Erdös–Turán Inequalities for Distance Functions on Spheres

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Dedicated to Professor B. Volkmann on the occasion of his 60th birthday

0. Introduction

When studying how much the distribution of an N-point subset ω_N of the unit interval $[0,1)$—or, equivalently, the unit circle in the complex plane—deviates from uniform distribution, we are led to a natural measure of deviation called the *discrepancy* $D(\omega_N)$ of ω_N. A classical result of Erdös and Turán relates this number to the maximum modulus $M(\omega_N)$ of the corresponding polynomial $p(z,\omega_N)$ on the unit circle, that is, the monic polynomial whose zeros are the points of ω_N.

Roughly speaking, the Erdös–Turán inequality states that a "small" value of $M(\omega_N)$ implies a certain degree of uniformity of the point set ω_N, expressed by a "small" value of $D(\omega_N)$. In the present paper we prove similar inequalities for a large class of distance functions defined for finite subsets ω_N of the unit sphere S^{d-1} in d-dimensional Euclidean space ($d \geq 2$). In essence, these inequalities relate the spherical cap discrepancy to certain "potentials" and also to certain "energy sums" generated by the set ω_N.

In Section 1 we prove two refinements of the classical Erdös–Turán inequality. First, by studying the L^1-norm of the function $\log|p(z,\omega_N)|$ instead of its maximum, an inequality is obtained which is best possible in some sense. Secondly, by considering the discrepancy *function* of the given set, we are in a position to account properly for irregularities of distribution that are "global" rather than "local".

In Section 2 we discuss the asymptotics of $\pi(\omega_N)$, the product of mutual distances between points of sets ω_N which are obtained by letting $\omega_N := \{z_1, \dots, z_N\}$, that is, the set of the first N terms of a fixed infinite sequence on the unit circle. We prove that, among all sequences, the van der Corput sequence essentially shows the best behaviour.

In Section 3 the classical Erdös–Turán inequality is generalized to an arbitrary dimension $d \geq 2$: We replace one-dimensional discrepancy by spherical

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cap discrepancy, and the function $\log|p(z,\omega_N)|$ by a rather general distance function U_α.

Finally, in Section 4 we study the relation between spherical cap discrepancy and the energy sums E_α which correspond to the distance functions U_α. From another point of view, E_α may be considered as a generalization of $\log \pi(\omega_N)$ as discussed in Section 2.

1. On the Classical Erdös–Turán Inequality

Let $\omega_N = \{z_1, \dots, z_N\}$, $|z_j| = 1$, be an N-point set on the unit circle. Denote by $A_N[\alpha,\beta)$ the number of points z_j satisfying the inequality $\alpha \le \arg z_j < \beta$, where $0 \le \alpha < \beta < 2\pi$. Let

$$D(\omega_N) = \sup_{0\le\alpha<\beta<2\pi} \left| A_N[\alpha,\beta) - \frac{\beta-\alpha}{2\pi} N \right|$$

denote the so-called discrepancy of the point set ω_N. The number $D(\omega_N)$ is a natural measure for the deviation of the distribution of the points of ω_N from uniform distribution.

With the set ω_N we associate the polynomial $p(z,\omega_N) := \prod_{j=1}^{N}(z-z_j)$. The following result relates the number $M(\omega_N) := \max_{|z|=1}|p(z,\omega_N)|$ to the discrepancy of ω_N.

THEOREM (Erdös and Turán [4]). *For some constant* $c>0$,

$$M(\omega_N) \ge \exp(c \cdot D^2(\omega_N)/N). \tag{1}$$

The original proof of (1) uses the theory of orthogonal polynomials and is rather intricate. A much simpler approach was discovered by Hlawka [6]. Furthermore, Ganelius [5] proved estimates for the logarithmic potential generated by a measure on the unit circle, which contain (1) as a special case.

In [4], Erdös and Turán conjectured that (1) might be best possible, but this has never been proved. We shall see, however, that a variant of (1) is indeed best possible in some sense.

It is more convenient to consider the function $\log|p(z,\omega_N)|$ instead of $p(z,\omega_N)$. Passing from the unit circle to the unit interval $[0,1)$, we introduce the notation $z=\exp(2\pi i t)$ and $z_j = \exp(2\pi i t_j)$, with $t, t_j \in [0,1)$. Thus we have

$$\log|p(z,\omega_N)| = \sum_{j=1}^{N} \log|2\sin\pi(t-t_j)| =: U(t,\omega_N).$$

Next we shall refine the concept of discrepancy by considering the so-called discrepancy function $\Delta(t,\omega_N)$ instead of the number $D(\omega_N)$:

$$\Delta(t,\omega_N) := \sum_{j=1}^{N} f(t-t_j),$$

where f is the 1-periodic sawtooth function

$$f(t)=\begin{cases} 1/2-t & \text{for } 0<t<1, \\ 0 & \text{for } t=0. \end{cases}$$

The function $\Delta(t,\omega_N)/N$ equals (up to a constant) the difference between the "empirical" distribution function associated with the discrete measure assigning the weight $1/N$ to each point t_j, and the "theoretical" uniform distribution on the unit interval. Some properties of the function $\Delta(t,\omega_N)$ are expressed by the following proposition. We write $\|f\|_1=\int_0^1|f(t)|\,dt$ and $\|f\|_\infty=\sup_{t\in[0,1]}|f(t)|$.

LEMMA 1.

(a) $\Delta(t,\omega_N)$ *is* 1*-periodic, piecewise linear with constant negative slope* $-N$*, and has jump discontinuities at the points* $t_1,\dots,t_N$ *with jump heights* $\Delta(t_j^+,\omega_N)-\Delta(t_j^-,\omega_N)\geq 1$ $(j=1,\dots,N)$. *Furthermore,* $\int_0^1\Delta(t,\omega_N)\,dt=0$.

(b) $\frac{1}{2}D(\omega_N)\leq\sup_{0\leq t<1}|\Delta(t,\omega_N)|\leq D(\omega_N)$.

(c) $\|\Delta(t,\omega_N)\|_1\geq 1/4$.

(d) *If we divide the interval* $[0,1)$ *into* $L=48N$ *subintervals* $I_1,\dots,I_L$ *of equal length, then there is a subset* $\mathfrak{A}$ *of* $\{1,\dots,L\}$ *such that*

$$2\inf_{t\in I_\mu}\Delta(t,\omega_N)\geq\sup_{t\in I_\mu}\Delta(t,\omega_N)>0 \quad \textit{for } \mu\in\mathfrak{A},$$

and

$$\sum_{\mu\in\mathfrak{A}}\int_{I_\mu}\Delta(t,\omega_N)\,dt\geq\frac{1}{16}\|\Delta(t,\omega_N)\|_1.$$

Proof. Assertions (a) and (b) are obvious from the definition of $\Delta(t,\omega_N)$. (The reader may wish to draw a picture.) Part (c) is an immediate consequence of Lemma 1 in [12].

