CLOSED IDEALS IN CONVOLUTION ALGEBRAS AND THE LAPLACE TRANSFORM

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Introduction. Let $L^1[0,1]^n$ denote the Banach algebra of integrable functions on $[0,1]^n$ with restricted convolution as multiplication. It is easy to prove that all closed ideals in $L^1[0,1]$ are of the form:

$$M_\beta = \{f : \inf(\text{essential support}(f)) \geq \beta\} \quad \beta \in [0,1].$$

(See Section 2.) Thus, a function f in $L^1[0,1]$ generates a dense ideal in $L^1[0,1]$ if and only if zero is in the essential support of f. We demonstrate that this is not true for $n>1$ and then describe a relationship between the closed ideals in $L^1[0,1]^n$ (for any finite n) and those in a quotient of an algebra of analytic functions.

Let $L^1((\mathfrak{R}^+)^n)$ be the Banach algebra of integrable functions on $(\mathfrak{R}^+)^n$ with the usual norm and convolution as multiplication. Notice that $L^1[0,1]^n \cong L^1((\mathfrak{R}^+)^n)/I$, where I is the closed ideal in $L^1((\mathfrak{R}^+)^n)$ of functions whose support is contained in the complement of $[0,1]^n$. Define $A_0^{(n)}$ as the Banach algebra of functions of n complex variables which are continuous on the n-fold Cartesian product of the closed right half-plane, analytic on the interior of this set, and which vanish at infinity. The Laplace transform is a continuous monomorphism of $L^1((\mathfrak{R}^+)^n)$ into (but not onto) $A_0^{(n)}$. Let $\mathbf{K}$ be the ideal $e^{-z_1}A_0^{(n)} + \cdots + e^{-z_n}A_0^{(n)}$. A function f in $L^1((\mathfrak{R}^+)^n)$ is in I if and only if $\mathfrak{L}(f)$ is in the closure of $\mathbf{K}$ (see Section 4). Thus, $\mathfrak{L}$ induces a continuous monomorphism $\tilde{\mathfrak{L}}$ from $L^1[0,1]^n$ into $A_0^{(n)}/\bar{\mathbf{K}}$. If M is any closed ideal in $A_0^{(n)}/\bar{\mathbf{K}}$ then $\tilde{\mathfrak{L}}^{-1}(M)$ is a closed ideal in $L^1[0,1]^n$. We prove (Theorem 4.6) that $\tilde{\mathfrak{L}}^{-1}$ actually implements a bijection between closed ideals in $L^1[0,1]^n$ and $A_0^{(n)}/\bar{\mathbf{K}}$, so that an ideal J in $L^1[0,1]^n$ is dense if and only if $\tilde{\mathfrak{L}}(J)$ is dense in $A_0^{(n)}/\bar{\mathbf{K}}$.

In 1950 Nyman proved that ideals in $L^1(\mathfrak{R}^+)$ are dense if and only if their image under the Laplace transform is dense in $A_0^{(1)}$ ([8], [3]). In 1981 Domar showed that under suitable conditions on a weight w, all closed ideals in $L^1(\mathfrak{R}^+, w)$ are of the form M_β (defined above) [4]. (It can be shown, using recent results of Thomas [10], that there are weights w on $\mathfrak{R}^+$ for which this is not true.) The key idea in both of these results is to show that functions belonging to the annihilator of certain ideals necessarily have compact support. It is reasonable to expect that some of our methods should be useful in extending Domar's and Nyman's results to higher dimensions.

There are two problems in function theory which are relevant here. It is not difficult to see that the ideal $\mathbf{K} = e^{-z_1}A_0^{(1)}$ is closed in $A_0^{(1)}$. Is the ideal $\mathbf{K}$ closed in $A_0^{(n)}$ for $n>1$?

The second problem is that of finding a characterization of dense ideals in $A_0^{(n)}$ or $A_0^{(n)}/\bar{\mathbf{K}}$ for $n>1$. In Section 3 we discuss certain examples of nondense

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ideals in $L^1[0,1]^2$ containing functions with zero in their support. We then show that these ideals correspond (via the Laplace transform) with ideals in $A_0^{(2)}$ whose restrictions to certain curves are not dense (Section 3). In order to determine whether an ideal $\mathbf{J}$ is actually dense in $A_0^{(n)}$, it is necessary to consider all restrictions of $\mathbf{J}$ to analytic varieties. If, for each analytic variety V, the restriction of $\mathbf{J}$ to V is dense, in the uniform norm, and in the set of all restrictions of elements of $A_0^{(n)}$ to V, then is $\mathbf{J}$ necessarily dense in $A_0^{(n)}$?

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1. Definitions. We begin by defining several Banach spaces and algebras of functions of several variables which are contained in $L^1_{\text{loc}}(\mathfrak{R}^n)$, the space of all complex valued locally integrable functions on $\mathfrak{R}^n$ (a function whose domain is a subset of $\mathfrak{R}^n$ is considered to be a function on all of $\mathfrak{R}^n$ which takes the value zero on the complement of its domain). $\mathfrak{R}^+$ is defined to be the ray $[0,\infty]$ and $L^1((\mathfrak{R}^+)^n)$ is the Banach space of integrable functions on $(\mathfrak{R}^+)^n$. $L^\infty((\mathfrak{R}^+)^n)$ will be the Banach space of essentially bounded measurable functions on $(\mathfrak{R}^+)^n$. The convolution of two functions f and g in $L^1_{\text{loc}}((\mathfrak{R}^+)^n)$ is the element $f\star g$ of $L^1_{\text{loc}}((\mathfrak{R}^+)^n)$ defined almost everywhere by

$$(f\star g)(t_1,\dots,t_n)=\int_0^{t_n}\cdots\int_0^{t_1} f(s_1,\dots,s_n)g(t_1-s_1,\dots,t_n-s_n)\,ds_1\cdots ds_n.$$

