2\|g_2\|_2^2 \le \left(2r\delta + \frac{2M}{\sum_{\substack{1\le i\le N\\ j\in A_i}} R_{i,j}}\right)|\hat{g}(0)|^2 < \epsilon^2|\hat{g}(0)|^2,$$

and hence $\|g/\hat{g}(0) - 1\|_2 < \epsilon$. Letting $\epsilon \to 0$ gives $1 \in \operatorname{span}(T^k f, k \ge 0)$. Lemma 1.7 provides that f is cyclic. □

REMARK. Let $\{n_k\}_{k\ge1}$ be a Hadamard lacunary sequence, and let $\{d_k\}_{k\ge1}$ be a sequence of positive integers. Denote by Λ the union of all segments

$\{m \in \mathbb{Z}: n_k \le m < n_k + d_k\}$, $k \ge 1$. It follows from Theorem 2.1 that if the sequence $\{d_k\}$ is bounded, and if $f \in H^2$ is such that $\sigma(f)$ is an infinite subset of Λ, then f is cyclic. On the other hand, Aleksandrov [2] showed that if the sequence $\{d_k\}$ is unbounded, then there exists an *inner* (and hence noncyclic; see [4]) function $f \in H^2$ with $\sigma(f) \subset \Lambda$. It seems to be an interesting problem to characterize all subsets Λ of $\mathbb{Z}_+$ such that the only noncyclic H^2 power series supported on Λ are the polynomials. (For instance, is it true for every Sidon set? For the set of all perfect squares?) Note that the analogous problem of describing all Λ containing the spectrum of an inner function (different from z^n) is also unsolved.

Before stating the next theorem, we mention one more result from [4]: If f is analytic in a neighborhood of the closed unit disk, then either f is cyclic in H^2 or f is a rational function. This class of functions consists of the ones whose Taylor coefficients decrease rapidly (exponentially), and it is natural to ask what happens with the cyclicity property when adding such an overconvergent perturbation to a lacunary function. Now we will show that the cyclicity property for the class of Hadamard lacunary series is stable in this sense.

THEOREM 2.2. *Let $f = \sum_{k=1}^{\infty} a_k z^{n_k} \in H^2$ with $n_{k+1}/n_k > \lambda > 1$ and $a_k \ne 0$ for all $k \ge 1$. Let h be a function analytic in a neighborhood of the closed unit disk and let $F = f + h$. Then either F is cyclic in the space H^2 or F is a rational function.*

Proof. By the hypothesis there exists $\rho < 1$ such that $|\hat{h}(k)| = O(\rho^k)$, $k \to \infty$. We may suppose, without loss of generality, that $|\hat{h}(k)| \le \rho^k$, $k \ge 0$. In what follows we also assume that

$$\limsup_{k \to \infty} \frac{|a_k|}{\rho^{n_k}} = \infty \tag{$*$}$$

(if this is not the case, then f and hence F are analytic in $\{z: |z| < \rho^{-1}\}$ and the assertion of the theorem follows from the mentioned result of Douglas, Shapiro and Shields). To prove that F is cyclic, it suffices, in view of Lemma 1.7 (applied to the set Ω of all overconvergent perturbations of Hadamard lacunary functions with the convergence radius 1), to show that $1 \in \operatorname{span}(T^k F,$ $k \ge 0)$.

Let $\epsilon > 0$. It is sufficient to construct a function $g \in \operatorname{Lin}(T^k F, k \ge 0)$ such that

$$\sum_{m=1}^{\infty} |\hat{g}(m)|^2 < \epsilon |\hat{g}(0)|^2.$$

We now let $\{R_k\}_{k=1}^{\infty}$ be the remainder sequence for the function f: $R_k = |a_k|^2 / \sum_{j>k} |a_j|^2$, $k \ge 1$. The series $\sum_{k=1}^{\infty} R_k$ diverges, by Lemma 1.1.

In view of Lemma 1.2, one can choose a number M such that no positive integer K has more than M representations of the form $K = n_j - n_k$, $j, k \ge 1$. Choose an integer N and a number $c > 2$ so large that

$$\left(\frac{M+1}{\sum_{1\le j\le N} R_j}+\frac{(M+1)\rho^2}{1-\rho^2}\left(\frac{2}{c}\right)^2\right)\left(\frac{c}{c-2}\right)^2<\epsilon, \tag{$**$}$$

and put $A_0=(k\colon k>N$ and $|a_k|/\rho^{n_k}>c\}$. The set A_0 is infinite, by $(*)$.

First, we prove that

$$\sum_{k\in A_0} R_k=\infty.$$

Put $b_k=|a_k|^2-(\rho^{n_k})^2$ for $k\in A_0$. Note that all b_k, $k\in A_0$, are positive since $c>1$. Applying Lemma 1.1 to the convergent series $\sum_{k\in A_0} b_k$ gives

$$\sum_{k\in A_0}\frac{b_k}{\sum_{\substack{j>k\\ j\in A_0}} b_j}=\infty,$$

and hence

$$\sum_{k\in A_0} R_k=\sum_{k\in A_0}\frac{\rho^{2n_k}+b_k}{\sum_{\substack{j>k\\ j\in A_0}}\rho^{2n_j}+\sum_{\substack{j>k\\ j\in A_0}} b_j}\ge\sum_{k\in A_0}\frac{\rho^{2n_k}+b_k}{\rho^{2n_k}/(1-\rho^2)+\sum_{\substack{j>k\\ j\in A_0}} b_j}$$

$$\ge\sum_{k\in A_0}\min\left(1-\rho^2,\frac{b_k}{\sum_{\substack{j>k\\ j\in A_0}} b_j}\right)=\infty,$$

as desired.

Now it follows from the definition of A_0 that one can choose $s>0$ so large that

$$\frac{\sum_{j=1}^{N} R_j\frac{\rho^{n_j}}{|a_j|}+\sum_{k\in A_0,k<s} R_k\frac{\rho^{n_k}}{|a_k|}}{\sum_{j=1}^{N} R_j+\sum_{k\in A_0,k<s} R_k}<\frac{2}{c}. \tag{$***$}$$

Define

$$A=\{j\colon 1\le j\le N\}\cup\{k\in A_0\colon k<s\}\quad\text{and}\quad g=\sum_{j\in A}\frac{R_j}{a_j}(T^{n_j}F).$$