For the proof of (d), we put

$$M_\mu=\sup_{t\in I_\mu}\Delta(t,\omega_N) \quad\text{and}\quad m_\mu=\inf_{t\in I_\mu}\Delta(t,\omega_N),$$

where $I_\mu=[(\mu-1)/L,\mu/L)$, $\mu=1,\dots,L$.

We partition the index set $\mathfrak{L}=\{1,\dots,L\}$ into three subsets as follows:

$$\begin{aligned}
\mathfrak{A}&:=\{\mu\in\mathfrak{L}\mid m_\mu\geq\tfrac{1}{2}M_\mu>0\},\\
\mathfrak{B}&:=\{\mu\in\mathfrak{L}\mid M_\mu>\tfrac{1}{16}\text{ and } m_\mu<\tfrac{1}{2}M_\mu\},\\
\mathfrak{C}&:=\mathfrak{L}\setminus(\mathfrak{A}\cup\mathfrak{B}).
\end{aligned}$$

Denoting the nonnegative part of $\Delta(t,\omega_N)$ by $\Delta^+(t,\omega_N)=\max(0,\Delta(t,\omega_N))$, we obtain, noting that $\int_0^1\Delta(t,\omega_N)\,dt=0$, the relation

$$\begin{aligned}
\|\Delta(t,\omega_N)\|_1&=2\int_0^1\Delta^+(t,\omega_N)\,dt\\
&=2\Big(\sum_{\mu\in\mathfrak{A}}\int_{I_\mu}\Delta^+(t,\omega_N)\,dt+\sum_{\mu\in\mathfrak{B}}\cdots+\sum_{\mu\in\mathfrak{C}}\cdots\Big).
\end{aligned}\tag{2}$$

By the definition of $\mathcal{C}$ we have the estimate

$$\sum_{\mu\in\mathcal{C}} \int_{I_\mu} \Delta^+(t,\omega_N)\,dt \le \frac{1}{16}. \tag{3}$$

Now we consider the case $\mu\in\mathcal{B}$. Writing $I_{L+1}=I_1$ for convenience, we obtain, using (a), the inequalities

$$M_{\mu+1}\ge M_\mu-\frac{N}{L}=M_\mu-\frac{1}{48}>\frac{1}{24}$$

and

$$m_{\mu+1}\ge M_{\mu+1}-\frac{N}{L}=M_{\mu+1}-\frac{1}{48}\ge\frac{1}{2}M_{\mu+1}.$$

This means that we have $\mu+1\in\mathcal{A}$. Moreover, $m_{\mu+1}\ge M_\mu-\frac{1}{24}>\frac{1}{3}M_\mu$. Thus, the estimate

$$\sum_{\mu\in\mathcal{B}} \int_{I_\mu} \Delta^+(t,\omega_N)\,dt \le 3\sum_{\mu\in\mathcal{A}} \int_{I_\mu} \Delta^+(t,\omega_N)\,dt \tag{4}$$

follows. Combining relations (2), (3), and (4) with (c) yields

$$\begin{aligned}\|\Delta(t,\omega_N)\|_1 &\le \frac{1}{8}+8\sum_{\mu\in\mathcal{A}} \int_{I_\mu} \Delta^+(t,\omega_N)\,dt\\ &\le \frac{1}{2}\|\Delta(t,\omega_N)\|_1+8\sum_{\mu\in\mathcal{A}} \int_{I_\mu} \Delta^+(t,\omega_N)\,dt,\end{aligned}$$

from which the assertion follows. □

The following modifications of (1) are true.

THEOREM 1.

$$\text{(a)}\quad \int_0^1 |U(t,\omega_N)|\,dt \gg \frac{1}{N}D^2(\omega_N). \tag{5}$$

$$\text{(b)}\quad \int_0^1 |U(t,\omega_N)|\,dt \gg \frac{1}{\log N}\int_0^1 |\Delta(t,\omega_N)|\,dt. \tag{6}$$

Before giving a proof of this theorem, we make the following remarks.

- Inequality (5) is slightly stronger than (1): Recall that $\int_0^1 U(t,\omega_N)\,dt=0$. Hence (5) implies that

$$\max_{t\in[0,1)} U(t,\omega_N)\ge\int_0^1 \max(0,U(t,\omega_N))\,dt=\frac{1}{2}\|U(t,\omega_N)\|_1\gg\frac{1}{N}D^2(\omega_N),$$

  which is (by taking logarithms) an equivalent version of (1).
- Inequality (5) is best possible, as is shown by the following example: Remove from the set $\{1,\zeta,\zeta^2,\dots,\zeta^{N-1}\}$, $\zeta=\exp(2\pi i/N)$, the points $1,\zeta,\dots,\zeta^{2[\sqrt{N}]}$; replace them by the single point $\zeta^{[\sqrt{N}]}$ with multiplicity $2[\sqrt{N}]+1$; and denote the resulting point set by ω_N. A straightforward calculation shows that $D(\omega_N)\gg\sqrt{N}$ but still $\int_0^1|U(t,\omega_N)|\,dt\ll 1$.

- Comparing (5) with (6), we see that the value of $\int_0^1 |U(t,\omega_N)|\,dt$ is rather stable with respect to "local" irregularities of distributions but very sensitive to "global" irregularities.
- For many point sets ω_N, relation (6) holds even if the factor $1/\log N$ is omitted, but we could not prove this in general.

Proof of (6). Using the same notation as in the proof of Lemma 1(d), we define the "test function" $T(t)$ on $[0,1)$: Let

$$\tau(t)=\begin{cases} 2Lt & \text{for } 0\le t\le 1/2L,\\ 2-2Lt & \text{for } 1/2L<t\le 1/L,\\ 0 & \text{otherwise;}\end{cases}$$

$$T(t)=\begin{cases} \tau(t-(\mu-1)/L) & \text{for } t\in I_\mu \text{ with } \mu\in\mathfrak{A},\\ 0 & \text{otherwise.}\end{cases}$$

From Lemma 1(d) and the defining property of $\mathfrak{A}$, it follows that

$$\begin{aligned}\int_0^1 \Delta(t,\omega_N)T(t)\,dt &= \sum_{\mu\in\mathfrak{A}}\int_{I_\mu}\Delta(t,\omega_N)T(t)\,dt \ge \sum_{\mu\in\mathfrak{A}} m_\mu\int_{I_\mu}T(t)\,dt\\ &\ge \frac{1}{2}\sum_{\mu\in\mathfrak{A}} M_\mu\frac{1}{2L} \ge \frac{1}{4}\sum_{\mu\in\mathfrak{A}}\int_{I_\mu}\Delta(t,\omega_N)\,dt\\ &\ge \frac{1}{64}\|\Delta(t,\omega_N)\|_1.\end{aligned}\tag{7}$$