Note that $(L^1((\mathfrak{R}^+)^n),\star)$ is, in fact, a Banach algebra, and a subalgebra of the group algebra $(L^1((\mathfrak{R}^n)),\star)$. We denote by $L^1[0,1]^n$ the Banach space of integrable functions on $[0,1]^n$. We define the product $f\tilde{\star} g$ of two functions f and g in $L^1[0,1]^n$ to be

$$(f\tilde{\star} g)(t_1,\dots,t_n)=(f\star g)\cdot\chi_{[0,1]^n},$$

where $\cdot$ signifies pointwise multiplication and $\chi_{[0,1]^n}$ is the characteristic function of $[0,1]^n$. Equipped with this product, $L^1[0,1]^n$ is a Banach algebra which is known as the Volterra algebra. Note that, if

$$I=\{f\in L^1((\mathfrak{R}^+)^n): f\equiv 0 \text{ on } [0,1]^n\},$$

then $L^1[0,1]^n\cong L^1((\mathfrak{R}^+)^n)/I$. Thus, we can consider the restriction map from $L^1((\mathfrak{R}^+)^n)$ to $L^1[0,1]^n$ as the canonical quotient map onto $L^1((\mathfrak{R}^+)^n)/I$.

Next we define $A_0^{(n)}$ to be the algebra of all continuous complex-valued functions on the set

$$\Pi^n=\{(z_1,\dots,z_n): \operatorname{Re}(z_1),\dots,\operatorname{Re}(z_n)\ge 0\}$$

whose restriction to the interior of Π^n is analytic, and whose values converge to zero whenever $\|z\|\to\infty$. With the supremum norm and pointwise multiplication, $A_0^{(n)}$ is a Banach algebra. Let $A(D^n)$ be the Banach algebra of continuous complex functions on the closed unit polydisc whose restriction to the open unit polydisc is holomorphic, normed with the supremum norm (this algebra is discussed in [9]). We note that $A_0^{(n)}$ is homeomorphic to the ideal of $A(D^n)$ of functions which vanish on the set $(1\times D^{n-1})\cup(D\times 1\times D^{n-2})\cup\cdots\cup(D^{n-1}\times 1)$.

Now let $\mathfrak{L}$ be the Laplace transform. If $f\in L^1_{\mathrm{loc}}((\mathfrak{R}^+)^n)$, we define $\mathfrak{L}f$ by

$$(\mathfrak{L}f)(z_1,\dots,z_n)=\int_0^{+\infty}\cdots\int_0^{+\infty} f(t_1,\dots,t_n)e^{-z_1t_1\cdots-z_nt_n}\,dt_1\cdots dt_n, \tag{1}$$

where the domain of $\mathfrak{L}f$ is defined to be those points in $\mathbf{C}^n$ for which the integral converges absolutely. If $\varphi\in A_0^{(n)}$ and φ is integrable over the Cartesian product of the imaginary axes, then we can define $\mathfrak{L}^{-1}(\varphi)$ by the formula

$$\mathfrak{L}^{-1}(\varphi)(t_1,\dots,t_n)=\frac{1}{(2\pi)^n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\varphi(iy_1,\dots,iy_n)e^{iy_1t_1+\cdots+iy_nt_n}.$$

It is well known that, for such φ, $\mathfrak{L}^{-1}(\varphi)\in L^\infty((\mathfrak{R}^+)^n)$ and $\mathfrak{L}(\mathfrak{L}^{-1}(\varphi))$ agrees with φ on Π^n.

Finally, let $\mathbf{K}$ be the ideal

$$\mathbf{K}=e^{-z_1}A_0^{(n)}+\cdots+e^{-z_n}A_0^{(n)}.$$

We shall see in Section 4 that the Laplace transform $\mathfrak{L}$ induces a continuous monomorphism $\tilde{\mathfrak{L}}$ from $L^1[0,1]^n$ into $A_0^{(n)}/\bar{\mathbf{K}}$. We will prove (Theorem 4.6) that $\tilde{\mathfrak{L}}$ implements a bijection between the closed ideals in $L^1[0,1]^n$ and the closed ideals in $A_0^{(n)}/\bar{\mathbf{K}}$.

2. The Titchmarsh Convolution Theorem and closed ideals in $L^1[0,1]$. For each f in $L^1_{\mathrm{loc}}(\mathfrak{R}^n)$, the support of f is the set

$$\operatorname{supp}(f)=\operatorname{cl}\{x\in\mathfrak{R}^n:\text{if } U \text{ is an open set containing } x \text{ then } m\{y\in U: f(y)\neq 0\}>0\}$$

(where m is Lebesgue measure on $\mathfrak{R}^n$). We write $\Gamma(f)$ for the closed convex hull of the support of f.

The following theorem (one form of the Titchmarsh Convolution Theorem) is the main tool used to characterize all closed ideals in $L^1[0,1]$.

THEOREM 2.1 ([5, §4.5]). *Let $f,g\in L^1(\mathfrak{R}^n)$ be functions whose support is compact. Then*

$$\Gamma(f\star g)=\Gamma(f)+\Gamma(g).$$

We now use Theorem 2.1 to obtain the following result.

THEOREM 2.2. *Let $f\in L^1[0,1]^n$ be such that $0\in\operatorname{supp} f$. Set*

$$T(n)=\{(x_1,\dots,x_n)\in[0,1]^n: x_1+\cdots+x_n\geq n-1\}.$$

Let

$$M=\{\phi\in L^1[0,1]^n:\operatorname{supp}(\phi)\subseteq T(n)\}.$$

Then the ideal M is contained in $\overline{f\star L^1[0,1]^n}$.

