The Null Blow-Up of a Surface in Minkowski 3-Space and Intersection in the Spacelike Grassman

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Introduction

A smoothly immersed compact orientable surface Q in 3-dimensional Minkowski space can be decomposed into a disjoint union $D^+ \cup D^0 \cup D^-$, where the induced metric is positive definite on D^+, degenerate on D^0, and indefinite on D^-. Along D^0, the line orthogonal to Q is also tangent to Q. Imposing natural transversality conditions on this configuration stratifies the surface so $\mathrm{SP} \subset D^0 \subset Q$, where the set SP of stall points is contained in the set D^0 of stall curves that is embedded in Q. The stratification is defined as the loci, where the orthogonal line bundle is tangent to the next lower stratum. In [K2] we constructed a Gauss map for Q into the 2-sphere, $g: Q \to S^2$, with degree $\pm\frac{1}{2}\chi(Q)$. Here we construct a Gauss map for D^0 into the compactified spacelike Grassman, $cg: D^0 \to S^1 \times S^1$. In this context points of SP correspond to intersection points of cg with the diagonal in $S^1 \times S^1$. In this paper we establish a formula relating: the degree of g, $\chi(D^+)$, and the interesection number of cg with the diagonal (Theorem 4). As a consequence we have two integral inequalities that can be used to characterize simple configurations (Theorem 6). We then construct the null blow-up NB of Q. This is a compact folded double cover of $D^- \cup D^0$, $\rho: \mathrm{NB} \to D^- \cup D^0$, with an oriented line field L whose zero points are exactly $\mathrm{SP} \subset D^0$. This null blow-up is completely determined by the first fundamental form on Q, and can be thought of as a completion space for null geodesics in Q (i.e., a blow-up space for the singularities in the null geodesic ODE). We then show that the sum of the indices at these zero points is the intersection number of cg with the diagonal in $S^1 \times S^1$ (Corollary 8). Since this line field ρ-projects to null subspaces in D^-, this corollary links purely extrinsic properties (a Gauss map for D^0) with purely intrinsic properties (the global dynamics of null pre-geodesics in $D^- \cup D^0$). Furthermore, since these zero points can be viewed as a degenerate type of "conjugate point", we have a new link between conjugacy in the null geodesic ODE and global properties of the underlying manifold.

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We will begin by reviewing the relevant transversality conditions on the immersions. Sections 2 and 3 contain discussions of D^0 and the associated $(S^1 \times S^1)$-valued Gauss map. In Section 4 we prove the degree formula. Section 5 contains several applications and examples; in particular, the null blow-up construction will be found in Section 5(D). In Section 5(E) we discuss the manner in which the shape of the surface controls the existence of simply connected L-invariant sets and Rheeb components on the null blow-up. In Section 5(F), we use these observations to find global restrictions on the stratifications $\mathrm{SP} \subset D^0 \subset Q$, which can be realized in Minkowski space.

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1. Preliminaries

Throughout this paper $\mathbf{M}^3$ will denote Minkowski space, the real 3-dimensional vector space equipped with bilinear form $\langle\, , \rangle$ of type $(2, 1)$ (i.e., the normal form has two plus-signs and one minus-sign). We will assume that $\mathbf{M}^3$ is oriented and time oriented, that is, a 3-volume form dV and *future direction*, FUT, have been chosen. The *light cone* $\{v \in \mathbf{M}^3 \mid \langle v, v\rangle = 0\}$ will be denoted by LC. We will use various orthogonal splittings of Minkowski space $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$, with associated projections

$$\pi_{\mathbf{E}} : \mathbf{E}^2 + \mathbf{E}^- \to \mathbf{E}^2 \quad \text{and} \quad t : \mathbf{E}^2 + \mathbf{E}^- \to \mathbf{E}^-.$$

Throughout this paper a compact hypersurface in $\mathbf{M}^3$ will be a smooth immersion of a connected compact orientable 2-manifold, $j : Q \to \mathbf{M}^3$.

Over such a hypersurface we have $\mathrm{ORTH}(Q) \to Q$, the *orthogonal* (*line*) *bundle*, with fiber over $x \in Q$, $\mathrm{ORTH}_x = \{v \in T_{j(x)}\mathbf{M}^3 \mid v \perp T_x Q\}$. A compact hypersurface Q in $\mathbf{M}^3$ decomposes into three disjoint nonempty subsets $Q = D^+ \cup D^0 \cup D^-$ on which the induced metric is (respectively) of type $(2, 0)$, $(1, 0)$, and $(1, 1)$. Now at a point $x \in D^0$, where the induced metric degenerates, there is a 1-dimensional radical subspace $\mathrm{RAD}_x \subset T_x Q$, which is orthogonal to *all* of $T_x Q$. Hence $T_x Q \cap \mathrm{ORTH}_x = \mathrm{RAD}_x$, and we see that $\mathrm{ORTH}(Q)$ and $T\mathbf{M}^3/TQ$ differ.

Now, by adapting Lemma 1 of [K2], we have that $\mathrm{ORTH}(Q)$ is trivial and that a choice of orientation on Q determines a connected component of $\mathrm{ORTH}(Q) - \{\text{zero section}\}$. Thus, given $j : Q \to \mathbf{M}^3$ and $x \in Q$, we have a well-defined direction in $T_{j(x)}\mathbf{M}^3$ which, upon translation to the origin, may be viewed as a point of S^2. This defines $g : Q \to S^2$, the Gauss map of $j : Q \to \mathbf{M}^3$. By adapting Proposition 3 of [K2], we now have that the degree of $g : Q \to S^2$ is given by $\pm\frac{1}{2}\chi(Q)$, where $\chi(Q)$ denotes the Euler characteristic of Q. Throughout this paper we will choose the orientations of $\mathrm{ORTH}(Q)$ and S^2 so that the sign above is negative. Our first goal is to express this degree in terms of a Gauss map defined on D^0 (Theorem 4). This requires several regularity assumptions on the immersion, which we will now describe.

First we require that the map $j^*\langle\, , \rangle : Q \to \circ^2 T^*Q$ be transverse to the discriminant stratification in the symmetric 2-tensor bundle over Q. This implies that D^0 is a finite union of compact 1-dimensional submanifolds that form the boundary of the open submanifolds D^+ and D^- in Q. We orient D^0 as the boundary of D^+. A consequence of this condition is that every component of D^0 bounds both D^+ and D^-; that is, the bilinear type of the induced metric must change when crossing D^0. Now if n is a nonzero section of $\mathrm{ORTH}(Q)$ representing the orientation of Q, we can further decompose D^+ into disjoint open submanifolds $D^+ = D^+_F \cup D^+_P$, where

$$x \in D^+_F \text{ if } \langle n(x), \mathrm{FUT}\rangle < 0 \quad \text{and} \quad x \in D^+_P \text{ if } \langle n(x), \mathrm{FUT}\rangle > 0.$$

The boundaries of D^+_F and D^+_P then decompose as $D^0 = D^0_F \cup D^0_P$.

