NORMAL EXTENSIONS OF SUBNORMAL COMPOSITION OPERATORS

Alan Lambert

0. Introduction. A composition operator is an operator C of the form $Cf = f\circ T$, where T is a transformation on the state space X of a σ-finite measure space (X, Σ, m). Various questions of normality and semi-normality of such operators have been addressed. Usually the Radon–Nikodym derivatives $dm\circ T^{-n}/dm$ play a central role in these investigations. In this article it is shown how a subnormal composition operator may be extended to a normal composition operator. Section 1 deals with general properties of composition operators and re-states some known properties of composition operators regarding normality and semi-normality. Section 2 is concerned with establishing the extension of a subnormal composition operator to a quasi-normal composition operator. It is shown that if T is invertible (and bi-measurable) then this construction yields the minimal normal extension of C. The material in Section 3 relies heavily on a modification of an ergodic theory technique for constructing an invertible transformation in terms of T. This material is then used to construct a minimal normal composition operator extension of an arbitrary subnormal composition operator.

1. Preliminaries. Let (X, Σ, m) be a σ-finite measure space and let T be a mapping of X onto X such that $T^{-1}\Sigma \subseteq \Sigma$. The linear transformation C on $L^2_m = L^2(X, \Sigma, m)$ given by $Cf = f\circ T$ is called the *composition operator* induced by T. General properties of composition operators may be found in [7]. In particular, C is a bounded operator on L^2_m if and only if $m\circ T^{-1}$ is absolutely continuous with respect to m and the Radon–Nikodym derivative $dm\circ T^{-1}/dm$ is essentially bounded. Let $h = dm\circ T^{-1}/dm$. These assumptions will be made throughout the remainder of this article. Conditions for composition operators to belong to certain specific classes of operators have been widely studied. Proposition 1.1 below lists those results pertinent to this article together with references. The following notation will be used.

(i) For $f\in L^2_m$ or $f\geq 0$ a.e. dm, $E(f) = E(f\mid T^{-1}\Sigma)$ is the conditional expectation of f with respect to $T^{-1}\Sigma$. For $f\in L^2_m$, $E_n(f)$ is then the orthogonal projection $E_n(f) = E(f\mid T^{-n}\Sigma)$.

(ii) $h_n = dm\circ T^{-n}/dm$, $h = h_1$.

1.1. PROPOSITION. (a) *C is normal if and only if $T^{-1}\Sigma = \Sigma$, T is invertible and bi-measurable, and $h = h\circ T$ a.e. dm* ([9], [12]).

(b) *C is quasi-normal if and only if $h = h\circ T$ a.e. dm* ([10], [12]).

(c) *The following are equivalent.*

(i) *C is subnormal.*

Received October 28, 1987. Revision received March 16, 1988.
Michigan Math. J. 35 (1988).

(ii) *For every* $f \in L^2_m$, *there is a finite measure* μ_f *on* $I = [0, \|h\|_\infty]$ *such that, for each* $n \geq 0$, $\int_X h_n |f|^2 \, dm = \int_I t^n \, d\mu_f(t)$. (Throughout this paper I is the interval above. Any sequence of the form $\{\int_I t^n \, d\mu(t)\}$ will simply be referred to as a "moment sequence over I".)

(iii) *For every* Σ*-set* A *of finite measure*, $\{mT^{-n}A\}$ *is a moment sequence over* I.

(iv) *For almost every* x *in* X, $\{h_n(x)\}$ *is a moment sequence over* I ([5]).

(d) C *is hyponormal if and only if* $h > 0$ *a.e. and* $E(1/h) \leq 1/h \circ T$ *a.e.* dm ([4]).

Throughout the remainder of this article we assume that C is subnormal. Then from Proposition 1.1(c)(iii), for almost every $x \in X$ there is a probability measure μ_x on I such that, for each $n \geq 0$,

$$h_n(x) = \int_I t^n \, d\mu_x(t).$$

We will have occasion to use the following relation between h_n and h_{n+1}. (This relation holds for any composition operator regardless of its state of normality.)

1.2. LEMMA. $h_{n+1} = h \cdot [Eh_n] \circ T^{-1}$. (Even though T may not be invertible, the expression $(Eg) \circ T^{-1}$ is well defined since T is surjective and Eg is by definition a $T^{-1}\Sigma$-measurable function.)