One has

$$|\hat{g}(0)|=\left|\sum_{j\in A}\frac{R_j}{a_j}(a_j+\hat{h}(n_j))\right|\ge\left|\sum_{j\in A}\frac{R_j}{a_j}a_j\right|-\left|\sum_{j\in A}\frac{R_j}{a_j}\hat{h}(n_j)\right|$$

$$\ge\sum_{j\in A}R_j-\sum_{j\in A}R_j\frac{\rho^{n_j}}{|a_j|}>\sum_{j\in A}R_j-\frac{2}{c}\sum_{j\in A}R_j\quad(\text{by }(***))=\frac{c-2}{c}\sum_{j\in A}R_j,$$

and so, by the choice of M,

$$\sum_{m=1}^{\infty}|\hat{g}(m)|^2=\sum_{m=1}^{\infty}\left|\sum_{\substack{j\in A\\ l\colon n_l-n_j=m}} R_j\frac{a_l}{a_j}+\sum_{j\in A}R_j\frac{\hat{h}(m+n_j)}{a_j}\right|^2$$

$$\le\sum_{m=1}^{\infty}(M+1)\left(\sum_{\substack{j\in A\\ l\colon n_l-n_j=m}} R_j^2\frac{|a_l|^2}{|a_j|^2}+\left|\sum_{j\in A}R_j\frac{\hat{h}(m+n_j)}{a_j}\right|^2\right)$$

$$\le (M+1)\sum_{j\in A}\left(R_j^2\sum_{l>j}\frac{|a_l|^2}{|a_j|^2}\right)+\frac{(M+1)\rho^2}{1-\rho^2}\left(\sum_{j\in A}R_j\frac{\rho^{n_j}}{|a_j|}\right)^2$$

$$<(M+1)\sum_{j\in A}R_j+\frac{(M+1)\rho^2}{1-\rho^2}\left(\frac{2}{c}\sum_{j\in A}R_j\right)^2 \quad \text{(by (***))}$$

$$\le\left(\frac{M+1}{\sum_{j\in A}R_j}+\frac{(M+1)\rho^2}{1-\rho^2}\left(\frac{2}{c}\right)^2\right)\left(\frac{c}{c-2}\right)^2|\hat{g}(0)|^2$$

$$<\epsilon|\hat{g}(0)|^2 \quad \text{(by (**))}.$$

The theorem follows. □

Observe that when proving the cyclicity of a lacunary function f, the main role is played by the remainder sequence $\{R_k\}_{k=1}^{\infty}$ for f; namely, one uses the fact that the series $\sum_{k\ge 0}R_k$ diverges. On the other hand, if we require that $\lim_{k\to\infty}R_k=\infty$, no assumption on the lacunarity of f is needed. The following theorem gives a scale of sufficient conditions on the sparseness of the spectrum of f and on the speed of decreasing of Taylor coefficients of f (in terms of R_k) in order for f to be a cyclic vector.

THEOREM 2.3. *Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of nonnegative integers, let $f=\sum_{k=1}^{\infty}a_k z^{n_k}\in H^2$ with $a_k\ne 0$ for $k\ge 1$, and let $R_k=|a_k|^2/\sum_{j>k}|a_j|^2$, $k\ge 1$. Suppose that one of the following four conditions holds:*

(i) $\limsup_{k\to\infty}R_k=\infty$;
(ii) $\liminf_{k\to\infty}R_k>0$ *and* $\limsup_{k\to\infty}(n_{k+1}-n_k)=\infty$;
(iii) $\limsup_{k\to\infty}R_k>0$ *and* $\lim_{k\to\infty}(n_{k+1}-n_k)=\infty$; *or*
(iv) $\{n_k\}_{k=1}^{\infty}$ *is a finite union of Hadamard lacunary sequences.*

Then f is cyclic.

Proof. If condition (iv) holds, then the required assertion is simply that of Theorem 2.1. In each of the other cases (i)–(iii) it suffices, in view of Lemma 1.7, to prove that $1\in\operatorname{span}(T^kf, k\ge 0)$ or, in other words, that for every $\epsilon>0$ there is a function $g\in\operatorname{Lin}(T^kf, k\ge 0)$ such that $\sum_{j=1}^{\infty}|\hat{g}(j)|^2\le\epsilon|\hat{g}(0)|^2$. We consider each of the cases (i)–(iii) separately.

(i) Let $\epsilon>0$. By the hypothesis there exists $m\ge 1$ such that $R_m>1/\epsilon$. Put $g=T^{n_m}f$. Then

$$\sum_{j=1}^{\infty}|\hat{g}(j)|^2=\sum_{j=m+1}^{\infty}|a_j|^2=\frac{|a_m|^2}{R_m}<\epsilon|a_m|^2=\epsilon|\hat{g}(0)|^2.$$

(Note that the assertion that $R_k\to\infty$ implies cyclicity is mentioned in [4].)

(ii) Let $\epsilon>0$ and take a number $c>0$ such that $R_k\ge c$ for $k\ge 1$. Choose a positive integer N and $\delta>0$ so that $2(1/cN+\delta^2)<\epsilon$. We will construct inductively two increasing integer sequences $\{k(i)\}_{i=1}^{N}$ and $\{m_i\}_{i=0}^{N}$ and two sequences $\{g_i\}_{i=1}^{N}$ and $\{h_i\}_{i=1}^{N}$ of H^2-functions as follows. Put $m_0=0$ and

$k(1)=1$. Suppose that $k(i)$ and m_{i-1} have been constructed. Then choose $m_i > m_{i-1}$ so large that the function h_i defined by the equality

$$h_i = \frac{1}{a_{k(i)}} \sum_{j=m_i+1}^{\infty} (T^{n_{k(i)}} f)\hat{}(j) z^j$$

satisfies the condition $\|h_i\|_2 < \delta$; set

$$g_i = \frac{1}{a_{k(i)}} \sum_{j=1}^{m_i} (T^{n_{k(i)}} f)\hat{}(j) z^j,$$

and take $k(i+1)$ such that $k(i+1) > k(i)$ and $n_{k(i+1)+1} - n_{k(i+1)} > m_i$ (such a choice is possible by the hypothesis).

All four sequences are thus well-defined. Note that $T^{n_{k(i)}} f / a_{k(i)} = 1 + g_i + h_i$, $1 \le i \le N$, and, by the construction, $\sigma(g_i)$ is contained in $\{n : m_{i-1} < n \le m_i\}$ for every i, $1 \le i \le N$.