Next we define the *stall points* $\mathrm{SP} \subset D^0$ to be the subset where RAD and TD^0 agree. Thus the metric induced on D^0 vanishes at SP and is positive definite on $D^0 - \mathrm{SP}$. Such an induced metric on D^0 cannot change type transversely on D^0 in the sense defined above. Thus we are compelled to impose secondary transversality conditions.

Notice that the second fundamental form of $j: Q \to \mathbf{M}^3$ must be viewed as a tensor $\mathrm{II}: TQ \times TQ \times \mathrm{ORTH}(Q) \to \mathbf{R}$ with $\mathrm{II}(X, Y, n) = \langle \nabla_X Y, n\rangle$, since there is no way to globally normalize the section n. In fact, over D^0 this tensor is intrinsic, that is, determined by $\mathrm{I} = j^*\langle\, ,\, \rangle$. We will *assume* that, for all $x \in \mathrm{SP} \subset D^0 \subset Q$, the bilinear form $\mathrm{II}_x(-, -, n)$ is nondegenerate. Now RAD is a line field along D^0 that is tangent to D^0 at $x \in \mathrm{SP}$. We will *assume* that the order of contact is 1 (see [GG, Def. 2.3, p. 146]). This latter assumption implies that the stall points are isolated. Both of these conditions are intrinsic, but they will also control extrinsic properties of D^0 as a curve in $\mathbf{M}^3$. This is the content of Lemma 3 in Section 4. In the next two sections we collect the relevant extrinsic properties of D^0.

In order to make these definitions more concrete, we present a local model illustrating both the transversality conditions introduced above and the sense in which they are intrinsic. Since $\mathrm{I} = j^*\langle\, ,\, \rangle$ has rank 1 at $x \in D^0$, there exist local coordinates on Q such that I is represented by $E(x,y)\,dx^2 + G(x,y)\,dy^2$ with $E(0,0) = 0$ and $G(0,0) \neq 0$. Thus D^0 is locally represented by $\{(x,y) \mid E(x,y) = 0\}$ and RAD is locally represented by the line field spanned by ∂x. In these coordinates the tensor II is represented by

$$\tfrac{1}{2}(E_x\,dx^2 + 2E_y\,dx\,dy - G_x\,dy^2) \otimes dx \quad \text{over } \{(x,y) \mid E(x,y) = 0\}.$$

Notice that we have written II in terms of I. The reader may check that this expression agrees with the usual second fundamental form *over* D^0. An explanation can be found in [Kl].

At a stall point ∂x must be tangent to D^0, so $E_x(0,0) = 0$ and we may write $E = (y - x^2 e(x))\bar{E}$ with $\bar{E}(0,0) \neq 0$. This latter condition on E is equivalent to the first transversality assumption. Now ∂x is a vector field defined on a neighborhood of $(0,0)$ that spans RAD over D^0, and $\mathrm{Det}\,\mathrm{I} = EG$ vanishes on D^0. Hence $R(\mathrm{Det}\,\mathrm{I}) = (-2xe - x^2 e')\bar{E} + (y - x^2 e)\bar{E}_x$. This function has a first-order zero if and only if $e(0) \neq 0$. This is equivalent to the last transversality condition and implies that the induced metric on D^0 is locally represented by $x^2 \bar{e}(x)\,dx^2$ with $\bar{e}(0) \neq 0$. Hence the stall points are isolated.

2. Spacelike Loops

A spacelike loop in $\mathbf{M}^3$ is a smooth immersion $c\colon S^1 \to \mathbf{M}^3$ such that the induced metric is positive definite. It follows that the normal bundle $(S^1)^\perp$ is a trivial rank-2 bundle with fibres carrying a type-$(1,1)$ metric. To describe the two S^1-valued Gauss maps of such a curve, we choose an orthogonal splitting of Minkowski space, $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$, and we consider the two unique sections of $(S^1)^\perp$ that satisfy the following:

(E1)
(a) $\langle U_s^i, U_s^i \rangle = 0$ for $i = F, P$;
(b) $t_*(U_s^i(x))$ is of unit length for all $x \in S^1$ for $i = F, P$;
(c) $\langle U_s^F, \mathrm{FUT} \rangle < 0$; and
(d) $dV(-, U_s^P, U_s^F) \equiv dl$.

Here dl is an orientation 1-form on S^1, and equivalence is in the sense of orientations. The subscript s refers to the choice of splitting. Translating these sections to the origin in $\mathbf{M}^3$ yields two maps, $U_s^i\colon S^1 \to S_i^1$, $i = F, P$, where

(E2) $$S_F^1 = \mathrm{LC} \cap \{t^{-1}(1)\} \quad \text{and} \quad S_P^1 = \mathrm{LC} \cap \{t^{-1}(-1)\}.$$

Notice that each S_i^1, $i = F, P$, in (E2) may be viewed as a splitting dependent model for the space of null lines in $\mathbf{M}^3$. Thus we obtain two well-defined Gauss maps into S_i^1 and we say that the *U_s^i represent the Gauss maps of g_i: $S^1 \to S_i^1$ relative to a splitting of* $\mathbf{M}^3$. We will write F-deg and P-deg to denote the degree of g_F and g_P, respectively.

Both of the S_i^1 carry preferred orientations defined by positive rotation about FUT. Each splitting $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$ induces positively oriented length 1-forms dl_s^i on the S_i^1 whose total lengths are 2π. We define the split curvatures of a spacelike curve by $(U_s^i)^* dl_s^i = k_s^i\, dl$, $i = F, P$, where dl is the induced length 1-form on the curve. The sign of these split curvatures depends on a choice of orientation for the curve.

However, if we set $\mathrm{TILT}_s = (1/\sqrt{2})\langle U_s^F, U_s^P \rangle\colon S^1 \to R$ with $0 < \mathrm{TILT}_s \le 1$, then

$$k = \frac{k_s^F k_s^P}{\mathrm{TILT}_s}$$

defines the *quadratic curvature* of the spacelike curve. The function $k\colon S^1 \to R$ is independent of the choice of splitting and of the orientation on the curve (see [K4]).

PROPOSITION 1. *Let $c\colon S^1 \to \mathbf{M}^3$ be a smoothly immersed, spacelike curve; then F*-deg $= P$-deg. *Furthermore, given $n \in \mathbf{Z}$, there exists an imbedded spacelike curve with F*-deg $= n = P$-deg.

Proof. For the first part see Theorem 1 of [K3]. To construct such a curve, choose an immersed curve in $\mathbf{E}^2 \subset \mathbf{M}^3$ with $\mathbf{E}^2$-Gauss map of degree n and transverse self-intersection points. Now eliminate the self-intersection points in the curve by deforming the curve into $\mathbf{M}^3$. We are finished. □

Notice that a null line in LC determines a unique point in each S_i^1. Thus S_F^1 and S_P^1 can be identified in an orientation-preserving manner, and the $g^i: S^1 \to S_i^1$ define the *completed* $S^1 \times S^1$*-valued Gauss map*, $cg: S^1 \to S^1 \times S^1$, of a spacelike curve. Since the diagonal Δ in $S^1 \times S^1$ can be identified with the set of null lines in LC, we can think of this $S^1 \times S^1$ as a *compactification* of the Grassman of spacelike lines in $\mathbf{M}^3$. Now, by Proposition 1, the completed Gauss image of a spacelike curve is an (n, n)-curve in $S^1 \times S^1$ that does not intersect the diagonal.

