Removable Sets for Harmonic Functions

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0. Introduction

Suppose that K is a compact subset of $\mathbf{R}^d$, $d \geq 2$. A theorem of Carleson [Ca, Thm. VII.2] states that K is removable for harmonic functions satisfying a Lip_α condition, $0<\alpha<1$, if and only if $m_{d-2+\alpha}(K)=0$; here m_β denotes β-dimensional Hausdorff measure.

Carleson's result fails for $\alpha=1$: While it is easy to see from Green's theorem that K is removable for Lip_1 harmonic functions if $m_{d-1}(K)=0$, Uy [Uy] has recently given an example of a compact subset of $\mathbf{R}^d$ that is removable for Lip_1 harmonic functions in spite of having positive $(d-1)$-dimensional measure. (As noted in [Uy], for $d=2$ this follows from the existence of a set of positive length that is removable for bounded holomorphic functions. Such an example was given by Vitushkin [Vt] and simplified by Garnett [Gt]; the example in [Uy] is a generalization to $\mathbf{R}^d$ of the example in [Gt].)

We shall show that K is removable for harmonic functions in the Zygmund class if and only if $m_{d-1}(K)=0$. (Definitions and a more precise statement follow.) The argument below may also be used to give a proof of Carleson's theorem for $0<\alpha<1$ which is perhaps somewhat simpler than the argument in [Ca].

Suppose Ω is an open subset of $\mathbf{R}^d$ and $u:\Omega\to\mathbf{R}$. We say that u is a *Zygmund function* on Ω ($u\in\Lambda_1(\Omega)$) if u is continuous on Ω and there exists $c<\infty$ such that

$$|u(x-y)-2u(x)+u(x+y)|\leq c|y| \tag{0}$$

whenever $x, x\pm y\in\Omega$.

Note that the hypothesis of continuity cannot be omitted here; (0) alone does not imply that u is measurable, even for $\Omega=\mathbf{R}^d$ [Kr]. However, it is easy to see that u must be continuous if it is upper semicontinuous and satisfies (0), so that in particular a subharmonic function satisfying (0) is a Zygmund function. (We should perhaps point out that the standard definition of Λ_1 requires that u be (globally) bounded [Kr]; we shall find it more convenient to omit this condition.)

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THEOREM 1. *Suppose K is a compact subset of* $\mathbf{R}^d$, $d \geq 2$.

(i) *If* $m_{d-1}(K)=0$, Ω *is an open set with* $K \subset \Omega$, $u \in \Lambda_1(\Omega)$, *and u is harmonic in* $\Omega \backslash K$, *then u is harmonic in* Ω.

(ii) *If* $m_{d-1}(K)>0$ *then there exists* $u \in \Lambda_1(\mathbf{R}^d)$ *such that u is harmonic in* $\mathbf{R}^d \backslash K$ *and* $\lim_{x \to \infty} u(x)=0$, *although u does not vanish identically (so that u is not harmonic in* $\mathbf{R}^d$).

The following apparently stronger result follows from Theorem 1.

THEOREM 2. *Suppose Ω is an open subset of* $\mathbf{R}^d$, *K is a compact subset of Ω, and* $m_{d-1}(K)=0$. *If* $u \in \Lambda_1(\Omega \backslash K)$ *and u is harmonic in* $\Omega \backslash K$, *then there exists a harmonic function* $u_1 \in \Lambda_1(\Omega)$ *with* $u_1|_{\Omega \backslash K}=u$.

To show that Theorem 1 implies Theorem 2 is not quite as trivial as is the analogous assertion for Lip_α, because it is not obvious that $u \in \Lambda_1$ implies any particular bound on the modulus of continuity of u. As we shall see in Section 2, there is a standard estimate on the modulus of continuity which follows from (boundedness and) membership in Λ_1, but it depends on convexity of the domain, and $\Omega \backslash K$ is certainly not convex (nor are we requiring that Zygmund functions be bounded). However, we shall see in Section 2 that if $m_{d-1}(K)=0$ then $\Omega \backslash K$ is close enough to convex to allow us to prove Proposition 2, from which Theorem 2 follows immediately.