Setting

$$g = \sum_{i=1}^{N} \frac{T^{n_{k(i)}} f}{a_{k(i)}} \in \operatorname{Lin}(T^k f, k \ge 0),$$

we obtain that

$$\begin{aligned}
\sum_{j=1}^{\infty} |\hat{g}(j)|^2 &= \Big\| \sum_{i=1}^{N} (g_i + h_i) \Big\|_2^2 \le 2 \Big\| \sum_{i=1}^{N} g_i \Big\|_2^2 + 2 \Big\| \sum_{i=1}^{N} h_i \Big\|_2^2 \\
&\le 2 \Big\| \sum_{i=1}^{N} \sum_{j=m_{i-1}+1}^{m_i} \frac{1}{a_{k(i)}} (T^{n_{k(i)}} f)\hat{}(j) z^j \Big\|_2^2 + 2 \Big(\sum_{i=1}^{N} \|h_i\|_2 \Big)^2 \\
&\le 2 \sum_{i=1}^{N} \sum_{j=1}^{\infty} \frac{1}{|a_{k(i)}|^2} |(T^{n_{k(i)}} f)\hat{}(j)|^2 + 2\delta^2 N^2 = 2 \sum_{i=1}^{N} \frac{1}{R_{k(i)}} + 2\delta^2 N^2 \\
&\le 2 \Big(\frac{1}{cN} + \delta^2 \Big) N^2 < \epsilon N^2 = \epsilon |\hat{g}(0)|^2,
\end{aligned}$$

as required.

(iii) Let $\epsilon > 0$ and $0 < c < \limsup_{k\to\infty} R_k$, and put $A = \{k \ge 1 : R_k \ge c\}$. The proof of the theorem in case (iii) is almost the same as in case (ii); the only modification is in the construction of $\{k(i)\}_{i=1}^N$. One must choose this sequence in such a way that $k(i) \in A$, $1 \le i \le N$, in addition to the conditions $k(i+1) > k(i)$ and $n_{k(i+1)+1} - n_{k(i+1)} > m_i$ (this is possible because $\lim_{k\to\infty}(n_{k+1} - n_k) = \infty$ and the set A is infinite). □

3. The Case of ℓ_A^p Spaces

In the following two theorems we characterize all cyclic and noncyclic lacunary functions of order at least 2 in the spaces ℓ_A^p, $1 \le p < \infty$. As will follow from Theorem 3.3, the interesting case is $1 \le p < 2$.

THEOREM 3.1. *Let $1<p<\infty$, and let $f=\sum_{k=1}^{\infty} a_k z^{n_k} \in \ell_A^p$ with $n_{k+1}/n_k>2$ and $a_k \neq 0$ for all $k \geq 1$. Then f is cyclic in ℓ_A^p if and only if*

$$\sum_{k=1}^{\infty} R_k^{1/(p-1)}=\infty,$$

where $R_k=|a_k|^p/\sum_{j>k}|a_j|^p$, $k \geq 1$.

Proof. We use the notation $f_k=T^k f$, $k \geq 0$. Suppose that $\sum_{k=1}^{\infty} R_k^{1/(p-1)}=\infty$. First we show that $1 \in \operatorname{span}(f_k, k \geq 0)$. To this end, it is sufficient to construct for a given $\epsilon>0$ a function $g \in \operatorname{Lin}(f_k, k \geq 0)$ such that

$$\sum_{k=1}^{\infty}|\hat{g}(k)|^p<\epsilon|\hat{g}(0)|^p.$$

Fix $\epsilon>0$ and set $\gamma_k=R_k^{1/(p-1)}$, $k \geq 1$. Now choose N so large that

$$\left(\sum_{j=1}^{N} \gamma_j\right)^{1-p}<\epsilon$$

and put $g=\sum_{j=1}^{N}(\gamma_j/a_j) f_{n_j}$. Note that the sets $\sigma(f_{n_j}) \backslash\{0\}$, $1 \leq j \leq N$, are mutually disjoint by Lemma 1.3, and hence no integer k has more than one representation of the form $k=n_l-n_j$. It follows that

$$\begin{aligned}
\sum_{k=1}^{\infty}|\hat{g}(k)|^p &=\sum_{k=1}^{\infty}\left|\sum_{\substack{1 \leq j \leq N \\ l: n_l-n_j=k}} \gamma_j \frac{a_l}{a_j}\right|^p=\sum_{\substack{1 \leq j \leq N \\ l>j}} \gamma_j^p \frac{|a_l|^p}{|a_j|^p}=\sum_{j=1}^{N} \frac{\gamma_j^p}{R_j} \\
&=\sum_{j=1}^{N} \gamma_j=\left(\sum_{j=1}^{N} \gamma_j\right)^{1-p}\left|\sum_{j=1}^{N} \frac{\gamma_j}{a_j} \hat{f}(n_j)\right|^p<\epsilon|\hat{g}(0)|^p .
\end{aligned}$$

Thus, $1 \in \operatorname{span}(f_k, k \geq 0)$, and Lemma 1.7 (applied to the class of all lacunary functions of order 2 such that the series $\sum_{k=1}^{\infty} R_k^{1/(p-1)}$ diverges) yields the cyclicity of f.

To prove the reverse implication, we will assume that

$$\left(\sum_{k=1}^{\infty} R_k^{1/(p-1)}\right)^{p-1}<2^{-p}.$$

(If this is not the case, we can apply the arguments given below to the function f_N $(=T^N f)$ for N sufficiently large to prove the noncyclicity of f_N, and then use Lemma 1.6.)

CLAIM 1. *Let m_i, $1 \leq i \leq r$, be distinct nonnegative integers and let $h=\sum_{i=1}^{r} \beta_i f_{m_i}$ with $\beta_i \neq 0$ for $1 \leq i \leq r$. Suppose that there exists an integer $s \geq 0$ such that $s \in \sigma(f_{m_i})$ for every i, $1 \leq i \leq r$. Then*

$$|\hat{h}(s)|^p \leq 2^{-p} \sum_{j \neq s}|\hat{h}(j)|^p .$$

Proof. The hypotheses of the claim imply that for every i, $1 \leq i \leq r$, there exists a positive integer k_i such that $n_{k_i}=s+m_i$. It follows that

$$
\begin{aligned}
|\hat{h}(s)|^p &= \left|\sum_{i=1}^{r} \beta_i a_{k_i}\right|^p \\
&\le \left(\sum_{i=1}^{r} R_{k_i}^{1/(p-1)}\right)^{p-1} \left(\sum_{i=1}^{r} \frac{|\beta_i|^p |a_{k_i}|^p}{R_{k_i}}\right) \quad \text{(by Hölder inequality)} \\
&\le 2^{-p} \sum_{i=1}^{r} \left(|\beta_i|^p \sum_{l\ne s} |\hat{f}_{m_i}(l)|^p\right) \\
&= 2^{-p} \sum_{j\ne s} |\hat{h}(j)|^p \quad \text{(by Lemma 1.3).} \qquad \square
\end{aligned}
$$

In order to show that f is noncyclic, it is sufficient to prove that

$$\sum_{j=1}^{\infty} |\hat{g}(j)|^p \ge |\hat{g}(0)|^p$$

for every $g \in \operatorname{Lin}(f_i,\, i \ge 0)$. So, let M be a finite set of $\mathbb{Z}_+$, let γ_i, $i \in M$, be nonzero complex numbers, and let $g = \sum_{i\in M} \gamma_i f_i$.