Now we use the basic fact that

$$\Delta(t,\omega_N)\sim\frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{j=1}^{N}\frac{1}{n}\sin 2\pi n(t-t_j)$$

and

$$U(t,\omega_N)\sim-\frac{1}{\pi}\sum_{n=1}^{\infty}\sum_{j=1}^{N}\frac{1}{n}\cos 2\pi n(t-t_j)$$

are conjugate functions. If $T(t)\sim a_0+\sum_{n=1}^{\infty}(a_n\cos 2\pi nt+b_n\sin 2\pi nt)$ is the Fourier expansion of the test function, and if $\tilde{T}(t)\sim\sum_{n=1}^{\infty}(-b_n\cos 2\pi nt+a_n\sin 2\pi nt)$ denotes its conjugate, we have

$$\int_0^1\Delta(t,\omega_N)T(t)\,dt=\int_0^1 U(t,\omega_N)\tilde{T}(t)\,dt\le\|U(t,\omega_N)\|_1\cdot\|\tilde{T}\|_\infty.\tag{8}$$

In view of (7) and (8), it suffices to prove the inequality $\|\tilde{T}\|_\infty \ll \log N$. Direct calculation shows that

$$\tau(t)=\frac{1}{2L}+2L\left(F(t)-2F\left(t-\frac{1}{2L}\right)+F\left(t-\frac{1}{L}\right)\right),$$

where $F(t)=-(1/2\pi^2)\sum_{n=1}^{\infty}(1/n^2)\cos 2\pi nt$ is the 1-periodic continuation of the polynomial

$$-\frac{1}{12}+\frac{1}{2}t-\frac{1}{2}t^2,\quad 0\le t<1.$$

Hence the conjugate $\tilde{\tau}(t)$ has the representation

$$\tilde{\tau}(t)=2L\left(\Phi(t)-2\Phi\left(t-\frac{1}{2L}\right)+\Phi\left(t-\frac{1}{L}\right)\right),$$

where $\Phi(t)=\tilde{F}(t)=-(1/2\pi^2)\sum_{n=1}^{\infty}(1/n^2)\sin 2\pi nt$ is a primitive of the function $\log|2\sin\pi t|$.

For arbitrary $t\in[0,1)$ we have the estimate

$$\tag{9} \begin{aligned} |\tilde{\tau}(t)| &= \frac{4L}{\pi^2}\left|\sum_{n=1}^{\infty}\frac{1}{n^2}\sin^2\frac{\pi n}{2L}\cdot\sin 2\pi n\left(t-\frac{1}{2L}\right)\right| \\ &\le \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin^2(\pi n/2L)}{n^2}=\frac{4L}{\pi^2}\left(\sum_{n\le L}\cdots+\sum_{n>L}\cdots\right) \\ &< \frac{4L}{\pi^2}\left(\frac{\pi^2}{4L}+\frac{1}{L}\right)\ll 1. \end{aligned}$$

For $2/L\le t<1-1/L$, and $\theta_1,\theta_2\in(0,1)$ suitably chosen, Taylor's formula yields the relation

$$\tag{10} \begin{aligned} |\tilde{\tau}(t)| &= \frac{1}{2L}\left|\Phi''\left(t-\frac{1}{2L}-\frac{\theta_1}{2L}\right)+\Phi''\left(t-\frac{1}{2L}+\frac{\theta_2}{2L}\right)\right| \\ &\ll \frac{1}{L}\left(\left|\cot\pi\left(t-\frac{1}{L}\right)\right|+|\cot\pi t|\right)\ll(L\cdot\|t\|)^{-1}, \end{aligned}$$

where $\|t\|$ denotes the distance from t to the nearest integer.

Combining (9) with (10) and using the definition of T in terms of τ, for $N\ge 2$ we obtain the estimate

$$\begin{aligned} |\tilde{T}(t)| &= \left|\sum_{\mu\in\mathfrak{A}}\tilde{\tau}\left(t-\frac{\mu-1}{L}\right)\right|\ll\sum_{\mu=1}^{L}\min\left(1,\left(L\cdot\left\|t-\frac{\mu-1}{L}\right\|\right)^{-1}\right) \\ &\ll 1+\log L\ll\log N, \end{aligned}$$

which completes the proof. □

Proof of (5). This result follows already from the proofs in [5] or [6]. For completeness, we sketch a proof based on the same idea as above.

By Lemma 1, (a) and (b), there is an interval $I=[a,b)\subset[0,1)$ of length $\ll D(\omega_N)/N$ such that $\Delta(t,\omega_N)$ does not change sign on I and also satisfies $|\int_I\Delta(t,\omega_N)\,dt|\gg D^2(\omega_N)/N$. Defining the test function

$$T(t)=\begin{cases}\tau((t-a)/L(b-a)) & \text{for } t\in I,\\ 0 & \text{otherwise,}\end{cases}$$

we proceed as above, obtaining

$$\frac{D^2(\omega_N)}{N}\ll\left|\int_I\Delta(t,\omega_N)T(t)\,dt\right|\le\|U(t,\omega_N)\|_1\cdot\|\tilde{T}\|_\infty.$$

Inequality (9), adapted to the modified test function, establishes the result. □

In [3] Erdös asked the following question: Is there an infinite sequence of points $\omega = (z_1, z_2, \ldots)$ on the unit circle such that for $\omega_N = \{z_1, \ldots, z_N\}$ the corresponding sequence $M(\omega_N)$ is bounded? As is well known, there exist sequences ω satisfying $D(\omega_N) \ll \log N$ and $\|\Delta(t, \omega_N)\|_1 \ll \log N$. Hence neither (5) nor the stronger inequality (6) can be used to solve this problem. Using a direct approach, the author [13] answered the question of Erdös in the negative by proving that, for any sequence ω, the inequality

$$M(\omega_N) \geq c_1 (\log N)^\delta \tag{11}$$

holds for infinitely many N and suitable positive constants c_1 and δ. It is believed, however, that this is true even with N instead of $\log N$ in (11).

In the next section we shall completely solve the corresponding problem for the product of mutual distances between the points of ω_N.

2. On the Product of Mutual Distances on the Unit Circle

For a given set $\omega_N = \{z_1, \ldots, z_N\}$ of points on the unit circle, let

$$\pi(\omega_N) = \prod_{j \neq k} |z_j - z_k| \quad \text{and} \quad \gamma(\omega_N) = \log \pi(\omega_N).$$

It is not too difficult to establish the natural inequality

$$\gamma(\omega_N) \leq N \log N, \tag{12}$$

with equality holding if and only if ω_N is geometrically congruent to the set of Nth roots of unity.

We shall study the relation between the distribution of the points of ω_N and the value of $\gamma(\omega_N)$. In analogy to Theorem 1, the following inequality of the Erdös-Turán type holds:

$$\gamma(\omega_N) \leq N \log N + c_1 N - c_2 \frac{D^2(\omega_N)}{\log(c_3 N / D(\omega_N))}. \tag{13}$$

Here $N \geq 2$, and c_1, c_2, c_3 are positive numerical constants. The proof will be given in Section 4. Note that, in view of (12), nontrivial estimates can be obtained from (13) only if $D(\omega_N)$ is of larger order than $\sqrt{N \log N}$.

Although (13) shows some similarity to the classical Erdös-Turán inequality (1), the behaviour of $\gamma(\omega_N)$ with respect to irregularities of the set ω_N is quite different from that of $\|U(t, \omega_N)\|_1$. The following examples may serve to illustrate this.