Proof. Let $A \in \Sigma$. Then

$$\begin{aligned} mT^{-(n+1)}A &= mT^{-n}(T^{-1}A) \\ &= \int_{T^{-1}A} h_n \, dm \\ &= \int_A h(Eh_n) \circ T^{-1} \, dm. \end{aligned}$$

But $mT^{-(n+1)}A = \int_A h_{n+1} \, dm$. Since A was chosen arbitrarily, $h_{n+1} = h(Eh_n) \circ T^{-1}$. □

Now let $B(I)$ be the σ-ring of Borel sets in I. For each $J \in B(I)$, define ϕ_J on X by $\phi_J(x) = \mu_x(J)$.

1.3. LEMMA. ϕ_J *is* Σ*-measurable.*

Proof. Let $p_n(t) = \Sigma a_{nk} t^k$ define a sequence of polynomials which is uniformly bounded over I and converges pointwise to χ_J. Note that

$$\Sigma a_{nk} h_k(x) = \Sigma a_{nk} \int_I t^k \, d\mu_x(t) = \int_I p_n(t) \, d\mu_x(t).$$

This defines a sequence $\hat{p}_n = \Sigma a_{nk} h_k$, which is uniformly bounded in $L^\infty(X, \Sigma)$ and which converges pointwise a.e. dm to $\mu_x(J) = \phi_J(x)$, showing that ϕ_J is Σ-measurable. □

We conclude this section with a slight generalization of [3]. This more general result will be applied later in this article.

1.4. LEMMA. *Suppose that A is an operator on a Hilbert space H such that, for some interval I, there is a dense subset D of the unit ball of H and a probability measure μ_x on I for each x in D satisfying $\|A^n x\|^2 = \int_I t^n \, d\mu_x(t)$ $(n \geq 0)$. Then A is subnormal.*

Proof. According to [3], we must show that $\{\|A^n x\|^2\}$ is a moment sequence for every $x \in H$. Let $\{x_n\}$ be a sequence in D converging to x. Then $\{\mu_{x_n}\}$ has a weak-* convergent subsequence $\{\mu_{x_{n_k}}\}$ converging to a (sub)probability measure μ. In particular, for each integer $m \geq 0$ we have

$$\begin{aligned} \operatorname*{limit}_{k \to \infty} \|A^m x_{n_k}\|^2 &= \operatorname*{limit}_{k \to \infty} \int_I t^m \, d\mu_{x_{n_k}}(t) \\ &= \int_I t^m \, d\mu(t). \end{aligned}$$

But $\operatorname{limit}_{k \to \infty} A^m x_{n_k} = A^m x$, and the desired result holds true. □

2. Constructions of normal and quasi-normal extensions. We are assuming that C is subnormal and, in light of Proposition 1.1., that $h_n(x) = \int_I t^n \, d\mu_x(t)$. Let $Y = X \times I$ and let Γ be the σ-ring $\Sigma \times B(I)$ over Y. For each rectangle $A \times J$ in Γ define

$$\nu(A \times J) = \int_A \phi_J \, dm = \int_A \mu_x(J) \, dm(x).$$

2.1. LEMMA. *ν extends to a σ-finite measure on Γ.*

Proof. It suffices to show that if $\{A_i \times J_i\}$ is a sequence of mutually disjoint rectangles in Γ whose union is a rectangle $A \times J$, then $\nu(A \times J) = \Sigma \nu(A_i \times J_i)$. Indeed, for such a sequence we see that

$$\begin{aligned} \Sigma \nu(A_i \times J_i) &= \Sigma \int_{A_i} \left[\int_{J_i} d\mu_x \right] dm \\ &= \Sigma \int_X \left[\chi_{A_i}(x) \int_I \chi_{J_i}(t) \, d\mu_x(t) \right] dm(x) \\ &= \int_X \left[\int_I \Sigma \chi_{A_i \times J_i} \, d\mu_x \right] dm(x) \\ &= \int_X \left[\int_I \chi_{A \times J} \, d\mu_x \right] dm(x) \\ &= \nu(A \times J). \end{aligned}$$ □

Verification of the next assertion follows similar lines and so the proof is omitted.

