A NOTE ON DISJOINT INVARIANT SUBSPACES

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Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}$ be a subspace of $\mathcal{L}(\mathcal{H})$. We denote by $Q_{\mathcal{B}}$ the weak* dual of $\mathcal{B}$, that is, the elements ϕ of $Q_{\mathcal{B}}$ are linear functionals on $\mathcal{B}$ which are continuous in the induced weak* topology of $\mathcal{B}$. If h and k are in $\mathcal{H}$ we can define an element $[h\otimes k]\in Q_{\mathcal{B}}$ by

$$[h\otimes k](T)=\langle Th,k\rangle,\quad T\in\mathcal{B},$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathcal{H}$. The notation $[h\otimes k]$ is justified by the fact that $Q_{\mathcal{B}}$ can also be regarded as a quotient of the space $\mathcal{C}_1(\mathcal{H})$ of trace-class operators on $\mathcal{H}$, and in this case $[h\otimes k]$ is the equivalence class of the rank-one operator $h\otimes k$. Observe that $\mathcal{C}_1(\mathcal{H})$ is a separable Banach space, and we deduce that $Q_{\mathcal{B}}$ is also a separable Banach space.

Fix an integer $n\geq 1$, and recall from [1] that $\mathcal{B}$ is said to have property $(\mathbf{A}_n^{\sim})$ if the following holds. Given $\epsilon>0$ there exists $\delta=\delta(n,\epsilon)>0$ such that for every array $\{\phi_{ij}:1\leq i,j\leq n\}\subset Q_{\mathcal{B}}$ and every family $\{h_1,h_2,\dots,h_n,k_1,k_2,\dots,k_n\}\subset\mathcal{H}$ satisfying the inequalities

$$\|\phi_{ij}-[h_i\otimes k_j]\|<\delta,\quad 1\leq i,j\leq n,$$

we can find $\{h_1',h_2',h_3',\dots,h_n',k_1',k_2',\dots,k_n'\}\subset\mathcal{H}$ such that

$$\phi_{ij}=[h_i'\otimes k_j'],\quad 1\leq i,j\leq n,$$

and

$$\|h_i'-h_i\|<\epsilon,\quad \|k_j'-k_j\|<\epsilon,\quad 1\leq i,j\leq n.$$

Suppose that $\mathcal{B}$ has property $(\mathbf{A}_n^{\sim})$, and denote $\delta_n=\delta(n,1)$. It is then easy to see that we may choose $\delta(n,\epsilon)=\epsilon^2\delta_n$ for all $\epsilon>0$.

Let us also recall that $\mathcal{B}$ is said to have property $(\mathbf{A}_{\aleph_0})$ if every array $\{\phi_{ij}: 1\leq i,j<\aleph_0\}$ in $Q_{\mathcal{B}}$ can be written as $\phi_{ij}=[h_i\otimes k_j]$, $1\leq i,j\leq\aleph_0$, for some families $\{h_i:1\leq i<\aleph_0\}$ and $\{k_j:1\leq j<\aleph_0\}$ of vectors in $\mathcal{H}$.

The main purpose of this paper is to prove that a nonzero space $\mathcal{B}$ that has property $(\mathbf{A}_n^{\sim})$ for every positive integer n possesses many disjoint cyclic spaces, that is, there exist $x,y\in\mathcal{H}$ such that $(\mathcal{B}x)^-\neq\{0\}\neq(\mathcal{B}y)^-$ but $(\mathcal{B}x)^-\cap(\mathcal{B}y)^-=\{0\}$. This result is obtained from an auxiliary result which shows that $\mathcal{B}$ has a property stronger than $(\mathbf{A}_{\aleph_0})$; a form of this stronger property was first used by Brown [5] to show the existence of full analytic subspaces for certain operators.

Let us note that our results imply that operators T in the class $\mathbf{A}_{\aleph_0}$ defined in [2], and hence operators in the class (BCP), have disjoint invariant subspaces. Indeed, it is known from [4] that the weak* closed algebra $\mathcal{A}_T$ generated by an

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operator T in $\mathbf{A}_{\aleph_0}$ has property $(\mathbf{A}_n^{\sim})$ for every integer n. Thus we answer, in particular, a problem posed in [3].

The problem of disjoint invariant subspaces for subnormal operators has been studied in great detail by Olin and Thomson in [6] and [7]. In their terminology, we show that operators in the class $\mathbf{A}_{\aleph_0}$ are cellular decomposable.