- If two neighbouring points of ω_N are identified, the product $\pi(\omega_N)$ collapses to its minimal value zero. Hence $\gamma(\omega_N)$ is very sensitive to "local" irregularities.
- Let $N = 2n$ and cut the unit circle into two half-circles C_1 and C_2. Arrange $n - [\sqrt{n}]$ points equidistantly on C_1, and do the same with $n + [\sqrt{n}]$ points on C_2. Obviously we have $\|\Delta(t, \omega_N)\|_1 \gg \sqrt{N}$. Nevertheless we ob-

tain (by an elementary but lengthy calculation) the inequality $\gamma(\omega_N) \gg N \log N - cN$ with a constant $c > 0$. Thus $\gamma(\omega_N)$ is more stable with respect to "global" irregularities.

Consider also the following situation. Let $\omega = (z_1, z_2, \ldots)$ denote the classical van der Corput sequence; that is, $z_n = \exp(2\pi i s_n)$ for $(s_n) = (0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \ldots)$ (see e.g. [7]). If $N \geq 2$ has the dyadic expansion $N = \epsilon_0 2^0 + \epsilon_1 2^1 + \cdots + \epsilon_r 2^r$, $\epsilon_j \in \{0, 1\}$, $\epsilon_r = 1$, then the following relation is easily established by induction:

$$\gamma(\omega_N) = \sum_{j=0}^{r} \epsilon_j 2^j \log 2^j + 2 \sum_{j<k} \epsilon_j \epsilon_k 2^j \log 2. \tag{14}$$

Hence we have, with some constant $c > 0$, the relation

$$\begin{aligned}\gamma(\omega_N) - N \log N &\geq \sum_{j=0}^{r} \epsilon_j 2^j \log 2^j - N(\log 2^r + \log 2) \\ &= 2^r \sum_{j=0}^{r} \epsilon_j 2^{j-r}(j-r) \log 2 - N \log 2 \geq -cN.\end{aligned}$$

On the other hand, for $N = 2^r + 2^{r-1}$, there exists a constant $c' > 0$ such that

$$\gamma(\omega_N) - N \log N = 2^{r-1} \log(16/27) \leq -c'N.$$

Now let $\omega = (z_1, z_2, \ldots)$ denote an arbitrary sequence on the unit circle, and consider the sequence $(\gamma(\omega_N))$ associated with the sections $\omega_N = \{z_1, \ldots, z_N\}$. What can be said about the behaviour of $\gamma(\omega_N)$ as N tends to infinity? It turns out—see the following theorem—that, as is often the case, no sequence can behave essentially better than the van der Corput sequence (14).

THEOREM 2. *There exists a constant $c > 0$ such that, for an arbitrary sequence $\omega = (z_1, z_2, \ldots)$ of points on the unit circle, the inequality*

$$\gamma(\omega_N) \leq N \log N - cN$$

holds for infinitely many values of N.

We shall need the following elementary inequality.

LEMMA 2. *Let $v_1, \ldots, v_n$ denote unit vectors of a (real or complex) inner product space V with inner product $\langle \cdot, \cdot \rangle$ and norm $\|v\| = \langle v, v \rangle^{1/2}$. Then the Gram determinant $G(v_1, \ldots, v_n)$ satisfies the inequality*

$$G(v_1, \ldots, v_n) = \det(\langle v_i, v_j \rangle)_{i,j=1}^{n} \leq \prod_{j=1}^{n-1} (1 - \sigma_j^2),$$

where $\sigma_j := |\langle v_j, v_{j+1} \rangle|$ for $j = 1, \ldots, n-1$.

Proof of Lemma 2. We proceed by induction. The assertion is trivial for $n = 1$, so let us assume that $n \geq 2$. By a Gram-Schmidt process we introduce an orthonormal basis $\{w_1, \ldots, w_n\}$ such that $v_i = \sum_{k=1}^{n} a_{ik} w_k$ for $i = 1, \ldots, n$, with a lower triangular matrix $A = (a_{ik})$.

From the induction hypothesis we obtain

$$G(v_1,\dots,v_n)=|\det A|^2=|a_{nn}|^2 G(v_1,\dots,v_{n-1})\le |a_{nn}|^2\prod_{j=1}^{n-2}(1-\sigma_j^2).$$

An application of the Cauchy-Schwarz inequality yields

$$\sigma_{n-1}^2=\left|\sum_{k=1}^{n-1}a_{n-1,k}\overline{a_{n,k}}\right|^2\le 1\cdot\sum_{k=1}^{n-1}|a_{nk}|^2=1-|a_{nn}|^2,$$

which completes the proof. □

Proof of Theorem 2. We show that for fixed $N\ge 2$ there is at least one value of $n\in\{N+1,\dots,2N\}$ such that $\gamma(\omega_n)\le n\log n-cn$, with a positive constant c appropriately chosen.

Consider the points $z_1,\dots,z_N$, noting that each of them occurs in each of the products $\pi(\omega_{N+1}),\dots,\pi(\omega_{2N})$. Without loss of generality we may assume that $z_j=\exp(2\pi i t_j)$ with $0\le t_1<t_2<\cdots<t_N<1$.

With each point z_j we associate the vector $v_j=(1,z_j,z_j^2,\dots,z_j^{n-1})\in V=\mathbf{C}^n$, endowed with the usual inner product $\langle v,w\rangle=\sum_{k=1}^n v_k\overline{w_k}$. There are at least $r\ge(N-1)/2$ indices μ_j, $1\le\mu_1<\mu_2<\cdots<\mu_r\le N-1$, for which the inequality $t_{\mu_j+1}-t_{\mu_j}\le 2/(N-1)$ holds.

We let $\sigma_j=|\langle v_{\mu_j},v_{\mu_j+1}\rangle|$ for $j=1,\dots,r$ and apply Lemma 2. Noting that $\|v_j\|=\sqrt{n}$ for all j and omitting some factors less than or equal to 1, we obtain

$$\pi(\omega_n)=\prod_{j\ne k}|z_j-z_k|=|\det(v_1,\dots,v_n)|^2=G(v_1,\dots,v_n)\le n^n\prod_{j=1}^r\left(1-\frac{\sigma_j^2}{n^2}\right).$$

Taking logarithms and using the inequality $\log(1-x)\le -x$, we get

$$\gamma(\omega_n)\le n\log n-\frac{1}{n^2}\sum_{j=1}^r\sigma_j^2. \tag{15}$$

An easy calculation shows, now explicitly denoting the dependence on n,

$$(\sigma_j^2)_n=K_n(t_{\mu_j+1}-t_{\mu_j}),$$

where $K_n(t)=\sin^2\pi nt/\sin^2\pi t$ is the Fejer kernel of degree n.

