Spectral Types of Unitary Operators Arising from Irrational Rotations on the Circle Group

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Introduction

$\mathbf{T}$ will denote the circle group written additively so that its elements are real numbers in $[0,1)$ and are added modulo 1. For each $y\in\mathbf{R}$, $\{y\}$ is the fractional part of y, so that $\{y\}\equiv y \bmod 1$ and $\{y\}\in\mathbf{T}$. Fix irrational $\alpha\in(0,1)$. Let $\phi:\mathbf{T}\to\mathbf{S}^1=\{z\in\mathbf{C}:|z|=1\}$ be a measurable function, and define U_ϕ: $L^2(\mathbf{T})\to L^2(\mathbf{T})$ by

$$(U_\phi f)(x)=\phi(x)f(\{x+\alpha\}).$$

Then U_ϕ is a unitary operator, and

$$(U_\phi^n f)(x)=\begin{cases}\left(\prod_{k=0}^{n-1}\phi(\{x+k\alpha\})\right)f(\{x+n\alpha\}) & \text{for } n\geq 0,\\ \left(\prod_{k=1}^{|n|}\bar{\phi}(\{x-k\alpha\})\right)f(\{x-|n|\alpha\}) & \text{for } n<0.\end{cases}$$

The spectral theorem says that there exists a spectral measure P_ϕ on the Borel sets of $\mathbf{T}$ such that

$$U_\phi^n=\int_T e^{2\pi i n x}\,dP_\phi(x)\quad\text{for } n\in\mathbf{Z}.$$

In this paper we study the spectral measure P_ϕ associated with U_ϕ when $\phi=e^{2\pi i g}$, where $g:\mathbf{T}\to\mathbf{R}$ is absolutely continuous except at one point.

The study of such spectral measures and related questions about multiplicative and additive cocycles and skew product transformations has generated much interest recently (e.g. [1; 2; 3; 4; 6; 8; 11; 12; 13]). More specifically, the case $g=x^a$, $a>0$ and α with bounded partial quotients, has been solved [3; 4]. And the case when ϕ is continuous (i.e., the size of the jump of g is in $\mathbf{Z}$) has been successfully studied [12]. The results we present here are related to work in [3; 4; 5; 6] and rely on a very simple technique: the direct calculation of the Fourier coefficients of the measure $d\langle P_\phi 1, P_\phi 1\rangle$.

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In Section 2, we introduce our technique by proving results when $\phi(x) = e^{2\pi i\lambda x}$, $\lambda \in \mathbf{R}$. The first result of the section was known [2] and the second improves a result in [6]. Section 3 contains the main results of the paper. We show that if g is absolutely continuous except at one point where it has a simple discontinuity, then P_ϕ is not of discrete type. The smoothness hypothesis on g is strengthened a bit, and we obtain necessary conditions on the irrational α and the size of the jump of g for P_ϕ not to be of Lebesgue spectral type.

The answers to many of the questions in this field, including those given here, involve number-theoretic properties of the irrational α. Any number-theoretic facts and terminology used here can be found in [7].

1. Preliminaries from Spectral Theory

Any spectral measure can be decomposed into three parts: a discrete part, one that lives on discrete points; an absolutely continuous part, one that is absolutely continuous with respect to Lebesgue measure; and a singular continuous part, one that vanishes on single points but lives on a set of Lebesgue measure zero.

It is well-known [9; 10] that the spectral measure associated with a U_ϕ is pure. That is, it has only one of the three parts just mentioned. And if it is absolutely continuous, it is equivalent to Lebesgue measure.

DEFINITION 1. We say that the unitary operator U_ϕ has *discrete, Lebesgue,* or *singular continuous spectrum* if its associated spectral measure P_ϕ is discrete, Lebesgue, or singular continuous, respectively.

Note that

$$\langle U_\phi^n 1, 1\rangle = \int_T e^{2\pi i n x}\, d\langle P_\phi(x)1, 1\rangle. \tag{1}$$

This identity suggests that studying the sequence $\langle U_\phi^n 1, 1\rangle$ will yield information about the spectral measure P_ϕ. This is indeed the case. The results of this paper are for the most part based on the following well-known proposition. For the reader's convenience, we include its proof.

PROPOSITION 1. *Let $\phi: \mathbf{T} \to \mathbf{S}^1$ be a measurable function, and define U_ϕ: $L^2(\mathbf{T}) \to L^2(\mathbf{T})$ by $(U_\phi f)(x) = \phi(x) f(\{x+\alpha\})$.*

(i) *U_ϕ has discrete spectrum if and only if there exists $\psi: \mathbf{T} \to \mathbf{S}^1$ and κ, $|\kappa| = 1$, such that $\phi(x) = \kappa(\psi(\{x+\alpha\})/\psi(x))$ a.e. In this case, if for $y \in \mathbf{R}$, $\|y\|$ is the distance from y to $\mathbf{Z}$, then*

$$\lim_{\substack{\|n\alpha\| \to 0 \\ n>0}} |\langle U_\phi^n 1, 1\rangle| = \left| \int_0^1 \prod_{m=0}^{n-1} \phi(\{x + m\alpha\})\, dx \right| = 1.$$

