A Converse Fatou Theorem

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1. Introduction

A well-known result of Fatou states that every positive solution of the Laplace equation on the upper half-space $\mathbf{R}_+^{n+1}$ has a finite nontangential limit at Lebesgue almost every point of the horizontal boundary $\mathbf{R}^n$ (cf. [2], [9]). It is also known that the nontangential approach region (a cone) in this result cannot be replaced by one which is bounded by a surface tangential to the boundary (cf. [4], [11]).

However, there are many surfaces which do not lie entirely in a cone, yet are not tangential to the boundary. Recently, Nagel and Stein [6] obtained a new Fatou theorem for solutions of the Laplace equation on $\mathbf{R}_+^{n+1}$. Their approach regions allow sequential approach to the boundary at any desired degree of tangency. Their results were generalized in [5] to improve the classical approach regions for certain parabolic equations on $\mathbf{R}_+^{n+1}$ and for the heat equation on the right half-space.

The most important condition on these new approach regions Ω involves the Lebesgue measure of their cross sections

$$\Omega(t)=\{x\in\mathbf{R}^n\colon (x,t)\in\Omega\}$$

for every height $t>0$. However, it is intuitively clear that boundary limits from within an approach region only involve the structure of the region close to the boundary point. It is shown in Section 2 that this is indeed true. There we obtain a Fatou theorem for "locally admissible" regions.

In Section 3 we show that these locally admissible regions are the only ones which permit every bounded solution to have finite limits from within them at almost every boundary point. The proof of this result is accomplished by reducing to the case of a sequence of convolution operators on a group with finite Haar measure and then applying techniques developed by Stein [8] and Sawyer [7]. It appears that this reduction to the case of finite measure could be avoided by invoking Stein's theorem as presented in Chapter 6 of [3].

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2. Preliminaries

We first establish some notation and assumptions. With n a positive integer, let $\mathbf{R}_+^{n+1}=\{(x,t)\in\mathbf{R}^{n+1}: x\in\mathbf{R}^n,\ t>0\}$, $Q_n=[-\frac{1}{2},\frac{1}{2})^n$, and let Z^n be the set of lattice points in $\mathbf{R}^n$. The Lebesgue measure of a set $E\subset\mathbf{R}^n$ is denoted by $|E|$. Let ρ be a translation-invariant pseudo-distance on $\mathbf{R}^n$. That is, for all x,y,z in $\mathbf{R}^n$, $\rho:\mathbf{R}^n\times\mathbf{R}^n\to[0,\infty)$ satisfies

$$\text{(2.1)}\qquad \begin{aligned} &\text{(a)}\ \ \rho(x,y)=0\ \text{ iff } x=y,\\ &\text{(b)}\ \ \rho(x,y)=\rho(y,x),\\ &\text{(c)}\ \ \rho(x,z)\le\gamma[\rho(x,y)+\rho(y,z)],\\ &\text{(d)}\ \ \rho(x+z,y+z)=\rho(x,y), \end{aligned}$$

where $\gamma\ge1$ is a constant independent of x,y,z. We denote $|x|=\rho(x,0)$, and $\|x\|$ denotes the usual Euclidean norm of x. We write

$$B(x,r)=\{y\in\mathbf{R}^n: |x-y|<r\}$$

for the ρ-ball with center x and radius r. It is further assumed that

$$\text{(2.2)}\qquad \text{(a)}\ \ \{B(0,r): r>0\}$$

is a base of open neighborhoods for the Euclidean topology of $\mathbf{R}^n$, and that

$$\text{(b)}\ \ \tau_n(\alpha)=\sup_{r>0}\frac{|B(0,r\alpha)|}{|B(0,r)|}<\infty\ \text{ for each } \alpha>0.$$

Assumptions similar to (2.2) as well as facts similar to the next lemma are treated in [1]. The following lemma is used in Section 3.

LEMMA 2.1.

