A Factorization Theorem for Smooth Crossed Products

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Introduction

By a remarkable theorem of Dixmier and Malliavin [DM, Thm. 3.3], it is known that the convolution algebra $C_c^\infty(G)$ of compactly supported C^∞-functions on a Lie group G satisfies the factorization property—namely, that every set of C^∞-vectors E for the action of G is equal to the finite linear span $C_c^\infty(G)E$. In this paper, we replace $C_c^\infty(G)$ by the smooth crossed products for transformation groups $G \rtimes \mathcal{S}(M)$ defined in [S1]. We define an appropriate notion of a differentiable $G \rtimes \mathcal{S}(M)$-module, which generalizes the notion of C^∞-vectors for actions of Lie groups. (This definition was first introduced by Du Cloux [D1; D2]). Under the assumption that the Schwartz functions $\mathcal{S}(M)$ vanish rapidly with respect to a continuous, proper map $\sigma: M \to [0, \infty)$, we then show that $G \rtimes \mathcal{S}(M)$ satisfies the factorization property—namely, that any differentiable $G \rtimes \mathcal{S}(M)$-module E is the finite span of elements of the form ae, where $a \in G \rtimes \mathcal{S}(M)$ and $e \in E$. In the course of doing this, we also show that if a Fréchet algebra A has the factorization property, then the smooth crossed product $G \rtimes A$ does also.

Other aspects of the representation theory of the smooth crossed products $G \rtimes \mathcal{S}(M)$ are studied in [S2]. I would like to thank Berndt Brenken for a pleasant stay at the University of Calgary, where I wrote the first draft of this paper.

1. Differentiable Representations and Multipliers

We define what it means for an algebra and a representation to be differentiable. We shall use representation and module terminology interchangeably throughout this paper. Everything we do will be for left modules, though similar statements are also true for right modules.

DEFINITION 1.1. By a *Fréchet algebra* we mean a Fréchet space with an algebra structure for which the multiplication is jointly continuous. (We do not assume that Fréchet algebras are m-convex.) Let A be a Fréchet algebra. By a *Fréchet A-module* we mean a Fréchet space E that is an A-module for

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which the map $(a, e) \mapsto ae$ is jointly continuous. An A-module E is *nondegenerate* (*differentiable*) if $\{v \in E \mid Av = 0\} = \{0\}$ and the image of the canonical map $A \hat{\otimes} E \to E$ is dense (onto) [D2, Déf. 2.3.1]. (All tensor products will be completed in the projective topology.) We make the same definitions for right Fréchet A-modules. If A is differentiable both as a left and right A-module, then we say that the Fréchet algebra A is *self-differentiable*.

(In [D2, Déf. 2.3.1], a self-differentiable Fréchet algebra is called a "differentiable Fréchet algebra". However, this terminology would suggest that the algebra has a derivation acting on it, or that it is a set of C^∞-vectors for the action of a Lie group. This is not the case; any C*-algebra is a "differentiable Fréchet algebra" since any element of the algebra can be written as a linear combination of four positive elements, each of which has a square root. So I prefer to say "self-differentiable" instead of "differentiable".)

If E is nondegenerate, we let $E_s(A)$ be the image of the canonical map $A \hat{\otimes} E \to E$. Then $E_s(A)$ inherits the quotient topology from $A \hat{\otimes} E$, making $E_s(A)$ a Fréchet A-module. When A is self-differentiable, the A-module $E_s(A)$ is always differentiable [D2, Lemme 2.3.4].

If G is a topological group, we say that a Fréchet space E is a *continuous G-module* if G acts on E by continuous automorphisms and if, for each $e \in E$ and each continuous seminorm $\| \;\|$ on E, the map $g \mapsto \|ge\|$ is continuous. If G is a Lie group, we say that E is a *differentiable* G-module if the action of G on E is differentiable in the usual sense.

We say that a self-differentiable Fréchet algebra A satisfies the *factorization property* if every differentiable A-module E is the finite span of elements of the form ae, where $a \in A$ and $e \in E$. Note that in particular A will be the finite span of products of elements of A. Note also that if A is a unital Fréchet algebra (this corresponds to the case of the group algebra of a discrete group), then A is self-differentiable, every A-module is differentiable, and A satisfies the factorization property.

If G is a compact Lie group and $A = C^\infty(G)$ is the convolution algebra of C^∞ functions on G, then an A-module E is differentiable if and only if the action of G on E is differentiable (see Theorem 5.3 below or [D2, Exemple 2.3.3]). It follows immediately from [DM, Thm. 3.3] that Schwartz functions $\mathcal{S}(\mathbf{R})$ on $\mathbf{R}$ with convolution multiplication is a self-differentiable Fréchet algebra, since the canonical map $\mathcal{S}(\mathbf{R}) \otimes \mathcal{S}(\mathbf{R}) \to \mathcal{S}(\mathbf{R})$ is onto. (Here $\otimes$ denotes the algebraic tensor product.)