(ii) *If U_ϕ has Lebesgue spectrum, then*

$$\lim_{n\to\infty} |\langle U_\phi^n 1, 1\rangle| = \left|\int_0^1 \prod_{m=0}^{n-1} \phi(\{x+m\alpha\})\,dx\right| = 0.$$

(iii) *If* $\{\langle U_\phi^n 1, 1\rangle\} \in l^2(\mathbf{Z})$, *then* U_ϕ *has Lebesgue spectrum.*

Proof. (i) ($\Rightarrow$) If U_ϕ has discrete spectrum then $U_\phi = \sum_{n=1}^\infty \kappa_n P_n$, where $\{P_n\}_{n=1}^\infty$ is a family of mutually orthogonal self-adjoint projections on $L^2(\mathbf{T})$ and $|\kappa_n| = 1$. Since $U_\phi \neq 0$, there exists n such that $P_n \neq 0$. Thus there is a $\psi \in L^2(\mathbf{T})$, $\|\psi\|_2 = 1$, such that $U_\phi \psi = \kappa_n \psi$, so that $\phi(x)\psi(\{x+\alpha\}) = \kappa_n \psi(x)$ a.e. After taking absolute values, ergodicity implies that $|\psi|$ is constant a.e. Since $\|\psi\|_2 = 1$, $|\psi| = 1$ a.e.

($\Leftarrow$) If there exist $\psi: \mathbf{T} \to \mathbf{S}^1$ and κ, $|\kappa| = 1$, such that $\phi(x) = \kappa(\psi(\{x+\alpha\})/\psi(x))$ a.e., then $\bar{\psi}$ is an eigenfunction of U_ϕ. Hence the spectral measure associated with U_ϕ has a discrete part. Since the spectrum is pure, it must be discrete. If $\phi(x) = \kappa(\psi(\{x+\alpha\})/\psi(x))$ a.e., then

$$|\langle U_\phi^n 1, 1\rangle| = \left|\int_0^1 \kappa^n \psi(\{x+n\alpha\})\bar{\psi}(x)\,dx\right| \quad \text{for } n > 0.$$

Since translation is continuous in $L^2(\mathbf{T})$,

$$\left|\int_0^1 \psi(\{x+n\alpha\})\bar{\psi}(x)\,dx\right| \to \left|\int_0^1 \psi(x)\bar{\psi}(x)\,dx\right| = 1 \quad \text{as } \|n\alpha\| \to 0.$$

(ii) If U_ϕ has Lebesgue spectrum, then (1) shows that $\{\langle U_\phi^n 1, 1\rangle\}$ is the set of Fourier–Stieltjes coefficients of an absolutely continuous measure. Mercer's theorem implies (ii).

(iii) If $\{\langle U_\phi^n 1, 1\rangle\} \in l^2(\mathbf{Z})$, then there exists an $f \in L^2(\mathbf{T})$ such that $\hat{f}(-n) = \langle U_\phi^n 1, 1\rangle$. Thus

$$\int_T f(x) e^{2\pi i n x}\,dx = \int_T e^{2\pi i n x}\,d\langle P_\phi(x)1, 1\rangle \quad \text{for all } n \in \mathbf{Z}.$$

This implies $f(x)\,dx = d\langle P_\phi(x)1, 1\rangle$. Thus P_ϕ has an absolutely continuous part and must therefore be Lebesgue. □

2. The Case When g Is a Linear Function

Fix $\lambda \in \mathbf{R}$, $\lambda \neq 0$, and define $V_\lambda: L^2(\mathbf{T}) \to L^2(\mathbf{T})$ by

$$(V_\lambda f)(x) = e^{2\pi i \lambda x} f(\{x+\alpha\}).$$

We will compute $\langle V_\lambda^n 1, 1\rangle$ for $n \in \mathbf{N}$:

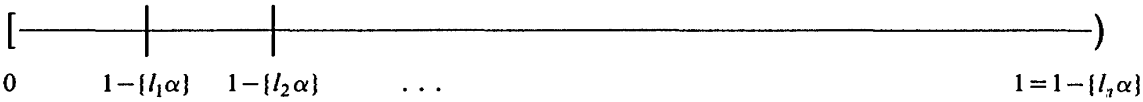
$$\langle V_\lambda^n 1, 1\rangle = \int_0^1 \prod_{m=0}^{n-1} e^{2\pi i \lambda\{x+m\alpha\}}\,dx;$$

$$\{x+m\alpha\} = \begin{cases} x+\{m\alpha\}, & 0 \le x < 1-\{m\alpha\}, \\ x+\{m\alpha\}-1, & 1-\{m\alpha\} \le x < 1. \end{cases}$$

Hence the function $\prod_{m=0}^{n-1} e^{2\pi i\lambda\{x+m\alpha\}}$ will have discontinuities at

$$x=0,1-\{\alpha\},1-\{2\alpha\},\dots,1-\{(n-1)\alpha\}.$$

Re-label these so that $0<1-\{l_1\alpha)<1-\{l_2\alpha\}<\cdots<1-\{l_n\alpha\}$, where $l_n=0$. We thus have the following picture.