(a) *There is a constant* $\epsilon>0$ *such that* $\{B(m,\epsilon): m\in Z^n\}$ *is a disjoint family.*

(b) *The number of lattice points in* $B(0,r)$ *is at most* $|B(0,\gamma(r+\epsilon))|\cdot|B(0,\epsilon)|^{-1}$.

(c) *For each* $r>0$, *there exist finite* $A(r)$ *and* $B(r)$ *such that*

$$\text{(2.3)}\qquad \|x\|<r\ \textit{ implies }\ |x|<A(r)$$

and

$$\text{(2.4)}\qquad |x|<r\ \textit{ implies }\ \|x\|<B(r).$$

Proof. (a) By (2.2)(a) we may choose $\epsilon>0$ so that $B(0,\epsilon)\subset\{x\in\mathbf{R}^n: \|x\|<\frac{1}{2}\}$. Then (a) follows by the translation invariance of ρ.

(b) This follows from (a) because if m is a lattice point in $B(0,r)$, then the triangle inequality for ρ implies that $B(m,\epsilon)$ is a subset of $B(0,\gamma(r+\epsilon))$.

(c) The proofs of (2.3) and (2.4) are nearly identical, so we prove (2.4). Suppose $|x_j|<r$ yet $\|x_j\|\to\infty$ as $j\to\infty$. By (2.2)(a) there exists $\beta(r)>0$ such that if $|x|<\beta(r)$ then $\|x\|<r$. By passing to a subsequence we may assume that $\|x_i-x_j\|\ge r$ if $i\ne j$; so $|x_i-x_j|\ge\beta(r)$ if $i\ne j$. It follows that the ρ-balls $B(x_j,\beta(r)/2\gamma)$, $j=1,2,3,\dots$ are disjoint; so their union has infinite Lebesgue measure, yet is a subset of $B(0,\gamma r+\beta(r)/2)$ which has finite measure, by (2.2). This contradiction establishes (2.4) and concludes our proof of Lemma 2.1. □

For each $t>0$ let K_t be a nonnegative measurable function on $\mathbf{R}^n$ satisfying

(2.5) (a) $\int K_t(x)\,dx\to 1$ as $t\to 0^+$.

(b) For all $(x,t)\in\mathbf{R}^{n+1}_+$, $K_t(x)\le|B(0,t)|^{-1}\cdot\phi(|x|/t)$, where ϕ is a bounded and decreasing real-valued function on $[0,\infty)$ for which $\sum_{k=1}^{\infty}\tau_n(2^{k+1})\phi(2^k)<\infty$.

(c) For each $x_0\in\mathbf{R}^n$, open $W\ni x_0$, and $0<T\le\infty$, there exist open sets $U\supset V\ni x_0$, $U\subset W$, and $(y_0,s_0)\in\mathbf{R}^n\times(0,T)$ such that, for all $x\in V$, $y\in\mathbf{R}^n\setminus U$, and t sufficiently close to 0, $K_t(x-y)\le\delta(t)K_{s_0}(y_0-y)$, where $\delta(t)\to 0$ as $t\to 0^+$.

(d) There is a constant $A>0$ such that $\int_{B(0,t)}K_t(x)\,dx>A$ for all $t>0$.

These conditions are satisfied by the kernels appearing in the integral representation of the positive solutions of Laplace's equation and certain parabolic equations (cf. [5]).

Let $\Omega\subset\mathbf{R}^{n+1}_+$, $t>0$, and $\alpha>0$. Define $\Omega(t)=\{x\in\mathbf{R}^n\colon (x,t)\in\Omega\}$, and

$$\Omega_\alpha=\left\{(x,t)\in\mathbf{R}^{n+1}_+\colon |x-x_0|<\alpha\left(t-\frac{t_0}{2}\right)\text{ for some }(x_0,t_0)\in\Omega\right\}.$$

Then we have the following properties:

(i) $\Omega\subset\Omega_\alpha$;
(ii) if $0<s<t$ then $\Omega_\alpha(s)\subset\Omega_\alpha(t)$;
(iii) if $(0,0)\in\bar\Omega$ then $(0,t)\in\Omega_\alpha$ for all $\alpha>0$ and $t>0$;
(iv) if $(y,s)\in\Omega_\alpha$ and $|x-y|<\alpha(t-s)$ then $(x,t)\in\Omega_{\gamma\alpha}$.