Proof. The dual space of $L^1[0,1]^n$ is $L^\infty[0,1]^n$ where the dual action is

$$\langle g, h\rangle = \int_{[0,1]^n} g(x)h(x)\,dx \quad (g\in L^\infty[0,1]^n,\ h\in L^1[0,1]^n).$$

For any $t\in((\mathfrak{R}^+)^n)$ and $f\in L^1((\mathfrak{R}^+)^n)$ we define the translation by t of f to be the function

$$f_t(x)=\begin{cases} f(x-t) & \text{if } x-t\in((\mathfrak{R}^+)^n);\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, $f_t\in L^1((\mathfrak{R}^+)^n)$ for all t in $((\mathfrak{R}^+)^n)$. It is well known ([2, §3.4.48]) that

$$\overline{\operatorname{span}\{f_t : t\in((\mathfrak{R}^+)^n)\}} = \overline{f\star L^1((\mathfrak{R}^+)^n)}$$

The arguments for $L^1((\mathfrak{R}^+)^n)$ are easily extended to $L^1[0,1]^n$. That is, if we define the translation by t of $f\in L^1[0,1]^n$ to be

$$S_t f = f_t\cdot\chi_{[0,1]^n} \quad (t\in[0,1]^n),$$

where $\chi_{[0,1]^n}$ is the characteristic function of $[0,1]^n$, then

$$\overline{\operatorname{span}\{S_t f : t\in[0,1]^n\}} = \overline{f\tilde{\star}L^1[0,1]^n}.$$

Let $g\in L^\infty([0,1]^n)$, g orthogonal to $\overline{f\tilde{\star}L^1[0,1]^n}$. Then

$$\langle g, S_t f\rangle = 0 \quad (t\in[0,1]^n)$$

and so

$$\int_{[0,1]^n} g(x)f(x-t)\,dx=0 \quad (t\in[0,1]^n). \tag{2}$$

Let $\hat{g}$ be the reflection of g, that is,

$$\hat{g}(-x)=g(x) \quad (x\in[0,1]^n).$$

After a change of variables, (2) yields

$$\begin{aligned}(f\star\hat{g})(-t) &= \int_{[0,1]^n}\hat{g}(-t-x)f(x)\,dx\\ &= \int_{\mathfrak{R}^n}\hat{g}(-t-x)f(x)\,dx = 0 \quad (t\in[0,1]^n).\end{aligned} \tag{3}$$

(We are now considering f and $\hat{g}$ to be functions in $L^1_{\text{loc}}(\mathfrak{R}^n)$.) Since $\operatorname{supp}(f)\subseteq [0,1]^n$ and $\operatorname{supp}(\hat{g})\subseteq[-1,0]^n$, $\operatorname{supp}(f\star\hat{g})\subseteq[-1,1]^n$ and so (3) implies that

$$\operatorname{supp}(f\star\hat{g})\subseteq[-1,1]^n\backslash[-1,0]^n.$$

This implies that

$$\Gamma(f\star\hat{g})\subseteq\{(x_1,\ldots,x_n)\in[-1,1]^n : x_1+\cdots+x_n\geq -(n-1)\}. \tag{4}$$

Since $0\in\operatorname{supp}(f)$, the Titchmarsh theorem together with (4) shows that

$$\Gamma(g)\cap T(n)=\phi,$$

which proves the theorem. □

NOTE. If, for $f \in L^1[0,1]$, one defines $\alpha(f) = \inf(\operatorname{supp}(f))$, then Theorem 2.1 implies that

$$\alpha(f \star g) = \alpha(f) + \alpha(g) \quad (f, g \in L^1[0,1]).$$

Using this property, and the same method as in the proof of Theorem 2.2, it is easy to show that all closed ideals in $L^1[0,1]$ are of the form

$$M_\beta = \{f \in L^1[0,1] : \alpha(f) \geq \beta\} \quad \text{for some } \beta \in [0,1],$$

as stated in the introduction. (For details, see [2].) Thus, if $f \in L^1[0,1]$ and $0 \in \operatorname{supp} f$, then the ideal generated by f in $L^1[0,1]$ is dense. In the next section, we demonstrate that no such simple characterization of closed ideals can be obtained in $L^1[0,1]^n$ for $n > 1$.

3. Unexpected ideals in $L^1[0,1]^2$. As shown in Section 2, if $f \in L^1[0,1]$ and $0 \in \operatorname{supp} f$, then f generates a dense ideal in $L^1[0,1]$. In this section we show that the ideal structure of $L^1[0,1]^2$ is much more complex, by giving examples of nondense ideals in $L^1[0,1]^2$ that contain functions with zero in their support. Some of our examples are nondense ideals in $L^1[0,1]^2$ which are inverse images by certain homomorphisms of nondense ideals in $L^1[0,1]$. All of our examples are related to ideals in $A_0^{(2)}$ which are not dense along certain curves in Π^2. This motivates Section 4, where we characterize density of ideals in $L^1[0,1]^2$ in terms of the density of the image of these ideals (by an induced Laplace transform) in the quotient algebra $A_0^{(n)}/\bar{\mathbf{K}}$.

The following proposition was communicated to the author by A. M. Sinclair.

