COMPLETE SPECTRAL AREA ESTIMATES AND SELF-COMMUTATORS

Takahiko Nakazi

Dedicated to Professor Shōzō Koshi on his sixtieth birthday

1. Introduction. Let A be a uniform algebra on a compact Hausdorff space X; that is, A is a closed subalgebra of $C(X)$ which contains the constant functions and which separates the points of X. The spectrum $\sigma(f)$ of $f \in A$ is the compact set of complex numbers λ such that $1/(\lambda - f)$ does not belong to A. The norm $\|u\|$ of $u \in C(X)$ is the supremum on X of the absolute value of u. Alexander [2, Lemma 2] proved the following theorem, using a quantitative version of the classical Hartogs–Rosenthal theorem on rational approximation in the complex plane.

ALEXANDER'S THEOREM. *If f is in A then*

$$\operatorname{dist}(\bar{f}, A) \leq \{\operatorname{Area}(\sigma(f))/\pi\}^{1/2},$$

where $\bar{f}$ is the complex conjugate of f and $\operatorname{dist}(\bar{f}, A) = \inf\{\|\bar{f} - g\| : g \in A\}$.

In Section 3 of this paper we give a new proof of this theorem. The proof we give is very abstract. We use a distance formula in a uniform algebra, which will be proved in Section 2, and we will need the famous Putnam inequality in operator theory. In Section 4, using Alexander's theorem, we will give an area estimate of a complete spectral set for the distance from the adjoint T^* of T to some norm closed algebra generated by T and $(T-\lambda)^{-1}$, where T is a bounded linear operator and λ is not in the complete spectral set. In Section 5, we will give an area estimate of the spectrum $\sigma(T)$ of a hyponormal operator T for the distance from T^* to some weakly closed algebra generated by T and $(T-\lambda)^{-1}$ for $\lambda \notin \sigma(T)$. In Section 6, we will show an area estimate of the complete spectrum set for the self-commutator of a bounded linear operator. This estimate looks like the Putnam inequality.

In this paper $\mathcal{H}$ denotes a Hilbert space and $\mathcal{L}(\mathcal{H})$ is the set of all bounded linear operators on $\mathcal{H}$. If T is in $\mathcal{L}(\mathcal{H})$ and $T^*T - TT^*$ is nonnegative then we call T a *hyponormal* operator. In Section 6 we shall show that, if T is a hyponormal operator and K is in $\mathcal{L}(\mathcal{H})$ with $KT = TK$, then

$$\|T^*K - KT^*\| \leq 2\{\operatorname{Area}(\sigma(T)\}^{1/2}\|K\|.$$

Moreover, for any T in $\mathcal{L}(\mathcal{H})$ we shall show the result above with a complete spectral set instead of $\sigma(T)$.

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2. Distance formula. Let A be a uniform algebra on X. Suppose $A^\perp$ denotes the set of all orthogonal measures on X to A. For any nonzero μ in $A^\perp$, $H^2(|\mu|)$ denotes the $L^2(|\mu|)$ closure of A; let P^μ be the orthogonal projection from $L^2(|\mu|)$ to $H^2(|\mu|)$. Then, for any ϕ in $C(X)$,

$$\operatorname{dist}(\phi, A) \geq \sup\{\|(1-P^\mu)M_\phi P^\mu\| : \mu \in A^\perp \text{ and } \|\mu\| \leq 1\},$$

where M_ϕ is the multiplication operator on $L^2(|\mu|)$. The following lemma shows that equality holds.

LEMMA 1. *For any ϕ in $C(X)$,*

$$\operatorname{dist}(\phi, A) = \sup\{\|(1-P^\mu)M_\phi P^\mu\| : \mu \in A^\perp \text{ and } \|\mu\| \leq 1\},$$

and the supremum is attained.