3. Spacelike Loops with Stall Points

A smooth immersed loop, $c: S^1 \to \mathbf{M}^3$, whose induced metric is positive or zero is called *spacelike with stall points.* Here we will *assume* that the stall points, where the induced metric is zero, are isolated. The orthogonal 2-plane is still well defined at every point of the curve. Here we will assume that for any splitting there are *two pairs* of smooth sections of $(S^1)^\perp$ that satisfy (a), (b), and (c) of (E1). There need not exist sections that satisfy (a) through (d) of (E1). If we write (U_s^F, U_s^P) to denote one pair satisfying (a), (b), and (c) of (E1), then the other pair is given by $(-U^P, -U^F)$. This defines a $(S^1 \times S^1)$-valued Gauss map whose image intersects the diagonal exactly at stall points. Here we will *assume* that the intersection of this Gauss image and the diagonal is transverse in $S^1 \times S^1$ and that *both* of the associated Gauss maps $g_i: S^1 \to S_i^1$ are immersive at stall points. These maps are represented by the chosen U_s^i and their degrees are again denoted by F-deg and P-deg. Although the functions k_s^i and k become unbounded at stall points, the 1-forms $k_s^i\, dl = (U_s^i)^* dl_s^i$ are smooth on the entire loop. Notice that we may define dl on all of $c(S^1)$ with the following convention: If $dl^2 = t^2 a(t)^2 dt^2$ and $a(0) > 0$ at a stall point, then $dl = |t| a(t)\, dt$. Thus, if $k_s^i\, dl = b^i(t)\, dt$ then we may define $|k_s^i|\, dl = |b^i(t)/ta(t)|\, dl = |b^i(t)|\, dt$. Implicit in this convention is the assumption that dl, when paired with a positively oriented vector tangent to $c(S^1)$, is nonnegative. As a contrast to Proposition 1 we have the following.

PROPOSITION 2. *Given nonnegative integers n and m, there is an imbedded spacelike curve with stall points* $c: S^1 \to \mathbf{M}^3$ *such that* $cg: S^1 \to S^1 \times S^1$ *is transverse to the diagonal and* $(F\text{-deg}, P\text{-deg}) = \pm(n+m, n-m) \in \mathbf{Z} \times \mathbf{Z}$.

Proof. Consider the arc in $\mathbf{M}^2 \subset \mathbf{M}^3$ given by $c(t) = (t, 0, \arctan(t))$. Then the induced metric is positive except at $t = 0$. With respect to the implicit splitting, the sections of $(S^1)^\perp$ defined in (E1) are given by

$$U_s^i = ((1+t^2)^{-1}, \pm t(2+t^2)^{1/2}(1+t^2)^{-1}, 1) \quad \text{for } i = F, P.$$

They are both immersive at $t = 0$. Because of the opposite signs in these sections, the $S^1 \times S^1$ Gauss map is transverse to the diagonal. Now we can smoothly glue together several such arcs in an $\mathbf{M}^2$ hyperplane of $\mathbf{M}^3$ to get a curve that has its $S^1 \times S^1$ Gauss map positively intersecting the diagonal m times. Let $c: S^1 \to \mathbf{M}^3$ be a spacelike loop whose completed Gauss map is a type-$\pm(n,n)$ curve, as described in Proposition 1, which contains a linear segment lying in an $\mathbf{M}^2 \subset \mathbf{M}^3$. Next, in this $\mathbf{M}^2$ take $\bar{c}$ to be the connected sum of c and the m-stall point arc in $\mathbf{M}^2$ described above. Then the resulting curve is of type $\pm(n+m, n-m)$ in $S^1 \times S^1$. We are finished. □

If $c: S^1 \to \mathbf{M}^3$ is an oriented spacelike loop with stall points satisfying the conditions of Proposition 2, we then define the *intersection number of the curve*, $\cap^{\#}(\Delta, cg)$, to be the intersection number of the diagonal Δ with $cg(S^1)$ in $S^1 \times S^1$. This intersection number can be expressed as the difference of the degrees of the $g_i: S^1 \to S^1_i$, $i = F, P$.

4. A Degree Formula

Now we are given a compact oriented surface $j: Q \to \mathbf{M}^3$ satisfying each of the transversality conditions in Section 1, and a section n of the orthogonal bundle determined by the orientation. We also have the decomposition $D^+ = D^+_F \cup D^+_P$, with $\partial D^+ = D^0 = D^0_F \cup D^0_P$ and $\mathrm{SP} \subset D^0$. Each component of D^0_i is oriented as the boundary of D^+_i for $i = F, P$. For each splitting $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$, we now have unique sections of $(D^0)^\perp$ defined as follows:

(E3)
- if $S^1 \subset D^0$ and $S^1 \cap \mathrm{SP} = \emptyset$, then use sections (U^F_s, U^P_s) satisfying (E1)(a)–(d);
- if $S^1 \subset D^0_F$ and $S^1 \cap \mathrm{SP} \neq \emptyset$, then use sections (U^F_s, U^P_s) satisfying (E1)(a)–(c) with U^F a nonzero multiple of $n \mid S^1$;
- if $S^1 \subset D^0_P$ and $S^1 \cap \mathrm{SP} \neq \emptyset$, then use sections (U^F_s, U^P_s) satisfying (E1)(a)–(c) with U^P a nonzero multiple of $n \mid S^1$.

We will write $g^j_i: D^0_j \to S^1_i$ for $i, j = F, P$ to denote the Gauss maps defined by these sections. The degree of g^j_i is denoted i-$\deg_j$ where $i, j = F, P$. We will write $cg_i: D^0_i \to S^1 \times S^1$ to denote the *completed Gauss map* defined by these sections. In defining the associated intersection number, $\cap^{\#}(\Delta, cg_i)$, we will impose the *orientation convention*; the target $S^1 \times S^1$ is oriented by $dl^F_s \wedge dl^P_s$ for cg_F and by $dl^P_s \wedge dl^F_s$ for cg_P.

LEMMA 3. *If $j: Q \to \mathbf{M}^3$ is stratified by* $\mathrm{SP} \subset D^0 \subset Q$ *and satisfies the conditions of Section 1, then:*

(a) *For all components $S^1 \subset D^0$, the set* $\mathrm{SP} \cap S^1$ *is empty or contains an even number of points.*
(b) *The hypersurface's second fundamental form is indefinite at every stall point.*
(c) *At any stall point, both S^1-valued Gauss maps are immersive and cg_i: $D^0 \to S^1 \times S^1$ is transverse to the diagonal.*
(d) *If a component $S^1 \subset D^0$ has* $\mathrm{SP} \cap S^1 = \emptyset$, *then there exists an $x \in S^1$ with $k(x) \geq 0$. If a component $S^1 \subset D^0$ has* $\mathrm{SP} \cap S^1 \neq \emptyset$, *then on some deleted neighborhood of $x \in \mathrm{SP}$ in D^0 we have $k < 0$.*