EXAMPLE 1.2. We give an example of a self-differentiable Fréchet (in fact Banach) algebra without the factorization property. Let $A = l_1(\mathbf{Z})$ with pointwise multiplication. Then $c_f(\mathbf{Z})$ is dense in A, so A is nondegenerate. Since $A \hat{\otimes} A \cong l_1(\mathbf{Z} \times \mathbf{Z})$, and the canonical map $\pi: A \hat{\otimes} A \to A$ is given by evaluation along the diagonal, A is self-differentiable.

A quick calculation shows that

$$\|\varphi * \psi\|_{1/2} \leq \|\varphi\|_1 \|\psi\|_1 \quad \text{and} \quad \|\varphi + \psi\|_{1/2} \leq 2(\|\varphi\|_{1/2} + \|\psi\|_{1/2}).$$

Hence the algebraic span A^2 is contained in $l_{1/2}(\mathbf{Z})$. Since $l_{1/2}(\mathbf{Z}) \neq A$, the algebra A does not have the factorization property. Similar arguments show that the Banach algebra $l_2(\mathbf{Z})$ with pointwise multiplication is an example of a nondegenerate but non-self-differentiable Banach algebra.

Definition 1.3. We say that T is a *multiplier* for a Fréchet algebra A if T acts as a continuous linear operator both on the left of A and the right of A, and the left and right actions commute. It follows that, for every seminorm $\| \|_d$ on A, there is some $C > 0$ and another seminorm $\| \|_m$ on A such that

$$\max(\|Ta\|_d, \|aT\|_d) \leq C\|a\|_m, \quad a \in A.$$

Example 1.4. In general, if T is a multiplier and E is a nondegenerate A-module, the action of T on A does not extend to an action on E. For example, let A be Schwartz functions $\mathcal{S}(\mathbf{R})$ on $\mathbf{R}$ with pointwise multiplication, and let T be multiplication by the function r^2. Let E be the nondegenerate A-module $C_0(\mathbf{R})$ with action of A given by pointwise multiplication. (Here $C_0(\mathbf{R})$ denotes the set of continuous functions on $\mathbf{R}$ which vanish at infinity.) Let f be any continuous function which vanishes like $1/r^2$ at infinity on $\mathbf{R}$. Then $f \in E$ and Tf does not vanish at infinity, so $Tf \notin E$.

Theorem 1.5. *Let A be a self-differentiable Fréchet algebra and let E be a differentiable A-module. Let T be a multiplier for A. Then there is a unique action of T on E as a continuous linear operator, which is consistent with the action of A on E.*

Proof. Since T is a continuous linear map from A to A, T also gives a continuous linear map of the projective completions $T: A \hat{\otimes} E \to A \hat{\otimes} E$ [Tr, Prop. 43.6]. Since E is differentiable, to see that this map induces a continuous linear map on E it suffices to show that T leaves the kernel of the canonical map $\pi: A \hat{\otimes} E \to E$ invariant. Assume that $\pi(\eta) = 0$ for $\eta \in A \hat{\otimes} E$. Let $b \in A$. Then

$$b\pi(T\eta) = \pi(bT\eta) = bT\pi(\eta) = 0,$$

since $bT \in A$. Hence $\pi(T\eta) \in E$ is annihilated by every element of A. Since E is a nondegenerate A-module, it follows that $\pi(T\eta) = 0$. Hence T leaves the kernel of π invariant. $\square$

2. Smooth Crossed Products

We recall the definitions of our smooth crossed products from [S1]. First, let H be a Lie group and let M be a locally compact space on which H acts. We say that a Borel measurable function $\sigma: M \to [0, \infty)$ is a *scale* on M if it is bounded on compact subsets of M. We say that a scale σ *dominates* another scale γ if there exists $C, D > 0$ and $d \in \mathbf{N}$ such that $\gamma(m) \leq C\sigma(m)^d + D$ for $m \in M$. We say that σ and γ are *equivalent* ($\sigma \sim \gamma$) if they dominate each other. We may always replace a scale σ with the equivalent scale $1 + \sigma$, so that we lose no generality by assuming $\sigma \geq 1$. From now on, we will assume this. If $h \in H$, define $\sigma_h(m) = \sigma(h^{-1}m)$. We say that σ is *uniformly H-translationally equivalent* if for every compact subset K of H there exists $C_K > 0$ and $d \in \mathbf{N}$ such that

$$\sigma_h(m) \leq C_K \sigma(m)^d, \quad m \in M, \ h \in K. \tag{2.1}$$

If σ is a uniformly H-translationally equivalent scale on M, we may define the *H-differentiable σ-rapidly vanishing functions* $\mathcal{S}_H^\sigma(M)$ by

$$\mathcal{S}_H^\sigma(M) \\ = \{ f \in C_0(M),\ f\ H\text{-differentiable} \mid \|\sigma^d X^\gamma f\|_\infty < \infty \text{ and } X^\gamma f \in C_0(M) \},$$

where X^γ ranges over all differential operators from the Lie algebra of H, and d ranges over all natural numbers. We topologize $\mathcal{S}_H^\sigma(M)$ by the seminorms $\|f\|_{d,\gamma} = \|\sigma^d X^\gamma f\|_\infty$. Then $\mathcal{S}_H^\sigma(M)$ is a Fréchet *-algebra under pointwise multiplication, with differentiable action of H [S1, §5].