So

$$\langle V_\lambda^n 1,1\rangle=\int_0^{1-\{l_1\alpha\}}\prod_{m=0}^{n-1} e^{2\pi i\lambda\{x+m\alpha\}}\,dx+\sum_{j=1}^{n-1}\int_{1-\{l_j\alpha\}}^{1-\{l_{j+1}\alpha\}}\prod_{m=0}^{n-1} e^{2\pi i\lambda\{x+m\alpha\}}\,dx.$$

The next step is the crucial one. We want to perform the integration and will therefore write the product as a function whose anti-derivative we can compute. On the interval $[0,1-\{l_1\alpha\})$,

$$\prod_{m=0}^{n-1} e^{2\pi i\lambda\{x+m\alpha\}}=\prod_{m=0}^{n-1} e^{2\pi i\lambda(x+\{m\alpha\})}=e^{2\pi i\lambda nx}\prod_{m=0}^{n-1} e^{2\pi i\lambda\{m\alpha\}}=e^{2\pi i\lambda nx}C,$$

where $C=\prod_{m=0}^{n-1} e^{2\pi i\lambda\{m\alpha\}}$ and hence $|C|=1$. On $[1-\{l_j\alpha\},1-\{l_{j+1}\alpha\})$,

$$\begin{aligned}\prod_{m=0}^{n-1} e^{2\pi i\lambda\{x+m\alpha\}}&=\prod_{m=1}^{n} e^{2\pi i\lambda\{x+l_m\alpha\}}\\&=\prod_{m=1}^{j} e^{2\pi i\lambda(x+\{l_m\alpha\}-1)}\prod_{m=j+1}^{n} e^{2\pi i\lambda(x+\{l_m\alpha\})}=e^{2\pi i\lambda nx}e^{-2\pi i\lambda j}C.\end{aligned}$$

Now perform the integration:

$$\begin{aligned}\langle V_\lambda^n 1,1\rangle&=\frac{C}{2\pi i\lambda n}\left[e^{2\pi i\lambda nx}\Big|_0^{1-\{l_1\alpha\}}+\sum_{j=1}^{n-1} e^{-2\pi i\lambda j}e^{2\pi i\lambda nx}\Big|_{1-\{l_j\alpha\}}^{1-\{l_{j+1}\alpha\}}\right]\\&=\frac{C}{2\pi i\lambda n}\Bigg[e^{2\pi i\lambda n(1-\{l_1\alpha\})}-1\\&\qquad+\sum_{j=1}^{n-1} e^{-2\pi i\lambda j}\left(e^{2\pi i\lambda n(1-\{l_{j+1}\alpha\})}-e^{2\pi i\lambda n(1-\{l_j\alpha\})}\right)\Bigg].\end{aligned}$$

Collect the $e^{2\pi i\lambda n(1-\{l_j\alpha\})}$ terms, remembering that $\{l_n\alpha\}=0$:

$$\begin{aligned}\langle V_\lambda^n 1,1\rangle&=\frac{1}{2\pi i\lambda n}\sum_{j=1}^{n} e^{2\pi i\lambda n(1-\{l_j\alpha\})}\left(e^{-2\pi i\lambda(j-1)}-e^{-2\pi i\lambda j}\right)\\&=\frac{Ce^{2\pi i\lambda n}(e^{2\pi i\lambda}-1)}{2\pi i\lambda n}\sum_{j=1}^{n} e^{-2\pi i\lambda n\{l_j\alpha\}}e^{-2\pi i\lambda j}.\end{aligned}\tag{2}$$

This equation is the key to the results of this section.

PROPOSITION 2. *Let* $\lambda\in\mathbf{R}$, $\lambda\neq 0$, *and define* $V_\lambda\colon L^2(\mathbf{T})\to L^2(\mathbf{T})$ *by*

$$(V_\lambda f)(x)=e^{2\pi i\lambda x}f(\{x+\alpha\}).$$

Then V_λ *does not have discrete spectrum.*

Proof. Using (2),

$$|\langle V_\lambda^n 1, 1\rangle| = \left| \frac{Ce^{2\pi i\lambda n}(e^{2\pi i\lambda}-1)}{2\pi i\lambda n} \sum_{j=1}^{n} e^{-2\pi i\lambda n\{l_j\alpha\}} e^{-2\pi i\lambda j} \right|$$

$$\le \frac{|2\sin \pi\lambda| n}{|2\pi\lambda n|} = \frac{|\sin \pi\lambda|}{|\pi\lambda|} = 1-\epsilon_\lambda \quad \text{for some } \epsilon_\lambda > 0.$$

Thus $\limsup_{n\to\infty} |\langle V_\lambda^n 1, 1\rangle| \le 1-\epsilon_\lambda < 1$. □

We now discuss the distribution of the set $\{\{k\alpha\}\}_{k=0}^{n-1}$ for certain special n.