Proof. Choose linear coordinates on $\mathbf{M}^3$ so that $\langle\ ,\ \rangle$ is represented by $dx^2 + 2\,dy\,dz$ and $j: Q \to \mathbf{M}^3$ is represented by the graph, $z = f(x, y)$, with $df = 0$ at $(0,0)$. Then the induced metric is locally represented by $\mathrm{I} = dx^2 + 2f_x\,dx\,dy + 2f_y\,dy^2$, and RAD at $(0,0)$ is represented by ∂y. Since $f \equiv \alpha x^2/2 + \beta xy + (\gamma y^2/2)\,\mathrm{MOD}\,M^3$ (i.e., modulo third-order terms), we have that

$$\mathrm{Det}\,\mathrm{I} = 2f_y - f_x^2 \equiv 2(\beta x + \gamma y) - (\alpha x + \beta y)^2\,\mathrm{MOD}\,M^3.$$

Thus D^0 corresponds to $\{\mathrm{Det}\,\mathrm{I} = 0\}$ and RAD is tangent to D^0 at $(0,0)$ if and only if $\gamma = 0$. Since II at $(0,0)$ is represented by a multiple of

$$(f_{xx}\,dx^2 + 2f_{xy}\,dx\,dy + f_{yy}\,dy^2) \otimes dy,$$

we have that II is nondegenerate at the stall point $(0,0)$ if and only if $\beta \neq 0$. Next, with an isometry of $\mathbf{M}^3$ fixing the null plane $(\partial y)^\perp$ at $(0,0,0)$, we may assume $\alpha = 0$. Now there exists a vector field R defined on a neighborhood of $(0,0)$ which spans RAD over D^0, and $R \equiv (\beta y)\,\partial x - \partial y\,\mathrm{MOD}\,M^2$. It follows that $R(\mathrm{Det}\,\mathrm{I}) \equiv 4\beta^2 y\,\mathrm{MOD}\,M^2$. This implies the order of contact between D^0 and RAD is 1 if and only if $\beta \neq 0$.

The image of $j: D^0 \subset Q \to \mathbf{M}^3$ near the stall point can be parameterized by $c(t) \equiv (\beta t^2/2, t, \beta^2 t^3/2)\,\mathrm{MOD}\,M^4$. Then the two local sections of the orthogonal 2-plane can be written as

$$(\beta t(1 \pm 2), 1, z(t))\,\mathrm{MOD}\,M^2. \tag{E4}$$

We now prove part (a). Let $c: S^1 \to \mathbf{M}^3$ be a component of D^0. The section $n \mid S^1$ does not vanish. Choose a splitting $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$ and consider $\pi_{\mathbf{E}*}n$ along the immersed curve, $\pi_{\mathbf{E}} \circ c: S^1 \to \mathbf{E}^2$. By the computations above, $\pi_{\mathbf{E}*}n$ is tangent to $\pi_{\mathbf{E}} \circ c(S^1)$ exactly at stall points, and must point to opposite sides of the tangent line to $\pi_{\mathbf{E}} \circ c(S^1)$; that is,

$$\pi_{\mathbf{E}} \circ c(t) \equiv \left(\frac{\beta t^2}{2}, t\right)\mathrm{MOD}\,M^4 \quad \text{and} \quad \pi_{\mathbf{E}*}n \equiv \mu(\beta t(1 \pm 2), 1)\,\mathrm{MOD}\,M^2,$$

$0 \neq \mu \in \mathbf{R}$. Thus the existence of a nonzero section of $(D^0)^\perp$ implies that there is an even number of stall points on each component of D^0.

The proofs of (b) and (c) are immediate consequences of the computation above. For the second part of (d) we note that, by comparing the signs in (E4), the two $g_i: S^1 \to S_i^1$ cannot both preserve or both reverse orientation near $t = 0$. This implies that k_s^F and k_s^P have opposite signs near $t = 0$. Hence $k < 0$ for t sufficiently near zero. For the first part of (d) we note that, since there are no stall points, we have $0 = \int_{S^1} k_s^F - k_s^P\,dl$ by Proposition 1. Hence there exists $t \in S^1$ with $k_s^F(t) = k_s^P(t)$ for each splitting, and $k(t) \geq 0$. This completes the proof. □

Notice that if $j: Q \to \mathbf{M}^3$ satisfies the first transversality condition of Section 1 then the degrees i-$\deg_j$, $j = F, P$, are well defined. Should $cg_i: D^0 \to S^1 \times S^1$ fail to globally immerse, then by Lemma 3(c) the secondary transversality conditions imply that the intersection numbers $\cap^\#(\Delta, cg_i)$ are well defined. The following is the central result of this paper. Recall that Q is oriented and we have chosen n, the section of $\mathrm{ORTH}(Q)$, so that the degree of $g: Q \to S^2$ is $-\frac{1}{2}\chi(Q)$.

THEOREM 4. *If $j: Q \to \mathbf{M}^3$ is an immersed oriented compact hypersurface satisfying the conditions of Section 1, then, for $i = F, P$,*

$$\begin{aligned}\tfrac{1}{2}\chi(Q) &= \chi(D_i^+) + \tfrac{1}{2}\cap^{\#}(\Delta, cg_i)\\ &= -\text{degree}(g).\end{aligned}$$

Proof. We view the S_i^1 for $i = F, P$ as subsets of the target sphere of $g: Q \to S^2$, so that $g^{-1}(S_i^1) = D_i^0$. The first transversality condition on $j: Q \to \mathbf{M}^3$ implies that a point in $\{S_i^1 - g(\mathrm{SP})\} \subset S^2$ is a regular value for $g: Q \to S^2$ if and only if it is a regular value for $g_i^i: D_i^0 \to S_i^1$. It follows that the degree of $g: Q \to S^2$ is the negative of the degree of $g_i^i: D_i^0 \to S_i^1$, where $i = F$ or P. Now choose a splitting $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$, and consider $\pi_{\mathbf{E}*}n$, a vector field along $\pi_{\mathbf{E}}(D_i^0)$ in $\mathbf{E}^2$. Notice that it is tangent to the immersed curve $\pi_{\mathbf{E}}(D_i^0)$ exactly at $\mathrm{SP} \cap D_i^0$. By (c) and (d) of Lemma 3, $\pi_{\mathbf{E}*}n$ must point to opposite sides of the tangent to $\pi_{\mathbf{E}}(D_i^0)$ in $\mathbf{E}^2$ when a stall point is transversed; see Figure 1.