We begin with a factorization theorem, related to Theorem 1 in [5].

THEOREM 1. *Let $\mathfrak{B} \subset \mathfrak{L}(\mathfrak{H})$ be a subspace which has property $(\mathbf{A}_n^{\sim})$ for every positive integer n. Assume that $\mathfrak{D} \subset Q_{\mathfrak{B}}$ is a dense set. For every array $\{\phi_{ij}: 1 \le i, j < \infty\}$ in $Q_{\mathfrak{B}}$ there exist sequences $\{x_i: 1 \le i < \infty\}$, $\{y_j: 1 \le j < \infty\}$, and $\{z_j: 1 \le j < \infty\}$ with the following properties:*

(i) $[x_i \otimes y_j] = \phi_{ij}$, $i, j \ge 1$;

(ii) $[x_i \otimes z_j] \in \mathfrak{D}$, $i, j \ge 1$; *and*

(iii) $\{z_j: 1 \le j < \infty\}$ *is dense in* $\mathfrak{H}$.

Proof. Let $\delta_n = \delta(n, 1)$ be derived, as before, from property $(\mathbf{A}_n^{\sim})$. Choose inductively positive numbers $a_1, a_2, \ldots$, satisfying the inequalities

$$a_i a_j \|\phi_{ij}\| < 4^{-n}\delta_{2n}, \quad n = \max(i, j), \quad i, j \ge 1.$$

Fix also a sequence $\{f_j: 1 \le j < \infty\}$ dense in $\mathfrak{H}$. We will determine inductively (on n) sequences of vectors $\{h_i^{(n)}: 1 \le i < \infty\}$, $\{k_j^{(n)}: 1 \le j < \infty\}$, $\{f_j^{(n)}: 1 \le j < \infty\}$ in $\mathfrak{H}$, and arrays $\{\psi_{ij}^{(n)}: 1 \le i, j < \infty\} \subset \mathfrak{D}$ satisfying the following properties for $n = 0, 1, 2, \ldots$:

$$h_i^{(n)} = k_j^{(n)} = 0, \quad i, j > n; \tag{2}$$

$$f_j^{(n)} = f_j, \quad j > n; \tag{3}$$

$$\psi_{ij}^{(n)} = 0 \quad \text{if } \min(i, j) > n; \tag{4}$$

$$[h_i^{(n)} \otimes k_j^{(n)}] = a_i a_j \phi_{ij}, \quad 1 \le i, j \le n; \tag{5}$$

$$[h_i^{(n)} \otimes f_j^{(n)}] = a_i \psi_{ij}^{(n)}, \quad 1 \le i, j \le n; \tag{6}$$

$$\psi_{ij}^{(n+1)} = \psi_{ij}^{(n)}, \quad \max(i, j) \le n; \tag{7}$$

$$\|h_i^{(n+1)} - h_i^{(n)}\| < 2^{-n-1}, \quad i \ge 1; \tag{8}$$

$$\|k_j^{(n+1)} - k_j^{(n)}\| < 2^{-n-1}, \quad j \ge 1; \tag{9}$$

and

$$\|f_j^{(n+1)} - f_j^{(n)}\| < 2^{-n-1}, \quad j \ge 1. \tag{10}$$

For $n = 0$ we have $h_i^{(0)} = k_j^{(0)} = f_j^{(0)} = 0$ and $\psi_{ij}^{(0)} = 0$ for all i and j, and the conditions (5) and (6) are vacuously satisfied. Suppose that for some $n \ge 0$ vectors $h_i^{(n)}$, $k_j^{(n)}$, $f_j^{(n)}$ and elements $\psi_{ij}^{(n)} \in \mathfrak{D}$ have been found satisfying (2)–(6). We can then choose elements $\psi_{i,j}^{(n+1)} \in \mathfrak{D}$, $\max(i, j) = n+1$, such that

$$\begin{aligned} &\|[h_i^{(n)} \otimes f_{n+1}^{(n)}] - a_i \psi_{i,n+1}^{(n+1)}\| < 4^{-n-1}\delta_{2(n+1)}, \quad 1 \le i \le n, \\ &\|a_{n+1}\psi_{n+1,j}^{(n+1)}\| < 4^{-n-1}\delta_{2(n+1)}, \quad 1 \le j \le n+1. \end{aligned} \tag{11}$$