Let $\{p_n/q_n\}$ be the sequence (or perhaps a subsequence) of convergents to α. There is a function $\xi_\alpha : \{q_n\} \to \mathbf{R}^+$ such that $\xi_\alpha \ge \sqrt{5}$ and

$$\left| \frac{p_n}{q_n} - \alpha \right| = \frac{1}{\xi_\alpha(q_n) q_n^2}. \tag{3}$$

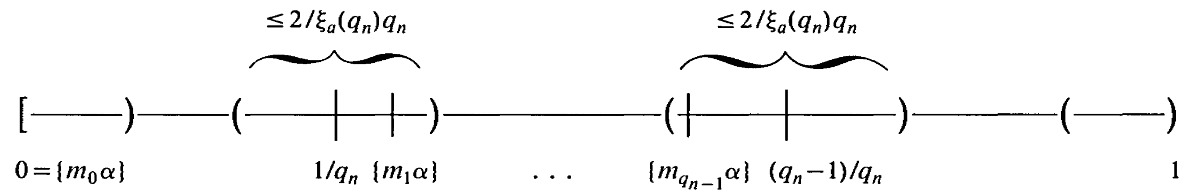
Multiplying (3) by $q \in \mathbf{N}$, $0 \le q < q_n$, we get

$$\left| \frac{qp_n}{q_n} - q\alpha \right| = \frac{q}{\xi_\alpha(q_n) q_n^2} < \frac{1}{\xi_\alpha(q_n) q_n}.$$

Since $\gcd(p_n, q_n) = 1$,

$$\left\{ \left\{ \frac{qp_n}{q_n} \right\} : 0 \le q < q_n \right\} = \left\{ 0, \frac{1}{q_n}, \frac{2}{q_n}, \ldots, \frac{q_n - 1}{q_n} \right\}.$$

Hence for each q in $\{0, 1, \ldots, q_n - 1\}$ there exists a unique integer m_q, $0 \le m_q < q_n - 1$, such that $|\{m_q\alpha\} - q/q_n| < 1/\xi_\alpha(q_n) q_n$. We thus have the following picture.



Proposition 3. *Let* $\lambda \in \mathbf{R}\backslash\mathbf{Z}$, $|\lambda| < \limsup_{n\to\infty}(\xi_\alpha(q_n)/4)$, *and define* V_λ: $L^2(\mathbf{T}) \to L^2(\mathbf{T})$ *by*

$$(V_\lambda f)(x) = e^{2\pi i\lambda x} f(\{x+\alpha\}).$$

Then V_λ *does not have Lebesgue spectrum.*

Proof. We will show that $\lim_{n\to\infty} |\langle V_\lambda^{q_n} 1, 1\rangle| \ne 0$. Rewrite (2) in the form

$$\langle V_\lambda^{q_n} 1, 1\rangle = \frac{e^{2\pi i\lambda}-1}{2\pi i\lambda} \frac{1}{q_n} \sum_{j=1}^{q_n} e^{-2\pi i\lambda(q_n\{l_j\alpha\}+j-q_n)}.$$

If $\lambda \notin \mathbf{Z}$, then $|(e^{2\pi i\lambda}-1)/2\pi i\lambda|$ is bounded away from zero. Thus we need only show that $(1/q_n)\sum_{j=1}^{q_n} e^{-2\pi i\lambda(q_n\{l_j\alpha\}+j-q_n)}$ is bounded away from zero for arbitrarily large q_n. To do this, it is enough to show that for any $j, j' \in \{1, 2, \ldots, q_n\}$,

$$2\pi|\lambda| \left| (q_n\{l_j\alpha\}+j-q_n) - (q_n\{l_{j'}\alpha\}+j'-q_n) \right| < \pi(1-\epsilon)$$

for some $\epsilon > 0$, because then we are averaging values on the unit circle that lie on an arc of length at most $\pi(1-\epsilon)$ and therefore cannot average to zero.

Since $0 < 1 - \{l_1\alpha\} < 1 - \{l_2\alpha\} < \cdots < 1 - \{l_{q_n}\alpha\} = 1$, and since we know the distribution of $\{\{l_j\alpha\}\}_{k=1}^{q_n}$, we have

$$\left| \{l_j\alpha\} - \frac{q_n - j}{q_n} \right| < \frac{1}{\xi_\alpha(q_n) q_n} \quad \text{for all } j \in \{1, 2, \ldots, q_n\},$$

and thus

$$|q_n\{l_j\alpha\} + j - q_n| < 1/\xi_\alpha(q_n).$$

Pick $\epsilon > 0$ so that $|\lambda| < (\limsup_{n\to\infty}(\xi_\alpha(q_n)/4))(1-\epsilon)$. There exists q_n arbitrarily large so that

$$|\lambda| < \frac{\xi_\alpha(q_n)}{4}\left(1 - \frac{\epsilon}{2}\right).$$

Thus, for any $j, j' \in \{1, 2, \ldots, q_n\}$,

$$2\pi|\lambda|\left|(q_n\{l_j\alpha\} + j - q_n) - (q_n\{l_{j'}\alpha\} + j' - q_n)\right| < 2\pi \frac{\xi_\alpha(q_n)}{4}\left(1 - \frac{\epsilon}{2}\right)\frac{2}{\xi_\alpha(q_n)} = \pi\left(1 - \frac{\epsilon}{2}\right). \quad \square$$