We wish to thank the referee for answering a question that arose in connection with Proposition 1.

1. Proof of Theorem 1

The two cases $d=2$ and $d \geq 3$ are identical except for one detail which is even more trivial for $d \geq 3$ than for $d=2$; thus we will restrict attention to the case $d=2$. We will let $D(z,r)$ denote the open disc in the plane with center z and radius r. The word "measure" will mean "Borel measure".

LEMMA 1. *Suppose* $\mu \geq 0$ *is a measure in the plane such that* $\mu(D(z,r)) \leq r$ *for all* $z \in \mathbf{C}$, $r>0$. *Then*

$$\int_{\mathbf{C}} \left| \log \left| \frac{z^2-h^2}{z^2} \right| \right| d\mu(z) \leq c|h| \quad (h \in \mathbf{C})$$

for some absolute constant c.

Proof. Observe first that it is sufficient to consider the case $h=1$; a dilation and rotation will then give the general case. Let $\chi(z)=|\log|1-z^{-2}||$.

If $|z| \leq 1/2$ then $\chi(z) \leq c+2\log(1/|z|)$. If we set

$$A_n=\{z: 2^{-(n+1)} \leq |z|<2^{-n}\},$$

it follows that

$$\int_{D(0,1/2)} \chi\, d\mu = \sum_{n=1}^{\infty} \int_{A_n} \chi\, d\mu \le c + c \sum_{n=1}^{\infty} n2^{-n} \le c.$$

The integral over $D(\pm 1, 1/2)$ may be estimated in a similar manner, so that $\int_{D(0,2)} \chi\, d\mu \le c$.

Now set $B_n = \{z : 2^n \le |z| < 2^{n+1}\}$. If $|z| \ge 2$ then $\chi(z) \le c|z|^{-2}$, so that

$$\int_{|z|\ge 2} \chi\, d\mu = \sum_{n=1}^{\infty} \int_{B_n} \chi\, d\mu \le c \sum_{n=1}^{\infty} 2^{-2n} 2^n \le c.$$

Thus $\int_{\mathbf{C}} \chi\, d\mu \le c$. □

The proof of part (ii) of Theorem 1 will begin with an application of Frostman's lemma: If $m_1(K) > 0$ then there exists a probability measure μ supported on K such that $\mu(D(z,r)) \le cr$ for every $z \in \mathbf{C}$, $r > 0$ [Ca, Thm. II.1]. The following proposition will be used in the proof of part (i) of the theorem; the proof of the proposition proceeds by showing that *if $m_1(K) = 0$ then there exists a probability measure μ supported on K such that*

$$\limsup_{r\to 0} r^{-1}\mu(D(z,r)) = \infty \quad \textit{for all } z \in K.$$

It is easy to construct an example showing that "lim sup" cannot be replaced by "lim" here. In fact "lim inf" cannot be replaced by "lim" in the statement of Proposition 1, although this is not quite so easy to see; a construction of the required example was provided by the referee.

PROPOSITION 1. *Suppose that K is a compact subset of the plane with $m_1(K) = 0$. There exists a function v, subharmonic in $\mathbf{C}$ and harmonic in $\mathbf{C}\setminus K$, such that*

$$\liminf_{r\to 0} r^{-1}\left\{ v(z) - (2\pi)^{-1}\int_0^{2\pi} v(z + re^{it})\, dt \right\} = -\infty \tag{1}$$

for every $z \in K$.

Proof. For $n = 1, 2, \ldots$ we may choose $z_j^n \in K$ and $r_j^n > 0$ $(1 \le j \le N_n)$ such that $K \subset \bigcup_{j=1}^{N_n} D(z_j^n, r_j^n)$ and

$$\sum_{j=1}^{N_n} r_j^n < 4^{-n}. \tag{2}$$

Let $v = \sum_{n=1}^{\infty} v_n$, where

$$v_n(z) = 2^n \sum_{j=1}^{N_n} r_j^n \log|z - z_j^n|. \tag{3}$$

It is clear that v is subharmonic in $\mathbf{C}$ and harmonic in $\mathbf{C}\setminus K$, because v is simply the (logarithmic) potential of a finite measure supported on K. (Note that $\sum_{n=1}^{\infty} 2^n \sum_{j=1}^{N_n} r_j^n < \infty$, by (2).)