The other values of $\psi_{i,j}^{(n+1)}$ are dictated by (4):

$$\psi_{i,j}^{(n+1)} = \psi_{i,j}^{(n)}, \quad 1 \le i, j \le n,$$
$$\psi_{i,j}^{(n+1)} = 0, \quad \min(i,j) > n+1.$$

Let us note now that we have the inequalities

$$\|[h_i^{(n)} \otimes k_j^{(n)}] - a_i a_j \phi_{ij}\| < 4^{-n-1}\delta_{2(n+1)}, \quad 1 \le i, j \le n+1, \tag{12}$$

$$\|[h_i^{(n)} \otimes f_j^{(n)} - a_i \psi_{ij}\| < 4^{-n-1}\delta_{2(n+1)}, \quad 1 \le i, j \le n+1. \tag{13}$$

Indeed, (12) and (13) follow at once from (5), (6), and (11). It follows now from property $(\mathbf{A}_{2(n+1)}^{\sim})$ (note that $4^{-n-1}\delta_{2(n+1)} = \delta(2(n+1), 2^{-n-1})$) that we can find vectors $h_i^{(n+1)}$, $k_j^{(n+1)}$, and $f_j^{(n+1)}$, $1 \le i, j \le n+1$, satisfying (8), (9), and (10) for $1 \le i, j \le n+1$, and (5) and (6) with n replaced by $n+1$. The values of $h_i^{(n+1)}$, $k_j^{(n+1)}$, and $f_j^{(n+1)}$ are prescribed by (2) and (3) for $i, j \ge n+2$, and the existence of $\{h_i^{(n)} : i \ge 1\}$, $\{k_j^{(n)} : j \ge 1\}$, $\{f_j^{(n)} : j \ge 1\}$, and $\{\psi_{ij}^{(n)} : i, j \ge 1\}$ for every value of n is proved by induction.

Let us set now $h_i = \lim_{n\to\infty} h_i^{(n)}$, $k_j = \lim_{n\to\infty} k_j^{(n)}$, $z_j = \lim_{n\to\infty} f_j^{(n)}$, and $\psi_{ij} = \lim_{n\to\infty} \psi_{ij}^{(n)} = \psi_{ij}^{(i+j)}$ for $i, j \ge 1$. These limits exist by (8), (9), (10), and (7). Note that $[h_i \otimes k_j] = a_i a_j \phi_{ij}$, $[h_i \otimes z_j] = a_i \psi_{ij}$, and

$$\begin{aligned} \|z_j - f_j\| &= \|\lim_{n\to\infty} f_j^{(n)} - f_j^{(j-1)}\| \\ &\le \sum_{k=j-1}^{\infty} \|f_j^{(k+1)} - f_j^{(k)}\| \\ &< \sum_{k=j-1}^{\infty} 2^{-k} = 2^{-j+2} \end{aligned}$$

for $j \ge 1$. These inequalities show that the sequence $\{z_j : j \ge 1\}$ is dense in $\mathcal{H}$. To conclude the proof it suffices now to set $x_i = a_i^{-1} h_i$ and $y_j = a_j^{-1} k_j$ for $i, j \ge 1$. □

We are now ready to prove the main result of this paper. If $\mathcal{B} \subset \mathcal{L}(\mathcal{H})$ is a linear subspace, a space of the form $(\mathcal{B}h)^-$ will be called, somewhat improperly, a cyclic space for $\mathcal{B}$.

THEOREM 14. *Assume that the subspace $\mathcal{B} \subset \mathcal{L}(\mathcal{H})$, $\mathcal{B} \ne \{0\}$, has property $(\mathbf{A}_n^{\sim})$ for every positive integer n. Then there exists a sequence $\{\mathfrak{M}_j : j \ge 1\}$ of nonzero cyclic subspaces for $\mathcal{B}$ with the following property. If $\{K_a : a \in A\}$ is a family of subsets of $\{1, 2, \ldots\}$ such that $\bigcap_{a\in A} K_a = \varnothing$ and at least one of the sets K_a is finite, then $\bigcap_{a\in A} (\bigvee_{j\in K_a} \mathfrak{M}_j) = \{0\}$.*