COROLLARY 4. *If* $\lambda \in \mathbf{R}\setminus\mathbf{Z}$ *and* $|\lambda| < \limsup_{n\to\infty}(\xi_\alpha(q_n)/4)$, *then* V_λ *has singular continuous spectrum.*

Proof. The proof follows from Propositions 2 and 3. □

REMARK 1. Proposition 3 is an improvement of a result of Choe [6]. In the worst case, $\xi_\alpha(q_n) = \sqrt{5}$ and the bound for $|\lambda|$ is still better than Choe's, which was $\limsup_{n\to\infty}(8(1 + 1/\xi_\alpha(q_n)))^{-1}$ for all α. (This bound was never stated as such, but is the bound given by the technique used there. A refinement of this technique will improve the bound by a factor of two, but this is still not as good as the bound given here even in the case when α has bounded partial quotients.) If α has unbounded partial quotients, then $\limsup_{n\to\infty}\xi_\alpha(q_n) = \infty$ and Proposition 3 holds for all $\lambda \in \mathbf{R}\setminus\mathbf{Z}$. Thus, in this case, the type of the spectral measure is completely known: discrete when $\lambda = 0$, Lebesgue when $\lambda \in \mathbf{Z}\setminus\{0\}$, and singular continuous otherwise.

We will now state Proposition 2 in the language of cocycles.

DEFINITION 2. A measurable function $\phi: \mathbf{T} \to \mathbf{S}^1$ is called an *α-multiplicative coboundary* if there is a measurable function $q: \mathbf{T} \to \mathbf{S}^1$ such that

$$\phi(x) = \frac{q(\{x+\alpha\})}{q(x)} \text{ a.e.}$$

Two measurable functions $\phi, \psi: \mathbf{T} \to \mathbf{S}^1$ are said to be *α-cohomologous* if ϕ/ψ is an α-multiplicative coboundary.

COROLLARY 5 (Baggett and Merrill [2]). *Let* $\lambda, \lambda', s, s' \in \mathbf{R}$. *Then* $e^{2\pi i\lambda x}e^{2\pi is}$ *is α-cohomologous to* $e^{2\pi i\lambda' x}e^{2\pi is'}$ *if and only if* $\lambda = \lambda'$ *and* $s = s' + k\alpha$, *where* $k \in \mathbf{Z}$.

Proof. The "if" part is true because $e^{2\pi ik\alpha}=e^{2\pi ik(x+\alpha)}/e^{2\pi ikx}$. If $e^{2\pi i(\lambda-\lambda')x}\times e^{2\pi i(s-s')}$ is an α-coboundary, Proposition 1 says that the operator V defined by

$$(Vf)(x)=e^{2\pi i(\lambda-\lambda')x}e^{2\pi i(s-s')}f(\{x+\alpha\})$$

has discrete spectrum. The spectrum of V is the same as the spectrum of κV for all $|\kappa|=1$. Proposition 2 implies that $\lambda=\lambda'$. Thus $e^{2\pi i(s-s')}$ is an α-coboundary. But then this says that $e^{2\pi i(s-s')}$ is an eigenvalue of the translation by an α operator, and this implies that $s-s'=k\alpha$ for some $k\in\mathbf{Z}$. □

REMARK 2. Corollary 5 also implies Proposition 2. The proof given in [2] relies on results on the cohomology of step functions. The techniques we have used here are attractive since they are direct and can be used to study more complicated operators. This is the topic of the next section.

3. The General Cases

We now study the operator $V_{\lambda,g}\colon L^2(\mathbf{T})\to L^2(\mathbf{T})$ defined by

$$(V_{\lambda,g}f)(x)=e^{2\pi i(\lambda x+g(x))}f(\{x+\alpha\}),$$

where $\lambda\in\mathbf{R}$, $\lambda\neq 0$, and $g\colon\mathbf{T}\to\mathbf{R}$ is absolutely continuous or smoother.

In [5], it is shown that $V_g\colon L^2(\mathbf{T})\to L^2(\mathbf{T})$ defined by

$$(V_gf)(x)=e^{2\pi ig(x)}f(\{x+\alpha\})$$

does not have Lebesgue spectrum if $g\in C^1(\mathbf{T},\mathbf{R})$. It is easy to prove that for some α, V_g does in fact have discrete spectrum. Since the operators studied in the previous section never have discrete spectrum, and in fact have Lebesgue spectrum when $\lambda\in\mathbf{Z}$, we should be able to make similar conclusions for all α for $V_{\lambda,g}$. This is indeed the case. The following proposition generalizes Proposition 2.