Next, let $G \subseteq H$ be a Lie group with differentiable inclusion map $\iota\colon G \hookrightarrow H$. Let $\omega \geq 1$ be a scale on G. Let E be any Fréchet space. We define the *differentiable ω-rapidly vanishing functions $\mathcal{S}^\omega(G, E)$ from G to E* to be the set of differentiable functions φ from G to E such that

$$\|\varphi\|_{d,\gamma,m} = \int_G \|\omega^d X^\gamma \varphi(g)\|_m \, dg < \infty, \tag{2.2}$$

where X^γ is any differentiable operator from the Lie algebra of G acting by left translation, $\| \ \|_m$ is any seminorm for E, and d is any natural number. We topologize $\mathcal{S}^\omega(G, E)$ by the seminorms (2.2).

We say that the action of G on a G-module E is *ω-tempered* if for every $m \in \mathbf{N}$ there exists $C > 0$, $d \in \mathbf{N}$, and $k \in \mathbf{N}$ such that

$$\|\alpha_g(e)\|_m \leq C\omega(g)^d \|e\|_k, \quad e \in E, \ g \in G.$$

Simple arguments show that every closed G-submodule and every quotient of a tempered G-module is again a tempered G-module. We say that ω is *subpolynomial* if there exist $C > 0$ and $d \in \mathbf{N}$ such that

$$\omega(gh) \leq C\omega(g)^d \omega(h)^d, \quad g, h \in G.$$

The *inverse scale* ω_- is defined by $\omega_-(g) = \omega(g^{-1})$. We say that ω *bounds Ad on H* if there exist $C > 0$ and $d \in \mathbf{N}$ such that

$$\|Ad_g\| \leq C\omega(g)^d, \quad g \in G,$$

where $\|Ad_g\|$ is the operator norm of Ad_g as an operator on the Lie algebra of H. Finally, if ω is a subpolynomial scale on G such that ω_- bounds Ad on H, and if σ satisfies

$$\sigma(gm) \leq C\omega(g)^d \sigma(m)^l, \quad g \in G, \ m \in M \tag{2.3}$$

for some $C > 0$ and $d, l \in \mathbf{N}$, then we say that (M, σ, H) is a *scaled (G, ω)-space*.

THEOREM 2.4 [S1, Thm. 2.2.6, Thm. 5.17]. *Let ω be a subpolynomial scale on a Lie group G such that ω_- bounds Ad on G. Assume that the action of G on a Fréchet algebra A is continuous and ω-tempered. Then $\mathcal{S}^{\omega}(G,A)$ is a Fréchet algebra under convolution, which we denote by $G \rtimes^{\omega} A$. Moreover, if (M, σ, H) is a scaled (G,ω)-space then the action of G on $\mathcal{S}^{\sigma}_{H}(M)$ is differentiable and ω-tempered. In particular, $G \rtimes^{\omega} \mathcal{S}^{\sigma}_{H}(M)$ is a Fréchet algebra under convolution.*

See [S1, §5] or [S2] for examples.

3. Differentiable Scales for M

THEOREM 3.1. *Every uniformly H-translationally equivalent scale σ on M is equivalent to an H-differentiable scale $\tilde{\sigma}$ on M for which there is some $d \in \mathbf{N}$ such that, for every differential operator X^{γ} from the Lie algebra of H, we have*

$$(\exists C_{\gamma} > 0) \quad X^{\gamma}\tilde{\sigma}(m) \le C_{\gamma}\tilde{\sigma}^{d}(m), \quad m \in M. \tag{3.2}$$

If σ is continuous to begin with, then the scale $\tilde{\sigma}$ produced in the proof is continuous also.

Proof. Let σ be any uniformly H-translationally equivalent scale on M. Let K be a compact neighborhood of e in H such that

$$\sigma_g(m) \le C\sigma(m)^d, \quad m \in M, \tag{3.3}$$

and

$$\sigma(m) \le C\sigma_g(m)^d, \quad m \in M,$$

for every $g \in K$. Let $\varphi \in C_c^{\infty}(H)$ be any nonnegative function with support contained in K such that $\int \varphi(g)\,dg = 1$. Define

$$\tilde{\sigma}(m) = \int \varphi(g)\sigma_g(m)\,dg.$$

Then $\tilde{\sigma}(m) \ge 1$ and $\tilde{\sigma}$ is Borel measurable on M. If σ is continuous, then taking limits inside the integral shows that $\tilde{\sigma}$ is also. We show that $\sigma \sim \tilde{\sigma}$. By (3.3), we have

$$\tilde{\sigma}(m) \le \int \varphi(g)C\sigma(m)^d\,dg = C\sigma(m)^d,$$

so σ dominates $\tilde{\sigma}$ (in particular, $\tilde{\sigma}$ is bounded on compact sets). Similarly,

$$\sigma(m)^{1/d} = \int \varphi(g)\sigma(m)^{1/d}\,dg \le \int \varphi(g)C^{1/d}\sigma_g(m)\,dg = C^{1/d}\tilde{\sigma}(m), \tag{3.4}$$

so $\sigma(m) \le C\tilde{\sigma}^d(m)$.