Proof. As we noted in the introduction, the space $Q_{\mathcal{B}}$ is separable, and hence we can choose a sequence $\mathfrak{D} = \{\phi_j : j \ge 1\}$ dense in $Q_{\mathcal{B}}$. An application of Theorem 1 implies the existence of vectors x_i, y_{jp}, z_j in $\mathcal{H}$, $i, j, p \ge 1$, with the following properties:

$$[x_i \otimes y_{jp}] = \delta_{ij}\phi_p, \quad i, j, p \ge 1; \tag{15}$$

$$\{z_j : j \ge 1\} \text{ is dense in } \mathcal{H}; \tag{16}$$

and

$$[x_i \otimes z_j] = \phi_{\pi(i,j)} \in \mathfrak{D}, \quad i, j \geq 1. \tag{17}$$

Here $\pi: N \times N \to N$ is some function provided by the fact that $[x_i \otimes z_j] \in \mathfrak{D}$, and, of course, $\delta_{ij} = 0$ or 1 according to whether $i \neq j$ or $i = j$. We will show that the spaces $\mathfrak{M}_i = (\mathfrak{B}x_i)^-$, $i \geq 1$, satisfy the conditions of the theorem. First we show that $\mathfrak{M}_i \neq \{0\}$ for each i. Indeed, $\mathfrak{B} \neq \{0\}$, so that there exists $p \geq 1$ with $\phi_p \neq 0$. Choose $T \in \mathfrak{B}$ such that $\phi_p(T) \neq 0$, and note that

$$\langle Tx_i, y_{ip} \rangle = \phi_p(T) \neq 0,$$

and hence $Tx_i \in \mathfrak{M}_i \setminus \{0\}$.

Let now $\{K_a : a \in A\}$ be a family of subsets of $\{1, 2, \ldots\}$ with K_{a_0} finite and $\bigcap_{a \in A} K_a = \varnothing$. Fix an element $x \in \bigcap_{a \in A} (\bigvee_{i \in K_a} \mathfrak{M}_i)$; to finish the proof it will suffice to show that $x = 0$. We show first that

$$\langle x, y_{jp} \rangle = 0, \quad j, p \geq 1. \tag{18}$$

Indeed, fix j, and choose $a_1 \in A$ with $j \notin K_{a_1}$. Since $x \in \bigvee_{i \in K_{a_1}} \mathfrak{M}_i$, there exist operators $T_i^{(n)}$ in $\mathfrak{B}$ such that for each n, $T_i^{(n)} \neq 0$ for finitely many values of i, and

$$x = \lim_{n \to \infty} \sum_{i \in K_{a_1}} T_i^{(n)} x_i.$$

We deduce from (15) that

$$\langle x, y_{jp} \rangle = \lim_{n \to \infty} \sum_{i \in K_{a_1}} [x_i \otimes y_{jp}](T_i^{(n)}) = \lim_{n \to \infty} \sum_{i \in K_{a_1}} \delta_{ij} \phi_p(T_i^{(n)}) = 0$$

for all $p \geq 1$.

Use now the fact that $x \in \bigvee_{i \in K_{a_0}} \mathfrak{M}_i$ to write

$$x = \lim_{n \to \infty} \sum_{i \in K_{a_0}} T_i^{(n)} x_i.$$

For $j \in K_{a_0}$ and $p \geq 1$ we deduce from (18) that

$$0 = \langle x, y_{jp} \rangle = \lim_{n \to \infty} \sum_{i \in K_{a_0}} \delta_{ij} \phi_p(T_i^{(n)}) = \lim_{n \to \infty} \phi_p(T_j^{(n)}). \tag{19}$$

Suppose that $x \neq 0$, and choose k such that $\|x - z_k\| < \frac{1}{2}\|x\|$; this is possible by (16). We have

$$|\langle x, z_k \rangle| = |\langle x, x \rangle - \langle x, x - z_k \rangle| \geq \|x\|^2 - \|x\| \, \|x - z_k\| > 0$$

but, on the other hand,

$$\begin{aligned} \langle x, z_k \rangle &= \lim_{n \to \infty} \sum_{i \in K_{a_0}} \langle T_i^{(n)} x, z_k \rangle \\ &= \lim_{n \to \infty} \sum_{i \in K_{a_0}} [x_i \otimes z_k](T_i^{(n)}) \\ &= \lim_{n \to \infty} \sum_{i \in K_{a_0}} \phi_{\pi(i,k)}(T_i^{(n)}) = 0 \end{aligned}$$

by (17) and (19). This contradiction shows that necessarily $x = 0$. The theorem is proved. □

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