Calculating the average,

$$\bar K_N(t)=\frac{1}{N}\sum_{n=N+1}^{2N}K_n(t)=\frac{N\sin\pi t-\sin\pi Nt\cdot\cos(3N+1)\pi t}{2N\sin^3\pi t},$$

shows that $\bar K_N(t)\ge c_1N^2$ for $|t|\le 2/(N-1)$, where $c_1>0$ is an absolute constant. Thus we have, by the definition of μ_j,

$$\frac{1}{N}\sum_{n=N+1}^{2N}\sum_{j=1}^r(\sigma_j^2)_n=\sum_{j=1}^r\bar K_N(t_{\mu_j+1}-t_{\mu_j})\ge c_1N^2r.$$

Hence, for at least one value of $n\in\{N+1,\dots,2N\}$, the estimate

$$\sum_{j=1}^r(\sigma_j^2)_n\ge c_1N^2r\gg n^3$$

holds. In view of (15), the assertion follows. □

3. Distance Functions on a Sphere

Let $\omega_N=\{x_1,\dots,x_N\}$ be an N-point set on the unit sphere $S=S^{d-1}=\{u\in E^d \mid |u|=1\}$ in Euclidean d-space E^d, $d\geq 2$. ($|\cdot|$ denotes the Euclidean distance in E^d.)

For a variable point $x\in S$ and a real parameter $\alpha>1-d$, $\alpha\neq 0,2,4,\dots$, consider the distance function

$$U_\alpha(x,\omega_N)=\sum_{j=1}^{N}|x-x_j|^\alpha-N\cdot m(\alpha,d).$$

Here $m(\alpha,d)$ is the mean value

$$m(\alpha,d)=\frac{1}{\sigma(S)}\int_S|x-x_0|^\alpha\,d\sigma(x),$$

where σ denotes the $(d-1)$-dimensional surface measure on S.

For $\alpha\in\{0,2,4,\dots\}$ the definition of U_α must be slightly modified as follows:

$$U_{2k}(x,\omega_N)=\sum_{j=1}^{N}|x-x_j|^{2k}\log|x-x_j|-N\cdot m(2k,d)\quad\text{for } k=0,1,2,\dots .$$

For the L^1-norms

$$\|U_\alpha(x,\omega_N)\|_1=\frac{1}{\sigma(S)}\int_S|U_\alpha(x,\omega_N)|\,d\sigma(x)$$

the author [15] established the "natural" (and best possible) lower bounds

$$\|U_\alpha(\omega_N)\|_1\geq c(\alpha,d)\cdot N^{-\alpha/(d-1)}. \tag{16}$$

Studying the relation between $\|U_\alpha(x,\omega_N)\|_1$ and the uniformity of distribution of the point set ω_N, we prove an inequality of the Erdös–Turán type which contains Theorem 1(a) as a special case. In the general case the rule still holds: a "small" value of $\|U_\alpha(x,\omega_N)\|_1$ implies a "uniform" distribution of the points $x_1,\dots,x_n$ over S^{d-1}.

First, we introduce the concept of so-called cap discrepancy for finite subsets of S^{d-1}. Let κ be a spherical cap on S^{d-1} (i.e., the intersection of S^{d-1} with some half-space) of area measure $\sigma(\kappa)$. Denote by $A_\kappa(\omega_N)$ the number of points x_j, $j=1,\dots,N$, lying in κ. Define the discrepancy $D(\omega_N)$ by

$$D(\omega_N)=\sup_{\kappa\subseteq S}\left|A_\kappa(\omega_N)-N\frac{\sigma(\kappa)}{\sigma(S)}\right|.$$

(This definition generalizes the one given in Section 1 for the special case $d=2$.) Then we have the following result.

THEOREM 3.

$$\|U_\alpha(x,\omega_N)\|_1\geq c(\alpha,d)\frac{D(\omega_N)^{d+\alpha}}{N^{d-1+\alpha}}. \tag{17}$$

Here $c(\alpha,d)$ is a positive constant which does not depend on N or ω_N.

Let us make a few remarks before proving this theorem.

Actually, we shall prove an inequality which is for $d \geq 3$ somewhat sharper than (17) in the sense that values of $D(\omega_N)$ which arise from "local" irregularities will be given stronger influence than values of $D(\omega_N)$ due to "global" irregularities of the set ω_N. This phenomenon should not be seen as a contradiction to our remark following Theorem 1. It is mainly due to the fact that the above definition of $D(\omega_N)$ attributes equal importance to each cap κ, regardless of its size. As it turns out, however, the size of the boundary $\partial\kappa$ —which for $d \geq 3$ depends on the size of κ itself—plays an important role.

Note that, in view of (16), inequality (17) yields nontrivial results only if $D(\omega_N)$ is of larger order than N^{β}, $\beta := (d-2)/(d-1) + 1/(d+\alpha)(d-1)$. We should mention the following facts:

- By a result of Beck [1] there exist point sets ω_N on S^{d-1} such that
$$D(\omega_N) \ll N^{1/2 \cdot (d-2)/(d-1)} \sqrt{\log N},$$
hence the critical exponent β is at least twice as large as the minimal exponent.
- The constructions in [16], however, show that there exist point sets ω_N such that $D(\omega_N) \gg N^{(d-2)/(d-1)}$ for which the opposite inequality
$$\|U_\alpha(x, \omega_N)\|_1 \leq \tilde{c}(\alpha, d) N^{-\alpha/(d-1)}$$
is still true, even with L^1-norm replaced by maximum norm (for $\alpha > 0$) or by one-sided maximum norm (for $\alpha \leq 0$).
- In the special case $\alpha = 2-d$ ("Newtonian case") Sjögren [9] proved very general results for sufficiently smooth closed surfaces which contain (16) and (17) (see also the remarks following the proof of Theorem 3).
- Some results involving the concept of cap discrepancy may also be found in [2, Chap. 7].

Proof of Theorem 3. We introduce spherical coordinates $(\theta_1, \theta_2, \ldots, \theta_{d-2}, \varphi)$ on S^{d-1} in the usual way. If $f(x) = f(\theta_1)$ is a function on S^{d-1} whose value at a point x depends on the distance between x and the north pole $\theta_1 = 0$ only, we denote by $f(x \mid y)$ its translation with the point y as its new "pole of reference". For any such function f and an arbitrary function g (or measure μ) on S^{d-1}, we define the convolutions $f * g$ and $f * d\mu$ by

$$(f*g)(x) = \int_S f(y \mid x) g(y)\, d\sigma(y) \quad \text{and} \quad (f*d\mu)(x) = \int_S f(y \mid x)\, d\mu(y).$$

Let $\omega_N = \{x_1, \ldots, x_N\}$ be the given set. Denote by $\kappa(\gamma, y)$ the spherical cap which is the intersection of S^{d-1} with a ball of radius $2\sin(\gamma/2)$, $0 < \gamma < \pi$, centered at $y \in S^{d-1}$. By the definition of $D(\omega_N)$, there exist an angle γ_0 and a point y_0 such that $D'_\kappa(\omega_N) := A_\kappa(\omega_N) - N(\sigma(\kappa)/\sigma(S))$ is $\geq \frac{1}{2} D(\omega_N)$ or $\leq -\frac{1}{2} D(\omega_N)$, where $\kappa := \kappa(\gamma_0, y_0)$. Without loss of generality, we may assume that

$$c_1 \left(\frac{D(\omega_N)}{N} \right)^{1/(d-1)} \leq \gamma_0 \leq \frac{\pi}{2}. \tag{18}$$

(The constants c_i introduced here and in the following may depend on d and α, but not on N or ω_N.)