Proof. Assuming $\phi \notin A$, by the Hahn-Banach theorem there exists a nonzero measure $\mu \in A^\perp$ with $\|\mu\| \leq 1$ such that

$$\operatorname{dist}(\phi, A) = \int \phi \, d\mu.$$

Let $F = d\mu/d|\mu|$; then $\bar{F} \in (1-P^\mu)L^2(|\mu|)$ and $1 \in P^\mu L^2(|\mu|) = H^2(|\mu|)$. Hence

$$\int \phi \cdot 1 \cdot F \, d\mu \leq \|(1-P^\mu)M_\phi P^\mu\|$$

and the lemma follows. □

Let Y be a compact subset of the plane whose complement has a finite number of components, say $n+1$, and let m be the harmonic measure of a point in the interior of Y. Let $X = \text{Boundary } Y = X_0 \cup X_1 \cup \cdots \cup X_n$, where X_0 is the component of X that is the boundary of the unbounded component of the complement of Y. Define $v_j \in C(X)$ to be 1 on X_j and 0 on $X \backslash X_j$, $1 \leq j \leq n$.

THEOREM 1. *Let $R(Y)$ be the uniform closure of the set of rational functions in $C(Y)$ and $A = R(Y)|X$. Then for any ϕ in $C(X)$,*

$$\operatorname{dist}(\phi, A) = \sup\left\{\|(1-P^v)M_\phi P^v\| : \right.$$
$$\left. v = \exp \sum_{j=1}^n t_j v_j = |F|,\ F dm \in A^\perp \text{ and } \int |F| \, dm \leq 1\right\},$$

and the supremum is attained.

Proof. By the theorem of F. and M. Riesz (cf. [7, Chap. 4, Thm. 3.3]), $A^\perp = A^\perp \cap L^1(m)$. If $f \in A^\perp \cap L^1(m)$ then $\log|f| \in L^1(m)$ and there exist $\{t_j\}_{j=1}^n$ such that $\log|f| - \sum_{j=1}^n t_j v_j$ has a single-valued harmonic conjugate [11, Chap. I, §10]. Set $u = \log|f| - \sum_{j=1}^n t_j v_j$ and $h^2 = \exp(u + i\,{}^*u)$, where ${}^*u$ is the harmonic conjugate of u. Then h is an outer function in the Hardy space $H^2(m)$, that is, the $L^2(m)$-closure of A. Let $F = fh^{-2}$; then $|F| = v = \exp \sum_{j=1}^n t_j v_j$ and $F \in A^\perp \cap L^1(m)$ because $A^\perp \cap L^1(m)$ has the form $s^{-1}H^1(m)$, where $H^1(m)$ is the Hardy space, s is an invertible function in L^∞, and h is an outer function. Since $|f| = v|h|^2$,

$$H^2(|f|dm)=h^{-1}H^2(vdm) \quad \text{and} \quad H^2(|f|dm)^\perp=\bar{h}^{-1}H^2(vdm)^\perp.$$

For $h^{-1}g\in H^2(|f|dm)$ and $\bar{h}^{-1}l\in H^2(|f|dm)^\perp$,

$$\int \phi h^{-1}g\cdot h^{-1}\bar{l}\,|f|\,dm=\int \phi g\cdot \bar{l}v\,dm;$$

also, $\int |h^{-1}g|^2|f|\,dm=\int |g|^2 v\,dm$ and $\int |\bar{h}^{-1}l|^2|f|\,dm=\int |l|^2 v\,dm$. This implies

$$\|(1-P^{|f|})M_\phi P^{|f|}\|=\|(1-P^v)M_\phi P^v\|$$

and the theorem follows from Lemma 1. □

The distance formula looks like Theorem 3 in [9] and relates with Theorem 1.3 in [8]. If A is the disc algebra then F is an inner function and $v=1$ in Theorem 1, and hence $\operatorname{dist}(\phi,A)=\|(1-P^1)M_\phi P^1\|$. Of course this is a special case of Nehari's theorem for Hankel operators [10]. We could prove a distance formula similar to Theorem 1 for a finite codimension subalgebra of the disc algebra.

Let $\mathcal{A}$ be an algebra of bounded linear operators on $\mathcal{H}$ and lat $\mathcal{A}$ denote the lattice of all $\mathcal{A}$-invariant projections. We shall write $l^2\otimes\mathcal{H}$ for the Hilbert space direct sum $\mathcal{H}\oplus\mathcal{H}\oplus\cdots$, and $1\otimes T$ will denote the operator $T\oplus T\oplus\cdots\in\mathcal{L}(l^2\otimes\mathcal{H})$ for each operator $T\in\mathcal{L}(\mathcal{H})$. The following lemma is due to Arveson [4, Lemma 2] and will be used in Section 5.