We show that $\tilde{\sigma}$ is differentiable, and that the derivatives satisfy the bounds (3.2). We have

$$X^\gamma \tilde{\sigma}(m) = \int X^\gamma \varphi(g) \sigma_g(m)\, dg.$$

Hence $\tilde{\sigma}(m)$ is an H-differentiable function on M. Using (3.3), we bound the derivative

$$|X^\gamma \tilde{\sigma}(m)| \leq \int |X^\gamma \varphi(g)| C\sigma(m)^d\, dg = C_\gamma C\sigma(m)^d.$$

Since $\tilde{\sigma}$ dominates σ (see (3.4)), we have (3.2). □

We say that a scale $\sigma: M \to [0, \infty)$ is *proper* if the inverse image $\sigma^{-1}(K)$ of every compact subset K of $[0, \infty)$ is relatively compact. The property of being proper is preserved under equivalence.

PROPOSITION 3.5. *Let σ be a continuous uniformly H-translationally equivalent H-differentiable scale on M with property* (3.2). *Then σ is a multiplier on $\mathcal{S}_H^\sigma(M)$. If σ is proper then there is a natural continuous algebra homomorphism $\mathcal{S}(\mathbf{R}) \to \mathcal{S}_H^\sigma(M)$ given by $\varphi \mapsto \varphi \circ \sigma$.*

Proof. To see that σ is a multiplier on $\mathcal{S}_H^\sigma(M)$, *let* $f \in \mathcal{S}_H^\sigma(M)$. Then

$$\|\sigma^l X(\sigma f)\|_\infty = \|\sigma^l((X\sigma)f + \sigma(Xf))\|_\infty \leq \|\sigma^l C\sigma^d f\|_\infty + \|\sigma^{l+1} Xf\|_\infty.$$

Similar arguments show that for higher derivatives we also have $\|\sigma^l X^\gamma(\sigma f)\|_\infty$ bounded by some linear combination of seminorms of f. The function $X^\gamma(\sigma f)$ is a continuous function on M, since σ and f are continuous and H-differentiable. Since for each $l \in \mathbf{N}$ and γ the function $\sigma^l X^\gamma f$ vanishes at infinity (see "Added in proof" below or [S2, Proof of Prop. 5.2]), and since $|X^\gamma \sigma| \leq C_\gamma \sigma^d$ by (3.2), we also have $X^\gamma(\sigma f) \in C_0(M)$ for all γ. Hence $\sigma f \in \mathcal{S}_H^\sigma(M)$ and σ is a multiplier on $\mathcal{S}_H^\sigma(M)$.

For the second statement, it suffices to show that the seminorms of $\varphi \circ \sigma$ in $\mathcal{S}_H^\sigma(M)$ are bounded by linear combinations of seminorms of φ in $\mathcal{S}(\mathbf{R})$, and also that $X^\gamma(\varphi \circ \sigma) \in C_0(M)$. We apply the chain rule. If X is in the Lie algebra of H, then

$$X(\varphi \circ \sigma)(m) = (\varphi' \circ \sigma)(m) X\sigma(m).$$

Thus

$$\begin{aligned} \|\sigma^l X \varphi \circ \sigma\|_\infty &= \sup_{m \in M} |\sigma(m)^l X(\varphi \circ \sigma)(m)| \\ &\leq \sup_{m \in M} |\sigma^l \varphi'(\sigma(m)) C\sigma(m)^d| \leq \sup_{r \in \mathbf{R}} |r^l \varphi'(r) C r^d|, \end{aligned}$$

where the last expression is a seminorm of φ in $\mathcal{S}(\mathbf{R})$. Since σ is proper and $r^d \varphi'(r)$ vanishes at infinity,

$$|X(\varphi \circ \sigma)(m)| = |\varphi'(\sigma(m)) X\sigma(m)| \leq |C\sigma(m)^d \varphi'(\sigma(m))|$$

implies that $X(\varphi \circ \sigma) \in C_0(M)$. Similar arguments work for higher derivatives. □

EXAMPLE 3.6. We consider the case when $M=\mathbf{R}^n$, and $\sigma(\vec{r})=r_1^2+\cdots+r_n^2$. Then σ is a differentiable scale on $\mathcal{S}(\mathbf{R}^n)$. The map $\mathcal{S}(\mathbf{R})\to\mathcal{S}(\mathbf{R}^n)$ in Proposition 3.5 is given by $\varphi(\vec{r})=\varphi(r_1^2+\cdots+r_n^2)$. The image of this map consists of radially symmetric functions on $\mathbf{R}^n$. In this sense, a differentiable scale can be regarded as a generalized "radial" coordinate for M, and the image of $\mathcal{S}(\mathbf{R})$ in $\mathcal{S}_H^\sigma(M)$ consists of functions that depend only on this radial coordinate.