If $D'_\kappa(\omega_N)>0$, then we replace κ by a larger cap $\kappa'(\gamma, y_0)$, where

$$\gamma := \gamma_0 + \Delta\gamma \quad \text{and} \quad \Delta\gamma = c_2 \frac{D(\omega_N)}{N\gamma_0^{d-2}}. \tag{19}$$

Choosing $c_2>0$ small enough we have, by (18) and (19),

$$N\cdot\sigma(\kappa'\backslash\kappa) < \tfrac{1}{4}D(\omega_N)\sigma(S). \tag{20'}$$

Similarly, if $D'_\kappa(\omega_N)<0$, we replace κ by a smaller cap $\kappa''(\gamma, y_0)$ with $\gamma := \gamma_0 - \Delta\gamma$ and $\Delta\gamma$ as above. Again we have

$$N\cdot\sigma(\kappa\backslash\kappa'') < \tfrac{1}{4}D(\omega_N)\sigma(S). \tag{20''}$$

We continue the proof using κ' or κ'', respectively, instead of κ, but writing again κ for simplicity.

For $l=1,2,\ldots$ we define the "test functions" $\tau_l(\theta_1)$ by

$$\tau_l(\theta_1) = \begin{cases} (\Delta\gamma)^{2l}\cos^{2l}(\pi/\Delta\gamma)\theta_1 & \text{for } 0 \le \theta_1 \le (\Delta\gamma)/2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Delta^l\tau_l \ll 1$ (where Δ denotes the spherical Laplace operator) and that $\Delta^\nu\tau_l$ has vanishing normal derivatives along the boundary $\{\theta_1 = (\Delta\gamma)/2\}$ for $\nu = 0, 1, \ldots, l-1$. Furthermore, for $l=1,2,\ldots$ we introduce the kernel $h_l(\theta_1)$ by the property

$$\Delta^l h_l(x) \equiv -1 \quad \text{for } x \in S^{d-1}\backslash\{\theta_1 = 0\}, \tag{21}$$

and the expansion as a series of spherical harmonics,

$$h_l(\theta_1) \sim c(\lambda, l) \sum_{n=1}^{\infty} \frac{n+\lambda}{(n(n+2\lambda))^l} P_n^{(\lambda)}(\cos\theta_1), \quad \lambda := \frac{d}{2} - 1. \tag{22}$$

Clearly, the kernel $h_l(\theta_1)$ is uniquely determined by (21) and (22). Its asymptotic behaviour near the value $\theta_1 = 0$ is like

$$\begin{cases} \text{const.}(\sin(\theta_1/2))^{2l-d+1} & \text{if } 2l-d+1 \ne 0, 2, 4, \ldots, \\ \text{const.}(\sin(\theta_1/2))^{2l-d+1}|\log\sin(\theta_1/2)| & \text{if } 2l-d+1 = 0, 2, 4, \ldots. \end{cases}$$

Let $H_l(x) = \sum_{j=1}^N h_l(x\,|\,x_j)$, and define the signed measure

$$\mu = \sum_{j=1}^{N} \delta_{x_j} - \frac{N}{\sigma(S)}\sigma,$$

where δ_{x_j} is the discrete measure assigning weight 1 to the point x_j. By (20′) and (20″), respectively, we obtain

$$\left|\int_\kappa (\tau_l * d\mu)(x)\, d\sigma(x)\right| \gg D(\omega_N)\int_S \tau_l(x)\, d\sigma(x). \tag{23}$$

On the other hand, using Green's formula and the asymptotic properties of h_l, we have

(24) $$(\tau_l * d\mu)(x) = (\Delta^l \tau_l * H_l)(x) \quad \text{for all } x \in S^{d-1}.$$

Now we use the fact that for $2l-1-d \le \alpha < 2l+1-d$ the kernel

$$k_\alpha(\theta_1) := \left(2 \sin \frac{\theta_1}{2}\right)^\alpha - m(\alpha, d)$$

has an inverse with respect to h_l (see [15]). More explicitly, there exists a kernel k_α^{-1} satisfying

(25) $$k_\alpha^{-1} * k_\alpha = h_l$$

and admitting a representation

(26) $$k_\alpha^{-1} = d_1 k_{2l+2-2d-\alpha} + \cdots + d_s k_{2l+s+1-2d-\alpha} + R_s,$$

with $\Delta^l R_s$ bounded and continuous if s is chosen sufficiently large.

From (25) we have $H_l = k_\alpha^{-1} * U_\alpha$, and thus from (23) and (24),

(27) $$D(\omega_N) \int_S \tau_l(x)\, d\sigma(x) \ll \left| \int_\kappa (\Delta^l \tau_l * k_\alpha^{-1}) * U_\alpha \, d\sigma \right| \ll \|U_\alpha\|_1 \cdot \|\Delta^l \tau_l * k_\alpha^{-1}\|_1.$$

An estimate for the norm $\|\Delta^l \tau_l * k_\alpha^{-1}\|_1$ will complete the proof. A straightforward computation, using (26) and Green's formula, yields

$$|(\Delta^l \tau_l * k_\alpha^{-1})(\theta_1)| \ll (\Delta\gamma)^{2l+d-1} \min\{(\Delta\gamma)^{2-2d-\alpha}, (2\sin(\theta_1/2))^{2-2d-\alpha}\};$$

hence

(28) $$\|\Delta^l \tau_l * k_\alpha^{-1}\|_1 \ll (\Delta\gamma)^{2l-\alpha}.$$

Now (27) yields

$$D(\omega_N) \int_S \tau_l(x)\, d\sigma(x) \ll \|U_\alpha\|_1 \cdot (\Delta\gamma)^{2l-\alpha}.$$

Noting that $\int_S \tau_l \, d\sigma \gg (\Delta\gamma)^{2l+d-1}$ by definition, and that $\Delta\gamma$ has been chosen subject to (19), we obtain

(29) $$\|U_\alpha(x, \omega_N)\|_1 \gg D(\omega_N) \left(\frac{D(\omega_N)}{N \gamma_0^{d-2}} \right)^{d-1+\alpha},$$

which is stronger than the assertion since $\gamma_0 \le \pi/2$. □

REMARKS. (1) The sharpening contained in (29) may be illustrated by the situation in which M of the N points x_j coincide. We may choose $\gamma_0 = c_1(M/N)^{1/(d-1)}$, thus obtaining from (29) the estimate

$$\|U_\alpha(x, \omega_N)\|_1 \gg M \left(\frac{M}{N} \right)^{1+\alpha/(d-1)}$$

instead of

$$\|U_\alpha(x, \omega_N)\|_1 \gg M \left(\frac{M}{N} \right)^{d-1+\alpha}.$$

(2) In the proof of Theorem 3 we may take any subset $B \subseteq S$ instead of considering a spherical cap κ, provided that the boundary ∂B is such as to

allow the transition from B to B' or B'' (as in (20′) or (20″), respectively) with a sufficiently small value of $\Delta\gamma$. For example, if we choose $B=\bigcup_{j=1}^{N}\kappa(\gamma,x_j)$, where $\gamma \ll N^{-1/(d-1)}$, we proceed in exactly the same way as above, obtaining again the natural bound (16). (See also [5], where this method yields estimates for logarithmic potentials on the unit circle.) In the same way we may prove the analogue of [9, Thm. 1] for the sphere with an arbitrary $\alpha>1-d$.