Define $M_0 = \{i \in M : 0 \in \sigma(f_i)\}$. We assume that $M_0 \ne \emptyset$ (otherwise the desired inequality is obvious). One can represent g as $g = g_* + \sum_{i\in M\setminus M_0} \gamma_i f_i$ where $g_* = \sum_{i\in M_0} \gamma_i f_i$. Applying Claim 1 to $h = g_*$ and $s = 0$, we obtain

$$|\hat{g}(0)|^p = |\hat{g}_*(0)|^p \le 2^{-p} \sum_{j=1}^{\infty} |\hat{g}_*(j)|^p. \tag{$*$}$$

Construct the sequence $\{M_l\}_{l=0}^{\infty}$ of subsets of $\mathbb{Z}_+$ and the sequence of functions $\{g_l\}_{l=0}^{\infty}$ of the space ℓ_A^p as follows. Define $g_0 = \sum_{j=1}^{\infty} \hat{g}_*(j) z^j$, and if M_l and g_l have been constructed then define

$$M_{l+1} = \left\{ i \in M \setminus \bigcup_{0\le j\le l} M_j : \sigma(f_i) \cap \sigma(g_l) \ne \emptyset \right\}; \qquad g_{l+1} = g_l + \sum_{i\in M_{l+1}} \gamma_i f_i.$$

One easily checks that for every positive integer k the sets $M_0, M_1, \dots, M_k$ satisfy the conditions on $\{D_i\}_{i=1}^{r}$ in Lemma 1.5.

CLAIM 2. *Let $l \ge 0$. Then*

$$\sum_{s\in\sigma(g_l)} |(g_l - g_{l+1})\hat{\ }(s)|^p \le 2^{-p} \sum_{s\in\sigma(g_{l+1})\setminus\sigma(g_l)} |\hat{g}_{l+1}(s)|^p.$$

Proof. For every $s \in \sigma(g_l)$ put

$$A_s = \left(\bigcup_{\substack{i:\, i\in M_{l+1},\\ \sigma(f_i)\ni s}} \sigma(f_i) \right) \Big\backslash \{s\}.$$

By Lemma 1.5 and Claim 1, for any $s \in \sigma(g_l)$ one has

$$
\begin{aligned}
|(g_l - g_{l+1})\hat{\ }(s)|^p &= \sum_{\substack{i:\, i\in M_{l+1},\\ \sigma(f_i)\ni s}} |\gamma_i \hat{f}_i(s)|^p \le 2^{-p} \sum_{j\ne s} \sum_{\substack{i:\, i\in M_{l+1},\\ \sigma(f_i)\ni s}} |\gamma_i \hat{f}_i(j)|^p \\
&= 2^{-p} \sum_{j\in A_s} |\hat{g}_{l+1}(j)|^p.
\end{aligned}
$$

It remains to observe that (again in view of Lemma 1.5) the sets A_s, $s\in\sigma(g_l)$, are mutually disjoint and that $\bigcup_{s\in\sigma(g_l)} A_s=\sigma(g_{l+1})\backslash\sigma(g_l)$. □

CLAIM 3. *Let $l\geq 0$. Then at least one of the two following assertions holds:*
(i) $\|g_l\|_p^p\leq\|g_{l+1}\|_p^p$;
(ii) $\|g_l\|_p^p\leq 2^p\sum_{s\in\sigma(g_l)}|\hat{g}_{l+1}(s)|^p$.

Proof. Suppose that both (i) and (ii) fail. Then

$$\begin{aligned}\|g_l\|_p^p &> \frac{1}{2}\|g_{l+1}\|_p^p+\frac{1}{2}2^p\sum_{s\in\sigma(g_l)}|\hat{g}_{l+1}(s)|^p\\ &\geq 2^{p-1}\sum_{s\in\sigma(g_l)}|(g_l-g_{l+1})\hat{}(s)|^p+2^{p-1}\sum_{s\in\sigma(g_l)}|\hat{g}_{l+1}(s)|^p \quad \text{(by Claim 2)}\\ &\geq \sum_{s\in\sigma(g_l)}|\hat{g}_l(s)|^p=\|g_l\|_p^p,\end{aligned}$$

a contradiction. □

Further, let t be the minimal nonnegative integer such that $M_{t+1}=\emptyset$ (and hence $g_{t+1}=g_t$). It follows from the construction of the sequence $\{g_l\}_{l=0}^{\infty}$ and from Lemma 1.5 that $\hat{g}(s)=\hat{g}_t(s)$ for $s\in\sigma(g_t)$. Therefore,

$$\|g_t\|_p^p\leq\sum_{j=1}^{\infty}|\hat{g}(j)|^p. \tag{**}$$

In view of Claim 3, one of the two following cases holds:
(i) $\|g_0\|_p^p\leq\|g_1\|_p^p\leq\cdots\leq\|g_t\|_p^p$; or
(ii) $\|g_0\|_p^p\leq\|g_1\|_p^p\leq\cdots\leq\|g_l\|_p^p\leq 2^p\sum_{s\in\sigma(g_l)}|\hat{g}_{l+1}(s)|^p$ for some l, $0\leq l<t$.

In the latter case we have

$$\|g_t\|_p^p\geq\sum_{s\in\sigma(g_l)}|\hat{g}_t(s)|^p=\sum_{s\in\sigma(g_l)}|\hat{g}_{l+1}(s)|^p\geq 2^{-p}\|g_l\|_p^p\geq 2^{-p}\|g_0\|_p^p.$$

Thus, in either case (i) or (ii), we obtain by use of (*) and (**) that

$$\sum_{j=1}^{\infty}|\hat{g}(j)|^p\geq\|g_t\|_p^p\geq 2^{-p}\|g_0\|_p^p=2^{-p}\sum_{j=1}^{\infty}|\hat{g}_*(j)|^p\geq|\hat{g}(0)|^p,$$

as required. □

REMARK. The sufficiency of the condition $\sum_{k\geq 1}R_k^{1/(p-1)}=\infty$ for cyclicity of f holds true under the weaker assumption that $\{n_k\}_{k=1}^{\infty}$ is a finite union of Hadamard lacunary sequences. We omit the proof, which is essentially the same as for $p=2$ in Theorem 2.1.

THEOREM 3.2. *Let $f=\sum_{k=1}^{\infty}a_k z^{n_k}\in\ell_A^1$, with $n_{k+1}/n_k>2$ and $a_k\neq 0$ for all $k\geq 1$. Then f is cyclic in ℓ_A^1 if and only if*

$$\prod_{k=1}^{\infty}\max(R_k,1)=\infty,$$

where $R_k=|a_k|/\sum_{j>k}|a_j|$, $k\geq 1$.