4. Factorization Property for $\mathcal{S}_H^\sigma(M)$

We recall some of the functions on $\mathbf{R}$ defined in the proof of the Dixmier-Malliavin theorem [DM]. Let $\lambda=(\lambda_0,\lambda_1,\dots,\lambda_k,\dots)$ be any subsequence of $(1,2,\dots,2^k,\dots)$. For $x\in\mathbf{R}$, let

$$\varphi_\lambda(x)=\prod_{k=0}^{\infty}\left(1+\frac{x^2}{\lambda_k^2}\right),\quad \chi_\lambda(x)=\varphi_\lambda(x)^{-1}.$$

We show that φ_λ is a well-defined function from $\mathbf{R}$ to $[1,\infty)$. For k sufficiently large we have $x^2<\lambda_k^2$, so

$$1+\frac{x^2}{\lambda_{k+p}^2}<1+\frac{1}{2^p}$$

for all $p\in\mathbf{N}$. Since

$$\log\left(\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\cdots\left(1+\frac{1}{2^p}\right)\cdots\right)=\sum_{p=0}^{\infty}\log\left(1+\frac{1}{2^p}\right)\le\sum_{p=0}^{\infty}\frac{1}{2^p}<\infty,$$

we see that $\varphi_\lambda(x)$ is well-defined for any $x\in\mathbf{R}$.

It is shown in [DM, §2.3] that χ_λ is in $\mathcal{S}(\mathbf{R})$. Also, it is shown in the proof of [DM, Lemme 2.5, pp. 309–310] that for any sequence $(\beta_0,\beta_1,\dots)$ of positive numbers, there exists a sequence $(\alpha_0,\alpha_1,\dots)$ of positive numbers and a sequence $(\lambda_0,\lambda_1,\dots)$ as above such that α_n occur in the expansion

$$\varphi_\lambda(x)=\sum_{n=0}^{\infty}\alpha_n x^{2n}$$

and satisfy

$$\alpha_n\le\min(\beta_n,1/n^2). \tag{4.1}$$

We use this to show that $\mathcal{S}_H^\sigma(M)$ has the factorization property.

THEOREM 4.2. *Assume that σ is continuous and proper. Then, for every function $\psi\in\mathcal{S}_H^\sigma(M)$, there are $\theta,\phi\in\mathcal{S}_H^\sigma(M)$ such that $\psi=\theta\phi$.*

Proof. Let $\psi\in\mathcal{S}_H^\sigma(M)$, and let σ be an H-differentiable scale as in Theorem 3.1. Define

$$M_{d,l,n}=\max_{|\gamma|\le l}\|\sigma^{(d+1)l}\sigma^{2n}X^\gamma\psi\|_\infty.$$

Choose $\lambda=(\lambda_0,\lambda_1,\dots)$ so that the sequence $(\alpha_0,\alpha_1,\dots)$ satisfies

$$\sum_{n=0}^{\infty} \alpha_n M_{d,l,n} < \infty, \quad d, l \in \mathbf{N}. \tag{4.3}$$

Recall that $\sum_{n=0}^{\infty} \alpha_n x^{2n}$ is the expansion for φ_λ. Define $\tilde{\varphi}_\lambda = \varphi_\lambda \circ \sigma : M \to \mathbf{R}$. Then the series for $\tilde{\varphi}_\lambda \psi$ converges absolutely in $\mathcal{S}_H^\sigma(M)$ to an element of $\mathcal{S}_H^\sigma(M)$. For if $|\gamma| \le l$, we have

$$\begin{aligned}
\left\| \sigma^d X^\gamma \left(\sum_{n=k}^{\infty} \alpha_n \sigma^{2n} \right) \psi \right\|_\infty &\le \sum_{n=k}^{\infty} \alpha_n \| \sigma^d X^\gamma (\sigma^{2n} \psi) \|_\infty \\
&\le \sum_{n=k}^{\infty} \alpha_n \sum_{|\beta_1| + \cdots + |\beta_{2n+1}| \le l} D \| \sigma^d X^{\beta_1} \sigma \cdots X^{\beta_{2n}} \sigma X^{\beta_{2n+1}} \psi \|_\infty \\
&\le \sum_{n=k}^{\infty} \alpha_n DC \max_{|\beta| \le l} \| \sigma^d \sigma^{dl} \sigma^{2n-l} X^\beta \psi \|_\infty \quad \text{(by (3.2))} \\
&\le \sum_{n=k}^{\infty} \alpha_n DC M_{d,l,n} \quad (\text{since } \sigma^{2n-l} \le \sigma^{2n}).
\end{aligned} \tag{4.4}$$

By our constraint on the α_ns (4.3), the last sum tends to zero as $k \to \infty$. Hence $\tilde{\varphi}_\lambda \psi$ converges to some well-defined element $\phi \in \mathcal{S}_H^\sigma(M)$. Let $\theta = \chi_\lambda \circ \sigma$. By Proposition 3.5 and since $\chi_\lambda \in \mathcal{S}(\mathbf{R})$, we know $\theta \in \mathcal{S}_H^\sigma(M)$. Since $\chi_\lambda(x) = \varphi_\lambda^{-1}(x)$, we have $\theta(m)\tilde{\varphi}_\lambda(m) = 1$ for each $m \in M$, where 1 denotes the identity multiplier on $\mathcal{S}_H^\sigma(M)$. It follows that $\theta(m)\phi(m) = 1\psi(m)$, and the theorem is proved. □

The following corollary was part of the motivation for this paper.