LEMMA 2. *Let $\mathcal{A}$ be an arbitrary ultra-weakly closed subalgebra of $\mathcal{L}(\mathcal{H})$ containing* 1, *and let $T\in\mathcal{L}(\mathcal{H})$. Then*

$$\operatorname{dist}(T,\mathcal{A})=\sup\{\|(1-P)(1\otimes T)P\|: P\in\operatorname{lat}(1\otimes\mathcal{A})\}.$$

In Lemma 2, if lat $\mathcal{A}$ is totally ordered then $\operatorname{dist}(T,\mathcal{A})=\sup\{\|(1-P)TP\|: P\in \operatorname{lat}\mathcal{A}\}$ [4, Thm. 1.1].

3. A new proof of Alexander's theorem. By Lemma 1, if $f\in\mathcal{A}$ then

$$\operatorname{dist}(\bar{f},A)=\sup\{\|(1-P^\mu)M_{\bar{f}}P^\mu\|: \mu\in A^\perp \text{ and } \|\mu\|\le 1\}.$$

For $\mu\in A^\perp$ with $\|\mu\|\le 1$ and $f\in A$,

$$\begin{aligned}P^\mu M_f(1-P^\mu)M_{\bar{f}}P^\mu &= P^\mu M_f M_{\bar{f}}P^\mu-P^\mu M_f P^\mu M_{\bar{f}}P^\mu\\ &= P^\mu M_{\bar{f}}P^\mu M_f P^\mu-P^\mu M_f P^\mu M_{\bar{f}}P^\mu\end{aligned}$$

because $H^2(|\mu|)$ is an A-invariant subspace. Set $T_f^\mu=P^\mu M_f\,|\,H^2(|\mu|)$; then

$$\operatorname{dist}(\bar{f},A)=\sup\{\|T_f^{\mu*}T_f^\mu-T_f^\mu T_f^{\mu*}\|^{1/2}: \mu\in A^\perp \text{ and } \|\mu\|\le 1\}.$$

Now we need an inequality due to Putnam [11].

PUTNAM'S THEOREM. *If T is a hyponormal operator in $\mathcal{L}(\mathcal{H})$ then*

$$\|T^*T-TT^*\|\le \operatorname{Area}(\sigma(T))/\pi.$$

By Putnam's theorem,

$$\operatorname{dist}(\bar{f},A)\le\sup\{(\operatorname{Area}(\sigma(T_f^\mu))/\pi)^{1/2}: \mu\in A^\perp \text{ and } \|\mu\|\le 1\}.$$

If f is invertible in A then T_f^μ is invertible in $\mathcal{L}(H^2(|\mu|))$ because $f^{-1}H^2(|\mu|)\subset H^2(|\mu|)$ and hence $\sigma(T_f^\mu)\subseteq\sigma(f)$. Thus

$$\operatorname{dist}(\bar{f},A)\leq\{\operatorname{Area}(\sigma(f))/\pi\}^{1/2}.$$

In the proof of Alexander's theorem we show there exists a measure μ in $A^\perp$ with $\|\mu\|\leq 1$ such that

$$\operatorname{dist}(\bar{f},A)=\|T_f^{\mu *}T_f^\mu-T_f^\mu T_f^{\mu *}\|^{1/2}.$$

For a special algebra as in Theorem 1, such a μ can be described. Moreover, then, $\sigma(T_f^\mu)=\sigma(f)$ by the proof of Lemma 2.2 in [1].

4. Complete spectral sets. Let T be in $\mathcal{L}(\mathcal{H})$ and let Y be a compact subset of the complex plane C which contains the spectrum of T. Y is called a *spectral set* for T if $\|f(T)\|\leq\|f\|$ for f in $\operatorname{rat}(Y)$, where $\operatorname{rat}(Y)$ denotes the set of all rational functions on Y. Arveson introduced a somewhat stronger definition (see [3, p. 277]). For each $k\geq 1$ let $\operatorname{rat}_k(Y)$ denote the algebra of all $k\times k$ matrices over $\operatorname{rat}(Y)$. Each element in $\operatorname{rat}_k(Y)$ is then a $k\times k$ matrix of rational functions $F=(f_{ij})$, and we may define a norm on $\operatorname{rat}_k(Y)$ in the obvious way:

$$\|F\|=\sup\{\|F(\lambda)\|:\lambda\in Y\}.$$

A compact plane set X is called a *complete spectral set* for T if X contains $\sigma(T)$ and

$$\|F(T)\|\leq\sup\{\|F(\lambda)\|:\lambda\in X\}$$

for every F in $\operatorname{rat}_k(X)$ and every $k\geq 1$.