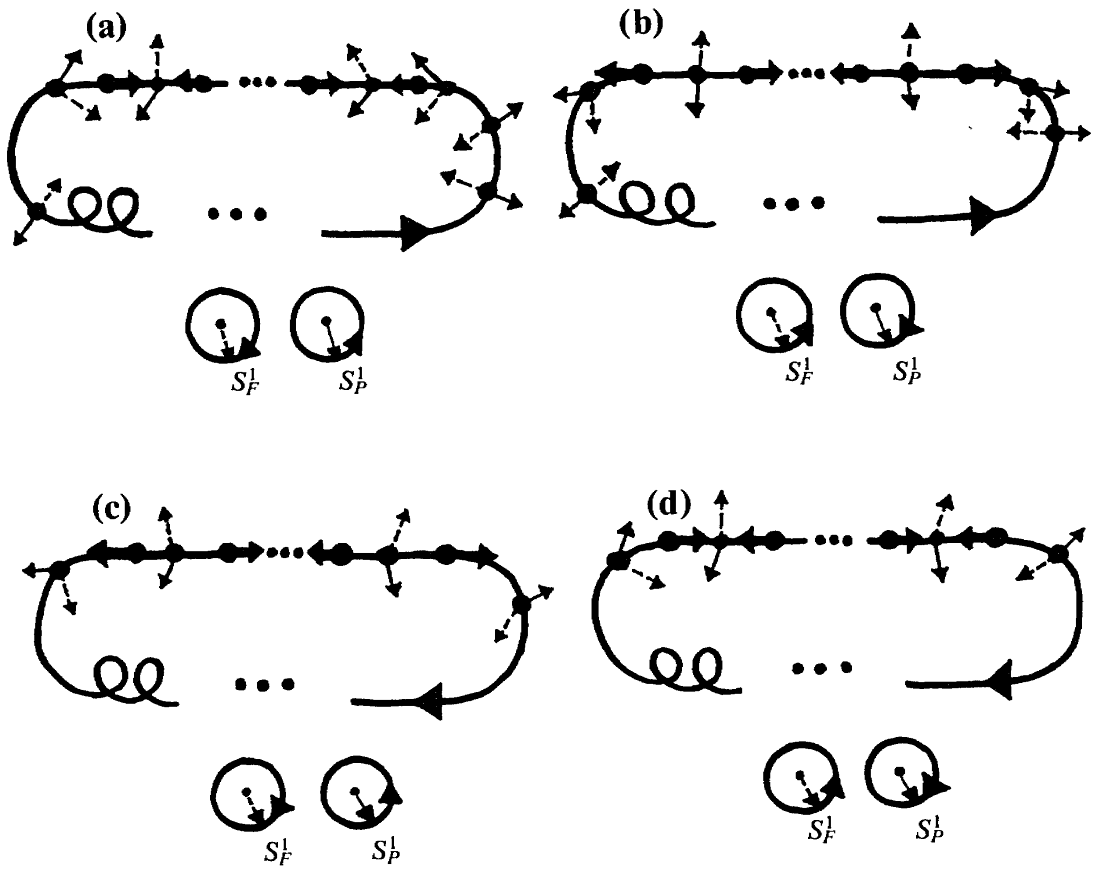

Figure 1

We need only show that the degree of $(\pi_{\mathbf{E}*}n)(\|\pi_{\mathbf{E}*}n\|)^{-1} = \tilde{n}: D_i^0 \to S^1 \subset \mathbf{E}^2$ is $\chi(D_i^0) + \tfrac{1}{2}\cap^{\#}(\Delta, cg_i)$ for $i = F$ or P. We will prove the $i = F$ case below. The $i = P$ case follows by reflecting the immersion through a splitting $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$, so that F and P are interchanged.

Consider the degree of the outward unit normal to the curve $\pi_{\mathbf{E}}(D_F^0)$. We view this normal as a map $D_F^0 \to S^1 \subset \mathbf{E}^2$ via translation to the origin in $\mathbf{E}^2$. We claim this degree is exactly $\chi(D_F^+)$. To see this, choose a C^∞-vector field X

on D_F^+ having nondegenerate, isolated zero points and pointing outward at the boundary. Let $B_1, \ldots, B_r$ be small disks containing the zero points that project diffeomorphically to $\mathbf{E}^2$ via $\pi_{\mathbf{E}}$. Now

$$(\pi_{\mathbf{E}^*}X)(\|\pi_{\mathbf{E}^*}X\|)^{-1} = \tilde{X}: D_F^0 - B_1 \cup \cdots \cup B_r \to S^1$$

is defined by a projection to $\mathbf{E}^2$ followed by a translation to the origin. If $d\theta$ denotes a fundamental class on S^1, then by Stokes's theorem the degree of the outward normal is

$$\text{(E5)} \qquad \int_{D_F^0} \tilde{X}^* d\theta = \sum \int_{\partial B_j} \tilde{X}^* d\theta = \chi(D_F^+),$$

since the middle term is the sum of the indices at the rest points.

Returning to the degree of $\tilde{n}: D_F^0 \to S^1 \subset \mathbf{E}^2$, consider a component of D_F^0 that contains stall points; then $\pi_{\mathbf{E}^*}n$ along $\pi_{\mathbf{E}}(D_F^0)$ is homotopic in $\mathbf{E}^2$ to one of the configurations in Figure 1. In these figures, one of the arrows corresponds to $\pi_{\mathbf{E}^*}n$ and the other corresponds to the projection of the negative of the companion section defined by (E3). Notice that we may assume that, at the SP, $\pi_{\mathbf{E}^*}n$ alternatively agrees or disagrees with the positive direction on D_F^0. Otherwise, nonalternating pairs could be cancelled with a homotopy. Now each stall point corresponds to an intersection point $cg_F(D_F^0) \cap \Delta$ in $S^1 \times S^1$. In (a) and (c) of Figure 1 each intersection is oriented positively, whereas in (b) and (d) each intersection is oriented negatively. Clearly, in (a) and (c) of Figure 1 the degree of $\tilde{n}$ is the degree of the outward normal plus one-half the number of stall points, while in (b) and (d) the degree of $\tilde{n}$ is the degree of the outward normal minus one-half the number of stall points. In other words, the degree of $\tilde{n}$ is $\chi(D_F^+) + \frac{1}{2}\cap^{\#}(\Delta, cg_F)$, and we are finished. $\square$

COROLLARY 5. *If $j: Q \to \mathbf{M}^3$ satisfies the conditions of Section 1, then*

$$\text{(a)} \qquad 2\chi(D_i^+) = F\text{-deg}_i + P\text{-deg}_i \quad \text{for } i = F, P,$$

and

$$\begin{aligned} \text{(b)} \qquad 2\chi(D^-) &= \cap^{\#}(\Delta, cg_F) + \cap^{\#}(\Delta, cg_P) \\ &= (F\text{-deg}_F) - (P\text{-deg}_F) + (P\text{-deg}_P) - (F\text{-deg}_P). \end{aligned}$$

Proof. For part (a) with $i = F$, we need only note that the degree of n restricted to D_F^0 is $F\text{-deg}_F$. Now, by Theorem 4,

$$F\text{-deg}_F = -\tfrac{1}{2}\chi(Q) = \chi(D_F^+) + \tfrac{1}{2}\cap^{\#}(\Delta, cg_F).$$

Since $\frac{1}{2}\cap^{\#}(\Delta, cg_F) = \frac{1}{2}(F\text{-deg}_F - P\text{-deg}_F)$, we have part (a). For part (b) we need only recall that

$$\chi(Q) = \chi(D_F^+) + \chi(D_P^+) + \chi(D^-),$$

completing the proof. $\square$

5. Examples and Applications

(A) *RAD is Transverse to* D^0. If $j: Q \to \mathbf{M}^3$ contains no stall points, then all of the components of D^- are annuli by Proposition 5 of [K2]. Any genus surface admits such an embedding. For example, the boundary of a small diameter $\mathbf{E}^3$-tube about a spacelike $c: S^1 \to \mathbf{M}^3$ will have four spacelike loops as D^0. We also note that the notion of partial curvature of Section 4 in [K2] is directly adaptable to this setting.