Now fix $z \in K$, and given $n = 1, 2, \ldots$ select a value of j such that $|z - z_j^n| < r_j^n < 4^{-n}$. Since each term in the definition of v is subharmonic it follows that

$$(2r_j^n)^{-1}\left\{v(z)-(2\pi)^{-1}\int_0^{2\pi} v(z+2r_j^n e^{it})\,dt\right\}$$
$$\leq 2^{n-1}\left\{\log|z-z_j^n|-(2\pi)^{-1}\int_0^{2\pi}\log|z+2r_j^n e^{it}-z_j^n|\,dt\right\}$$
$$=2^{n-1}\left\{\log|z-z_j^n|-(2\pi)^{-1}\int_0^{2\pi}\log|2r_j^n+e^{-it}(z-z_j^n)|\,dt\right\}$$
$$=2^{n-1}\{\log|z-z_j^n|-\log(2r_j^n)\}\leq -2^{n-1}\log(2). \qquad \square$$

Proof of Theorem 1. We begin with part (ii): Suppose $m_1(K)>0$. Choose two disjoint compact sets K_1 and K_2 with $K_j\subset K$ and $m_1(K_j)>0$ $(j=1,2)$. As mentioned above, we may choose a probability measure μ_j supported on K_j such that $\mu_j(D(z,r))\leq cr$ for all $z\in\mathbf{C}$ and $r>0$. Define

$$u_j(z)=\int\log|z-w|\,d\mu_j(w).$$

It follows that u_j is subharmonic in $\mathbf{C}$ and harmonic in $\mathbf{C}\backslash K$, and Lemma 1 shows that u_j satisfies inequality (0) above. Hence $u_j\in\Lambda_1(\mathbf{C})$. Furthermore, $u_j(z)=\log|z|+o(1)$ as $z\to\infty$, so that if we define $u=u_1-u_2$ then $\lim_{z\to\infty}u(z)=0$. (Here the argument is slightly simpler for $d\geq 3$; in that case $\lim_{x\to\infty}u_1(x)=0$.) Finally, the function u is certainly not identically zero; for example, $\Delta u=\mu_1-\mu_2$ in the sense of distributions.

We turn now to part (i). Suppose K is compact, $m_1(K)=0$, Ω is an open neighborhood of K, $u\in\Lambda_1(\Omega)$, and u is harmonic in $\Omega\backslash K$. The fact that $u\in\Lambda_1(\Omega)$ has the following consequence:

$$\text{(4)}\qquad \left|u(z)-(2\pi)^{-1}\int_0^{2\pi}u(z+re^{it})\,dt\right|\leq cr \quad (z\in\Omega,\ 0<r<\operatorname{dist}(z,\partial\Omega)).$$

Indeed, a bit of rearrangement shows that

$$u(z)-(2\pi)^{-1}\int_0^{2\pi}u(z+re^{it})\,dt$$
$$=(4\pi)^{-1}\int_0^{2\pi}\{2u(z)-u(z+re^{it})-u(z-re^{it})\}\,dt,$$

so that (4) follows from (0).

Now choose a bounded open set Ω_1 with $\bar\Omega_1\subset\Omega$, $K\subset\Omega_1$, and such that Ω_1 is regular for the Dirichlet problem. Given $f\in C(\partial\Omega_1)$, let $P[f]$ denote the harmonic function in Ω_1 that tends to f at the boundary of Ω_1.

Define a function $\tilde u$ in Ω_1 by $\tilde u=u-P[u|_{\partial\Omega_1}]$; we need only show that $\tilde u=0$ in Ω_1. Let v be as in Proposition 1, and set $\tilde v=v-P[v|_{\partial\Omega_1}]$ in Ω_1. For $\delta>0$ define $u_\delta=\tilde u+\delta\tilde v$.