Proof. First suppose that $\prod_{k=1}^{\infty}\max(R_k, 1) = \infty$. Fix $\epsilon > 0$. To prove that $1 \in \operatorname{span}(T^k f,\ k \geq 0)$, it is sufficient to construct a function $g \in \operatorname{Lin}(T^k f,\ k \geq 0)$ such that

$$\sum_{j=1}^{\infty} |\hat{g}(j)| < \epsilon |\hat{g}(0)|.$$

It is easy to see that there exists an integer sequence $\{k(i)\}_{i=1}^{\infty}$ such that $1 < k(1) < k(2) < \cdots$, $R_{k(i)} > 1$ for $i \geq 1$, and that $\prod_{j=2}^{\infty} R_{k(2j)} = \infty$. We introduce some further notation. Put

$$A_j = \{m : n_{k(j-1)} < m \leq n_{k(j)}\}, \quad j \geq 2; \qquad B_j = \bigcup_{i>j} A_i, \quad j \geq 1.$$

Construct inductively a sequence $\{g_s\}_{s=1}^{\infty}$ of functions of ℓ_A^1 satisfying the following five conditions (for every $s \geq 1$):

(a) $$g_s \in \operatorname{Lin}(T^k f,\ k \geq 1),$$

(b) $$\hat{g}_s(0) = 1,$$

(c) $$\sigma(g_s) \backslash \{0\} \subset B_{2s-1},$$

(d) $$\sum_{m \in A_t} |\hat{g}_s(m)| \geq \sum_{m \in B_t} |\hat{g}_s(m)|, \quad t > 2s,$$

(e) $$\sum_{m=1}^{\infty} |\hat{g}_{s+1}(m)| \leq \frac{1 + R_{k(2s+2)}}{2R_{k(2s+2)}} \sum_{m=1}^{\infty} |\hat{g}_s(m)|.$$

First, put

$$g_1 = \frac{1}{\hat{f}(n_{k(1)})} T^{n_{k(1)}} f.$$

Conditions (a) and (b) for the function g_1 hold obviously; (c) follows from the assumption that $n_{k+1} - n_k > n_k$, $k \geq 1$. In order to show that (d) holds, take an integer $t > 2$ and observe that

$$\sum_{m \in A_t} |\hat{g}_1(m)| \geq \frac{|\hat{f}(n_{k(t)})|}{|\hat{f}(n_{k(1)})|} > \frac{1}{R_{k(t)}} \frac{|\hat{f}(n_{k(t)})|}{|\hat{f}(n_{k(1)})|} = \sum_{l > k(t)} \frac{|\hat{f}(n_l)|}{|\hat{f}(n_{k(1)})|} = \sum_{m \in B_t} |\hat{g}_1(m)|.$$

Further, if g_s has been constructed, we put

$$g_{s+1} = g_s - \sum_{m \in A_{2s} \cup A_{2s+1}} \frac{\hat{g}_s(m)}{\hat{f}(n_{k(2s+2)})} T^{n_{k(2s+2)} - m} f.$$

Assume that g_s satisfies the conditions (a)–(d) and show that these four conditions hold for g_{s+1}. (a) is obvious. To verify (b) and (c), it suffices to observe that if $m \in A_{2s} \cup A_{2s+1}$ then $\sigma(T^{n_{k(2s+2)} - m}) \backslash \{m\} \subset B_{2s+1}$. For (d), one can apply Lemma 1.5 to show that all the sets

$$\sigma(g_s) \quad \text{and} \quad \sigma(T^{n_{k(2s+2)} - m} f) \backslash \{m\}, \quad m \in A_{2s} \cup A_{2s+1},$$

are mutually disjoint. Using this together with condition (d) for the function g_s, for $t > 2s + 2$ we obtain

$$\begin{aligned}\sum_{m\in A_t}|\hat{g}_{s+1}(m)| &= \sum_{m\in A_t}|\hat{g}_s(m)| + \sum_{m\in A_{2s}\cup A_{2s+1}} \frac{|\hat{g}_s(m)|}{|\hat{f}(n_{k(2s+2)})|}\sum_{k(t-1)<l\le k(t)}|\hat{f}(n_l)|\\
&\ge \sum_{m\in B_t}|\hat{g}_s(m)| + \sum_{m\in A_{2s}\cup A_{2s+1}} \frac{|\hat{g}_s(m)|}{|\hat{f}(n_{k(2s+2)})|}\frac{|\hat{f}(n_{k(t)})|}{R_{k(t)}}\\
&= \sum_{m\in B_t}|\hat{g}_s(m)| + \sum_{m\in A_{2s}\cup A_{2s+1}} \frac{|\hat{g}_s(m)|}{|\hat{f}(n_{k(2s+2)})|}\sum_{l>k(t)}|\hat{f}(n_l)|\\
&= \sum_{m\in B_t}|\hat{g}_{s+1}(m)|.\end{aligned}$$

Finally, we must verify that g_s satisfies (e) for every $s\ge 1$. Fix s, and set $\Sigma_1=\Sigma_{m\in A_{2s}\cup A_{2s+1}}|\hat{g}_s(m)|$ and $\Sigma_2=\Sigma_{m\in B_{2s+1}}|\hat{g}_s(m)|$. We have

$$\begin{aligned}\sum_{m=1}^{\infty}|\hat{g}_{s+1}(m)| &= \frac{1}{R_{k(2s+2)}}\Sigma_1+\Sigma_2 \quad \text{(by Lemma 1.5)}\\
&\le \left(\frac{1}{2R_{k(2s+2)}}+\frac{1}{2}\right)(\Sigma_1+\Sigma_2) \quad (\text{since } \Sigma_1\ge\Sigma_2 \text{ by (d)})\\
&= \frac{1+R_{k(2s+2)}}{2R_{k(2s+2)}}\sum_{m=1}^{\infty}|\hat{g}_s(m)| \quad \text{(by (c))}.\end{aligned}$$

Thus, conditions (a)–(e) hold for all functions g_s, $s\ge 1$.

Now we are in a position to prove the cyclicity of f. The assumptions that $R_{k(2i)}>1$ ($i\ge 1$) and $\prod_{s=2}^{\infty}R_{k(2s)}=\infty$ imply that $\prod_{s=2}^{\infty}(1+R_{k(2s)})/2R_{k(2s)}=0$. Choose N so large that

$$\prod_{s=2}^{N}\frac{1+R_{k(2s)}}{2R_{k(2s)}}\cdot\sum_{m=1}^{\infty}|\hat{g}_1(m)|<\epsilon.$$

In view of (e) and (b), we have

$$\sum_{m=1}^{\infty}|\hat{g}_N(m)|\le\prod_{s=2}^{N}\frac{1+R_{k(2s)}}{2R_{k(2s)}}\sum_{m=1}^{\infty}|\hat{g}_1(m)|<\epsilon=\epsilon|\hat{g}_N(0)|.$$

This completes the proof of the inclusion $1\in\operatorname{span}(T^kf,\ k\ge 0)$. Lemma 1.7 provides the cyclicity of f (note that the condition $\prod_{k\ge 1}\max(R_k,1)=\infty$ is invariant under T).