PROPOSITION 6. *If $\lambda\in\mathbf{R}$, $\lambda\neq 0$, and $g\colon\mathbf{T}\to\mathbf{R}$ is absolutely continuous, then $V_{\lambda,g}$ does not have discrete spectrum.*

Proof. We will show that for all large n, $|\langle V_{\lambda,g}^n1,1\rangle|$ is bounded away from 1. We have

$$\langle V_{\lambda,g}^n1,1\rangle=\int_0^1\prod_{m=0}^{n-1}e^{2\pi i\lambda(\{x+m\alpha\}+g(\{x+m\alpha\}))}\,dx.$$

Subdividing $[0,1)$ as in the computation that led to (2), we see that

$$\begin{aligned}\langle V_{\lambda,g}^n1,1\rangle&=\int_0^{1-\{l_1\alpha\}}\prod_{m=0}^{n-1}e^{2\pi i\lambda(\{x+m\alpha\}+g(\{x+m\alpha\}))}\,dx\\&\quad+\sum_{j=1}^{n-1}\int_{1-\{l_j\alpha\}}^{1-\{l_{j+1}\alpha\}}\prod_{m=0}^{n-1}e^{2\pi i\lambda(\{x+m\alpha\}+g(\{x+m\alpha\}))}\,dx\\&=\int_0^{1-\{l_1\alpha\}}Ce^{2\pi i\lambda nx}e^{2\pi i\sum_{m=0}^{n-1}g(\{x+m\alpha\})}\,dx\\&\quad+\sum_{j=1}^{n-1}\int_{1-\{l_j\alpha\}}^{1-\{l_{j+1}\alpha\}}Ce^{2\pi i\lambda nx}e^{-2\pi i\lambda j}e^{2\pi i\sum_{m=0}^{n-1}g(\{x+m\alpha\})}\,dx,\end{aligned}$$

where $|C|=1$. We integrate by parts in the same way on all integrals:

$$u=e^{2\pi i\sum_{m=0}^{n-1}g(\{x+m\alpha\})},\quad dv=e^{2\pi i\lambda nx}dx,$$

$$du=2\pi iu\sum_{m=0}^{n-1}g'(\{x+m\alpha\})\,dx,\quad v=\frac{1}{2\pi i\lambda n}e^{2\pi i\lambda nx};$$

$$\langle V_{\lambda,g}^{n}1,1\rangle=\left[Cuv\Big|_{0}^{1-\{l_1\alpha\}}+C\sum_{j=1}^{n-1}e^{-2\pi i\lambda j}uv\Big|_{1-\{l_j\alpha\}}^{1-\{l_{j+1}\alpha\}}\right]$$
$$-\left[\int_{0}^{1-\{l_1\alpha\}}Cv\,du+C\sum_{j=1}^{n-1}e^{-2\pi i\lambda j}\int_{1-\{l_j\alpha\}}^{1-\{l_{j+1}\alpha\}}v\,du\right]=S_1-S_2. \tag{4}$$

The first quantity $|S_1|$ can be computed as before, and yields an equality similar to (2):

$$|S_1|=\left|\frac{Ce^{2\pi i\lambda n}(e^{2\pi i\lambda}-1)}{2\pi i\lambda n}\sum_{j=1}^{n}e^{-2\pi i\lambda n\{l_j\alpha\}}e^{-2\pi i\lambda j}e^{2\pi i\sum_{m=0}^{n-1}g(\{-l_j\alpha+m\alpha\})}\right|. \tag{5}$$

The same calculation as in the proof of Proposition 2 yields

$$|S_1|\le\frac{|\sin\pi\lambda|}{|\pi\lambda|}=1-\epsilon_\lambda\quad\text{for some } \epsilon_\lambda>0.$$

It is easy to see that

$$|S_2|\le\int_0^1\frac{1}{|\lambda|}\left|\frac{1}{n}\sum_{m=0}^{n-1}g'(\{x+m\alpha\})\right|dx.$$

Since g is absolutely continuous, $g'\in L^1(\mathbf{T})$ and $\int_0^1 g'(x)\,dx=0$. The L^1 ergodic theorem says that

$$\left\|\frac{1}{n}\sum_{m=0}^{n-1}g'(\{x+m\alpha\})\right\|_1\to 0\quad\text{as } n\to\infty.$$

Thus, $|S_2|\le\epsilon_\lambda/2$ for all large n. Therefore $|\langle V_{\lambda,g}^{n}1,1\rangle|$ is bounded away from 1 for all large n. □

DEFINITION 3. A function $g:\mathbf{T}\to\mathbf{R}$ is said to be *absolutely continuous except at one point* if there exists an $x_0\in\mathbf{T}$ such that:

(i) $g'\in L^1(\mathbf{T}\setminus\{x_0\})$ and g is the indefinite integral of g' on this interval; and
(ii) $\lim_{x\to x_0^+}g(x)\neq\lim_{x\to x_0^-}g(x)$.