(3) Sjögren [9] starts from a relation similar to (23). As the solution of a Dirichlet problem with boundary values on the given closed surface, a "test function" is obtained which corresponds to our function $\Delta^l\tau_l * k_\alpha^{-1}$. Then Sjögren uses deeper results on the asymptotic behaviour of this solution near the boundary which are rather easy to prove in the case of a sphere. An extension of his results to exponents satisfying $1-d<\alpha<3-d$ (the "Riesz case") seems possible.

4. Mutual Distances on a Sphere

Recalling the notation introduced in Section 3, we consider the functionals

$$E_\alpha(\omega_N)=\sum_{j=1}^{N}\sum_{k=1}^{N}(|x_j-x_k|^\alpha-m(\alpha,d)) \quad \text{for } 0<\alpha<2,$$

$$E_\alpha(\omega_N)=\sum_{j\neq k}\sum(|x_j-x_k|^\alpha-m(\alpha,d)) \quad \text{for } 1-d<\alpha<0,$$

$$E_0(\omega_N)=\sum_{j\neq k}\sum(\log|x_j-x_k|-m(0,d)) \quad \text{for } \alpha=0.$$

Natural lower and upper bounds for these quantities were derived in [15]. Again we shall establish inequalities of the Erdös–Turán type. In the unbounded case $\alpha\le 0$, however, we restrict ourselves to the "Newtonian case" $\alpha=2-d$. In principle, our method works for any $\alpha\le 0$, but certain technical problems seem difficult to surmount.

THEOREM 4. *The following inequalities are valid:*

(a) $$E_\alpha(\omega_N)\le -c(\alpha,d)D^2(\omega_N)\left(\frac{D(\omega_N)}{N}\right)^{d+\alpha-2} \qquad \textit{for } 0<\alpha<2,$$

(b) $$E_{2-d}(\omega_N)\ge -c_1(d)N^{1+(d-2)/(d-1)} + c_2(d)\frac{D^2(\omega_N)}{\log(c_3(d)N/D(\omega_N))} \qquad \textit{for } \alpha=2-d<0,$$

(c) $$E_0(\omega_N)\le N\log N+c_1N-c_2\frac{D^2(\omega_N)}{\log(c_3N/D(\omega_N))} \qquad \textit{for } d=2.$$

(*All the constants are positive and do not depend on N or ω_N.*)

REMARKS. (1) Again the reader should compare these results with the corresponding natural bounds

$$E_\alpha(\omega_N) \le -c'(\alpha, d) N^{1-\alpha/(d-1)} \quad \text{for } 0<\alpha<2,$$
$$E_{2-d}(\omega_N) \ge -c'(d) N^{1+(d-2)/(d-1)} \quad \text{for } d \ge 3,$$
$$E_0(\omega_N) \le N \log N \quad \text{for } d=2.$$

(2) At this point we should mention a direct relation between $E_1(\omega_N)$ and the distribution of the points of ω_N: We introduce the discrepancy function

$$\Delta_\gamma(x, \omega_N) = \sum_{j=1}^{N} v_\gamma(x \mid x_j) - N\sigma(\kappa) \quad (x \in S,\ 0<\gamma<\pi),$$

where $v_\gamma(\cdot \mid x_j)$ denotes the indicator function of the spherical cap $\kappa(\gamma, x_j)$. For $d \ge 3$, Stolarsky [10] proved the remarkable identity

(30) $$-E_1(\omega_N) = c(d) \int_{\gamma=0}^{\pi} \int_S \Delta_\gamma^2(x, \omega_N)\, d\sigma(x) \sin^{d-2}\gamma\, d\gamma,$$

which in a somewhat different form is also valid on the unit circle. One might attempt to use this equation in order to prove Theorem 4(a) in the case $\alpha=1$, but the connection between (30) and (a) does not seem to be straightforward.

Proof of (a). For $0<\alpha<2$, we have the identity

(31) $$-E_\alpha(\omega_N) = \int_S \left(\sum_{j=1}^{N} \delta_\alpha(x \mid x_j) \right)^2 d\sigma(x),$$

where $\delta_\alpha(\theta_1)$ is a distance kernel which for $\theta_1 \to 0$ behaves asymptotically like the kernel $k_\beta(\theta_1)$ with $\beta = \beta(\alpha) = \frac{1}{2}(1+\alpha-d)$ introduced in Section 3 (see [15]).

We use the expansion as a series of spherical harmonics,

(32) $$\delta_\alpha(\theta_1) \sim \sum_{n=1}^{\infty} b_n(\alpha) P_n^{(\lambda)}(\cos\theta_1), \quad \lambda = \frac{d}{2}-1,$$

observing that the coefficients $b_n(\alpha)$ satisfy $\lim_{n\to\infty}(b_n(\alpha)/n^{(3-\alpha-d)/2}) > 0$, and the expansion

(33) $$v_\gamma(\theta_1) - \sigma(\kappa) \sim \sum_{n=1}^{\infty} a_n P_n^{(\lambda)}(\cos\theta_1).$$

An explicit calculation yields (integrating by parts in the case $d \ge 4$)

$$\int_S v_\gamma(\theta_1) P_n^{(\lambda)}(\cos\theta_1)\, d\sigma \ll \gamma^\lambda n^{\lambda-2},$$

by use of classical results on ultraspherical polynomials (see e.g. [11, §4.7, §7.33]). Noting that $\int_S (P_n^{(\lambda)}(\cos\theta_1))^2\, d\sigma \gg n^{2\lambda-2}$, we obtain

(34) $$|a_n| \ll \left(\frac{\gamma}{n}\right)^\lambda.$$

In the case $d=2$, $\lambda=0$, the $P_n^{(\lambda)}$'s being the Chebyshev polynomials of the first kind, (34) may be established as well; in the case $d=3$, $\lambda=\frac{1}{2}$, where the

$P_n^{(\lambda)}$'s are the Legendre polynomials, an explicit calculation yields $|a_n| \ll \gamma^{\lambda-1} n^{-\lambda}$ instead.