Next, suppose that $C=\prod_{k=1}^{\infty}\max(R_k,1)<\infty$. To prove that f is noncyclic it is sufficient to show that

$$\sum_{j=1}^{\infty}|\hat{g}(j)|>\frac{1}{C^2}|\hat{g}(0)|$$

for every function $g\in\operatorname{Lin}(f_k,\ k\ge 0)$ (we denote by f_k the function T^kf). Let M be a finite subset of $\mathbb{Z}_+$, let γ_i ($i\in M$) be nonzero complex numbers, and let $g=\Sigma_{i\in M}\gamma_i f_i$.

Define $M_0=\{i\in M\colon 0\in\sigma(f_i)\}$. We may assume that $M_0\ne\emptyset$. One has $g=g_0+\Sigma_{i\in M\setminus M_0}\gamma_i f_i$ where $g_0=\Sigma_{i\in M_0}\gamma_i f_i$. For every $i\in M_0$, denote by $k(i)$ the unique integer such that $n_{k(i)}=i$. Using Lemma 1.3 we obtain

$$\sum_{j=1}^{\infty}|\hat{g}_0(j)| = \sum_{i\in M_0} \gamma_i \frac{|a_{k(i)}|}{R_{k(i)}} \geq \frac{1}{C} \sum_{i\in M_0} \gamma_i |a_{k(i)}| \geq \frac{1}{C}|\hat{g}_0(0)| = \frac{1}{C}|\hat{g}(0)|. \qquad (*)$$

Construct inductively the sequences $\{M_l\}_{l=0}^{\infty}$, $\{B_{l,s}\}_{l,s=1}^{\infty}$, $\{A_l\}_{l=0}^{\infty}$ of subsets of $\mathbb{Z}_+$ and the sequence $\{g_l\}_{l=0}^{\infty}$ of functions of ℓ_A^1 as follows. Set $A_0 = \sigma(g_0)\setminus\{0\}$ (note that M_0 and g_0 have been already defined). If M_l, A_l, and g_l have been constructed, then define

$$M_{l+1} = \left\{ i \in M \setminus \bigcup_{j=0}^{l} M_j : n_{l+1} - i \in A_l \right\};$$

$$B_{l+1,s} = \begin{cases} \emptyset & \text{if } n_{l+1} - s \notin M_{l+1}, \\ \sigma(f_{n_{l+1}-s}) \cap \{j : j > s\} & \text{if } n_{l+1} - s \in M_{l+1}, \end{cases} \quad s \geq 1;$$

$$A_{l+1} = A_l \cup \bigcup_{s\in A_l} B_{l+1,s}; \qquad g_{l+1} = g_l + \sum_{i\in M_{l+1}} \gamma_i f_i.$$

CLAIM 1. *Let $l \geq 0$. Then*

$$\sum_{s\in A_l} |\hat{g}_l(s)| \leq \max(R_{l+1}, 1) \sum_{s\in A_{l+1}} |\hat{g}_{l+1}(s)|.$$

Proof. It is easy to verify that for every $k \geq 1$ the sets $M_0, M_1, \dots, M_k$ satisfy the conditions on $\{D_i\}_{i=1}^{r}$ in Lemma 1.5. Using this lemma, one can obtain that the sets A_l and $B_{l+1,s}$, $s \in A_l$, are mutually disjoint, and

$$\begin{aligned} \frac{1}{\max(R_{l+1}, 1)} \sum_{s\in A_l} |\hat{g}_l(s)| &\leq \sum_{s\in A_l} \frac{1}{R_{l+1}} |(g_{l+1} - g_l)\hat{\ }(s)| + \sum_{s\in A_l} |\hat{g}_{l+1}(s)| \\ &= \sum_{s\in A_l} \sum_{j\in B_{l+1,s}} |\hat{g}_{l+1}(j)| + \sum_{s\in A_l} |\hat{g}_{l+1}(s)| \\ &= \sum_{s\in A_{l+1}} |\hat{g}_{l+1}(s)|. \end{aligned}$$
□

Now let t be the minimal integer for which $g_t = g_{t+1} = g_{t+2} = \cdots$.

CLAIM 2. $\sum_{s\in A_t} |\hat{g}_t(s)| \leq \sum_{s=1}^{\infty} |\hat{g}(s)|$.

Proof. Fix $s \in A_t$ and show that $\hat{g}(s) = \hat{g}_t(s)$. For this, it suffices to prove that if $s = n_m - j$ for some $m \geq 1$ and $j \geq 0$, then $j \in M_r$ for some $r \geq 0$. If $s \in A_0$ then, by the definition of M_m, either $j \in M_m$ or $j \in \bigcup_{r=0}^{m-1} M_r$. Let now $s \in A_l \setminus A_{l-1}$, $l \geq 1$. Then, by the construction, there exist $i \in M$ and $k > l$ that $n_k - i = s$ and $n_l - i \in A_{l-1}$ (and hence $n_l \geq i$). It follows that $n_m \geq s = n_k - i > n_k - n_l > n_l$. Thus, $m > l$ and hence $n_m - j = s \in A_l \subset A_{m-1}$. So, by the definition of M_m, again either $j \in M_m$ or $j \in \bigcup_{r=0}^{m-1} M_r$, as required. □

Finally, Claims 1 and 2 together with $(*)$ imply the desired inequality:

$$\begin{aligned} \sum_{s=1}^{\infty} |\hat{g}(s)| \geq \sum_{s\in A_t} |\hat{g}_t(s)| &\geq \frac{1}{\prod_{1\leq j\leq t} \max(R_j, 1)} \sum_{s\in A_0} |\hat{g}_0(s)| \\ &= \frac{1}{C} \sum_{s=1}^{\infty} |\hat{g}_0(s)| \geq \frac{1}{C^2} |\hat{g}(0)|. \end{aligned}$$
□

REMARK. The statement of Theorem 3.2 can be considered as a limit case for the one of Theorem 3.1. Indeed, for any p, $1 \le p < \infty$, the cyclicity condition for a lacunary function $f \in \ell_A^p$ is equivalent to $\prod_{k=1}^{\infty}(R_k, 1)_{1/(p-1)} = \infty$, where $\{R_k\}_{k\ge 1}$ denotes the remainder sequence for f and $(\cdot,\cdot)_r$ stands for the ℓ^r-norm of the vector $(\cdot,\cdot)$ in the space $\mathbb{R}^2$.

Now we are able to state the following theorem, which shows the difference between the cases $p \ge 2$ and $p < 2$.