We now introduce the concept of locally α-admissible (which is to be compared with that of α-admissible in [5]).

DEFINITION 2.2. Let $\Omega\subset\mathbf{R}^{n+1}_+$ be open and let $\alpha>0$. Then Ω is said to be *locally α-admissible* if

(a) $0\in\Omega(t)$ for all $t>0$;
(b) $0<s<t$ implies $\Omega(s)\subset\Omega(t)$; and
(c) there exists open $\Omega'\supset\Omega$ such that
 (i) $|\Omega'(t)|=O(|B(0,t)|)$ as $t\to 0^+$, and
 (ii) if $(y,s)\in\Omega$ and $|x-y|<\alpha(t-s)$ then $(x,t)\in\Omega'$.

If ρ is a metric then in Definition 2.2 we take $\Omega'=\Omega$ and replace (a) by $(0,0)\in\bar\Omega$. Observe that if (i) is modified to read

$$|\Omega'(t)|\le c|B(0,t)|\ \text{ for a constant } c \text{ and all } t>0,$$

then Ω is α-admissible.

Now let $\Omega \subset \mathbf{R}_+^{n+1}$ have (0,0) as a limit point. For each regular Borel measure μ on $\mathbf{R}^n$ and $y \in \mathbf{R}^n$, define

$$M_\Omega^* \mu(y) = \limsup_{\Omega \ni (x,t) \to 0} \frac{\mu(\beta(y+x,t))}{|B(0,t)|},$$

$$N^*\mu(y) = \limsup_{t \to 0^+} \frac{\mu(y+Q(t))}{|Q(t)|},$$

where

$$Q(t) = \bigcup \{B(x,t) : x \in \Omega(t)\}.$$

Then, by carefully examining the proof that the maximal function

$$N\mu(y) = \sup_{t>0} \frac{\mu(y+Q(t))}{|Q(t)|}$$

is weak type (1,1) if Ω is α-admissible ([5, Lemma 1.7], [6, Thm. 1]), one sees that N^* is weak type (1,1) if Ω is locally α-admissible. Hence we obtain the following.

THEOREM 2.3. *Let Ω be locally α-admissible. Then there is a constant $c>0$ such that, for any finite Borel measure μ on $\mathbf{R}^n$,*

$$|\{x \in \mathbf{R}^n : M_\Omega^* \mu(x) > \lambda\}| \le c \frac{|\mu|}{\lambda} \quad \textit{for all } \lambda > 0.$$

Then, as in [5], we obtain the following general Fatou theorem.

THEOREM 2.4. *If Ω is locally α-admissible with $(0,0)$ as a limit point, and if μ is a signed measure such that*

$$K\mu(x,t) = \int_{\mathbf{R}^n} K_t(x-y)\, d\mu(y)$$

is finite on $\mathbf{R}^n \times (0,T)$ for some $0 < T \le \infty$, then

$$\lim_{\Omega \ni (x,t) \to 0} K\mu(x+x_0,t) = \frac{d\mu}{dm}(x_0) \quad \textit{for Lebesgue a.e. } x_0 \in \mathbf{R}^n.$$

3. A Converse Fatou Theorem

In this section our principal result is Theorem 3.2. The main tool is Theorem 2 of [7]; our problem in applying it is that Lebesgue measure on $\mathbf{R}^n$ is not totally finite. We therefore restrict ourselves to Q_n and periodize our kernel K_t to obtain H_t. We then show that a certain maximal operator associated with H_t satisfies a weak type inequality.

For this section we assume that $\Omega \subset \mathbf{R}_+^{n+1}$ is open, that $\Omega \subset Q_n \times (0, \frac{1}{2})$, and that Ω has the origin as its only limit point in the boundary $\mathbf{R}^n$. We remark that conditions (a) and (c) of (2.5) on the kernel K_t are not used in what follows.