Note that

$$\tilde u(z)-(2\pi)^{-1}\int_0^{2\pi}\tilde u(z+re^{it})\,dt=u(z)-(2\pi)^{-1}\int_0^{2\pi}u(z+re^{it})\,dt$$

for $z \in \Omega_1$ and $0 < r < \operatorname{dist}(z, \partial\Omega_1)$; similarly for $\tilde{v}$ and v. Thus it follows from (1) and (4) that

$$\liminf_{r\to 0} r^{-1}\left\{u_\delta(z) - \int_0^{2\pi} u_\delta(z+re^{it})\,dt\right\} = -\infty$$

for every $z \in K$. In particular, if $z \in K$ then there exist arbitrarily small $r > 0$ such that

$$u_\delta(z) \le (2\pi)^{-1}\int_0^{2\pi} u_\delta(z+re^{it})\,dt. \tag{5}$$

On the other hand, if $z \in \Omega_1 \backslash K$ then (5) holds for all sufficiently small r because u_δ is harmonic in a neighborhood of z.

This shows that u_δ is subharmonic, so that $u_\delta \le 0$ in Ω_1, by the maximum principle. Let δ approach zero; it follows that $\tilde{u} \le 0$ on $\Omega_1 \backslash K$, because $\tilde{v} > -\infty$ there. But $\tilde{u}$ is continuous, so that $\tilde{u} \le 0$ in all of Ω_1. The same argument shows that $-\tilde{u} \le 0$. □

2. Proof of Theorem 2

Theorem 2 follows directly from Theorem 1 and the following proposition.

PROPOSITION 2. *Suppose Ω is an open subset of $\mathbf{R}^d$, K is a compact subset of Ω with $m_{d-1}(K) = 0$, and $u \in \Lambda_1(\Omega\backslash K)$. Then there exists a (unique) $u_1 \in \Lambda_1(\Omega)$ with $u_1|_{\Omega\backslash K} = u$.*

Note that Proposition 2 depends on the fact that $m_{d-1}(K) = 0$: Two examples are given in [Kr] of bounded open sets O and functions $v \in L^\infty \cap \Lambda_1(O)$ with $v \notin \mathrm{Lip}_{1/2}(O)$. Suppose that we are given such an example, with $|x| \le 1$ $(x \in O)$. Define $K = \{x \in \mathbf{R}^d : |x| \le 2,\ x \notin O\}$ and $\Omega = \mathbf{R}^d$, and set $u(x) = 0$ $(|x| > 2)$ and $u(x) = v(x)$ $(x \in O)$. Then $u \in \Lambda_1(\Omega\backslash K)$ but $L^\infty \cap \Lambda_1(\mathbf{R}^d) \subset \mathrm{Lip}_{1/2}(\mathbf{R}^d)$ (see (6) below), so that u cannot be extended to $u_1 \in \Lambda_1(\mathbf{R}^d)$. Of course, the K given here is quite large; one might ask precisely which sets K are removable for Λ_1 in the sense of Proposition 2.

In order to prove Proposition 2 it is sufficient to show that u extends to a function $u_1 \in C(\Omega)$; the fact that u_1 satisfies (0) then follows by continuity. Thus we need only show that u is uniformly continuous near K. For this we will use the following standard estimate [Kr, Lemma 2.8]: *Suppose that $u \in L^\infty \cap \Lambda_1(\Omega)$ and Ω is convex. Then*

$$|u(x+y) - u(x)| \le c\psi(|y|) = c|y|\left(1 + \log\left(\frac{1}{|y|}\right)\right) \tag{6}$$

whenever $x, x+y \in \Omega$ and $|y| \le 1$.

The following lemma shows that if $m_{d-1}(K) = 0$ then $\Omega\backslash K$ contains enough straight lines to allow one to derive Proposition 2 from (6). Given $x, y \in \mathbf{R}^d$, let $[x, y] \subset \mathbf{R}^d$ denote the closed line segment with endpoints x and y.