THEOREM 3.3. (1) *Let $2 \le p \le \infty$. If a function $f \in \ell_A^p$ is such that its spectrum $\sigma(f)$ is a finite union of Hadamard lacunary sequences, then f is cyclic in the space ℓ_A^p.*

(2) *Let $1 \le p < 2$. Then in the space ℓ_A^p there are Hadamard lacunary functions that are noncyclic. Moreover, for every infinite subset $\Lambda \subset \mathbb{Z}_+$ there is a noncyclic function $f \in \ell_A^p$ such that $\sigma(f) \in \Lambda$ and* $\operatorname{card}\sigma(f) = \infty$.

Proof. (1) Let $2 \le p < \infty$, and let $f = \sum_{k=1}^{\infty} a_k z^{n_k} \in \ell_A^p$, where the increasing sequence $\{n_k\}_{k=1}^{\infty}$ is a finite union of Hadamard lacunary sequences. Put $R_k = |a_k|^p / \sum_{j>k}|a_j|^p$, $k \ge 1$. By Lemma 1.1, the series $\sum_{k=1}^{\infty} R_k$ diverges, so that $\sum_{k=1}^{\infty} R_k^{1/(p-1)} = \infty$ since $p \ge 2$. Theorem 3.1 and the Remark following its proof imply that f is cyclic.

For $p = \infty$, the assertion of the theorem can be proved by a similar method as for $p = 2$ in Theorem 2.1.

(2) Now let $1 \le p < 2$. Take a sequence $\{n_k\}_{k=1}^{\infty} \subset \Lambda$ so that $n_{k+1}/n_k > 2$, $k \ge 1$. Consider the function $f = \sum_{k=1}^{\infty}(1/k^2) z^{n_k} \in \ell_A^p$. A simple estimation shows that $R_k = k^{-2p}/(\sum_{j>k} j^{-2p}) = O(k^{-1})$ as $k \to \infty$. It remains to apply Theorem 3.1 (or Theorem 3.2 if $p = 1$) to show that f is noncyclic. □

EXAMPLES. Here we give some examples of cyclic and noncyclic lacunary functions in the spaces ℓ_A^p, $1 \le p < 2$. We omit the proofs, which are mostly simple calculations using Theorems 3.1 and 3.2.

Let $\Lambda = \{2^k,\ k \ge 0\} \subset \mathbb{Z}_+$. Although Λ is not a lacunary sequence of order 2, we can, by virtue of Lemma 1.6, apply Theorems 3.1 and 3.2 to a function f with the spectrum $\sigma(f) = \Lambda$ (note that the function Tf is lacunary of order 2).

(1) $1 < p < 2$. Then the function

$$f_\alpha = \sum_{n\in\Lambda} \frac{1}{n^{\log^\alpha n}} z^n$$

is cyclic in ℓ_A^p for $\alpha \ge 1 - p$ and is noncyclic for $-1 < \alpha < 1 - p$.

(2) $p = 1$. Then the function

$$f_\alpha = \sum_{n\in\Lambda} \frac{1}{n^\alpha} z^n$$

is cyclic in ℓ_A^1 for $\alpha > 1$ and is noncyclic for $0 < \alpha \le 1$. For an increasing sequence of positive integers $\phi(n)$, the function

$$f_\phi = \sum_{n\in\Lambda} \frac{1}{n\phi(n)} z^n$$

is cyclic in ℓ^1_A if and only if the sequence $\phi(n)$ is unbounded.

Considering the cyclicity problem in the spaces ℓ^p_A, $1 \le p \le \infty$, it is natural to ask what the situation is for the space

$$c_A = \left\{ f = \sum_{k=0}^{\infty} \hat{f}(k) z^k : \lim_{k\to\infty} \hat{f}(k) = 0 \right\}$$

endowed with the norm $\|f\|_\infty = \sup_{k\ge 0}|\hat{f}(k)|$. The following theorem shows that a function $f \in c_A$ is cyclic under a much weaker (than Hadamard lacunarity) assumption on the sparseness of the spectrum $\sigma(f)$.

THEOREM 3.4. *Let $\{n_k\}_{k=1}^\infty$ be an increasing sequence of nonnegative integers such that $n_{k+1} - n_k \to \infty$ as $k \to \infty$ and let $f = \sum_{k=1}^\infty a_k z^{n_k} \in c_A$ with $a_k \ne 0$ for $k \ge 1$. Then f is cyclic in c_A.*

Proof. Put $R_k = |a_k|/\max_{j>k}|a_j|$, $k \ge 1$, and denote by A the set $\{k \ge 1 : R_k \ge 1\}$. Obviously, A is infinite. We construct inductively an increasing integer sequence $\{k(i)\}_{i=1}^\infty$ and a sequence $\{g(i)\}_{i=1}^\infty$ of functions of c_A as follows. Choose an arbitrary $k(1) \in A$ and put $g_1 = (1/a_{k(1)})T^{n_{k(1)}}f$. Further, if $k(i)$ have been constructed, take $k(i+1) \in A$ such that for all $j \ge n_{k(i+1)+1} - n_{k(i+1)}$ the inequality $|\hat{g}_i(j)| < 1$ holds, and put

$$g_{k+1} = g_k + \frac{1}{a_{k(i+1)}} T^{n_{k(i+1)}} f.$$

Clearly, $\hat{g}_N(0) = N$ for every $N \ge 1$. We use induction to prove that

$$\max_{1\le j\le\infty} |\hat{g}_N(j)| \le 2.$$

This inequality holds for $N = 1$ since $k(1) \in A$. Suppose that

$$\max_{1\le j\le\infty} |\hat{g}_i(j)| \le 2$$

and fix $j \ge 1$. If $j \ge n_{k(i+1)+1} - n_{k(i+1)}$, then $|\hat{g}_i(j)| \le 1$ by the construction, and hence

$$|\hat{g}_{i+1}(j)| \le 1 + \frac{1}{a_{k(i+1)}} (T^{n_{k(i+1)}} f)\hat{}\,(j) \le 2,$$

since $k(i+1) \in A$. On the other hand, if $1 \le j < n_{k(i+1)+1} - n_{k(i+1)}$, then $(T^{n_{k(i+1)}}f)\hat{}\,(j) = 0$ and so $|\hat{g}_{i+1}(j)| = |\hat{g}_i(j)| \le 2$. Thus, the desired inequality is proved.

Now fix $\epsilon > 0$. Choose N such that $2/N < \epsilon$. Considering the function g_N which lies in $\mathrm{Lin}(T^k f,\, k \ge 0)$, we obtain that

$$\left\| \frac{g_N}{\hat{g}_N(0)} - 1 \right\|_\infty \le \frac{2}{N} < \epsilon.$$

Therefore, $1 \in \operatorname{span}(T^k f,\, k \ge 0)$, and f is cyclic because Lemma 1.7 remains true for the space c_A. □

REMARK. One can prove that, if the moduli $|a_k|$ tend to 0 monotonically (or, more generally, if $\liminf_{k\to\infty} R_k > 0$), then it suffices to require

$$\limsup_{k\to\infty} (n_{k+1} - n_k) = \infty$$

for the cyclicity of $f = \sum_{k=1}^{\infty} a_k z^{n_k}$ in the space c_A.