PROPOSITION 3.1. *Let* $\lambda \in \mathfrak{R}^+$, $u \in \mathfrak{R}$, *and let* $E_{\lambda,u} : L^1(\mathfrak{R}^+)^2 \mapsto L^1(\mathfrak{R}^+)$ *be defined by*

$$E_{\lambda,u} f(\xi) = \lambda \int_0^\xi f(\lambda t, \xi - t) e^{-iut}\, dt \quad (\xi \in (\mathfrak{R}^+)).$$

Then $E_{\lambda,u}$ *is a continuous homomorphism with dense range.*

Proof. Fix $\lambda \in \mathfrak{R}^+$, $u \in \mathfrak{R}$. Clearly $E_{\lambda,u}$ is a continuous linear map. Let

$$f, g \in L^1((\mathfrak{R}^+)^2).$$

Then

$$E_{\lambda,u}(f \star g)(\xi) = \lambda \int_0^\xi \int_0^{\lambda t} \int_0^{\xi - t} f(\lambda t - x, \xi - t - y) g(x,y) e^{-iut}\, dy\, dx\, dt$$

$$= \lambda \int_0^{\lambda\xi} \int_0^{\xi - x/\lambda} g(x,y) \int_{x/\lambda}^{\xi - y} f(\lambda t - x, \xi - t - y) e^{-iut}\, dt\, dy dx.$$

Let $s = t - x/\lambda$, $w = y + x/\lambda$, and $v = x/\lambda$. Then, with a bit of work, the formula above reduces to

$$E_{\lambda,u}(f \star g)(\xi) = \int_0^\xi E_{\lambda,u} f(\xi - w) E_{\lambda,u} g(w) \quad dw = (E_{\lambda,u} f \star E_{\lambda,u} g)(\xi)$$

and so $E_{\lambda,u}$ is, in fact, a continuous homomorphism.

Finally, if $g \in L^1(\mathfrak{R}^+)$ with compact support, and we set

$$f(x, y) = \frac{g(x/\lambda + y)}{(x + \lambda y)} e^{iu(x/\lambda)}$$

then $f \in L^1((\mathfrak{R}^+)^2)$ and $E_{\lambda, u} f = g$. This completes the proof. $\square$

Let $\beta \in \mathfrak{R}^+$ and let $M_\beta = \{f \in L^1(\mathfrak{R}^+) : \inf \operatorname{supp}(f) \geq \beta\}$. Let $\lambda, \beta \in \mathfrak{R}^+$, $u \in \mathfrak{R}$. We define the ideal $I_{\beta, \lambda, u}$ by

$$\begin{aligned} I_{\beta, \lambda, u} &= E_{\lambda, u}^{-1}(M_\beta) \\ &= \left\{ f \in L^1(\mathfrak{R}^+)^2 : \int_0^\xi f(\lambda t, \xi - t) e^{-iut}\, dt = 0 \text{ a.e. } 0 \leq \xi \leq \beta \right\}. \end{aligned}$$

Clearly, none of the ideals $I_{\beta, \lambda, u}$ are dense in $L^1(\mathfrak{R}^+)^2$. But, for fixed $u \in \mathfrak{R}$, $\lambda \in \mathfrak{R}^+$ the function

$$f(x, y) = \begin{cases} e^{i(u/\lambda)x} & \text{if } x < \lambda y \\ -e^{i(u/\lambda)x} & \text{if } x \geq \lambda y \end{cases}$$

is in $I_{\beta, \lambda, u}$ for all appropriate β. Clearly, zero is in the support of all such functions.

Next, we characterize functions in the ideals $I_{\beta, \lambda, u}$ by their Laplace transforms.

Let $A_0^{(n)}$ be the algebra of analytic functions in n variables defined in Section 1. We begin with the observation that the map $\theta_{\lambda, u}$ ($\lambda \in \mathfrak{R}^+, u \in \mathfrak{R}$) defined by

$$\theta_{\lambda, u} : A_0^{(2)} \mapsto A_0^{(1)},$$

$$\theta_{\lambda, u}(\varphi)(z) = \varphi\left(\frac{z + iu}{\lambda}, z\right) \quad (\varphi \in A_0^{(2)})$$

is a continuous homomorphism with dense range. Thus, if an ideal J is dense in $A_0^{(2)}$, then $\theta_{\lambda, u}(J)$ is dense in $A_0^{(1)}$ for all $\lambda > 0$, $u \in \mathfrak{R}$. Note that, if $\lambda > 0$, $u \in \mathfrak{R}$, $f \in L^1((\mathfrak{R}^+)^n)$, and $E_{\lambda, u}$ is the map defined in Proposition 3.1, then

$$\theta_{\lambda, u}(\mathcal{L}(f)) = \mathcal{L}(E_{\lambda, u}(f)). \tag{5}$$

Now, it is well known that a function $g \in L^1(\mathfrak{R}^+)$ is in the ideal M_β defined above if and only if $\mathcal{L}g \in e^{-\beta z} A_0^{(1)}$ [3, Thm. 3.7]. Thus, if we set $N_\beta = e^{-\beta z} A_0^{(1)}$ and $R_\beta = \theta_{\lambda, u}^{-1}(N_\beta)$, we have $I_{\beta, \lambda, u} = \mathcal{L}^{-1}(R_\beta)$. So, the nondensity of $I_{\beta, \lambda, u}$ in $L^1((\mathfrak{R}^+)^2)$ follows from the nondensity of $\mathcal{L}(I_{\beta, \lambda, u})$ along the line $z_1 = (z_2 + iu)/\lambda$.