LEMMA 2. *Suppose* Ω *is a bounded open subset of* $\mathbf{R}^d$ *and* K *is a compact subset of* Ω *with* $m_{d-1}(K)=0$. *Let* $\delta=\operatorname{dist}(K,\partial\Omega)$.

(i) *If* $x,y\in\Omega\backslash K$ *with* $\operatorname{dist}(x,K)<\delta/2$ *and* $|x-y|<\delta/2$, *then there exists* z *such that* $[x,z]\subset\Omega\backslash K$, $[z,y]\subset\Omega\backslash K$, *and*

$$|x-z|+|z-y|\leq 2|x-y|.$$

(ii) *There exists a compact set* $E\subset\Omega\backslash K$ *such that if* $x\in\Omega$ *and* $\operatorname{dist}(x,K)\leq\delta/2$ *then there exist* $y,z\in E$ *with* $x=(y+z)/2$.

Proof. Suppose x and y are as in (i). Let D denote the $(d-1)$-dimensional "disc" with center $(x+y)/2$ and radius $|x-y|/2$, lying in the hyperplane passing through the point $(x+y)/2$ and orthogonal to the line segment $[x,y]$. The fact that $m_{d-1}(K)=0$ shows that $([x,z]\cup[z,y])\cap K$ is empty for m_{d-1}-almost all $z\in D$.

For (ii), let $K_1=\{x\in\Omega:\operatorname{dist}(x,K)\leq\delta/2\}$. For each $x\in K_1$ there exists $w(x)$ such that $|w(x)|\leq\delta/4$ and $x\pm w(x)\notin K$, again because $m_{d-1}(K)=0$. Now the fact that K is closed allows us to find $r(x)>0$ such that

$$(\bar{B}(x+w(x),r(x))\cup\bar{B}(x-w(x),r(x)))\subset\Omega\backslash K\quad(z\in K_1).$$

(Here $B(p,r)$ is the open ball with center p and radius r.) The compactness of K_1 shows that there exist finitely many $x_1,\dots,x_N\in K_1$ with

$$K_1\subset\bigcup_{j=1}^{N}B(x_j,r(x_j)).$$

Let

$$E=\bigcup_{j=1}^{N}(\bar{B}(x_j+w(x_j),r(x_j))\cup\bar{B}(x_j-w(x_j),r(x_j))).\qquad\Box$$

Proof of Proposition 2. We may suppose that Ω is bounded. Let

$$\delta=\operatorname{dist}(K,\partial\Omega)\quad\text{and}\quad\Omega_1=\{x\in\Omega\backslash K:\operatorname{dist}(x,K)<\delta/2\};$$

as noted before, we are done if we can show that u is uniformly continuous on Ω_1. We shall show that (6) holds for $x,x+y\in\Omega_1$, $|y|\leq 1$.

First, let E be as in Lemma 2(ii). The fact that E is compact shows that u is bounded on E (by definition, $\Lambda_1(\Omega\backslash K)\subset C(\Omega\backslash K)$). Hence u is bounded in Ω_1, by (0). Now suppose $x,y\in\Omega_1$ and $|x-y|\leq 1/2$. Choose z as in Lemma 2(i). We may apply (6) to obtain an upper bound for $|u(x)-u(z)|+|u(z)-u(y)|$, showing that in fact $|u(x)-u(y)|\leq c\psi(|x-y|)$, as required. $\Box$

References

[Ca] L. Carleson, *Selected problems on exceptional sets,* Van Nostrand, Princeton, N.J., 1967. (Reprint: Wadsworth, Belmont, Cal., 1983.)

[Gt] J. Garnett, *Positive length but zero analytic capacity,* Proc. Amer. Math. Soc. 24 (1970), 696–699.

[Kr] S. Krantz, *Lipschitz spaces, smoothness of functions, and approximation theory,* Exposition. Math. 1 (1983), 193–260.

[Uy] N. X. Uy, *A removable set for Lipschitz harmonic functions,* Michigan Math. J. 37 (1990), 45–51.

[Vt] A. G. Vitushkin, *Example of a set of positive length but zero analytic capacity,* Dokl. Akad. Nauk SSSR 127 (1959), 246–249 (Russian).

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