4. An Application to Invariant Subspaces

One of the consequences of the main theorem in [4] is that, in the Hardy space H^2, the sum of two noncyclic vectors is always noncyclic. Using the results of Section 3 one can show that this is not true for the backward shift operator on the spaces ℓ_A^p for $1 \le p < 2$.

Moreover, the following statement is valid.

PROPOSITION 4.1. *Let $1 \le p < 2$. Then in the space ℓ_A^p there exist a noncyclic vector f and a cyclic vector g such that $f + \lambda g$ is noncyclic for every complex number λ.*

Proof. First, let $1 < p < 2$. Set, for example,

$$f = \sum_{j=1}^{\infty} \frac{1}{j} z^{3^{2j-1}} \quad \text{and} \quad g = \sum_{j=1}^{\infty} \frac{1}{2^j} z^{3^{2j}}.$$

Fix $\lambda \in \mathbb{C}$. Obviously, the functions f and $f + \lambda g$ belong to ℓ_A^p and are lacunary of order 2. Let $\{R_k'\}_{k=1}^{\infty}$ and $\{R_k''\}_{k=1}^{\infty}$ denote the remainder sequences for the functions f and $f+\lambda g$, respectively. Then $R_k' = 2^p - 1$, $k \ge 1$, and since $\sum_{j>k} j^{-p} > \text{const.}/k^{p-1}$ it is easy to verify that $R_k'' = O(k^{-1})$ as $k \to \infty$, so that $\sum_{k=1}^{\infty} (R_k'')^{1/(p-1)} < \infty$. By Theorem 3.1, f is cyclic and $f + \lambda g$ is noncyclic.

Next, if $p = 1$, then one easily checks by using Theorem 3.2 that the functions

$$f = \sum_{j=1}^{\infty} \frac{1}{j^2} z^{3^{2j-1}} \quad \text{and} \quad g = \sum_{j=1}^{\infty} \frac{1}{3^j} z^{3^{2j}}$$

provide the necessary example. □

Now we obtain the following theorem as a dual fact to this proposition.

THEOREM 4.2. *Let $2 < q \le \infty$. Then in the space ℓ_A^q (endowed with the weak* topology in the case $q = \infty$) there exists a family $\{E_\lambda\}_{\lambda \in \mathbb{C}}$ of nonzero S-invariant subspaces such that $E_\lambda \cap E_\mu = \{\mathbb{O}\}$ whenever $\lambda \ne \mu$.*

Proof. Let p be the conjugate exponent of q. Since $1 \le p < 2$, we can apply Proposition 4.1 and construct two functions f and g in the space ℓ^p_A such that g is T-cyclic and $f+\lambda g$ is noncyclic (for T) for every complex number λ.

Denote by I_λ the smallest T-invariant subspace of ℓ^p_A containing $f+\lambda g$:

$$I_\lambda = \operatorname{span}(T^k(f+\lambda g),\, k \ge 0), \quad \lambda \in \mathbb{C},$$

and define the subspace E_λ of $\ell^q_A = (\ell^p_A)^*$ as the annihilator of I_λ:

$$E_\lambda = \{\phi \in \ell^q_A \colon \langle h, \phi\rangle = 0 \text{ for every } h \in I_\lambda\}, \quad \lambda \in \mathbb{C}.$$

Clearly, for every $\lambda \in \mathbb{C}$, E_λ is an S-invariant subspace of ℓ^q_A that is nonzero since $I_\lambda \ne \ell^p_A$.

Now let $\lambda \ne \mu$ be complex numbers. Since

$$g = \frac{(f+\lambda g)-(f+\mu g)}{\lambda-\mu} \in \operatorname{span}(I_\lambda, I_\mu)$$

and $\operatorname{span}(I_\lambda, I_\mu)$ is a T-invariant subspace of ℓ^p_A, it follows that $\operatorname{span}(T^k g,\, k \ge 0) \subset \operatorname{span}(I_\lambda, I_\mu)$. The cyclicity of g yields $\operatorname{span}(I_\lambda, I_\mu) = \ell^p_A$ and, by duality, $E_\lambda \cap E_\mu = \{\mathbb{O}\}$. □

ACKNOWLEDGMENTS. Most of this work was accomplished when I was a Ph.D. student at Bordeaux University. I thank the Analysis Group and the Graduate School in Mathematics of this university for their warm hospitality. I wish to express my sincere gratitude to Professor N. Nikolski for his guidance and encouragement over the years when the thesis was carried out. I would also like to thank Professor M. S. Ramanujan, Professor D. Sarason, and the referee for useful remarks and suggestions to improve the paper.

References

[1] E. V. Abakumov, *Approximation ability of lacunary series in weighted spaces* ℓ^p (to appear).

[2] A. B. Aleksandrov, *Inner functions on compact spaces,* Funktsional Anal. i Prilozhen. 18 (1984), 1–13 (Russian); translation in Functional Anal. Appl. 18 (1984), 87–98.

[3] A. Beurling, *On two problems concerning linear transformations in Hilbert space,* Acta Math. 81 (1949), 239–255.

[4] R. G. Douglas, H. S. Shapiro, and A. L. Shields, *Cyclic vectors and invariant subspaces for the backward shift operator,* Ann. Inst. Fourier (Grenoble) 20 (1970), 37–76.

[5] V. Havin and B. Jöricke, *The uncertainty principle in harmonic analysis,* Springer, Berlin, 1994.

[6] K. Hoffman, *Banach spaces of analytic functions,* Prentice-Hall, Englewood Cliffs, NJ, 1962.

[7] N. Nikolski, *Pseudocontinuation of the Borel transform and mean aperiodic functions,* Algebra i Analiz (St. Petersburg Math. J.) (to appear).

[8] ———, *Treatise on the shift operator,* Springer, Berlin, 1986.
[9] W. Rudin, *Principles of mathematical analysis,* 3rd ed., McGraw-Hill, New York, 1987.
[10] V. I. Smirnov, *Sur les formules de Cauchy et de Green et quelques problèmes qui s'y rattachent,* Izv. Akad. Nauk SSSR 3 (1932), 338–372.

Steklov Institute of Mathematics
St. Petersburg Branch
Fontanka 27
191011 St. Petersburg
Russia

Current Address:
Department of Applied Mathematics and Analysis
University of Barcelona
Gran Via 585
08071 Barcelona
Spain