THEOREM 2. *If X is a complete spectral set for T in $\mathcal{L}(\mathcal{H})$ then*

$$\operatorname{dist}(T^*,\mathfrak{A})\leq\{\operatorname{Area}(X)/\pi\}^{1/2},$$

where $\mathfrak{A}$ denotes the norm closure of $\{f(T):f\in\operatorname{rat}(X)\}$.

Proof. By a dilation theorem due to Arveson [3, p. 279], there exists a Hilbert space $\mathcal{K}$ and a normal operator N on $\mathcal{K}$ such that $\sigma(N)\subseteq X$ and $f(T)=V^*f(N)V$ for every $f\in\operatorname{rat}(X)$, where V is an isometric imbedding of $\mathcal{H}$ in $\mathcal{K}$. Since N is a normal operator, $\|u(N)\|=\sup\{|u(z)|:z\in\sigma(N)\}$ for any $u\in C(\sigma(N))$. Hence, for any $f\in\operatorname{rat}(X)$,

$$\|N^*-f(N)\|\leq\sup\{|\bar{z}-f(z)|:z\in X\}.$$

Let B be the norm closure of $\{f(N):f\in\operatorname{rat}(X)\}$; then

$$\operatorname{dist}(N^*,B)\leq\operatorname{dist}(\bar{z},\operatorname{rat}(X)).$$

Hence, by Alexander's theorem,

$$\operatorname{dist}(T^*,\mathfrak{A})\leq\{\operatorname{Area}(X)/\pi\}^{1/2}$$

because $\operatorname{dist}(T^*,\mathfrak{A})\leq\operatorname{dist}(N^*,B)$. ❐

In the theorem above, it would be nice if we could take a spectral set instead of a complete spectral set of T; unfortunately, we cannot do so. If a spectral set of T has a connected complement then it is a complete spectral set (cf. [3, Prop. 1.2.1]). However, when a spectral set of T has a disconnected complement we do not know whether it is a completely spectral set or not. Douglas and Paulsen [6, Cor. 2.4] showed that if a spectral set X of T has a complement with only finitely many components then there exists an invertible operator S such that X is a complete spectral set of $S^{-1}TS$. In general, if T is a subnormal operator then the spectrum is always a complete spectral set. Hence we can show the following two corollaries, using Theorem 2.

COROLLARY 1. *If T is in $\mathcal{L}(\mathcal{H})$ and if a spectral set X of T has a complement with only finitely many components, then there exists an invertible operator S and*

$$\operatorname{dist}((S^{-1}TS)^*, S^{-1}\mathcal{A}S) \le \{\operatorname{Area}(X)/\pi\}^{1/2}.$$

Moreover, if X is simply connected then

$$\operatorname{dist}(T^*, \mathcal{A}) \le \{\operatorname{Area}(X)/\pi\}^{1/2}.$$

(Here $\mathcal{A}$ denotes the norm closure of $\{f(T): f \in \operatorname{rat}(X)\}$.)

COROLLARY 2. *If T is a subnormal operator in $\mathcal{L}(\mathcal{H})$ then*

$$\operatorname{dist}(T^*, \mathcal{A}) \le \{\operatorname{Area}(\sigma(T))/\pi\}^{1/2},$$

where $\mathcal{A}$ denotes the norm closure of $\{f(T): f \in \operatorname{rat}(\sigma(T))\}$.

5. Hyponormal operators. A subnormal operator is hyponormal, but the converse is not true. We could not show Corollary 2 for hyponormal operators; however, the following theorem is still valid. Instead of Alexander's theorem we use Lemma 2.

THEOREM 3. *If T is a hyponormal operator in $\mathcal{L}(\mathcal{H})$ then*

$$\operatorname{dist}(T^*, \mathcal{B}) \le \{\operatorname{Area}(\sigma(T))/\pi\}^{1/2},$$

where $\mathcal{B}$ denotes the strong closure of $\{f(T): f \in \operatorname{rat}(\sigma(T))\}$.