2.2. LEMMA. *For $F \in L^1_\nu$,*

$$\int_Y F \, d\nu = \int_X \left[\int_I F(x, t) \, d\mu_x(t) \right] dm(x).$$

As a consequence of Lemma 2.2, we note that the inner product in L^2_ν is given by

$$(F, G)_\nu = \int_X \left[\int_I F\bar{G} \, d\mu_x(t) \right] dm(x).$$

Also, if $F \in L^2_\nu$ depending only on x (i.e., $F(x, t) = f(x)$ for some Σ-measurable function f), then

$$\begin{aligned} \|F\|_\nu{}^2 &= \int_X \left[\int_I |f(x)|^2 \, d\mu_x(t) \right] dm(x) \\ &= \int_X |f^2(x)| \, dm(x) \quad \text{(each } \mu_x \text{ is a prob. meas.)} \\ &= \|f\|^2_m. \end{aligned}$$

Thus, if H is the set of all such L^2_ν functions F then H is isometrically isomorphic to L^2_m. We shall refer to L^2_m as a subspace of L^2_ν.

2.3. LEMMA. *The orthogonal projection of L^2_ν onto L^2_m is given by*

$$(PF)(x) = \int_I F(x, t) \, d\mu_x(t).$$

Proof.

$$\begin{aligned} \|PF\|^2_m &\le \int_X \left[\int_I |F|^2 \, d\mu_x \right] dm \\ &= \|F\|^2_\nu. \end{aligned}$$

Also, since each μ_x is a probability measure, $P^2 = P$. Finally, for each $F \in L^2_\nu$,

$$\begin{aligned} (PF, F) &= \int_X \left[\int_I (PF)(x)\overline{F(x, t)} \, d\mu_x \right] dm \\ &= \int_X |PF|^2 \, dm = \|PF\|^2. \end{aligned}$$

Thus $P = P^*P$, completing the proof. □

We now define the transformation S on Y by $S(x, t) = (Tx, t)$. Then S is a surjection, and since $S^{-1}(A \times J) = (T^{-1}A) \times J$, S is Γ-measurable. Since we are about to examine the composition operator induced by S, we must compute the appropriate Radon–Nikodym derivative.

2.4. LEMMA. $(d\nu \circ S^{-1}/d\nu)(x, t) = t$ *a.e.* $d\nu$.

Proof. Let $A \times J$ be a measurable rectangle. Then

$$\begin{aligned} \nu \circ S^{-1}(A \times J) &= \nu(T^{-1}A \times J) \\ &= \int_{T^{-1}A} \phi_J \, dm = \int_A h(E\phi_J) \circ T^{-1} \, dm. \end{aligned}$$

As before, let $P_n(t) = \Sigma a_{nk} t^k$ define a uniformly bounded sequence of polynomials on I converging pointwise to χ_J, and let $\hat{P}_n = \Sigma a_{nk} h_k$. Then $\{E\hat{P}_n\}$ is uniformly bounded and converges a.e. dm to $E(\phi_J)$. Thus

$$\text{limit} \int_A (E\hat{P}_n)\circ T^{-1} h\, dm = \int_A h(E\phi_J)\circ T^{-1}\, dm$$
$$= \nu\circ S^{-1}(A\times J).$$

Now, it follows from Lemma 1.2 and the definition of the measures μ_x that

$$h(x)(E\hat{P}_n)\circ T^{-1}(x) = \Sigma a_{nk} h(x) E(h_k)\circ T^{-1}(x)$$
$$= \Sigma a_{nk} h_{k+1}(x) = \Sigma a_{nk} \int_I t^{k+1}\, d\mu_x(t)$$
$$= \int_I t P_n(t)\, d\mu_x(t),$$

so that

$$\lim h(x)(E\hat{P}_n)\circ T^{-1}(x) = \int_I t\chi_J(t)\, d\mu_x(t)$$
$$= \int_J t\, d\mu_x(t).$$

It then follows that

$$\nu\circ S^{-1}(A\times J) = \int_A \left[\int_J t\, d\mu_x(t)\right] dm(x)$$
$$= \int_{A\times J} t\, d\nu(x,t),$$

and so $d\nu\circ S^{-1}/d\nu = t$. □

Now define the operator Q on L^2_ν by $QF = F\circ S$; that is, $QF(x,t) = F(Tx,t)$.