(B) *Simplest* D^+ *Configurations.* Recall, by Lemma 2 of [K2], that any $j: Q \to \mathbf{M}^3$ can be homotoped via a family of ambient diffeomorphisms (i.e., time dilation in a Morse splitting $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$) to an immersion that satisfies the transversality conditions of Section 1 and for which D^+ is a disjoint union of disks. Such disks can be further deformed into one of the following local models. (See Figure 2.)

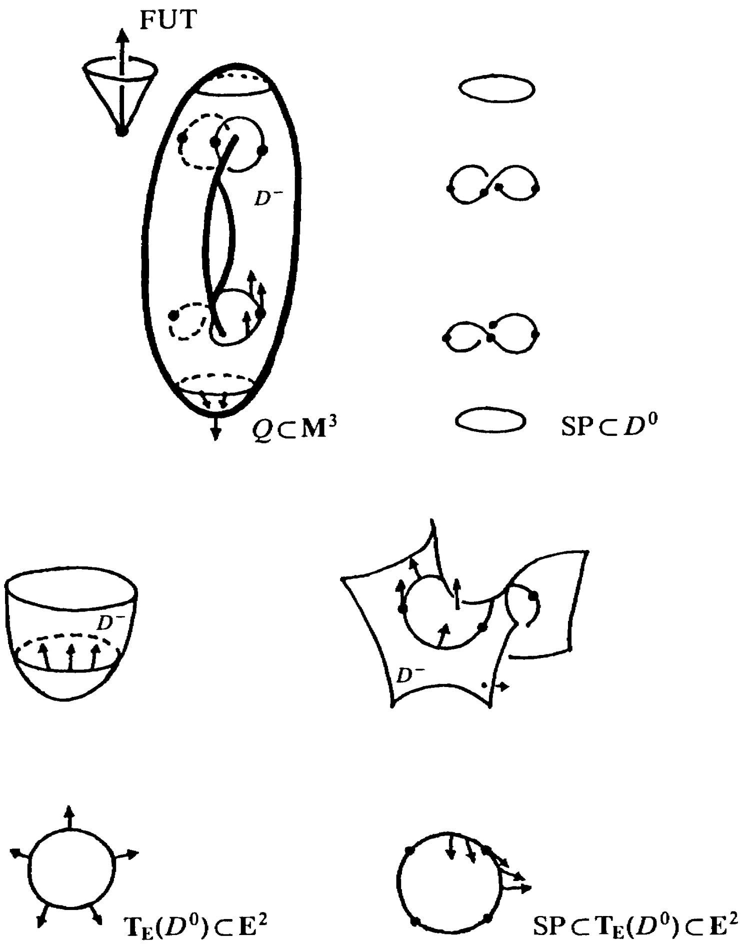

Figure 2

(i) Consider the graphs of $\pm z = x^2 + y^2$ in coordinates where $\langle\,,\rangle$ is represented by $dx^2 + dy^2 - dz^2$; then D^+ is an open disk which graphs over $\{x^2 + y^2 < 1\}$. Since this is a surface of revolution, RAD is everywhere transverse to D^0. The Gauss maps of D^0 are represented by

$$U_s^F(t) = (\cos(t), \sin(t), 1) = n \quad \text{and} \quad U_s^P(t) = (\cos(t), \sin(t), -1).$$

Hence both k_s^i are positive and the Gauss maps $g_i : S^1 \to S_i^1$ have degree 1.

(ii) Consider the graph of $z = xy$ in the above coordinates; then D^+ graphs over $\{x^2 + y^2 < 1\}$, $\mathrm{SP} = \{(\pm 1, 0, 0) \cup (0, \pm 1, 0)\}$, and D^0 is parameterized by $c : S^1 \to \mathbf{M}^3$, $c(t) = (\cos(t), \sin(t), \cos(t)\sin(t))$, with

$$U_s^F(t) = (\sin(t)(\sin^2(t) - 3\cos^2(t)), \cos(t)(\cos^2(t) - 3\sin^2(t)), 1) = n,$$

and

$$U_s^P(t) = (\sin(t), \cos(t), 1).$$

One computes that $k_s^F > 0$ and $k_s^P < 0$ on S^1, and that F-deg $= 1$ and P-deg $= -3$. Hence $\pm \cap^{\#}(\Delta, cg) = 4$ and $\pm 2\chi(D^+) = F\text{-deg} + P\text{-deg} = 2$. See Figure 2.

(C) *Two Integral Inequalities.* Because the degree terms in Corollary 5 can be expressed as integration of split curvature, we have the following inequalities.

THEOREM 6. *If $j : Q \to \mathbf{M}^3$ satisfies the conditions of Section 1, then the following hold for all splittings $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$:*

(a) $$4\pi |\chi(D_i^+)| \leq \int_{D_i^0} |k_s^F| + |k_s^P| \, dl \quad \textit{for } i = F, P.$$

Equality for some splitting implies that $k \geq 0$ on D_i^0; hence $\mathrm{SP} \cap D_i^0 = \emptyset$. If equality holds for both $i = F, P$, then D^- is a union of annuli.

(b) $$2\pi |\cap^{\#}(\Delta, cg_F)| \leq \int_{D_F^0} |k_s^F - k_s^P| \, dl \qquad \textit{and}$$

$$2\pi |\cap^{\#}(\Delta, cg_P)| \leq \int_{D_P^0} |k_s^P - k_s^F| \, dl.$$

Equality for some splitting and fixed $i = F, P$ imply that each component of D_i^0 either contains a pair of stall points, or is a spacelike loop lying in a spacelike hyperplane $\mathbf{E}^2 \subset \mathbf{M}^3$.

REMARK. Examples of the type discussed in (A) and (B) illustrate equality in parts (a) and (b) above.

Proof of Theorem 6. For part (a) we use Corollary 5 to write, for $i = F, P$,

$$4\pi |\chi(D_i^+)| \leq \left| \int_{D_i^0} k_s^F + k_s^P \, dl \right| \leq \int_{D_i^0} |k_s^F| + |k_s^P| \, dl.$$

Equality for a splitting implies that k_s^F and k_s^P have the same sign on D_i^0. Hence $k \geq 0$ on D_i^0, and by (d) of Lemma 3 this implies $\mathrm{SP} \cap D_i^0 = \emptyset$. If $\mathrm{SP} \cap (D_F^0 \cup D_P^0) = \emptyset$ then RAD is transverse to TD^0 and hence D^- is a union of annuli.