COROLLARY 4.5. *If σ is continuous and proper, then the Fréchet algebra $\mathcal{S}_H^\sigma(M)$ is a self-differentiable Fréchet algebra.*

Proof. This follows directly from Theorem 4.2. □

THEOREM 4.6. *Let E be any differentiable representation of $\mathcal{S}_H^\sigma(M)$, and assume that σ is continous and proper. Then, for every $e \in E$, there exist $\theta \in \mathcal{S}_H^\sigma(M)$ and $f \in E$ such that $e = \theta f$. In particular, $\mathcal{S}_H^\sigma(M)$ satisfies the factorization property.*

Proof. We proceed very much as in the proof of Theorem 4.2. Let $e \in E$, and let σ be an H-differentiable scale as in Theorem 3.1. Since E is a differentiable $\mathcal{S}_H^\sigma(M)$-module, $\sigma^{2n} e$ is a well-defined element of E for each n (see Proposition 3.5 and Theorem 1.5). Define

$$M_{m,n} = \| \sigma^{2n} e \|_m,$$

where $\| \ \|_m$ is an increasing family of seminorms for E. Choose $\lambda = (\lambda_0, \lambda_1, \ldots)$ so that the sequence $(\alpha_0, \alpha_1, \ldots)$ satisfies

$$\sum_{n=0}^{\infty} \alpha_n M_{m,n} < \infty, \quad m \in \mathbf{N}. \tag{4.7}$$

Recall that $\sum_{n=0}^{\infty} \alpha_n x^{2n}$ is the expansion for φ_λ. Define $\tilde{\varphi}_\lambda = \varphi_\lambda \circ \sigma$. Then the series for $\tilde{\varphi}_\lambda e$ converges absolutely in E to an element of E. For if $m \in \mathbf{N}$, we have

$$\left\| \left(\sum_{n=k}^{\infty} \alpha_n \sigma^{2n} \right) e \right\|_m \leq \sum_{n=k}^{\infty} \alpha_n \|\sigma^{2n} e\|_m \leq \sum_{n=k}^{\infty} \alpha_n M_{m,n}. \tag{4.8}$$

By our constraint on the α_ns (4.7), the last sum tends to zero as $k \to \infty$. Hence $\tilde{\varphi}_\lambda e$ converges to some well-defined element $f \in E$. The remainder of the proof is just as in Theorem 4.2. □

5. Factorization Property for the Crossed Product

DEFINITION 5.1. Let ω be a scale on a Lie group G, and let A be a Fréchet algebra on which G acts by algebra automorphisms α_g. Assume that we have representations of G and A on a Fréchet space E such that the action of G on E is ω-tempered and differentiable, the action of A is differentiable, and the covariance condition

$$g(ae) = \alpha_g(a) ge, \quad g \in G, \ a \in A, \ e \in E, \tag{5.2}$$

is satisfied. We call such a representation an *ω-tempered differentiable covariant representation of* (G, A).

THEOREM 5.3. *Let G be a Lie group with subpolynomial scale ω such that ω_- bounds Ad on G, and let A be a self-differentiable Fréchet algebra with an ω-tempered, differentiable action α_g of G. Assume that we have an ω-tempered differentiable covariant representation of (G, A) on a Fréchet space E. Then we may integrate this representation to obtain a differentiable representation of the smooth crossed product $G \rtimes^\omega A$ on E.*

Conversely, if we have a differentiable representation of $G \rtimes^\omega A$ on E, then there is an ω-tempered differentiable covariant representation of (G, A) on E whose integrated form gives back the original action of $G \rtimes^\omega A$ on E. It follows from the proof that the smooth crossed product $G \rtimes^\omega A$ is a self-differentiable Fréchet algebra.

Proof. To simplify notation, we let $B = G \rtimes^\omega A$. We define an action of the algebra B on E by

$$Fe = \int_G F(g)(ge)\, dg. \tag{5.4}$$

We estimate

$$\begin{aligned} \|Fe\|_d &\leq \int_G \|F(g) ge\|_d \, dg \\ &\leq \int_G \|F(g)\|_m \|ge\|_k \, dg \quad (E \text{ is a continuous } A\text{-module}) \\ &\leq C \int_G \|F(g)\|_m \omega(g)^l \|e\|_n \, dg \quad (E \text{ is a tempered } G\text{-module}) \\ &\leq C \|F\|_{l,m} \|e\|_n, \end{aligned} \tag{5.5}$$

where $\|F\|_{l,m}$ are seminorms for B. So (5.4) is well-defined and continuous. By the covariance condition (5.2), it follows that $(F_1 * F_2)e = F_1(F_2 e)$, so E is a continuous B-module.