We next see that, for appropriate (β, λ, u), the restrictions of the $I_{\beta, \lambda, u}$ to $L^1[0,1]^2$ are ideals in $L^1[0,1]^2$. Let I be the ideal in $L^1((\mathfrak{R}^+)^2)$ of functions whose support is contained in the complement of $[0,1]^2$, so that $L^1[0,1]^2 \cong L^1((\mathfrak{R}^+)^2)/I$. Let P be the restriction map from $L^1((\mathfrak{R}^+)^2)$ onto $L^1[0,1]^2$ and let S be the subset of $\mathfrak{R}^3$ defined by

$$S = \{(\beta, \lambda, u) : \lambda \in \mathfrak{R}^+,\ u \in \mathfrak{R},\ 0 < \beta \leq \min\{1, \lambda^{-1}\}\}.$$

If $(\beta, \lambda, u) \in S$ then $I_{\beta, \lambda, u} \supset I$, so $\tilde{I}_{\beta, \lambda, u} = P(I_{\beta, \lambda, u})$ is a proper closed ideal in $L^1[0,1]^2$ containing functions with zero in their support. Notice that, if we set

$\tilde{E}_{\lambda,u}f = PE_{\lambda,u}f$ for $f \in L^1[0,1]^2$, then $\tilde{E}_{\lambda,u}: L^1[0,1]^2 \mapsto L^1[0,1]$ is a homomorphism. If $\beta \in [0,1]$ and we set $\tilde{M}_\beta = PM_\beta$ then $\tilde{I}_{\beta,\lambda,u} = \tilde{E}_{\lambda,u}^{-1}(\tilde{M}_\beta)$. Note that, if N_β and R_β are defined as above, then $\tilde{I}_{\beta,\lambda,u} = P(\mathfrak{L}^{-1}(R_\beta))$. Similar ideals can be constructed in $L^1[0,1]^n$ for $n > 2$. We leave the details to the reader.

Let

$$\mathbf{K} = e^{-z_1}A_0^{(n)} + \cdots + e^{-z_n}A_0^{(n)}.$$

It is not difficult to see that $\mathfrak{L}(I) \subseteq \mathbf{K}$. (In fact, $I = \mathfrak{L}^{-1}(\mathbf{K}) = \mathfrak{L}^{-1}(\bar{\mathbf{K}})$; see Section 4.) Thus, if $f \in L^1[0,1]^2$, then

$$\mathfrak{L}f + \mathbf{K} = \mathfrak{L}(Pf) + \mathbf{K} \tag{6}$$

and, if J is a dense ideal in $L^1[0,1]^2$, then $\mathfrak{L}(J) + \mathbf{K}$ generates a dense ideal in $A_0^{(n)}$. If $\beta, \lambda, u \in S$, then $\mathfrak{L}(\tilde{I}_{\beta,\lambda,u}) + \mathbf{K} \subseteq R_\beta$. Since R_β is not dense in $A_0^{(2)}$ we see again that $\tilde{I}_{\beta,\lambda,u}$ is not dense in $L^1[0,1]^2$.

We shall prove (Theorem 4.6) that, if J is an ideal in $L^1[0,1]^n$, then the condition that $\mathfrak{L}(J) + \mathbf{K}$ generate a dense ideal in $A_0^{(n)}$ is not only necessary but also sufficient for J to be dense in $L^1[0,1]^n$.

CONJECTURES. P. Dixon conjectured that, if $0 \in \operatorname{supp}(f)$ and f is not an element of any $\tilde{I}_{\beta,\lambda,u}$, then f generates a dense ideal in $L^1[0,1]^2$. Recently, H. Hedenmalm communicated the following counterexample to the author. Let

$$f(t_1, t_2) = (-t_2 + t_1 - \tfrac{1}{2}t_1^2)e^{-t_1 - t_2}.$$

Then f has zero in its support and is not an element of any $I_{\beta,\lambda,u}$. So, Pf is not in any $\tilde{I}_{\beta,\lambda,j}$. However, its Laplace transform is

$$\mathfrak{L}f(z_1, z_2) = \frac{z_2 - z_1 - (1/(z_1+1))}{(z_1+1)^2(z_2+1)^2}$$

and so, if $\theta: A_0^{(2)} \mapsto A_0^{(1)}$ is the continuous homomorphism with dense range defined by

$$(\theta\varphi)(z) = \varphi\left(z, z + \frac{1}{z+1}\right),$$

then $(\theta \circ \mathfrak{L})(f) \equiv 0$. But clearly,

$$\theta(\mathbf{K}) \subseteq e^{-z}A_0^{(1)}$$

and so $\mathfrak{L}f + \mathbf{K}$ does not generate a dense ideal in $A_0^{(2)}$. By (6), this implies that $\mathfrak{L}Pf + \mathbf{K}$ does not generate a dense ideal in $A_0^{(2)}$. Thus

$$\overline{Pf \,\tilde{\star}\, L^1[0,1]^2} \neq L^1[0,1]^2$$

and the conjecture is false. The problem here is that there is no known characterization of dense ideals in either $A_0^{(n)}$ or $A_0^{(n)}/\bar{\mathbf{K}}$. (A characterization of dense ideals in certain subalgebras of the polydisc algebra which contain isometrically $A_0^{(n)}$ can be found in [7].) Perhaps a reasonable conjecture about density of ideals in $A_0^{(n)}$ is that, if $\mathbf{J}$ is an ideal in $A_0^{(n)}$, and, for each analytic variety V in Π^n, the restriction of $\mathbf{J}$ to V is dense (in the uniform norm) in the set of all restrictions of elements of $A_0^{(n)}$ to V, then $\mathbf{J}$ is dense in $A_0^{(n)}$. If we could prove that this conjecture were true at least for those ideals in $A_0^{(n)}$ containing $\bar{\mathbf{K}}$, we would have a characterization of ideals in $A_0^{(n)}/\bar{\mathbf{K}}$, and thereby (using Theorem 4.6) a characterization of dense ideals in $L^1[0,1]^n$.

4. Characterization of closed ideals in $L^1[0,1]^n$. The existence of a bijection between the closed ideals in $L^1[0,1]^n$ and $A_0^{(n)}/\bar{\mathbf{K}}$ will be established as a consequence of the following more general theorem about closed ideals in certain Banach algebras.

If M is a subset in a Banach algebra A, we write $\mathbf{J}(M)$ for the smallest closed ideal in A containing M. If $u \in A$ we write $u \cdot A = \{u \cdot a : a \in A\}$.