For part (b) we use the same observations to write, for $i = F$,

$$2\pi |\cap^{\#}(\Delta, cg_F)| \leq \left| \int_{D_F^0} k_s^F - k_s^P \, dl \right| \leq \int_{D_F^0} |k_s^F - k_s^P| \, dl.$$

Equality for some splitting implies that on each component of D_F^0 either $k_s^F \geq k_s^P$ with $k_s^F \neq k_s^P$, or $k_s^F = k_s^P$. In the first case we must have $\deg_F > \deg_P$. Thus by Proposition 1 this component contains stall points. In the second case we recall from (E1) that $k_s^i\, dl = (U_s^i)^* dl_s^i$; hence the maps $U_s^F, -U_s^P : S^1 \to S_F^1$ differ by a fixed rotation of the target. This implies that TILT_s is constant on the component. Hence this component lies in a fiber of $t: \mathbf{M}^3 \to \mathbf{E}^-$ by Lemma 2 of [K3]. This completes the proof. □

(D) *The Null Blow-Up.* Given an immersed $j: Q \to \mathbf{M}^4$ that satisfies the transversality conditions of Section 1, we will construct a canonical smooth compact 2-dimensional manifold NB and a smooth folded double cover ρ: $\mathrm{NB} \to D^- \cup D^0$. The subset $\mathrm{NB} - \rho^{-1}(D^0)$ submersively double-covers D^-, and ρ has standard fold singularities along $\rho^{-1}(D^0)$. On NB we will find a canonical smooth orientable line field that ρ-projects onto null vectors and has isolated zeros at exactly $\rho^{-1}(\mathrm{SP}) \subset \rho^{-1}(D^0)$. The behavior of the line field near these zeros is determined by high-order information in the induced metric $\mathrm{I} = j^*\langle\, ,\rangle$.

Let $\rho: G(Q) \to Q$ denote the Grassman bundle of oriented lines in the tangent spaces to Q, and let (θ) denote its tautological differential ideal. $G(Q)$ is a circle bundle. Now fix a Morse splitting $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$, and let ∂t denote the unit length future-pointing parallel vector field tangent to the fibers of $\pi_{\mathbf{E}}: \mathbf{M}^3 \to \mathbf{E}^2$. Next, artificially impose an $\mathbf{E}^3$ structure on $\mathbf{M}^3$ by changing the sign of the bilinear form $\langle\, ,\rangle$ in the ∂t summand. We denote this bilinear form by $\mathbf{E}_s\langle\, ,\rangle$. Now $\mathbf{E}_s\langle\, ,\rangle$ orthogonally projects ∂t into the tangent spaces of Q to get a vector field FUT_Q that has zeros only on D^+ and is tangent to RAD on all of D^0. At each point $x \in D^-$ we can choose two unique null vectors l_x^F, l_x^P in T_xQ so that $\langle \mathrm{FUT}_Q, l^F\rangle_x > 0$, $\langle \mathrm{FUT}_Q, l^P\rangle_x < 0$, and $(l_x^F, -l_x^P)$ is positively oriented relative to the orientation on T_xQ. We define $\Sigma^i \subset G(Q)$ to be the subsets $\{\mathrm{span}(l_x^i) \in G(Q) \mid x \in D^-\}$ for $i = F, P$. These are disjoint embedded submanifolds in $\rho^{-1}(D^-)$, each of which submersively ρ-covers D^-.

Now construct a smooth vector field $(\mathrm{FUT}_Q)^\perp$ tangent to Q that is spacelike and nonzero over $D^+ \cup D^0$. Let $\rho: J(Q) \to Q - \{\text{zeros of } (\mathrm{FUT}_Q)\}$ denote the open dense subset of $G(Q)$ consisting of lines transverse to the span $(\mathrm{FUT}_Q)^\perp$ in TQ. This $J(Q)$ has two connected components, one of which contains $\Sigma^F \cup \Sigma^P$; this component is a line bundle. Now Q is compact. Hence $(\mathrm{FUT}_Q)^\perp$ has a flow ψ_t that extends to a flow $\bar{\psi}_t$ on $G(Q)$. Let $\overline{\mathrm{FUT}}_Q^\perp$ be the vector field on $G(Q)$ that generates this flow.

LEMMA 7. *Let $j: Q \to \mathbf{M}^3$ satisfy the conditions of Section 1. Then the closure of $\Sigma^F \cup \Sigma^P$ in $G(Q)$ is a smooth embedded manifold* NB *that is homeomorphic to $D^0 \cup \Sigma^F \cup \Sigma^P$, where the two $\Sigma^i \cong D^-$ are identified along their boundary D^0 (i.e., the "double" of D^-). Furthermore, there exists a smooth vector field L on* NB *with zeros exactly at* $\mathrm{SP} \subset D^0 \subset \mathrm{NB}$ *such that, for all $x \in \mathrm{NB}$, $\rho_*(L(x))$ is a null vector.*

Proof. At any point of D^0 we may choose local coordinates on $U \subset Q$ so that $\mathrm{I} = j^*\langle\, ,\rangle$ is represented by $E\,dx^2 + G\,dy^2$ with $E(0,0) = 0 \neq G(0,0)$ and

$dE(0, 0) \neq 0$. Then $D^0 \cap U$ is represented by $\{E(x, y) = 0\}$ and RAD is spanned by ∂x over $\{E = 0\}$. A neighborhood of $\mathrm{RAD}_{(0,0)}$ in $G(Q) \cap \rho^{-1}(U)$ admits local coordinates (x, y, p). In these coordinates a point (x, y, p) corresponds to the subspace annihilated by $(dy - p\,dx)$ in $T_{(x,y)}U$. Now the null subspaces in $G(Q) \cap \rho^{-1}(U)$ must satisfy the relation $0 = E + Gp^2$. Thus $\Sigma^F \cup \Sigma^P \cap \rho^{-1}(U)$ is the set $\{(x, y, p) \mid 0 = E + Gp^2, \ \rho \neq 0\}$, and its closure in $G(Q) \cap \rho^{-1}(U)$ is obtained by adjoining $\{(x, y, p) \mid p = 0, E = 0\}$. These closure points can be identified with $D^0 \cap U$ when to each point (x, y) of $D^0 \cap U$ we assign the subspace $\mathrm{RAD}_{(x,y)}$, which is represented by $(x, y, 0)$.

To construct the vector field on NB we use a construction from contact geometry. First note that $\mathrm{NB} \cap J(Q)$ is exactly $\mathrm{NB} \cap G(Q)$. This is because null directions in Q are transverse to $(\mathrm{FUT}_Q)^\perp$ on all of $D^+ \cup D^0$. Now $\overline{\mathrm{FUT}_Q^\perp}$ defines a vector field on $J(Q)$. Hence there exists a unique representative θ of (θ) that satisfies $1 = \theta(\overline{\mathrm{FUT}}_Q)$ and the Lie derivative $\mathcal{L}\,\theta_{\overline{\mathrm{FUT}_Q^\perp}} = 0$. (If in local coordinates on Q the vector field $\mathrm{FUT}_Q^\perp$ is represented by ∂y, then θ is represented by $dy - p\,dx$.) Since NB is compact, orientable, and embedded in $J(Q)$, it is the regular value of some function $h\colon J(Q) \to \mathbf{R}$. Now define a vector field L on NB by

$$d\theta(L, -) = \overline{\mathrm{FUT}_Q^\perp}(h) \cdot \theta - dh, \quad \theta(L) = 0. \tag{E6}$$