2.5. THEOREM. (a) *Q is a bounded quasi-normal composition operator leaving L^2_m invariant. The restriction of Q to L^2_m is C.*

(b) *If T is invertible and bi-measurable, with the real-valued function $m\circ T$ mutually absolutely continuous with respect to m, then Q is the minimal normal extension of C.*

Proof. Q is bounded since its norm is $\|d\nu\circ S^{-1}/d\nu\|_\infty^{1/2}$. Let $g = d\nu\circ S^{-1}/d\nu$. Then $g(x,t) = t$, and so $g\circ S(x,t) = g(Tx,t) = t = g(x,t)$; that is, $g\circ S = g$. But this is precisely the characterization of quasi-normality given in Proposition 1.1. Let $f \in L^2_m$ $(\subseteq L^2_\nu)$. Then $(Qf)(x,t) = f(Tx,t)$, so $QL^2_m \subseteq L^2_m$. Moreover, $Q|_{L^2_m} = C$.

Now suppose that T is a bi-measurable bijection. Then so is S, and because $g\circ S = g$, Q is in fact normal. We now compute Q^*: Let f and F be in L^2_ν. Then

$$(Qf, \bar{F}) = \int_X \left[\int_I f(Tx,t)F(x,t)\, d\mu_x(t)\right] dm(x)$$
$$= \int_X h(x)\left[\int_I f(x,t)F(T^{-1}x,t)\, d\mu_{T^{-1}x}(t)\right] dm(x).$$

Since T is invertible, Lemma 1.2 reduces to $h_{n+1} = h\cdot h_n\circ T^{-1}$. It then follows by an argument similar to that given in Lemma 2.4 that $h(x)\, d\mu_{T^{-1}x}(t) = t\, d\mu_x(t)$, so that

$$(Qf,\bar{F})=\int_X\left[\int_I tf(x,t)F(T^{-1}x,t)\,d\mu_x(t)\right]dm(x),$$

and so $Q^*F(x,t)=tF(T^{-1}x,t)$. More generally, $Q^{*k}F(x,t)=t^kF(T^{-k}x,t)$. Let $f\in L^2_m$. Then $Q^{*k}f(x,t)=t^kf(T^{-k}x)$. Now, because T is invertible, C has dense range and so for each $k\geq 0$, $C^kL^2_m$ is dense in L^2_m. It then follows that $Q^{*k}L^2_m=t^kL^2_m$ (where t^k is in actuality the corresponding multiplication operator). This shows that the closed linear span $\bigvee_{k=0}^{\infty}Q^{*k}L^2_m$ contains all functions of the form $\chi_{A\times J}$ with $m(A)<\infty$, so $\bigvee_{k=0}^{\infty}Q^{*k}L^2_m=L^2_\nu$. This in turn shows that Q is the minimal normal extension of C ([1, p. 128]). □

2.6. LEMMA. *Suppose that there is a finite Borel measure μ over I such that μ_x is absolutely continuous with respect to μ for almost every x. Let $u(x,t)=d\mu_x/d\mu$, and for any function $f(x,t)$ let $f_t(x)=f(x,t)$. Suppose that $E(u_t)>0$ a.e. dm for all $t\in I$. Then*

$$(Q^*f)(x,t)=t[(Eu_t)\circ T^{-1}(x)]^{-1}[E(u_tf_t)]\circ T^{-1}(x).$$

Proof.

$$\begin{aligned}(Qf,\bar{g})&=\int_X\left[\int_I f(Tx,t)g(x,t)u(x,t)\,d\mu(t)\right]dm(x)\\&=\int_X h(x)\left[\int_I f(x,t)E(u_tg_t)\circ T^{-1}(x)\,d\mu(t)\right]dm(x)\\&=\int_X\left[\int_I (f(x,t))h(x)\frac{E(u_tg_t)}{E(u_t)}\circ T^{-1}(x)E(u_t)\circ T^{-1}(x)\,d\mu(x)\right]dm(x).\end{aligned}$$