THEOREM 4.1. *Let A and B be commutative Banach algebras and let φ be a continuous monomorphism from B into A. Suppose that:*

(i) *$\varphi(B)$ is dense in A.*

(ii) *B has a bounded approximate identity $\{e_\alpha\}_{\alpha \in \Lambda}$ (Λ an index set) such that:*

$$\varphi(e_\alpha) \cdot A \subseteq \varphi(B) \quad (\alpha \in \Lambda).$$

Then the map $\theta_1 : J \mapsto \varphi^{-1}(J)$ (J a closed ideal in A) is a bijection between the closed ideals in A and the closed ideals in B. Its inverse θ_2 is defined by:

$$\theta_2(I) = \mathbf{J}(\varphi(I)) \quad (I \text{ a closed ideal in } B).$$

Proof. Notice that the closed graph theorem implies that the map defined by

$$\psi(a) = \varphi^{-1}(a \cdot \varphi(e_\alpha)) \quad (a \in A)$$

is a continuous map from A into B for each $\alpha \in \Lambda$.

Assume (i) and (ii) and let I be a closed ideal in B. Since $\{e_\alpha\}_{\alpha \in \Lambda}$ is an approximate identity for B, $\overline{\varphi(I)} \subseteq \overline{\varphi(IB)} \subseteq \overline{\varphi(I)A}$. Thus, if $\varphi(b) \in \mathbf{J}(\varphi(I))$,

$$\varphi(b) = \lim_{n \to \infty} \varphi(m_{1,n})a_{1,n} + \cdots + \varphi(m_{k_n,n})a_{k_n,n}$$

for some $m_{i,n} \in I$, and $a_{i,n} \in A$ ($i = 1, \ldots, k_n$; $n = 1, 2, \ldots$). For each fixed $\alpha \in \Lambda$, there exist $b_{i,n} \in B$ ($i = 1, \ldots, k_n$; $n = 1, 2, \ldots$) such that

$$\varphi(b \cdot e_\alpha) = \lim_{n \to \infty} \varphi(m_{1,n})\varphi(b_{1,n}) + \cdots + \varphi(m_{k_n,n})\varphi(b_{k_n,n})$$

and the continuity of ψ implies that

$$b \cdot e_\alpha^2 = \lim_{n \to \infty} m_{1,n}b_{1,n}e_\alpha + \cdots + m_{k_n,n}b_{k_n,n}e_\alpha \quad (\alpha \in \Lambda).$$

Thus, for each $\alpha \in \Lambda$, $b \cdot e_\alpha^2$ is the limit of a sequence of elements of I and so

$$b \cdot e_\alpha^2 \in I \quad (\alpha \in \Lambda)$$

so that $b \in I$. Thus $I \subseteq \varphi^{-1}(\mathbf{J}(\varphi(I)) \subseteq I$, $I = \varphi^{-1}(\mathbf{J}(I))$, and θ_1 is a left inverse for θ_2. To finish the proof, we need the fact that $\mathbf{J}(\varphi(\varphi^{-1}(J))) = J$. It is enough to show that φ^{-1} implements a one-to-one map between the closed ideals in A and B. Let J be a closed ideal in A. Condition (ii) implies that

$$J \cdot \varphi(e_\alpha) \subseteq \varphi(\varphi^{-1}(J)).$$

This, together with the fact that $\{\varphi(e_\alpha)\}_{\alpha\in\Lambda}$ is a bounded approximate identity for A, means that

$$J \subseteq \overline{\varphi(\varphi^{-1}(J))} \subseteq J$$

and so $J = \overline{\varphi(\varphi^{-1}(J))}$, which completes the proof. $\square$

REMARK 1. Notice that Theorem 4.1 implies that an element $b \in B$ generates a dense ideal in B if and only if $\varphi(b)$ generates a dense ideal in A.

REMARK 2. The first part of the proof shows that, if we drop the hypothesis that $\{e_\alpha\}_{\alpha\in\Lambda}$ is bounded, then it is still true that

$$I = \varphi^{-1}(\mathbf{J}(\varphi(I))$$

for each closed ideal I in B. Remark 1 is also valid in this case.

Now, let $I \subseteq L^1((\mathfrak{R}^+)^n)$ be the closed ideal

$$I = \{f \in L^1((\mathfrak{R}^+)^n) : f \equiv 0 \text{ on } [0,1]^n\}$$

and $\mathbf{K} \subseteq A_0^{(n)}$ the ideal

$$\mathbf{K} = e^{-z_1}A_0^{(n)} + \cdots + e^{-z_n}A_0^{(n)}.$$

(The ideal $\mathbf{K}$ is closed for $n=1$, but the author has been unable to determine whether or not $\mathbf{K}$ is closed for $n>1$.)

We shall prove that Theorem 4.1 holds for $A = A_0^{(n)}/\bar{\mathbf{K}}$ and $B = L^1((\mathfrak{R}^+)^n)/I \cong L^1[0,1]^n$. The following facts about the function

$$u(t_1,\ldots,t_n) = e^{-t_1-\cdots-t_n}$$

and its Laplace transform $\mathfrak{L}(u)$ will be quite useful.

FACT 1. The function u generates a dense ideal in $L^1((\mathfrak{R}^+)^n)$. This is a consequence of the fact that, if

$$u_m = u \cdot \chi_{[0,1/m]^n},$$

then $w_m = u_m/\|u_m\|$ is a bounded approximate identity for $L^1((\mathfrak{R}^+)^n)$, and $w_m \in \operatorname{span}\{u_t\}_{t\in\mathfrak{R}^+} \subseteq \overline{u \star L^1((\mathfrak{R}^+)^n)}$ $(m=1,2,\ldots)$. Here, u_t is the translate of the function u defined in Section 2.

Note that, as a consequence of Fact 1, if L is a closed ideal in $L^1((\mathfrak{R}^+)^n)$ and $f \in L^1((\mathfrak{R}^+)^n)$, then $f \star u \in L$ implies that $f \in L$.