Proof. By Lemma 2,

$$\operatorname{dist}(T^*, \mathcal{B}) = \sup\{\|(1-P)(1\otimes T)P\| : P \in \operatorname{lat}(1\otimes\mathcal{B})\}.$$

Since $1\otimes T$ is also hyponormal, by Putnam's theorem we have

$$\begin{aligned}\|(1-P)(1\otimes T)^*P\|^2 &= \|P(1\otimes T)(1\otimes T)^*P - P(1\otimes T)P(1\otimes T)^*P\| \\ &\le \|P(1\otimes T)^*(1\otimes T)P - P(1\otimes T)P(1\otimes T)^*P\| \\ &\le \operatorname{Area}(\sigma(P(1\otimes T)P))/\pi.\end{aligned}$$

From $P \in \operatorname{lat}(1\otimes\mathcal{B})$, it follows that $\sigma(1\otimes T) \supset \sigma(P(1\otimes T)P)$. Because $\sigma(T) = \sigma(1\otimes T)$, the theorem follows. $\square$

6. Commutators and spectrum area estimates. Axler and Shapiro [5, Thm. 3.1] proved Putnam's theorem for subnormal operators by using Alexander's theorem. We will prove it using Theorem 2.

Let T be a subnormal operator on $\mathcal{H}$, and let N be a minimal normal extension on the Hilbert space $\mathcal{K}$ such that $\mathcal{H} \subset \mathcal{K}$. P is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$, and $f(T) = Pf(N) \mid \mathcal{H}$ for every $f \in \operatorname{rat}(\sigma(T))$. Since $N^*N = NN^*$,

$$\|T^*T - TT^*\| = \|PN^*NP - PNPN^*P\| = \|(1-P)N^*P\|^2 \leq \operatorname{dist}(N^*, \mathcal{A})^2,$$

where $\mathcal{A}$ denotes the norm closure of $\{f(N) : f \in \operatorname{rat}(\sigma(T))\}$. By Theorem 2,

$$\operatorname{dist}(N^*, \mathcal{A})^2 \leq \operatorname{Area}(\sigma(T))/\pi.$$

Hence, Putnam's theorem follows for subnormal operators. Unfortunately we could not prove Putnam's theorem for hyponormal operators using Theorem 3. However, the proof above shows that if there exists a hyponormal operator $\tilde{T}$ for T such that $P\tilde{T}^*\tilde{T}P = P\tilde{T}\tilde{T}^*P$ then $P\tilde{T}P = T$ and $P \in \operatorname{lat} \mathcal{B}$, where $\mathcal{B}$ denotes the strong closure of $\{f(\tilde{T}) : f \in \operatorname{rat}(\sigma(\tilde{T}))\}$.

Now we will show an inequality for a hyponormal operator using Theorem 3 (hence using Putnam's theorem).

THEOREM 4. *If T is a hyponormal operator in $\mathcal{L}(\mathcal{H})$ and if K is in $\mathcal{L}(\mathcal{H})$ with $KT = TK$, then*

$$\|T^*K - KT^*\| \leq 2\{\operatorname{Area}(\sigma(T))/\pi\}^{1/2}\|K\|.$$

Proof. Let $\mathcal{B}$ denote the strong closure of $\{f(T) : f \in \operatorname{rat}(\sigma(T))\}$. For any $J \in \mathcal{B}$,

$$\begin{aligned}\|T^*K - KT^*\| &= \|(T^* - J)K + JK - KT^*\| \\ &\leq 2\|T^* - J\|\,\|K\|.\end{aligned}$$

From Theorem 3, the theorem follows. □

If we use Theorem 2 then a version of Theorem 4 can be shown for any bounded operator.

THEOREM 5. *If T and K are in $\mathcal{L}(\mathcal{H})$, $TK = KT$, and X is a complete spectral set, then*

$$\|T^*K - KT^*\| \leq 2\{\operatorname{Area}(X)/\pi\}^{1/2}\|K\|.$$

Instead of $\|K\|$ in Theorems 4 and 5 we can take $\operatorname{dist}(K, \mathcal{B} \cap \mathcal{B}^*)$, where $\mathcal{B}$ denotes the double commutant of T.

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Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060
Japan