As in previous arguments, we see that $h(x)(Eu_t)\circ T^{-1}(x)=tu_t(x)$ so that

$$(Qf,\bar{g})=\int_X\left[\int_I f(x,t)t\frac{E(u_tg_t)}{E(u_t)}\circ T^{-1}(x)\,d\mu(t)\right]dm(x),$$

yielding the indicated formula for Q^*g. □

One attempt to generate a measure μ appropriate for application in the preceding lemma is as follows: Let r be a strictly positive function on X of L^1_m norm one. Since each ϕ_J is measurable, we define $\mu(J)=\int_X r(x)\phi_J(x)\,dm(x)$. Then μ is a probability measure. Moreover, for each J with $\mu(J)=0$, $\mu_x(J)=0$ for almost every x. However, without some extra conditions imposed on $\{\mu_x\}$ it is possible that for every x, $\mu_x\perp\mu$. For example, if m is Lebesgue measure on $[0,1]$ and μ_x is the point mass at x, then $\phi_J=\chi_J$ and $\mu=m$. However, this technique is applicable in at least one general situation of some interest, as shown below.

Suppose $X=\{x_i\}$ is countable. Then there exists a dominating measure μ for $\{u_{x_i}\}$. Indeed, m consists of point masses $\{m_i\}$. Let $\mu(J)=\Sigma(m_k/2^{|k|})\mu_{x_k}(J)$. Then $\mu(J)=0$ if and only if $\mu_{x_k}(J)=0$ for all k.

3. Construction of a normal composition operator extension of a quasi-normal composition operator. It was shown in Section 2 that if C is subnormal but T is not invertible, then C has a quasi-normal composition operator extension. Several constructions of normal extensions of quasi-normal operators are known (see

[1, p. 135], [1, p. 116]) but it is not clear if these give rise to composition operators. In [6] a construction is given of a bi-measurable bijection associated with a non-invertible transformation. That construction is modeled after the result of Rohlin [8] for measure-preserving transformations (see [2, p. 239] for a discussion of Rohlin's result and related topics). We will outline the construction in [6].

Let $T\Sigma \subset \Sigma$, $T^{-1}\Sigma \subset \Sigma$, $h>0$, $TX=X$, and the real-valued mapping $m\circ T$ mutually absolutely continuous with respect to m (in the sense of having the same null sets). Let Z be the inverse limit space

$$Z=\{z=\langle z_0, z_1, \ldots\rangle : \text{each } z_i \in X \text{ and } Tz_{i+1}=z_i\}.$$

Then Z is nonempty. For $A\in\Sigma$ and $n\geq 0$ let $(A)_n=\{z\in Z: z_n\in A\}$. Let Δ be the σ-field generated by $\{(A)_n : A\in\Sigma,\ n\geq 0\}$. The mapping $\lambda((A)_n)=\int_A H_n\,dm$ is well defined where $H_0=1$ and $H_n=1/[h\circ T\cdots h\circ T^n]$ for $n\geq 1$. Then λ extends to a σ-finite measure on Δ. The transformation R on Z given by $R\langle z_0, z_1, \ldots\rangle = \langle Tz_0, Tz_1, \ldots\rangle$ is a bi-measurable bijection ($R^{-1}\langle z_0, z_1, \ldots\rangle = \langle z_1, z_2, \ldots\rangle$). Moreover, if $G=d\lambda\circ R^{-1}/d\lambda$, then $G(\langle z_0, z_1, \ldots\rangle)=h(z_0)$. Also, the composition operator W induced by R on L^2_λ leaves L^2_m invariant, where L^2_m is identified (isometrically) with the set of L^2_λ functions depending on z_0 only. $W|_{L^2_m}$ is C.

3.1. THEOREM. *Let C and W be as above. If C is quasi-normal, then W is a normal composition operator extension of C.*

Proof. If C is quasi-normal, then $h\circ T=h$. It then follows, from the characterization of G above, that $G\circ R=G$. But R is a bi-measurable bijection, so (via Proposition 1.1) W is normal. □

Theorem 3.1 can now be combined with the results of Section 2 to construct minimal normal extensions.