We prove that E is a nondegenerate B-module. Let $\pi: B \hat{\otimes} E \to E$ be the canonical map. Let $\Psi_n \in C_c^\infty(G)$ be a sequence of positive functions such that $\operatorname{supp} \Psi_n \to 0$ and $\int_G \Psi_n(g)\,dg = 1$. Let $\Psi_n \otimes a$ denote the function $g \mapsto \Psi_n(g)a$ in B. To see that E is a nondegenerate B-module it suffices to show that $(\Psi_n \otimes a)e$ converges to ae in E for every $a \in A$ and $e \in E$, since then π will have dense image and the null space for the action of B on E will be contained in the null space for the action of A on E. We estimate

$$\begin{aligned}
\|(\Psi_n \otimes a)e - ae\|_d &\leq \int_G \Psi_n(g)\|age - ae\|_d\,dg \\
&\leq \sup_{g \in \operatorname{supp} \Psi_n} \|age - ae\|_d \\
&\leq \|a\|_m \sup_{g \in \operatorname{supp} \Psi_n} \|ge - e\|_k,
\end{aligned} \tag{5.6}$$

which tends to zero by the strong continuity of the action of G on E. Hence E is a nondegenerate B-module.

Now we show that $\pi: B \hat{\otimes} E \to E$ is onto. Since G acts differentiably on E, any element e is a finite sum of elements $\alpha_f(\tilde{e}) \in E$, where $f \in C_c^\infty(G)$ and $\tilde{e} \in E$ [DM, Thm. 3.3]. Thus it suffices to show that elements of the form $\alpha_f(\tilde{e})$ are in the image of π.

Let $\tilde{\pi}: A \hat{\otimes} E \to E$ be the canonical map for the action of A on E. Since $\tilde{\pi}$ is onto by assumption, using [Tr, Thm. 45.1] we can write

$$\tilde{e} = \tilde{\pi}\left(\sum_{n=0}^{\infty} \lambda_n a_n \otimes e_n \right),$$

where $\sum |\lambda_n| < 1$, $a_n \to 0$ in A, and $e_n \to 0$ in E. Then

$$\begin{aligned}
\alpha_f(\tilde{e}) &= \alpha_f \tilde{\pi}\left(\sum_{n=0}^{\infty} \lambda_n a_n \otimes e_n \right) \\
&= \sum_{n=0}^{\infty} \lambda_n \alpha_f(\tilde{\pi}(a_n \otimes e_n)).
\end{aligned} \tag{5.7}$$

Since G acts differentiably on A, the function $b_n(g) = f(g)\alpha_g(a_n)$ is in B. A simple calculation shows that

$$\begin{aligned}
\alpha_f(\tilde{\pi}(a_n \otimes e_n)) &= \int_G f(g)(g a_n e_n)\,dg \\
&= \int_G f(g)\alpha_g(a_n)(g e_n)\,dg = b_n e_n = \pi(b_n \otimes e_n).
\end{aligned} \tag{5.8}$$

By the product rule for differentiation, and since $f \in C_c^\infty(G)$, we have

$$
\begin{aligned}
\|b_n\|_{m,\gamma,d} &= \int_G \omega(g)^m \|X^\gamma b_n(g)\|_d \, dg \\
&= \int_G \omega(g)^m \|X^\gamma (f(g)\alpha_g(a_n))\|_d \, dg \\
&\leq C \sup_{|\beta| \leq |\gamma|,\, g \in \mathrm{supp}(f)} \|X^\beta \alpha_g(a_n)\|_d \quad (\omega \text{ is bounded on compact sets}) \\
&\leq C \sup_{g \in \mathrm{supp}(f)} \|\alpha_g(a_n)\|_k \quad (G \text{ acts differentiably on } A) \\
&\leq D\|a_n\|_l.
\end{aligned}
$$

So $b_n \to 0$ in B as $n \to \infty$. Hence the sum

$$\sum_{n=0}^{\infty} \lambda_n b_n \otimes e_n$$

converges absolutely in $B \hat{\otimes} E$, and by (5.7) and (5.8) its image under π is $\alpha_f(\tilde{e})$. We have proved that π is onto. Thus E is a differentiable B-module.

Proof of the converse: We assume that E is a differentiable B-module. Then E is a quotient of the B-module $B \hat{\otimes} E$, where B acts on the left factor. If we let G act on B by

$$(gF)(h) = \alpha_g(F(g^{-1}h)), \quad g, h \in G, \ F \in B, \tag{5.10}$$

then the corresponding action of G on $B \hat{\otimes} E$ on the left factor gives rise to an action of G on the quotient E. Since the action (5.10) of G on B is both differentiable and tempered, so is the action of G on E.