Finally, we introduce the Poisson kernel p_r, $0 \le r < 1$,

$$p_r(\cos\theta_1) = \lambda(1-r^2)(1+r^2-2r\cos\theta_1)^{-\lambda-1} = \sum_{n=0}^{\infty} (n+\lambda) r^n P_n^{(\lambda)}(\cos\theta_1). \tag{35}$$

Let $\omega_N = \{x_1, \ldots, x_N\}$ be the given point set with discrepancy $D(\omega_N)$. By an argument quite similar to the one given in the proof of Theorem 3, there are angles

$$\gamma \ge \epsilon_1 \left(\frac{D(\omega_N)}{N}\right)^{1/(d-1)} \quad \text{and} \quad \Delta\gamma = \epsilon_2 \frac{D(\omega_N)}{N\gamma^{d-2}}$$

(where ϵ_1, ϵ_2 are suitable positive constants) and a point $\xi \in S^{d-1}$ such that the inequality $|\Delta_\gamma(x, \omega_N)| \ge \frac{1}{2} D(\omega_N)$ holds for all x in the cap $\kappa(\Delta\gamma, \xi)$. Then we have, using decay properties of the Poisson kernel,

$$\left| \int_S \Delta_\gamma(x, \omega_N) p_r(x \mid \xi)\, d\sigma(x) \right| \gg D(\omega_N), \tag{36}$$

provided r is chosen so that $1 - r = \epsilon_3 \Delta\gamma$ with $\epsilon_3 > 0$ sufficiently small.

Using expansions (32), (33), and (35), and proceeding quite technically, we obtain the relation

$$\begin{aligned} &\left(\int_S \Delta_\gamma(x, \omega_N) p_r(x \mid \xi)\, d\sigma(x) \right)^2 \\ &= \left(\int_S \sum_{j=1}^{N} \delta_\alpha(x \mid x_j) \sum_{n=1}^{\infty} \frac{(n+\lambda) a_n}{b_n(\alpha)} r^n P_n^{(\lambda)}(x \mid \xi)\, d\sigma(x) \right)^2 \\ &\ll \int_S \left(\sum_{j=1}^{N} \delta_\alpha(x \mid x_j) \right)^2 d\sigma(x) \cdot \sum_{n=1}^{\infty} n^{d-2} r^{2n} \frac{a_n^2}{b_n^2(\alpha)}. \end{aligned} \tag{37}$$

The last sum can be estimated as

$$\begin{aligned} \sum_{n=1}^{\infty} n^{d-2} r^{2n} \frac{a_n^2}{b_n^2(\alpha)} &\ll \sum_{n=1}^{\infty} n^{d-2} r^{2n} \left(\frac{\gamma}{n}\right)^{2\lambda} n^{\alpha+d-3} \\ &= \gamma^{d-2} \sum_{n=1}^{\infty} \frac{r^{2n}}{n^{3-\alpha-d}} \ll \gamma^{d-2} \left(\frac{D(\omega_N)}{N\gamma^{d-2}}\right)^{2-d-\alpha}, \end{aligned}$$

by the definitions of $\Delta\gamma$ and r, in the case $d \ne 3$, and with the last expression replaced by $\gamma^\alpha (D(\omega_N)/N)^{-1-\alpha}$ if $d = 3$.

Hence from (31), (36), and (37) we obtain the estimates

$$-E_\alpha(\omega_N) \gg D^2(\omega_N) \gamma^{2-d} \left(\frac{D(\omega_N)}{N\gamma^{d-2}}\right)^{d-2+\alpha} \quad \text{for } d \ne 3,$$

$$-E_\alpha(\omega_N) \gg D^2(\omega_N) \gamma^{-\alpha} \left(\frac{D(\omega_N)}{N}\right)^{1+\alpha} \quad \text{for } d = 3,$$

which in either case is a sharpened version of (a).

The formal steps in the derivation of (37) may be justified by first replacing Δ_γ and δ_α by their harmonic continuations

$$\sum a_n \rho^n P_n^{(\lambda)}(x \mid x_j) \quad \text{and} \quad \sum b_n(\alpha) \rho^n P_n^{(\lambda)}(x \mid x_j),$$

respectively, and then letting $\rho \to 1$, using well-known properties of the solution of the Dirichlet problem with boundary values on S^{d-1}. □

Proof of (b). We make use of an approximate representation of $E_\alpha(\omega_N)$ which is derived in [15]. For $1-d<\alpha<0$ and arbitrary $\rho \in (0,1)$, one has

$$E_\alpha(\omega_N) \geq -N g_\rho(0) - N^2(m_1 - m_\rho) + \sum_{j=1}^{N} \sum_{k=1}^{N} (g_\rho(x_j \mid x_k) - m_\rho), \tag{38}$$

where $g_\rho(\theta_1) := (\rho + \rho^{-1} - 2\cos\theta_1)^{\alpha/2}$ and $m_\rho := (\sigma(S))^{-1} \int_S g_\rho(x)\, d\sigma(x)$.

It is essential for our proof to estimate the asymptotic behaviour of the coefficients in the spherical harmonics expansion of $g_\rho(\theta_1)$, and it is for this reason only that we restrict ourselves to the case $\alpha = 2-d$. In this case we have

$$g_\rho(\theta_1) = \rho^\lambda \sum_{n=0}^{\infty} \rho^n P_n^{(\lambda)}(\cos\theta_1).$$

As all the coefficients $\rho^{n+\lambda}$, $n \geq 1$, are positive, the last sum in (38) may be replaced by

$$\int_S \left(\sum_{j=1}^{N} G_\rho(x \mid x_j) \right)^2 d\sigma(x),$$

where

$$G_\rho(\theta_1) = \text{const.} \cdot \sum_{n=1}^{\infty} \sqrt{n+\lambda}\, \rho^{(n+\lambda)/2} P_n^{(\lambda)}(\cos\theta_1).$$

Proceeding as in part (a), we may derive the inequality

$$\begin{aligned} D^2(\omega_N) &\ll \left(\sum_{j,k} (g_\rho(x_j \mid x_k) - m_\rho) \right) \cdot \gamma^{d-2} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r^2}{\rho} \right)^n \\ &= (\cdots) \gamma^{d-2} \left| \log\left(1 - \frac{r^2}{\rho}\right) \right|. \end{aligned} \tag{39}$$

We choose $\rho = 1 - N^{-1/(d-1)}$ and $r = 1 - \epsilon_2(D(\omega_N)/N\gamma^{d-2})$, thereby tacitly assuming that $\epsilon_2(D(\omega_N)/N\gamma^{d-2}) > N^{-1/(d-1)}$. (Otherwise inequality (b) is trivial in view of the natural bounds for $E_{2-d}(\omega_N)$.)

Relations (38) and (39) together yield

$$E_{2-d}(\omega_N) \geq -c_1(d) N^{1+(d-2)/(d-1)} + c_2(d) \frac{D^2(\omega_N)}{\gamma^{d-2} \left| \log\left(c_3(d) \dfrac{D(\omega_N)}{N\gamma^{d-2}} \right) \right|},$$

which proves the assertion (b).

Case (c) $d=2$ may be handled in a completely analogous way. □

FINAL REMARKS. (1) The method used in the proof of (a) also applies to the situation of Theorem 3 and yields the same result. However, the proof involves certain rather cumbersome estimations.

(2) On the other hand, we may apply Theorem 3 (which also holds for the modified kernels δ_α) directly to the representation of $-E_\alpha(\omega_N)$, thereby obtaining an inequality that is slightly weaker than assertion (a) of Theorem 4.

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