3.2. THEOREM. *Let C be a subnormal composition operator with $T\Sigma\subseteq\Sigma$ and $T^{-1}\Sigma\subseteq\Sigma$, and such that $m\circ T^{-1}$ and $m\circ T$ are mutually absolutely continuous with respect to m. Then the minimal normal extension of C is a composition operator.*

Proof. If T is invertible then Theorem 2.5(b) yields the required result. Assume then that T is not invertible. The constructions from this and the preceding section yield $L^2(X,\Sigma,m)\subseteq L^2(Y,\Gamma,\nu)\subseteq L^2(Z,\Delta,\lambda)$ (isometric embeddings) and $C\subseteq Q\subseteq W$. We must show that W is the minimal normal extension of C. We will show that Q is the minimal quasi-normal extension of C, and that W is the minimal normal extension of Q. It then follows from Embry-Wardrop's theorem [11] that W is the minimal normal extension of C. Let $k\geq 0$ and let $f\in L^2_m$. Let $F\in L^2_\nu$ be given by $F(x,t)=f\circ T^k(x)$. Then $Q^{*k}F(x,t)=t^k f(x)$. Thus $Q^{*k}L^2_m\supseteq t^k L^2_m$. From the construction of L^2_ν, $\bigvee_{k=0}^{\infty} Q^{*k}L^2_m=L^2_\nu$. This shows the minimality of Q over C. Now let $g=d\nu\circ S^{-1}/d\nu$ and $G=d\lambda\circ R^{-1}/d\lambda$. Then $G\langle z_0, z_1\ldots\rangle = g(z_0)$. For $f\in L^2_\nu$ let $F\langle z_0, z_1, \ldots\rangle = f(z_0)$. Then $W^{*k}F\langle z_0, z_1, \ldots\rangle = [g(z_0)]^k f(z_k)$. Now any function on Z depending only on the variables $z_0, \ldots, z_k$ depends only on z_k, because $z_0=S^k z_k, \ldots, z_{k-1}=Sz_k$. The set of all such functions forms a closed subspace L_k of L^2_λ. Moreover, the function $G(z)=g(z_0)$ is strictly positive and bounded, and $G\cdot L_k \subset L_k$, so GL_k is dense in L_k. Thus $W^{*k}L_\nu{}^2 \supset L_k$. But $\bigvee_{k=0}^{\infty} L_k = L^2_\lambda$ (i.e., the set of all L^2_λ-functions depending on only a finite number of variables is dense in L^2_λ). This shows that W is the minimal normal extension of Q, and thus also of C. □

REFERENCES

1. J. Conway, *Subnormal operators,* Research Notes in Math., 51, Pitman, Boston, 1981.
2. I. Cornfield, S. Fomin, and Y. Sinai, *Ergodic theory,* Springer, New York, 1982.
3. A. Lambert, *Subnormality and weighted shifts,* J. London Math. Soc. (2) 14 (1976), 476–480.
4. ———, *Hyponormal composition operators,* Bull. London Math. Soc. 18 (1986), 395–400.
5. ———, *Subnormal composition operators,* Proc. Amer. Math. Soc. 103 (1988), 750–754.
6. ———, *A bi-measurable bijection generated by a non-measure preserving transformation,* Rocky Mountain J. Math., to appear.
7. E. Nordgren, *Composition operators in Hilbert space; Hilbert space operators,* Lecture Notes in Math., 693, Springer, Berlin, 1978.
8. P. V. Rohlin, *Exact endomorphisms of Lebesgue spaces,* Izv. Akad. Nauk SSR Ser. Mat. 2 (1961), 499–530.
9. R. Singh and A. Kumar, *Characterization of invertible, unitary, and normal composition operators,* Bull. Austral. Math. Soc. 19 (1978), 81–86.
10. R. Singh, A. Kumar, and D. Gupta, *Quasinormal composition operators on l^2_p,* Indian J. Pure Appl. Math. 11 (1980), 904–907.
11. M. R. Embry-Wardrop, *Quasinormal extensions of subnormal operators,* Houston J. Math. 7 (1981), 191–204.
12. R. Whitley, *Normal and quasinormal composition operators,* Proc. Amer. Math. Soc. 76 (1978), 114–118.

Department of Mathematics
University of North Carolina-Charlotte
Charlotte, NC 28223