Similarly, the algebra A acts on B via

$$(aF)(h) = aF(h), \quad a \in A, \ F \in B, \ h \in G. \tag{5.11}$$

Using our hypothesis that A is a self-differentiable Fréchet algebra, we show that the action (5.11) makes B into a differentiable A-module. The action (5.11) makes the L^1 ω-rapidly vanishing functions $L_1^\omega(G, A)$ [S1, §2] into an A-module. By [Sc, §5], we may write $L_1^\omega(G, A) \cong L_1^\omega(G) \hat{\otimes} A$. Since

$$A \hat{\otimes} L_1^\omega(G, A) \cong L_1^\omega(G) \hat{\otimes} A \hat{\otimes} A,$$

and since the map $A \hat{\otimes} A \to A$ is onto, we see that the canonical map

$$A \hat{\otimes} L_1^\omega(G, A) \to L_1^\omega(G, A)$$

is onto (the projective tensor product of surjective maps is surjective [Tr, Prop. 43.9]); hence $L_1^\omega(G, A)$ is a differentiable A-module. It follows from [S1, Thm. A.8] that if a G-module F is a differentiable A-module such that the action of G on F commutes with the action of A on F, then the set of C^∞-vectors F^∞ for the action of G is also a differentiable A-module. Thus B is a differentiable A-module, since it is the set of C^∞-vectors for the action $(gF)(h) = F(g^{-1}h)$ of G on $L_1^\omega(G, A)$.

Because $A\hat{\otimes}B\to B$ is onto, the map $A\hat{\otimes}B\hat{\otimes}E\to B\hat{\otimes}E$ is onto [Tr, Prop. 43.9]. Hence $B\hat{\otimes}E$ is a differentiable A-module and so, by passing to the quotient, we obtain a differentiable action of A on E.

To see the covariance of the actions of G and A on E, it suffices to notice that (5.10) and (5.11) give covariant actions of G and A on B. Also, if we integrate (5.10) and (5.11) via formula (5.4), we get B acting on B by left multiplication. So the integrated form of our covariant actions of G and A on E will give back the original action of B on E. This proves the converse.

Since we saw that (5.10) and (5.11) give an ω-tempered differentiable covariant representation of (G, A) on B, we know by the first part of the theorem that B is differentiable as a left module over itself. To see that B is self-differentiable it suffices to show that, for every nonzero $b\in B$, $b*B\neq\{0\}$. Since A is nondegenerate, find $a\in A$ such that $ba\neq 0$ and let $\Psi_n\in C_c^\infty(G)$ be as above. Then $b*(\Psi_n\otimes a)\to ba$ in B, so there must be some n such that $b*(\Psi_n\otimes a)\neq 0$. □

THEOREM 5.12. *Let G be a Lie group with subpolynomial scale ω such that ω_- bounds Ad on G, and let A be a self-differentiable Fréchet algebra with an ω-tempered, differentiable action of G. If A satisfies the factorization property, then $G\rtimes^\omega A$ is self-differentiable and satisfies the factorization property.*

Proof. Self-differentiability follows from the previous theorem. Let E be a differentiable $(G\rtimes^\omega A)$-module. Then there is an associated covariant representation of (G, A) on E by the previous theorem. Since G acts differentiably on E, we may apply [DM, Thm. 3.3] to see that E is the finite span of $\alpha_f(e)$, where $f\in C_c^\infty(G)$ and $e\in E$. By assumption, every $e\in E$ may be written as a finite sum of elements of the form $a\tilde{e}$, where $a\in A$ and $\tilde{e}\in E$. Define $b(g)=f(g)\alpha_g(a)$. Since G acts differentiably on A, $b\in G\rtimes^\omega A$. Also, $b\tilde{e}=\alpha_f(a\tilde{e})$, so every element of E is a finite sum of elements of the form $b\tilde{e}$. □

COROLLARY 5.13. *Let (M, σ, H) be any scaled (G, ω)-space, with σ a continuous, proper scale. Then the smooth crossed product $G\rtimes^\omega \mathcal{S}_H^\sigma(M)$ is self-differentiable and satisfies the factorization property.*

Proof. By Theorem 4.6, we know that $\mathcal{S}_H^\sigma(M)$ satisfies the factorization property. Hence, by Theorem 5.12, the smooth crossed product $G\rtimes^\omega \mathcal{S}_H^\sigma(M)$ does also. □

It follows that many of the examples of smooth dense subalgebras of transformation group C*-algebras in [S1, §5] satisfy the factorization property. In particular, see Examples 5.14, 5.18–20, 5.23–24, and 5.26 if M is compact. See also [S2, §10, §18, Examples 3.15, 12.25].

Added in proof. We include the portion of the proof of Proposition 3.5 which is cited from [S2]. It suffices to show that $\sigma^d f\in C_0(M)$ if $f\in \mathcal{S}_H^\sigma(M)$. Let $C_d=\|\sigma^{d+1}f\|_\infty$. Using $f\in C_0(M)$, let K be a sufficiently large compact subset of M such that $|f(m)| \leq \epsilon^{d+1} C_d^d$ for $m \notin K$. Then $|\sigma^d f(m)| \leq \epsilon$ for $m \notin K$ as long as $\sigma(m) \leq C_d/\epsilon$. But if $\sigma(m) > C_d/\epsilon$ then

$$|\sigma^d f(m)| < \epsilon C_d^{-1} \|\sigma^{d+1} f\|_\infty = \epsilon,$$

so $\sigma^d f$ vanishes at infinity.

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