THEOREM 3.1. *Let $1 \le p < \infty$ and suppose that, for each $f \in L^p(\mathbf{R}^n)$,*

(3.1) $$\sup_{(x,t)\in\Omega} |(K_t * f)(x+x_0)| < \infty \quad \textit{for a.e. } x_0 \in \mathbf{R}^n.$$

Then, for each $\alpha > 0$,

(3.2) $$|\Omega_\alpha(t)| = O|B(0,t)| \quad \textit{as } t \to 0^+.$$

Proof. Assumption (2.5)(b) implies that $K_t \in L^1(\mathbf{R}^n)$, so the convolution in (3.1) is well defined. As in [9, Ch. 7.2], let

(3.3) $$H_t(x) = \sum_{m \in Z^n} K_t(x+m) \quad \text{for } x \in \mathbf{R}^n,\ t > 0.$$

Then $H_t \in L^1(Q_n)$ and $H_t(x+m) = H_t(x)$ for all x, t, and m.

We want to establish that, for each $f \in L^p(Q_n)$,

(3.4) $$\sup_{(x,t)\in\Omega} \left| \int_{Q_n} f(s) H_t(x+x_0-s)\, ds \right| < \infty \quad \text{for a.e. } x_0 \in Q_n.$$

By Lemma 2.1(c) we choose $a > 0$ so that $|y| < a$ whenever $\|y\| < \frac{3}{2}\sqrt{n}$ ($\sqrt{n}$ is the Euclidean diameter of Q_n). Now fix $b > \gamma(a + \frac{1}{2})$ and let $(x,t) \in \Omega$, $x_0 \in Q_n$. We note that

$$\begin{aligned} \left| \int_{Q_n} f(s) H_t(x+x_0-s)\, ds \right| &\le \sum_{|m|<b} \int_{Q_n - m} |f(m+s)| K_t(x+x_0-s)\, ds \\ &\quad + \int_{Q_n} |f(s)| \sum_{|m| \ge b} K_t(x+x_0+m-s)\, ds \\ &= I + II. \end{aligned}$$

By Lemma 2.1(b), the set of lattice points m with $|m| < b$ is finite; so, by (3.1), $\sup_{(x,t)\in\Omega} I < \infty$ for a.e. $x_0 \in \mathbf{R}^n$. We now estimate the sum that appears in II using $y = x + x_0 - s$ (which guarantees that $\|y\| < \frac{3}{2}\sqrt{n}$). For $0 < t < \frac{1}{2}$, let $k_0(t)$ be the (nonnegative) integer defined by

$$2^{k_0(t)} \le \frac{b - a\gamma}{\gamma t} < 2^{k_0(t)+1}.$$

Then, since $|y| < a$, we use (mainly) the triangle inequality for ρ, the fact that ϕ decreases, and Lemma 2.1(b) to obtain that

$$\begin{aligned} \sum_{|m| \ge b} K_t(m+y) &\le \sum_{|m| \ge b} |B(0,t)|^{-1} \cdot \phi\left(\frac{|m+y|}{t} \right) \\ &\le \sum_{|m| \ge b} |B(0,t)|^{-1} \cdot \phi\left(\frac{\gamma^{-1}|m| - a}{t} \right) \\ &\le \sum_{k=k_0(t)}^{\infty} |B(0,t)|^{-1} \cdot \sum_{2^k \le (|m| - a\gamma)/\gamma t < 2^{k+1}} \phi\left(\frac{|m| - a\gamma}{\gamma t} \right) \\ &\le \sum_{k=k_0(t)}^{\infty} |B(0,t)|^{-1} \cdot \phi(2^k) \\ &\qquad |B(0, \gamma(2^{k+1}\gamma t + a\gamma + \epsilon))|\,|B(0,\epsilon)|^{-1} \le \end{aligned}$$

$$\leq \sum_{k=k_0(t)}^{\infty} |B(0,t)|^{-1}\cdot\phi(2^k)\cdot|B(0,2^{k+1}t\delta)|\,|B(0,\epsilon)|^{-1},$$

where

$$\delta=\gamma(\gamma+2a\gamma+2\epsilon)$$

$$\leq \tau_n(\delta)\,|B(0,\epsilon)|^{-1}\sum_{k=0}^{\infty}\phi(2^k)\tau_n(2^{k+1}),$$

which is a finite constant c_n by (2.5)(b). Thus $II \leq c_n \int_{Q_n} |f(s)|\,ds$ and (3.4) is established.