FACT 2. *If* $\varphi \in A_0^{(n)}$ then

$$\varphi \cdot \mathfrak{L}(u)^2 \in \mathfrak{L}(L^\infty((\mathfrak{R}^+)^n)$$

and the map

$$\varphi \mapsto \mathfrak{L}^{-1}(\varphi \cdot \mathfrak{L}(u)^2)$$

is a continuous map from $A_0^{(n)}$ to $L^\infty((\mathfrak{R}^+)^n)$. (This follows from the fact that $\mathfrak{L}(u)^2$ is integrable over the Cartesian product of the imaginary axes, and from the formula for the inverse Laplace transform given in Section 1.)

We begin by establishing a correspondence between the ideals I and $\mathbf{K}$.

LEMMA 4.2. *Let* $f \in L^1((\Re^+)^n)$. *Then* $f \in I$ *if and only if* $\mathcal{L}f \in \mathbf{K}$.

Proof. "If": Let $f \in I$. Clearly, f can be written

$$f = \sum_{i=1}^{n} g_i$$

where $g_i \equiv 0$ on $(\Re^+)^{i-1} \times [0,1] \times (\Re^+)^{n-i}$. Set

$$\phi_i(x_1, \dots, x_n) = g_i(x_1, \dots, x_i + 1, \dots, x_n) \quad (i = 1, \dots, n).$$

Then $\phi_i \in L^1((\Re^+)^n)$, so $\mathcal{L}(\phi_i) \in A_0^{(n)}$ $(i = 1, \dots, n)$. Also, by direct computation

$$(\mathcal{L}\phi_i)(z_1, \dots, z_n) = e^{z_i}(\mathcal{L}g_i),$$

and $\mathcal{L}f = \sum e^{-z_i}\mathcal{L}\phi_i \in \mathbf{K}$.

"Only if": Suppose $\mathcal{L}f = \sum_{i=1}^{n} e^{-z_i}\varphi_i$, where $\mathcal{L}f \in \mathbf{K}$ and $\varphi_i \in A_0^{(n)}$, $i = 1, \dots, n$. Then (because of Fact 2)

$$\mathcal{L}(f \star u^{\star 2}) = \mathcal{L}f \cdot (\mathcal{L}u)^2 = \sum_{i=1}^{n} e^{-z_i}\mathcal{L}h_i,$$

where $h_i \in L^\infty((\Re^+)^n)$ $(i = 1, \dots, n)$. Also:

$$\begin{aligned} \mathcal{L}((f \star u^{\star 2}) \cdot u)(z_1, \dots, z_n) &= \mathcal{L}(f \star u^{\star 2})(z_1 + 1, \dots, z_n + 1) \\ &= e^{-z_1 - 1}\mathcal{L}(h_1 \cdot u) + \cdots + e^{-z_n - 1}\mathcal{L}(h_n \cdot u) \\ &= \mathcal{L}(m_1 + \cdots + m_n) \end{aligned}$$

where

$$m_i \in L^1((\Re^+)^n) \text{ and } m_i \equiv 0 \text{ on } (\Re^+)^{i-1} \times [0,1] \times (\Re^+)^{n-i}.$$

Thus $(f \star u^{\star 2}) \cdot u \equiv 0$ on $[0,1]^n$, which implies that $f \star u^{\star 2} \in I$, and so $f \in I$. □

COROLLARY 4.3. *If* $f \in L^\infty((\Re^+)^n)$ *and* $\mathcal{L}f \in \mathbf{K}$, *then* $f \equiv 0$ *on* $[0,1]^n$.

Proof. Notice that $\mathcal{L}(f \cdot u) \in \mathbf{K}$. Since $f \cdot u \in L^1((\Re^+)^n)$ this implies that $f \cdot u \in I$, so that $f \equiv 0$ on $[0,1]^n$.

LEMMA 4.4. *Let* $f \in L^\infty((\Re^+)^n)$ *with* $\mathcal{L}f \in A_0^{(n)}$. *If* f *is equivalent to* 0 *on* $[0,1]^n$ *then* $\mathcal{L}f \in \bar{\mathbf{K}}$.

Proof. Suppose that $f \in L^\infty((\Re^+)^n)$, $\mathcal{L}f \in A_0^{(n)}$ and f is equivalent to zero on $[0,1]^n$. For each positive integer m, the function f_m defined by

$$f_m = f \cdot e^{-t_1/m - \cdots - t_n/m}$$

is an element of $L^1((\Re^+)^n)$. By Lemma 4.2, $\mathcal{L}f_m \in \mathbf{K}$ $(m = 1, 2, \dots)$. But $\mathcal{L}(f_m)$ is just the translation of the function $\mathcal{L}f$ by the vector $(1/m, \dots, 1/m)$, so the fact that functions in $A_0^{(n)}$ are uniformly continuous implies that $\{\mathcal{L}(f_m)\}_{m=1}^{\infty}$ converges to $\mathcal{L}f$ in $A_0^{(n)}$. Thus, $\mathcal{L}f \in \bar{\mathbf{K}}$.

COROLLARY 4.5. $\mathcal{L}(L^1((\Re^+)^n)) \cap \bar{\mathbf{K}} = \mathbf{K} \cap \mathcal{L}(L^1((\Re^+)^n))$.