Note that if g is a function satisfying (i) in the definition, then both of the limits in (ii) will exist and be finite.

We are now ready to state the main theorem about the operators that do not have discrete spectrum. (This generalizes work in [3].)

THEOREM 1. *Let $g:\mathbf{T}\to\mathbf{R}$ be absolutely continuous except at one point, and define $V_g:L^2(\mathbf{T})\to L^2(\mathbf{T})$ by*

$$(V_gf)(x)=e^{2\pi ig(x)}f(\{x+\alpha\}).$$

Then V_g does not have discrete spectrum.

Proof. Without loss of generality, we may assume that the discontinuity of g is at 0 and that $g(0)=\lim_{x\to 0^+} g(x)=0$. Let $\lim_{x\to 1^-} g(x)=\lambda$ $(\neq 0)$. Then

$$g(x)=\lambda x+(g(x)-\lambda x) \quad \text{for all } x\in \mathbf{T},$$

and $g(x)-\lambda x$ is absolutely continuous. Proposition 6 gives the desired conclusion. □

As a corollary, we obtain a result about linear combinations of positive powers of x which again has no hypothesis on the irrational α (cf. [4]).

COROLLARY 7. *If $g(x)=\sum_{j=1}^{n} b_j x^{a_j}$ where $b_j\in \mathbf{R}$ and $a_j\geq 0$, and if $g(0)\neq \lim_{x\to 1^-} g(x)$, then V_g does not have discrete spectrum.*

REMARK 3. In [13], it is shown that for any $a\in \mathbf{R}$, $a<0$, x^a is not a *trivial α-additive cocycle* [10]. This result together with Corollary 7 gives: *For any irrational α and for any $a\in \mathbf{R}$, $a\neq 0$, x^a is not a trivial α-additive cocycle.*

REMARK 4. If α has unbounded partial quotients then the hypothesis of Theorem 1 cannot be weakened much more, because in [12] it is shown that in this case there exists a function $g:\mathbf{T}\to\mathbf{R}$, continuous except at one point where it has a jump of size 1, such that V_g has discrete spectrum.

The following lemma will help in finding cases where $V_{\lambda,g}$ does not have Lebesgue spectrum.

LEMMA 8 (Choe). *Let $\lambda\in\mathbf{R}$ and $g\in C^1(\mathbf{T},\mathbf{R})$. For any $\epsilon>0$, there exists $h\in C^1(\mathbf{T},\mathbf{R})$ such that $|h'(x)|<\epsilon$ for all $x\in\mathbf{T}$, and such that $V_{\lambda,g}$ and $V_{\lambda,f}$ have the same type of spectrum.*

Proof. See the proof of Proposition 4 in [5]. □

PROPOSITION 9. *If $\lambda\in\mathbf{R}\setminus\mathbf{Z}$, $|\lambda|<\limsup_{n\to\infty}(\xi_\alpha(q_n)/4)$, and $g\in C^1(\mathbf{T},\mathbf{R})$, then $V_{\lambda,g}$ does not have Lebesgue spectrum.*

Proof. As in the proof of Proposition 3, we will show $\lim_{n\to\infty}|\langle V_{\lambda,g}^{q_n}1,1\rangle|\neq 0$, where $\{q_n\}$ is the sequence of denominators to the convergents of α. Write $\langle V_{\lambda,g}^{q_n}1,1\rangle=S_1-S_2$ as in (4). We have already seen that for any $\epsilon>0$, $|S_2|<\epsilon$ for all large n. Thus we need only show that $|S_1|$ is bounded away from zero for arbitrarily large q_n. Rewrite (5) as follows:

$$|S_1|=\left|\frac{C(e^{2\pi i\lambda}-1)}{2\pi i\lambda}\frac{1}{q_n}\sum_{j=1}^{q_n} e^{-2\pi i\lambda(q_n\{l_j\alpha\}+j-q_n)+\sum_{m=0}^{q_n-1} g(\{-l_j\alpha+m\alpha\})}\right|.$$

Again, we will show that we are averaging values on the unit circle which lie on an arc of length strictly less than π.

Pick $\epsilon>0$ so that $|\lambda|<(\limsup_{n\to\infty}(\xi_\alpha(q_n)/4))(1-\epsilon)$. Because of Lemma 8, we may assume that $|g'(x)|<\epsilon/16\pi$ for all $x\in\mathbf{T}$.