Now let $\{(x_j,t_j): j=1,2,\ldots\}$ be a dense subset of Ω. For $f\in L^p(Q_n)$ and $x\in Q_n$, we define

(3.5) $$T_j f(x)=\int_{Q_n} f(s)H_{t_j}(x+x_j-s)\,ds$$

and

(3.6) $$T^*f(x)=\sup_{j\geq 1}|T_j f(x)|.$$

Then, since $H_{t_j}\geq 0$ is periodic and $H_{t_j}\in L^1(Q_n)$, each T_j may be viewed as a positive and bounded convolution operator on $L^p(T^n)$, where T^n is the n-dimensional torus. Statement (3.4) above verifies that, for each $f\in L^p(T^n)$, $T^*f(x)$ is finite for a.e. $x\in T^n$. We conclude from Theorem 2 of [7] that

(3.7) $$|\{x\in T^n: T^*f(x)>\lambda\}|\leq\left(\frac{c\|f\|_{L^p(T^n)}}{\lambda}\right)^p$$

for all $f\in L^p(T^n)$ and for all $\lambda>0$, where c is independent of f and λ.

Now let $c_0=(4+\alpha)\gamma^2$. Choose t_0, $0<t_0<\frac{1}{2}$, so that $B(0,c_0t)\cup\Omega_\alpha(t)\subset Q_n$ whenever $0<t<t_0$. (This is possible because the origin is the only limit point of Ω which lies in the boundary $\mathbf{R}^n$.) Now fix t, $0<t<t_0$. Define $g(s)=1$ for $s\in B(0,c_0t)$ and define $g(s)=t$ for $s\in Q_n\backslash B(0,c_0t)$, and extend g periodically.

Next let $x\in\Omega_\alpha(t)$. Then $|x-y|<\alpha(t-s/2)$ for some $(y,s)\in\Omega$. Choose $(x_j,t_j)\in\Omega$ so that $|x_j-y|<t$ and $|t_j-s|<t$. It follows from the triangle inequality for ρ that $B(0,t_j)\subset B(x_j-x,c_0t)$. Hence we obtain that

$$T^*g(-x)\geq\int_{B(0,c_0t)}H_{t_j}(x_j-x-s)\,ds=\int_{B(x_j-x,\,c_0t)}H_{t_j}(s)\,ds$$

$$\geq\int_{B(0,t_j)}H_{t_j}(s)\,ds\geq\int_{B(0,t_j)}K_{t_j}(s)\,ds>A$$

by (2.5)(d).

What we have just shown is that $\Omega_\alpha(t)\subset\{x\in Q_n: (T^*g)(-x)>A\}$ for $0<t<t_0$. Applying (3.7), we obtain that

$$|\Omega_\alpha(t)|\leq\left(\frac{c}{A}\right)^p|B(0,c_0t)|\leq\left(\frac{c}{A}\right)^p\tau_n(c)\,|B(0,t)|$$

for $0<t<t_0$, which completes the proof of the theorem. □

We can now prove the main result of this section.

THEOREM 3.2. *Suppose for some* $1 \leq p < \infty$ *and every* $f \in L^p(\mathbf{R}^n)$ *that*

$$\lim_{\substack{(x,t)\to 0 \\ (x,t)\in\Omega}} (K_t * f)(x+x_0) \text{ exists for a.e. } x_0 \in \mathbf{R}^n. \tag{3.8}$$

Then, for each $\alpha > 0$, Ω *is contained in a locally* α*-admissible set.*

The proof is immediate because (3.8) (along with (2.5)(b)) implies (3.1), so we may apply Theorem 3.1 and take Ω_α for the required set.

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