Proof. Suppose $f \in L^1((\mathfrak{R}^+)^n)$ and $\mathfrak{L}f \in \bar{\mathbf{K}}$. Let $\{\phi_n\}$ be a sequence in $\mathbf{K}$, $\phi_n \to \mathfrak{L}f$. Then, by the inversion formula,

$$\|\mathfrak{L}^{-1}(\phi_n \cdot \mathfrak{L}u^2) - f \star u^{\star 2}\|_\infty \to 0. \tag{7}$$

But, by Corollary 4.3, $\mathfrak{L}^{-1}(\phi_n \cdot \mathfrak{L}u^2) \equiv 0$ on $[0,1]^n$, so (7) implies that $f \star u^{\star 2} \equiv 0$ on $[0,1]^n$. This means that $f \star u^{\star 2} \in I$ so that $f \in I$ and $\mathfrak{L}f \in \mathbf{K}$, and the proof is complete. □

NOTE. The author has mentioned above that she is unable to determine whether the ideal $\mathbf{K}$ is closed in $A_0^{(n)}$ for $n > 1$. Corollary 4.5 supports the conjecture that $\mathbf{K}$ is closed in $A_0^{(n)}$ for all positive integers n.

Corollary 4.5 together with Lemma 4.2 shows that, if $f \in L^1((\mathfrak{R}^+)^n)$, then $f \in I$ if and only if $\mathfrak{L}(f) \in \bar{\mathbf{K}}$. Thus, the Laplace transform $\mathfrak{L}$ induces a continuous monomorphism $\tilde{\mathfrak{L}}$ from $L^1[0,1]^n$ into $A_0^{(n)}/\bar{\mathbf{K}}$. If Q is the map from $L^1[0,1]^n$ into $L^1((\mathfrak{R}^+)^n)$ defined by

$$Qf(x) = \begin{cases} f(x) & x \in [0,1]^n, \\ 0 & \text{otherwise} \end{cases}$$

and ν is the canonical quotient map from $A_0^{(n)}$ to $A_0^{(n)}/\bar{\mathbf{K}}$, then we have

$$\tilde{\mathfrak{L}} = \nu \circ \mathfrak{L} \circ Q.$$

Now we are ready to prove our main result, which shows that $\tilde{\mathfrak{L}}$ implements a bijection between the closed ideals in $L^1[0,1]^n$ and the closed ideals in $A_0^{(n)}/\bar{\mathbf{K}}$.

THEOREM 4.6.
(i) *The map* $\mathbf{J} \mapsto \tilde{\mathfrak{L}}^{-1}(\mathbf{J})$ *is a bijection from the set of all closed ideals in* $A_0^{(n)}/\bar{\mathbf{K}}$ *onto the set of all closed ideals in* $L^1[0,1]^n$.
(ii) *A function* $f \in L^1[0,1]^n$ *generates a dense ideal in* $L^1[0,1]^n$ *if and only if* $\tilde{\mathfrak{L}}f$ *generates a dense ideal in* $A_0^{(n)}/\bar{\mathbf{K}}$.

Proof. Let $B = L^1[0,1]^n$, $A = A_0^{(n)}/\bar{\mathbf{K}}$, and $\tilde{\mathfrak{L}}$ be the monomorphism from B into A defined above. We show that conditions (i) and (ii) of Theorem 4.1 are satisfied.

Condition (i) is just a consequence of the well-known fact that $\mathfrak{L}(L^1((\mathfrak{R}^+)^n))$ is dense in $A_0^{(n)}$.

Let $\{e_m\}_{m=1}^\infty$ be any bounded approximate identity for $L^1((\mathfrak{R}^+)^n)$ (say $e_m = m^n \cdot \chi_{[0,1/m]^n}$). By Fact 1, we can find $w_m \in L^1((\mathfrak{R}^+)^n)$ such that

$$\|u^{\star 2} \star w_m - e_m\| < 1/m \quad (m = 1, 2, \ldots)$$

so that $(u^{\star 2} \star w_m)$ is another bounded approximate identity for $L^1((\mathfrak{R}^+)^n)$. Now let P denote the projection map from $L^1((\mathfrak{R}^+)^n)$ to $L^1[0,1]^n$. Then $(P(u)^{\tilde{\star} 2} \tilde{\star} P(w_m))$ is a bounded approximate identity for $L^1[0,1]^n$. Denote again by ν the quotient map from $A_0^{(n)}$ to $A_0^{(n)}/\bar{\mathbf{K}}$. We need to see that, for each $\varphi \in A_0^{(n)}$ and each positive integer m,

$$\nu(\varphi) \cdot \tilde{\mathfrak{L}}(P(u)^{\tilde{\star} 2}) \cdot \tilde{\mathfrak{L}}(P(w_m)) \in \tilde{\mathfrak{L}}(L^1[0,1]^n).$$

But, by Lemma 4.4, if $\varphi \in A_0^{(n)}$ then

$$\varphi \cdot \mathfrak{L}(u)^2 \cdot \mathfrak{L}(w_n) - \mathfrak{L}(P(\mathfrak{L}^{-1}(\varphi \cdot \mathfrak{L}(u)^2 \cdot \mathfrak{L}(w_m)))) \in \bar{\mathbf{K}}$$

so

$$\begin{aligned} \tilde{\mathfrak{L}}(P(\mathfrak{L}^{-1}(\varphi \cdot \mathfrak{L}(u)^2 \cdot \mathfrak{L}(w_m)))) &= \nu(\varphi \cdot \mathfrak{L}(u)^2 \cdot \mathfrak{L}(w_m)) \\ &= \nu(\varphi) \cdot \tilde{\mathfrak{L}}(u)^2 \cdot \tilde{\mathfrak{L}}(w_m) \end{aligned}$$

so that $\nu(\varphi) \cdot \tilde{\mathfrak{L}}(u)^2 \cdot \tilde{\mathfrak{L}}(w_m) \in \tilde{\mathfrak{L}}(L^1[0,1]^n)$. Hence, $P(u)^{\tilde{\star}2} \,\tilde{\star}\, P(w_m)$ satisfies condition (ii) and Theorem 4.6 follows from Theorem 4.1. □

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