The 1-form on the right-hand side of (E6) is zero when paired with $\overline{\mathrm{FUT}_Q^\perp}$. Hence there exists a unique vector field tangent to NB that satisfies (E6). The only points of NB where L can vanish are points where the subspace annihilated by the tautological ideal (θ) agrees with the tangent space to NB. This cannot occur at points over D^- since the tangent space to NB is transverse to the fibers of ρ. At a point of $D^0 - \mathrm{SP}$ we may choose coordinates (x, y, p) on $J(Q)$ so that NB is represented by $x\tilde{E} + p^2 = 0$, $\tilde{E}(0,0) \neq 0$, and (θ) is represented by $(dy - p\,dx)$. Clearly, the above subspaces to not agree. At a stall point, choose coordinates as at the end of Section 1 and observe that the two subspaces agree. Finally, $\rho_* L$ is null by construction and we are finished. □

We now define the *intrinsic index* or I-index at an isolated stall point of $j\colon Q \to \mathbf{M}^3$ to be the index of the associated zero point in L on NB. As examples, consider the graph of $f(x, y) = \beta xy + \gamma y^3/3$ discussed in the proof of Lemma 3. The induced metric is represented by

$$dx^2 + 2\beta y\,dx\,dy + (2\beta x + 2\gamma y^2)\,dy^2.$$

We may use ∂x for $\mathrm{FUT}_Q^\perp$. Then the associated surface in $G(Q) \cap \rho^{-1}(U)$ is represented by

$$\{(x, y, p) \mid p^2 + 2\beta yp + (2\beta x + 2\gamma y^2) = 0\}$$

and (θ) by $(dx - p\,dy)$. Near the stall point $(0, 0, 0)$ we may parameterize NB as $x = (-1/2\beta)(p^2 + 2\beta yp + \gamma y^2)$. In this case the line field $\pm L$ is represented by the span of $(-2p - 2\beta y)\partial y + (p + 2\beta p + 2\gamma y)\partial p$. The linear part of the rest point has eigenvalues $\frac{1}{2}(1 \pm (1 + 4\beta^2 + 2\beta - 4\gamma)^{1/2})$. Hence either index ± 1 can be locally realized by a surface that satisfies the transversality conditions of Section 1. Since the I-index of a stall point is determined by the induced metric, and since the intersection number in $S^1 \times S^1$ is determined by extrinsic properties of D^0 in $\mathbf{M}^3$, the following corollary links intrinsic and extrinsic properties.

COROLLARY 8. *If $j: Q \to \mathbf{M}^3$ satisfies the conditions of Section 1, then*

$$\sum_{\mathrm{SP}} \text{I-Index} = \chi(\mathrm{NB}) = \cap^{\#}(\Delta, cg_F) + \cap^{\#}(\Delta, cg_P).$$

Proof. Since NB is the double of D^-, we have

$$\sum \text{I-index} = \chi(\mathrm{NB}) = 2\chi(D^-),$$

and by Theorem 4 we are done. □

We note in passing that the components of D^0 in NB are independent in the first homology group $H_1(\mathrm{NB}, \mathbf{R})$. Thus, if 2 genus(NB) denotes the dimension of $H_1(\mathrm{NB}, \mathbf{R})$, we then have

$$\text{(E7)} \quad \begin{aligned} 2(\#\text{components of } D^0) &\leq 2(\#\text{components NB}) + 2\,\text{genus(NB)} \\ &\leq \#\mathrm{SP} + 4(\#\text{components NB}). \end{aligned}$$

(E) *Dynamical Properties of Null Geodesics.* The null blow-up NB and associated line field L can be viewed as a completion space for the null pre-geodesics in Q. Near D^0 these pre-geodesics are "F, P-reflected". (The behavior of nonnull pre-geodesics is more subtle; see [K1].) The dynamics of trajectories of L on NB can be quite complicated. For example, a 2-sphere of SO(2, 0)-revolution will have no stall points and hence NB will be a union of tori. The L-flows on these tori will be the familiar winding flow whose "slope" is determined by the geometry of the sphere. With a small deformation in the D^- component, limit cycles can be introduced into the L-flow. Next consider $j: Q \to \mathbf{M}^3$, depicted in Figure 2. Here we have the minimum number of stall points, hence $|\chi(\mathrm{NB})| = \#\mathrm{SP}$ and every stall point must have I-index equal to -1. In this case the unstable and stable manifolds of L do not coincide and there will be more complicated recurrence in the dynamics of the L-flow.

Next, we identify a simple link between the shape of $j: Q \to \mathbf{M}^3$ and the dynamics of the L-flow. First notice that the image of $n \circ \rho: \mathrm{NB} \to S^2$ (n is an orientation section of ORTH(Q)) retracts onto an $S^1 \subset S^2$. This defines a cohomology class $[\gamma]$ in $H^1(\mathrm{NB}, \mathbf{Z})$. Upon restriction to $D_i^0 \subset \mathrm{NB}$, this class agrees with $[k_s^i\, dl] \in H^1(D_i^0, \mathbf{Z})$ for $i = F, P$. Indeed, for any choice of splitting $\mathbf{M}^3 = \mathbf{E}^2 + \mathbf{E}^-$, there is a *natural* 1*-form* γ_s *on NB* that represents $[\gamma]$ and agrees with $k_s^i\, dl$ on D_i^0. Over D^-, the section n can be normalized to unit length, say un. We have a well-defined third fundamental form on NB given by $\mathrm{III} = \langle \nabla un, \nabla un \rangle$. The *condition* that $\mathrm{III}(L, L) \geq 0$ on $\mathrm{NB} - D^0$ does not locally restrict the signature of II on D^-, and implies that $\gamma_s(L) \neq 0$ on $\mathrm{NB} = D^0$. On D^0 the line field L is tangent to the ρ-fiber. Hence $\gamma_s(L) = 0$. Furthermore, if II is nonzero on TD^0 over $D^0 - \mathrm{SP}$, then $\gamma_x(L) \geq 0$ or $\gamma_x(L) \leq 0$ on each L-orbit. This latter condition does not locally restrict the signature of II on TQ over $D^0 - \mathrm{SP}$, and is entirely determined by $\mathrm{I} = j^*\langle\, ,\,\rangle$. On such a surface, no L-invariant set in NB can both fail to contain a stall point and have boundary oriented by L. Thus there can be no Rheeb components and no simply connected invariant sets with nonempty boundary.

(F) *Restrictions on Configurations* $SP \subset D^0 \subset Q \subset \mathbf{M}^3$. Theorem 4 imposes restrictions on the possible configurations of D^0 in Q. For example, if a type-(n, m) loop, $c: S^1 \to \mathbf{M}^3$, is both a component of D^0 for $j: Q \to \mathbf{M}^3$ and bounds a disk that is a componnt of D^+, then Corollary 5 implies $|n+m| = 2$. If it bounds a disk that is a component of D^- then $|n-m| = 2$. Similarly, (E7) imposes restrictions; for example, there exists no compact $j: Q \to \mathbf{M}^3$ with D^0 having five components, NB having one component, and $\mathrm{SP} = \emptyset$. Finally, $[\gamma] \in H^1(\mathrm{NB}, \mathbf{Z})$ can also be used to impose restrictions. For example, if D^- is a union of disks then $[\gamma] = 0$. But this implies that the degree of $g: Q \to S^2$ is zero; hence Q must be a torus.

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