Fix q_n. For $x\in\mathbf{T}$ and $0\leq i\leq q_n-1$, define

$$I_x^i=\left[\left\{x+\frac{i}{q_n}-\frac{1}{2q_n}\right\},\left\{x+\frac{i}{q_n}+\frac{1}{2q_n}\right\}\right).$$

Thus $I_x^i \cap I_x^{i'} = \emptyset$ for $i \neq i'$ and $\bigcup_{i=0}^{q_n-1} I_x^i = \mathbf{T}$. Each I_x^i contains one and only one $\{x+m\alpha\}$ because each interval $[i/q_n - 1/2q_n, i/q_n + 1/2q_n)$ contains one and only one $\{m\alpha\}$. Let $\{x+m_i\alpha\}$ be the one contained in I_x^i. We have

$$\int_0^1 g(t)\,dt = \sum_{i=0}^{q_n-1} \int_{I_x^i} g(t)\,dt = \sum_{i=0}^{q_n-1} g(x_i^*)\frac{1}{q_n}, \quad \text{where } x_i^* \in I_x^i.$$

Thus

$$\left|\frac{1}{q_n}\sum_{m=0}^{q_n-1} g(\{x+m\alpha\}) - \int_0^1 g(t)\,dt\right| = \left|\frac{1}{q_n}\sum_{i=0}^{q_n-1} g(\{x+m_i\alpha\}) - g(x_i^*)\right|$$
$$\leq \frac{1}{q_n}\left(\frac{\epsilon}{16\pi}\frac{1}{q_n}q_n\right) = \frac{\epsilon}{16\pi q_n}.$$

This inequality holds for all $x \in \mathbf{T}$ and all q_n. Therefore, for any $j, j' \in \{1, \ldots, q_n\}$ we have

$$2\pi\left|\sum_{m=0}^{q_n-1} g(\{-l_j\alpha + m\alpha\}) - \sum_{m=0}^{q_n-1} g(\{-l_{j'}\alpha + m\alpha\})\right| \leq 2\pi\frac{\epsilon}{8\pi} = \pi\frac{\epsilon}{4}.$$

In the proof of Proposition 3 we saw that we can find q_n arbitrarily large; hence for any $j, j' \in \{1, \ldots, q_n\}$ we have

$$2\pi|\lambda|\left|(q_n\{l_j\alpha\} + j - q_n) - (q_n\{l_{j'}\alpha\} + j' - q_n)\right| < \pi(1 - \epsilon/2). \qquad \square$$

DEFINITION 4. Let $\lambda \in \mathbf{R}\backslash\mathbf{Z}$. A function $g: \mathbf{T} \to \mathbf{R}$ is said to be *continuously differentiable except at one point with jump* λ if there exists an $x_0 \in \mathbf{T}$ such that:

(i) g is continuously differentiable on the interval $\mathbf{T}\backslash\{x_0\}$;
(ii) $\lim_{x\to x_0^+} g(x) - \lim_{x\to x_0^-} g(x) = \lambda$; and
(iii) $\lim_{x\to x_0^+} g'(x) = \lim_{x\to x_0^-} g'(x) \neq \pm\infty$.

THEOREM 2. *Let* $\lambda \in \mathbf{R}\backslash\mathbf{Z}$. *Let* $g: \mathbf{T} \to \mathbf{R}$ *be continuously differentiable except at one point with jump* λ, *and define* $V_g: L^2(\mathbf{T}) \to L^2(\mathbf{T})$ *by*

$$(V_g f)(x) = e^{2\pi i g(x)} f(\{x+\alpha\}).$$

If $|\lambda| < \limsup_{n\to\infty}(\xi_\alpha(q_n)/4)$, *then* V_g *does not have Lebesgue spectrum.*

Proof. The proof is the same as that of Theorem 1. Condition (iii) in Definition 4 guarantees that Proposition 9 can be applied to $g(x) - \lambda x$. $\square$

References

[1] L. Baggett, *On functions that are trivial cocycles for a set of irrationals,* Proc. Amer. Math. Soc. 104 (1988), 1212–1215.

[2] L. Baggett and K. Merrill, *Representations of the Mautner group and cocycles of an irrational rotation,* Michigan Math. J. 33 (1986), 221–229.

[3] ———, *Equivalence of cocycles under an irrational rotation,* Proc. Amer. Math. Soc. 104 (1988), 1050–1053.

[4] ———, *On the cohomological equivalence of a class of functions under an irrational rotation of bounded type,* Proc. Amer. Math. Soc. 111 (1991), 787–793.
[5] G. Choe, *Spectral types of skewed irrational rotations,* preprint.
[6] ———, *Products of operators with singular continuous spectra,* Proc. Sympos. Pure Math., 51, pp. 65–68, Amer. Math. Soc., Providence, RI, 1990.
[7] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers,* 5th ed., Clarendon Press, London, 1979.
[8] P. Hellekalek and G. Larcher, *On the ergodicity of a class of skew products,* Israel J. Math. 54 (1986), 301–306.
[9] H. Helson, *Cocycles on the circle,* J. Operator Theory 16 (1986), 189–199.
[10] ———, *The spectral theorem,* Lecture Notes in Math., 1227, Springer, New York, 1986.
[11] H. Helson and K. Merrill, *Cocycles on the circle II,* Oper. Theory: Adv. Appl., 28, pp. 121–124, Birkhäuser, Basel, 1988.
[12] A. Iwanik, M. Lemańczyk, and D. Rudolph, *Absolutely continuous cocycles over irrational rotations,* preprint.
[13] H. Medina, *Hilbert space operators arising from irrational rotations on the circle group,* Ph.D. thesis, Univ. of California, Berkeley, 1992.

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